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Konstantinos Gkikas, Laurent Veron

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To Patrizia Pucci, with friendship and high esteem

Abstract. Let V be a nonnegative locally bounded function defined in $Q_\infty := \mathbb{R}^n \times (0,\infty)$. We study under what conditions on V and on a Radon measure μ in $\mathbb{R}^d$ does it exist a function which satisfies $\partial_t u - \Delta u + Vu = 0$ in Q_∞ and $u(.,0) = \mu$. We prove the existence of a subcritical case in which any measure is admissible and a supercritical case where capacitary conditions are needed. We obtain a general representation theorem of positive solutions when $tV(x,t)$ is bounded and we prove the existence of an initial trace in the class of outer regular Borel measures.

Contents

1. Introduction

In this article we study the initial value problem for the heat equation

$$\begin{array}{rl} \partial_t u - \Delta u + V(x,t)u = 0 & \text{in } Q_T := \mathbb{R}^n \times (0,T) \\ u(.,0) = \mu & \text{in } \mathbb{R}^n, \end{array} \tag{1.1}$$

where $V \in L^\infty_{loc}(Q_T)$ is a nonnegative function and μ a Radon measure in $\mathbb{R}^n$. By a (weak) solution of (1.1) we mean a function $u \in L^1_{loc}(\overline{Q}_T)$ such that $Vu \in L^1_{loc}(\overline{Q}_T)$, satisfying

$$-\iint_{Q_T} (\partial_t \phi + \Delta\phi)\, u dx dt + \iint_{Q_T} Vu\phi dx dt = \int_\Omega \zeta d\mu \tag{1.2}$$

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for every function $\zeta \in C_c^{1,1;1}(\overline{Q}_T)$ which vanishes for $t = T$. Besides the singularity of the potential at $t = 0$, there are two main difficulties which appear for constructing weak solutions : the growth of the measure at infinity and the concentration of the measure near some points in $\mathbb{R}^n$. Diffusion equations with singular potentials depending only on x have been studied in connection with the stationnary equation (see e.g. [**13**]). The particular case of Hardy's potentials $v(x,t) = c|x|^{-2}$ has been thorougly investigated since the early work of Baras and Goldstein [**5**], in connection with the problem of instantaneous blow-up. For time dependent singular potentials most of the works are concentrated on the well posedness and the existence of a maximum principle; this is the case if $V \in L_t^\infty L_x^{\frac{n}{2},\infty}$, see e.g. [**16**]. In the case of time-singular potentials, a notion of non-autonomous Kato class have been introduced in [**18**] in order to prove that the evolution problem associated to the equation is well posed in $L^1(\mathbb{R}^n)$. This class is the extension to diffusion equations of the Kato's class in Schrödinger operators. Other studies have been performed by probabilistic methods in order to analyze the $L^p - L^q$ regularizing effect [**12**]. To our knowledge, no work dealing with the initial value problems with measure data for singular operators has already been published. We present here an extension to evolution equations of a series of questions raised and solved in the case of Schrödinger stationary equations in particular by [**2**], [**3**], [**19**], having in mind that one of the aim of this present work is to develop a framework adapted to the construction of the precise trace of solutions of semilinear heat equations. This aspect will appear in a forthcoming work [**11**]).

We denote by $H(x,t) = \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ the Gaussian kernel in $\mathbb{R}^n$ and by $\mathbb{H}[\mu]$ the corresponding heat potential of a measure $\mu \in \mathfrak{M}(\mathbb{R}^n)$. Thus

$$\mathbb{H}[\mu](x,t) = \left(\tfrac{1}{4\pi t}\right)^{\frac{n}{2}} \int e^{-\frac{|x-y|^2}{4t}} d\mu(y), \tag{1.3}$$

whenever this expression has a meaning: for example it is straightforward that if $\mu \in \mathfrak{M}(\mathbb{R}^n)$ satisfies

$$\|\mu\|_{\mathfrak{M}_T} := \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4T}} d|\mu|(y) < \infty, \tag{1.4}$$

then (1.3) has a meaning as long as $t < T$, and let be $\mathfrak{M}_T(\mathbb{R}^n)$ the set of Radon measures in $\mathbb{R}^n$ satisfying (1.4). If $G \subset \mathbb{R}^n$, let Q_T^G be the cylinder $G \times (0,T)$, $B_R(x)$ the ball of center x and radius R and $B_R = B_R(0)$. We prove

Theorem A *Let the measure μ verifies*

$$\iint_{Q_T^{B_R}} \mathbb{H}[|\mu|](x,t) V(x,t) dxdt \leq M_R \qquad \forall R > 0. \tag{1.5}$$

Then (1.1) admits a solution in Q_T.

A measure which satisfies (1.5) is called an *admissible measure* and a measure for which there exists a solution to problem (1.1) is called a *good measure*. Notice that even when $V = 0$, uniqueness without any restriction on u is not true, however the next uniqueness result holds:

Theorem B *Let u be a weak solution of (1.1) with $\mu = 0$. If u satisfies*

$$\iint_{Q_T} (1 + V(x,t))\, e^{-\lambda|x|^2} |u(x,t)| dxdt < \infty \tag{1.6}$$

for some $\lambda > 0$, then $u = 0$.

We denote by $\mathcal{E}_\nu(Q_T)$ the set of functions $u \in L^1_{loc}(Q_T)$ for which (1.6) holds for some $\lambda > 0$. The general result we prove is the following.

Theorem C *Let $\mu \in \mathfrak{M}(\mathbb{R}^n)$ be an admissible measure satisfying (1.4). Then there exists a unique solution $u_\mu \in \mathcal{E}_\nu(Q_T)$ to problem (1.1). Furthermore*

$$\iint_{Q_T} \left(\frac{n}{2T} + V\right) |u| e^{-\frac{|x|^2}{4(T-t)}} dxdt \le \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4T}} d|\mu|(y). \tag{1.7}$$

We consider first the *subcritical case*, which means that any positive measure satisfying (1.4) is a good measure and we prove that such is the case if for any $R > 0$ there exist $m_R > 0$ such that

$$\iint_{Q_T^{B_R}} H(x-y,t)V(x,t)dxdt \le m_R e^{-\frac{|y|^2}{4T}}. \tag{1.8}$$

Moreover we prove a stability result among the measures satisfying (1.4): if V verifies for all $R > 0$

$$\sup_{y\in\mathbb{R}^n} e^{\frac{|y|^2}{4T}} \iint_E H(x-y,t)V(x,t)dxdt \to 0 \quad \text{when } |E| \to 0\,,\ E \text{ Borel subset of } Q_T^{B_R}, \tag{1.9}$$

then if $\{\mu_k\}$ is a sequence of Radon measures bounded in $\mathfrak{M}_{_T}(\mathbb{R}^n)$ which converges in the weak sense of measures to μ, then $\{(u_{\mu_k}, Vu_{\mu_k})\}$ converges to (u_μ, Vu_μ) in $L^1_{loc}(\overline{Q}_T)$.

In the *supercritical case*, that is when not all measure in $\mathfrak{M}_{_T}(\mathbb{R}^n)$ is a good measure, we develop a capacitary framework in order to characterize the good measures. We denote by $\mathfrak{M}^V(\mathbb{R}^n)$ the set of Radon measures such that $V\mathbb{H}[\mu] \in L^1(Q_T)$ and $\|\mu\|_{\mathfrak{M}^V} := \|V\mathbb{H}[\mu]\|_{L^1}$. If $E \subset Q_T$ is a Borel set, we set

$$C_V(E) = \sup\{\mu(E) : \mu \in \mathfrak{M}^V_+(\mathbb{R}^n), \mu(E^c) = 0, \|\mu\|_{\mathfrak{M}^V} \le 1\}. \tag{1.10}$$

This defines a capacity. If

$$C^*_V(E) = \inf\{\|f\|_{L^\infty} : \check{H}[f](y) \ge 1 \quad \forall y \in E\}, \tag{1.11}$$

where

$$\check{H}[f](y) = \iint_{Q_T} H(x-y,t)V(x,t)f(x,t)dxdt = \int_0^T \mathbb{H}[Vf](y,t)dt \qquad \forall y \in \mathbb{R}^n, \tag{1.12}$$

then $C^*_V(E) = C_V(E)$ for any compact set. Denote by Z_V *the singular set of* V, that is the largest set with zero C_V capacity. Then

$$Z_V = \{x \in \mathbb{R}^n : \iint_{Q_T} H(x-y,t)V(y,t)dxdt = \infty\}, \tag{1.13}$$

and the following result characterizes the good measures.

Theorem D *If μ is an admissible measure then $\mu(Z_V) = 0$. If $\mu \in \mathfrak{M}_{_T}(\mathbb{R}^n)$ satisfies $\mu(Z_V) = 0$, then it is a good measure. Furthermore μ is a positive good measure if and only if there exists an increasing sequence of positive admissible measures $\{\mu_k\}$ which converges to μ in the weak $*$ topology.*

Since many important applications deal with the nonlinear equation

$$\partial_t u - \Delta u + |u|^{q-1}u = 0 \qquad \text{in } Q_\infty := \mathbb{R}^n \times (0,\infty), \tag{1.14}$$

where $q > 1$ and due to the fact that *any* solution defined in Q_∞ satisfies

$$|u(x,t)|^{q-1} \leq \frac{1}{t(q-1)} \qquad \forall (x,t) \in Q_\infty, \tag{1.15}$$

we shall concentrate on potentials V which satisfy

$$0 \leq V(x,t) \leq \frac{C_1}{t} \qquad \forall (x,t) \in Q_T, \tag{1.16}$$

for some $C_1 > 0$. For such potentials we prove the existence of a representation theorem for positive solutions of

$$\partial_t u - \Delta u + V(x,t)u = 0 \qquad \text{in } Q_T. \tag{1.17}$$

If u is a positive solution of (1.1) in Q_T with $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$, it is the increasing limit of the solutions $u = u_R$ of

$$\begin{array}{rl} \partial_t u - \Delta u + V(x,t)u = 0 & \text{in } Q_T^{B_R} \\ u = 0 & \text{in } \partial B_R \times (0,T) \\ u(.,0) = \chi_{B_R}\mu & \text{in } B_R, \end{array} \tag{1.18}$$

when $R \to \infty$, thus there exists a positive function $H_V \in C(\mathbb{R}^n \times \mathbb{R}^n \times (0,T))$ such that

$$u(x,t) = \int_{\mathbb{R}^N} H_V(x,y,t) d\mu(y). \tag{1.19}$$

Furthermore we show how to construct H_V from V and we prove the following formula

$$H_V(x,y,t) = \int_{\mathbb{R}^N} e^{\psi(x,t)} \Gamma(x,\xi,t) d\mu_y(\xi), \tag{1.20}$$

where μ_y is a Radon measure such that

$$\delta_y \geq \mu_y, \tag{1.21}$$

(δ_y is the Dirac measure concentrated at y),

$$\psi(x,t) = \int_t^T \int_{\mathbb{R}^n} \left(\frac{1}{4\pi(s-t)} \right)^{\frac{n}{2}} e^{-\frac{|x-y|^2}{4(s-t)}} V(y,s) dy ds \tag{1.22}$$

and Γ satisfies the following estimate

$$c_1 t^{-\frac{n}{2}} e^{-\gamma_1 \frac{|x-y|^2}{t}} \leq \Gamma(x,y,t) \leq c_2 t^{-\frac{n}{2}} e^{-\gamma_2 \frac{|x-y|^2}{t}} \tag{1.23}$$

where A_i, c_i depends on T, d and V. Conversely, we first prove the following representation result

Theorem E *Assume V satisfies (1.16). If u is a positive solution of (1.1) in Q_T, there exists a positive Radon measure μ in $\mathbb{R}^n$ such that (1.19) holds.*

If $\mu \in \mathfrak{M}_{_T}(\mathbb{R}^n)$ is positive, we can define for any $k > 0$ the solution u_k of

$$\begin{array}{rl} \partial_t u - \Delta u + V_k(x,t)u = 0 & \text{in } Q_T \\ u(.,0) = \mu & \text{in } \mathbb{R}^n, \end{array} \tag{1.24}$$

where $V_k(x,t) = \min\{k, V(x,t)\}$, and

$$u_k(x,t) = \int_{\mathbb{R}^N} H_{V_k}(x,y,t) d\mu(y). \tag{1.25}$$

Moreover $\{H_{V_k}\}$ and $\{v_k\}$ decrease respectively to H_V and u^* there holds

$$u^*(x,t) = \int_{\mathbb{R}^N} H_V(x,y,t)d\mu(y). \tag{1.26}$$

However u^* is not a solution of (1.1), but of a relaxed problem where μ is replaced by a smaller measure μ^* called *the reduced measure associated to* μ. If we define the *zero set of* V by

$$\mathcal{S}ing_V := \{y \in \mathbb{R}^N : H_V(x,y,t) = 0\}, \tag{1.27}$$

we prove

Theorem F *If*

$$\limsup_{t\to 0} \int_t^T \int_{\mathbb{R}^n} \left(\frac{1}{4\pi(s-t)}\right)^{\frac{n}{2}} e^{-\frac{|\xi-y|^2}{4(s-t)}} V(y,s)dyds = \infty, \tag{1.28}$$

then

$$\xi \in \mathcal{S}ing_V, \text{ i.e. } H_V(x,\xi,t) = 0, \ \forall\ (x,t) \in \mathbb{R}^n \times (0,\infty).$$

We note here that if V satisfies (1.28) then δ_ξ is not admissible measure and the reduced measure $(\delta_\xi)^* = \mu_\xi$ associated to δ_ξ is zero.

Theorem G *Assume V satisfies (1.15) and $\mu \in \mathfrak{M}_{_T}(\mathbb{R}^n)$. Then*

(i) $supp(\mu - \mu^*) \subset \mathcal{S}ing_V$.

(ii) If $\mu(\mathcal{S}ing_V) = 0$, then $\mu^* = 0$.

(iii) $\mathcal{S}ing_V = Z_V$.

The last section is devoted to the initial trace problem: to any positive solution u of (1.1) we can associate an open subset $\mathcal{R}(u) \subset \mathbb{R}^n$ which is the set of points y which possesses a neighborhood U such that

$$\iint_{Q_T^U} V(x,t)u(x,t)dxdt < \infty. \tag{1.29}$$

There exists a positive Radon measure μ_u on $\mathcal{R}(u)$ such that

$$\lim_{t\to 0} \int_{\mathbb{R}^n} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^n} \zeta d\mu \qquad \forall \zeta \in C_c(\mathcal{R}(u)). \tag{1.30}$$

The set $\mathcal{S}(u) = \mathbb{R}^n \setminus \mathcal{R}(u)$ is the set of points y such that for any open set U containing y, there holds

$$\iint_{Q_T^U} V(x,t)u(x,t)dxdt = \infty. \tag{1.31}$$

If V satisfies (1.17), $\mathcal{S}(u)$ it has the property that

$$\limsup_{t\to 0} \int_U u(x,t)dx = \infty. \tag{1.32}$$

Furthermore, if is satisfies (1.9), then $\mathcal{S}(u) = \emptyset$.

An alternative construction of the initial trace based on the sweeping method is also developed.

Precise definitions of the different notions used in the introduction will be given in the next sections.

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2. The subcritical case

Let $Q_T = \mathbb{R}^n \times (0,T]$. In this section we consider the linear parabolic problem

$$\begin{array}{rll} \partial_t u - \Delta u + Vu = 0 & \text{in} & Q_T \\ u(.,0) = \mu & \text{in} & \mathbb{R}^n \times \{0\}, \end{array} \tag{2.1}$$

where $V \in L^1_{loc}(Q_T)$ is nonnegative and μ is a Radon measure.

Definition 2.1. *We say that $\mu \in \mathfrak{M}(\mathbb{R}^n)$ is a good measure if problem (2.1) has a weak solution u i.e. there exists a function $u \in L^1_{loc}(\overline{Q}_T)$, such that $Vu \in L^1_{loc}(\overline{Q}_T)$ which satisfies*

$$-\iint_{Q_T} u(\partial_t\phi + \Delta\phi)dxdt + \iint_{Q_T} Vu\phi dxdt = \int_{\mathbb{R}^n} \phi(x,0)d\mu \quad \forall \phi \in X(Q_T), \tag{2.2}$$

where $X(Q_T)$ is the space of test functions defined by

$$X(Q_T) = \{\phi \in C_c(\overline{Q}_T),\ \partial_t\phi + \Delta\phi \in L^\infty_{loc}(\overline{Q}_\infty),\ \phi(x,T) = 0\}$$

Definition 2.2. *Let $H(x,t)$ be the heat kernel of heat equation in $\mathbb{R}^n$, we say that $\mu \in \mathfrak{M}(\mathbb{R}^n)$ is an admissible measure if*
(i)

$$||V\mathbb{H}[|\mu|]||_{L^1(Q_T^{B_R})} = \iint_{Q_T^{B_R}} \left(\int_{\mathbb{R}^n} H(x-y,t)d|\mu(y)|\right) V(x,t)dxdt < M_{R,T}$$

where $M_{R,T}$ is a positive constant.

Definition 2.3. *A function $u(x,t)$ will be said to belong to the class $\mathcal{E}_{\mathcal{V}}(Q_T)$ if there exists $\lambda > 0$ such that*

$$\iint_{Q_T} e^{-\lambda|x|^2}|u(x,t)|(1+V(x,t))dxdt < \infty.$$

A measure in $\mathbb{R}^n$ belongs to the class $\mathfrak{M}_{_T}(\mathbb{R}^n)$ if

$$\|\mu\|_{\mathfrak{M}_T} := \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4T}} d|\mu| < \infty.$$

Lemma 2.4. *There exists at most one weak solution of problem (2.1) in the class $\mathcal{E}_{\mathcal{V}}(Q_T)$.*

Proof. Let u_1 and u_2 be two solutions in the class $\mathcal{E}_{\mathcal{V}}(Q_T)$ then $w = u_1 - u_2$ is a solution with initial data 0. Choose a standard mollifier $\rho : B(0,1) \mapsto [0,1]$ and define

$$w_j(x,t) = j^n \int_{B_{\frac{1}{j}}(x)} \rho(j(x-y))w(y,t)dy \equiv \int_{B_{\frac{1}{j}}(x)} \rho_j(x-y)w(y,t)dy.$$

Then $w_j(.,t)$ is C^∞ and from the equation satisfied by w, it holds

$$\partial_t w_j - \Delta w_j + \int_{B_{\frac{1}{j}}(x)} V(y,t)\rho_j(x-y)w(y)dy = 0,$$

where $\partial_t w_j$ is taken in the weak sense.
First we consider the case $\lambda > 0$ and $t \le \min\{\frac{1}{16\lambda}, T\} = T'$.

Set $\phi(x,t) = \xi(x,t)\zeta(x)$, where $\xi(x,t) = e^{-\frac{|x|^2}{4(\frac{1}{8\lambda}-t)}}$ and $\zeta \in C_c^\infty(\mathbb{R}^n)$. Given $\varepsilon > 0$ we define

$$g_j = \sqrt{w_j^2 + \varepsilon}.$$

Because $\partial_t(g_j\phi) = \frac{\partial_t w_j}{\sqrt{w_j^2+\varepsilon}}\phi + g_j\partial_t\phi,$, by a straightforward calculation we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\Big[g_j\phi(.,s)\Big]_{s=t}^{s=0}dx &= \iint_{Q_t}\frac{w_j}{\sqrt{w_j^2+\varepsilon}}\phi\Delta w_j dxds \\
&- \iint_{Q_t}\frac{w_j(x,s)}{\sqrt{w_j^2(x,s)+\varepsilon}}\phi(x,s)\left(\int_{B_{\frac{1}{j}}(x)}V(y,t)\rho_j(x-y)w(y,s)dy\right)dxds \\
&+ \iint_{Q_t} g_j\phi_s dxds \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

By integration by parts, we obtain

$$\begin{aligned}
I_1 &= -\iint_{Q_t}\frac{|\nabla w_j|^2}{\sqrt{w_j^2(x,s)+\varepsilon}}\phi dxds + \iint_{Q_t}\frac{|\nabla w_j|^2 w_j^2}{(w_j^2(x,s)+\varepsilon)^{\frac{3}{2}}}\phi dxds - \iint_{Q_t}\frac{w_j}{\sqrt{w_j^2+\varepsilon}}\nabla w_j.\nabla\phi dxds \\
&\le -\iint_{Q_t}\frac{w_j}{\sqrt{w_j^2+\varepsilon}}\nabla w_j.\nabla\phi dxds \\
&\le -\iint_{Q_t}\nabla g_j.\nabla\phi dxds \\
&= -\iint_{Q_t}\zeta\nabla g_j.\nabla\xi dxds - \iint_{Q_t}\xi\nabla g_j.\nabla\zeta dxds \\
&= \iint_{Q_t}\zeta g_j\Delta\xi dxds + \iint_{Q_t} g_j\nabla\zeta.\nabla\xi dxds.
\end{aligned}$$

Since $t \le T$, there holds $\xi|\nabla g_j| \in L^1(Q_{T'})$, $\xi g_j \in L^1(Q_{T'})$, $|\Delta\xi|g_j \in L^1(Q_{T'})$, $\partial_s\xi g_j \in L^1(Q_{T'})$ and

$$\iint_{Q_t}\frac{w_j(x,s)}{\sqrt{w_j^2(x,s)+\varepsilon}}\left(\int_{B_{\frac{1}{j}}(x)}V(y,t)\rho_j(x-y)w(y,s)dy\right)\xi dxds < \infty.$$

The reason for which $\xi|\nabla g_j| \in L^1(Q_{T'})$ follows from the next inequality

$$\begin{aligned}
\iint_{Q_{T'}}|\nabla g_j|\xi dxds &= \iint_{Q_{T'}}\frac{|\nabla w_j|}{\sqrt{\varepsilon+w_j^2}}\xi dxds \\
&\le \iint_{Q_{T'}} e^{-\frac{|x|^2}{4(\frac{1}{8\lambda}-t)}}\left(\int_{B_{\frac{1}{j}}(x)}|\nabla\rho_j(x-y)|w(y,s)dy\right)dxds.
\end{aligned}$$

Since $\forall y \in B_{\frac{1}{j}}(x)$, we have $|x|^2 \ge (|y|^2 - \frac{1}{j})^2 = |y|^2 + \frac{1}{j^2} - 2\frac{|y|}{j} \ge \frac{|y|^2}{2} - (C-1)\frac{1}{j^2}$, for some positive constant $C > 0$ independent on j, y and x. Thus we have, using

the fact that $e^{-\lambda|y|^2}w \in L^1(Q_T)$,

$$\iint_{Q_{T'}} |\nabla g_j|\xi dxds \leq C(j,\lambda)\iint_{Q_{T'}}\int_{B_{\frac{1}{j}}(x)} e^{-\frac{|y|^2}{8(\frac{1}{8\lambda}-t)}}|\nabla\rho_j(x-y)|w(y,s)dydxds < \infty.$$

Also

$$\iint_{Q_t}\frac{w_j(x,s)}{\sqrt{w_j^2(x,s)+\varepsilon}}\xi\left(\int_{B_{\frac{1}{j}}(x)} V(y,t)\rho_j(x-y)w(y,s)dy\right)dxds \to_{j\to\infty}\iint_{Q_t}\frac{w^2(x,s)}{\sqrt{w^2(x,s)+\varepsilon}}\xi V(y,t)dxds$$

and

$$\int_{\mathbb{R}^n}\sqrt{w_j^2(x,s)+\varepsilon}(\xi_s+\Delta\xi)dxds \to_{j\to\infty}\int_{\mathbb{R}^n}\sqrt{w^2(x,s)+\varepsilon}(\xi_s+\Delta\xi)dxds.$$

We choose $\zeta_R = 1$ in B_R, $0 \leq \zeta_R \leq 1$ in $B_{R+1}\setminus B_R$ and 0 otherwise. Letting successively $j\to\infty$, $R\to\infty$ and finally $\varepsilon\to 0$, we derive

$$\int_{\mathbb{R}^n}|w(x,t)|\xi(x,t)dx \leq \iint_{Q_t}|w|(\xi_s+\Delta\xi)dxds - \iint_{Q_t} w(x,s)\xi V(y,t)dxds.$$

Since

$$\xi_s+\Delta\xi = -\frac{n}{2(\frac{1}{8\lambda}-s)},$$

and $V\geq 0$, we have $w(x,t)=0\ \forall\ (x,t)\in Q_{T'}$. If $T'=T$ this complete the proof for $\lambda\geq 0$, otherwise the proof can be completed by a finite number of interations of the same argument on $\mathrm{R}^n\times(T',2T')$, $\mathrm{R}^n\times(2T',3T')$, etc. If $\lambda=0$ we set $\xi=1$ and the result follows by similar argument □

THEOREM 2.5. *If $\mu\in\mathfrak{M}_{_T}(\mathbb{R}^n)$ is an admissible measure, there exists a unique $u=u_\mu\in\mathcal{E}_{\mathcal{V}}(Q_T)$ solution of (2.1). Furthermore the following estimate holds*

$$\frac{n}{2T}\iint_{Q_T}|u|e^{-\frac{|x|^2}{4(T-t)}}dxds + \iint_{Q_T}|u|Ve^{-\frac{|x|^2}{4(T-t)}}dxds \leq \int_{\mathbb{R}^n}e^{-\frac{|x|^2}{4T}}d|\mu|. \tag{2.3}$$

Proof. First we assume that $\mu\geq 0$. Let $\mu_R=\chi_{B_R}\mu$. It is well known that the heat kernel $H^{B_R}(x,y,t)$ in $\Omega=B_R$ is increasing with respect to R and $H^{B_R}\to H$, as $R\to\infty$ in $L^1(Q_T)$ for any $T>0$. Thus μ_R is an admissible measure in B_R and by Proposition 5.4, there exists a unique weak solution u_R of problem 5.2 on $\Omega=B_R$. By (ii) of Proposition 5.5 we have

$$-\iint_{Q_T}|u_R|(\partial_t\phi+\Delta\phi)dxdt + \iint_{Q_T}|u_R|V\phi dxdt \leq \int_{B_R}\phi(x,0)d|\mu_R|.$$

If we set $\phi_\varepsilon(x,t)=e^{-\frac{|x|^2}{4(T+\varepsilon-t)}};\ \varepsilon>0$, then

$$\partial_t\phi+\Delta\phi = -\frac{n}{2(T+\varepsilon-t)}e^{-\frac{|x|^2}{4(T+\varepsilon-t)}},$$

thus we have

$$\iint_{Q_T}|u_R|\frac{n}{2(T+\varepsilon-t)}e^{-\frac{|x|^2}{4(T+\varepsilon-t)}}dxdt + \iint_{Q_T}|u_R|Ve^{-\frac{|x|^2}{4(T+\varepsilon-t)}}dxdt \leq \int_{B_R}e^{\frac{-|x|^2}{4T+4\varepsilon}}d\mu_R,$$

which implies

$$\frac{n}{2T+\varepsilon}\int_0^T\int_{B_R}|u_R|e^{-\frac{|x|^2}{4(T+\varepsilon-t)}}dxdt + \int_0^T\int_{B_R}|u_R|Ve^{-\frac{|x|^2}{4(T+\varepsilon-t)}}dxdt \leq \int_{\mathbb{R}^n}e^{\frac{-|x|^2}{4T+4\varepsilon}}d\mu_R.$$

Letting $\varepsilon \to 0$, we derive

$$\frac{n}{2T}\iint_{Q_T}|u_R|e^{-\frac{|x|^2}{4(T-t)}}dxdt+\iint_{Q_T}|u_R|Ve^{-\frac{|x|^2}{4(T-t)}}dxdt \le \int_{\mathbb{R}^n} e^{\frac{-|x|^2}{4T}}d\mu_R \le \int_{\mathbb{R}^n} e^{\frac{-|x|^2}{4T}}d\mu.$$

Now by the maximum principle $\{u_R\}$ is increasing with respect to R and converges to some function u. By the above inequality $u \in \mathcal{E}_{\mathcal{V}}(Q_T)$ satisfies the estimate (2.5) and u is a weak solution of problem (2.1). By Lemma 2.4 it is unique. In the general case we write $\mu = \mu^+ - \mu^-$ and the result follows by the above arguments and Lemma 2.4. *In the sequel we shall denote by u_μ this unique solution.* □

DEFINITION 2.6. *A potential V is called subcritical in Q_T if for any $R > 0$ there exists $m_R > 0$ such that*

$$\iint_{Q_T^{B_R}} H(x-y,t)V(x,t)dxdt \le m_R e^{-\frac{|y|^2}{4T}} \qquad \forall y \in \mathbb{R}^n. \tag{2.4}$$

It is called strongly subcritical if moreover

$$e^{\frac{|y|^2}{4T}}\iint_E H(x-y,t)V(x,t)dxdt \to 0 \quad \text{when } |E| \to 0\,,\ E \text{ Borel subset of } Q_T^{B_R}, \tag{2.5}$$

uniformly with respect to $y \in \mathbb{R}^n$

THEOREM 2.7. *Assume V is subcritical. Then any measure in $\mathfrak{M}_T(\mathbb{R}^n)$ is admissible. Furthermore, if V is strongly subcritical and $\{\mu_k\}$ is a sequence of measures uniformly bounded in $\mathfrak{M}_T(\mathbb{R}^N)$ which converges weakly to μ, then the corresponding solutions $\{u_{\mu_k}\}$ converge to u_μ in $L^1_{loc}(\overline{Q}_T)$, and $\{Vu_{\mu_k}\}$ converges to Vu_μ in $L^1_{loc}(\overline{Q}_T)$.*

Proof. For the first statement we can assume $\mu \ge 0$ and there holds

$$\begin{aligned}\iint_{Q_T^{B_R}} H(x-y,t)d\mu(y)V(x,t)dxdt &= \int_{\mathbb{R}^n}\left(\iint_{Q_T^{B_R}} H(t,x-y)V(x,t)dxdt\right)d\mu(y)\\ &\le m_R\int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4T}}d\mu(y)\\ &\le m_R\,\|\mu\|_{\mathfrak{M}_T}\,.\end{aligned}$$

Thus μ is admissible. For the second statement, we assume first that $\mu_k \ge 0$. By lower semicontinuity $\mu \in \mathfrak{M}_T(\mathbb{R}^N)$ and $\|V\mathbb{H}[\mu]\|_{L^1(Q_T^{B_R})} \le M_{R,T}$ for any k. Since $0 \le u_{\mu_k} \le \mathbb{H}[\mu_k]$ and $\mathbb{H}[\mu_k] \to \mathbb{H}[\mu]$ in $L^1_{loc}(\overline{Q}_T)$, the sequence $\{u_{\mu_k}\}$ is uniformly integrable and thus relatively compact in $L^1_{loc}(\overline{Q}_T)$. Furthermore $0 \le Vu_{\mu_k} \le V\mathbb{H}[\mu_k]$. Let $E \subset Q_T^{B_R}$ be a Borel subset, then

$$\begin{aligned}\iint_E V\mathbb{H}[\mu_k]dxdt &= \int_{\mathbb{R}^n}\left(\iint_E VH(x-y,t)dxdt\right)d\mu_k(y)\\ &= \int_{\mathbb{R}^n}\left(e^{\frac{|y|^2}{4T}}\iint_E V(x)H(x-y,t)dxdt\right)e^{-\frac{|y|^2}{4T}}d\mu_k(y)\\ &\le \epsilon(|E|)\,\|\mu_k\|_{\mathfrak{M}_T}\,,\end{aligned}$$

where $\epsilon(r) \to 0$ as $r \to 0$. Thus $\{(u_{\mu_k}, Vu_{\mu_k})\}$ is locally compact in $L^1_{loc}(\overline{Q}_T)$ and, using a diagonal sequence, there exist $u \in L^1_{loc}(\overline{Q}_T)$ with $Vu \in L^1_{loc}(\overline{Q}_T)$ and a subsequence $\{k_j\}$ such that $\{(u_{\mu_{k_j}}, Vu_{\mu_{k_j}})\}$ converges to (u_μ, Vu_μ) a.e. and in $L^1_{loc}(\overline{Q}_T)$. From the integral expression (2.2) satisfied by the u_{μ_k}, u is a weak solution of problem (2.1). Since the u_{μ_k} satisfy (2.3), the property holds for u, therefore$u = u_\mu$ is the unique solution of (2.1), which ends the proof. □.

As a variant of the above result which will be useful later on we have

PROPOSITION 2.8. *Assume V satisfies*

(2.6)
$$e^{\frac{|y|^2}{4T}} \iint_E H(x-y,t)V(x,t+\tau)dxdt \to 0 \quad \text{when } |E| \to 0\,,\ E \text{ Borel subset of } Q_T^{B_R},$$

uniformly with respect to $y \in \mathbb{R}^n$ and $\tau \in [0, \tau_0]$. Let $\tau_k > 0$ with $\tau_k \to 0$ and $\{\mu_k\}$ be a sequence uniformly bounded in $\mathfrak{M}_T(\mathbb{R}^N)$ which converges weakly to μ. Then the solutions $\{u_{\tau_k,\mu_k}\}$ of

$$\begin{aligned} \partial_t u - \Delta u + Vu = 0 \qquad & \text{on} \quad \mathbb{R}^n \times (\tau_k, T) \\ u(.,\tau_k) = \mu_k \qquad & \text{on} \quad \mathbb{R}^n \times \{\tau_k\} \end{aligned} \tag{2.7}$$

(extended by 0 on $(0,\tau_k)$) converge to u_μ in $L^1_{loc}(Q_T)$, and $\{Vu_{\mu_k}\}$ converges to Vu_μ in $L^1_{loc}(Q_T)$.

Condition (2.5) may be very difficult to verify and we give below a sufficient condition for it to hold.

PROPOSITION 2.9. *Assume V satisfies*

$$\lim_{\lambda \to 0} e^{\frac{|y|^2}{4T}} \lambda^{-n} \int_0^\lambda \int_{B_{\lambda^2}(y)} V(x,t)dxdt = 0 \tag{2.8}$$

uniformly with respect to $y \in \mathbb{R}^n$, then V is strongly subcritical.

Proof. Let $E \subset Q_T^{B_R}$ be a Borel set. For $\delta > 0$, we define the weighted heat ball of amplitude $\delta e^{-\frac{|y|^2}{4T}}$ by

$$P_\delta = P_\delta(y,T) = \left\{(x,t) \in Q_T : H(x-y,t) \geq \delta e^{-\frac{|y|^2}{4T}}\right\}.$$

By an straightforward computation, one sees that

$$P_\delta(y,T) \subset B_{a_n \delta^{-\frac{1}{n}} e^{\frac{|y|^2}{4nT}}}(y) \times [0, b_n \delta^{-\frac{2}{n}} e^{\frac{|y|^2}{2nT}}] := R_\delta(y,T),$$

for some $a_n, b_n > 0$. We write

$$\iint_E H(x-y,t)V(x,t)dxdt = \iint_{E \cap P_\delta} H(x-y,t)V(x,t)dxdt + \iint_{E \cap P_\delta^c} H(x-y,t)V(x,t)dxdt.$$

Then

$$\iint_{E \cap P_\delta^c} H(x-y,t)V(x,t)dxdt \leq \delta e^{-\frac{|y|^2}{4T}} \iint_E V(x,t)dxdt,$$

and

$$\begin{aligned}\iint_{E\cap P_\delta} H(x-y,t)V(x,t)dxdt &\le \int_0^\delta \int_{\{(x,t)\in Q_T^{B_R}:H(x-y,t)=\tau e^{-\frac{|y|^2}{4T}}\}} V(x,t)dS_\tau(x,t)\tau d\tau \\ &\le \left[\tau\int_0^\tau \int_{\{(x,t)\in Q_T^{B_R}:H(x-y,t)=\sigma e^{-\frac{|y|^2}{4T}}\}} V(x,t)dS_\sigma(x,t)d\sigma\right]_{\tau=0}^{\tau=\delta} \\ &\quad -\int_0^\delta\int_0^\tau \int_{\{(x,t)\in Q_T^{B_R}:H(x-y,t)=\sigma e^{-\frac{|y|^2}{4T}}\}} V(x,t)dS_\sigma(x,t)d\sigma d\tau \\ &\le \delta\int_0^\delta \int_{\{(x,t)\in Q_T^{B_R}:H(x-y,t)=\sigma e^{-\frac{|y|^2}{4T}}\}} V(x,t)dS_\sigma(x,t)d\sigma.\end{aligned}$$

The first integration by parts is justified since $V\in L^1(Q_T^{B_R})$. Notice that

$$\delta\int_0^\delta \int_{\{(x,t)\in Q_T^{B_R}:H(x-y,t)=\sigma e^{-\frac{|y|^2}{4T}}\}} V(x,t)dS_\sigma(x,t)d\sigma = \delta\iint_{Q_T^{B_R}\cap P_\delta} V(x,t)dxdt$$

and

$$\begin{aligned}\delta\iint_{Q_T^{B_R}\cap P_\delta} V(x,t)dxdt &\le \delta\iint_{Q_T^{B_R}\cap R_\delta(y,T)} V(x,t)dxdt \\ &\le \beta r^{-n}\int_0^{\alpha r}\int_{B_R\cap B_{(\alpha r)^2}(y)} V(x,t)dxdt,\end{aligned}$$

for some $\alpha,\beta>0$ and if we have set $r=\delta^{-\frac{1}{n}}$. Notice also that $B_R\cap B_{(\alpha r)^2}(y)=\emptyset$ if $|y|\ge R+(\alpha r)^2$, or, equivalently, if $|y|\ge R+\alpha^2\delta^{-\frac{2}{n}}$.

(i) If $|y|\ge R+\alpha^2$, we fix δ such that $1<\delta$, then

$$e^{\frac{|y|^2}{4T}}\iint_E H(x-y,t)V(x,t)dxdt \le \delta\iint_E V(x,t)dxdt,$$

which can be made smaller than ϵ provided $|E|$ is small enough.

(ii) If $|y|<R+\alpha^2$, then

$$\begin{aligned}e^{\frac{|y|^2}{4T}}\iint_{E\cap P_\delta^c} H(x-y,t)V(x,t)dxdt &\le e^{\frac{R^2+\alpha^4}{2T}}\iint_{E\cap P_\delta^c} H(x-y,t)V(x,t)dxdt \\ &\le \delta e^{\frac{R^2+\alpha^4}{2T}}\iint_E V(x,t)dxdt.\end{aligned}$$

Given $\epsilon>0$, we fix $\delta=r^{-n}$ such that

$$e^{\frac{R^2+\alpha^4}{2T}}\iint_{E\cap P_\delta} H(x-y,t)V(x,t)dxdt \le \beta e^{\frac{R^2+\alpha^4}{2T}} r^{-n}\int_0^{\alpha r}\int_{B_R\cap B_{(\alpha r)^2}(y)} V(x,t)dxdt \le \frac{\epsilon}{2},$$

and then $\eta>0$ such that $|E|\le\eta$ implies

$$e^{\frac{|y|^2}{4T}}\iint_{E\cap P_\delta^c} H(x-y,t)V(x,t)dxdt \le \delta e^{\frac{R^2+\alpha^4}{2T}}\iint_E V(x,t)dxdt \le \frac{\epsilon}{2}.$$

Therefore

$$e^{\frac{|y|^2}{4T}}\iint_E H(x-y,t)V(x,t)dxdt \le \epsilon,$$

which is (2.5). □

Remark In Theorem 2.7 and Proposition 2.9, the assumption of uniformity with respect to $y \in \mathbb{R}^n$ in (2.5), (2.6) and (2.8) can be replaced by uniformity with respect to $y \in B_{R_0}$ if all the measures μ_k have their support in B_{R_0}. A extension of these assumptions, valid when the convergent measures μ_k have their support in a fixed compact set is to assume that V is **locally strongly subcritical**, which means that (2.5) holds uniformly with respect to y in a compact set. Similar extension holds for (2.8).

3. The supercritical case

3.1. Capacities. All the proofs in this subsection are similar to the ones of [**19**] and inspired by [**9**]; we omit them. We assume also that there exists a positive measure μ_0 such that $\mathbb{H}[\mu_0]V \in L^1(Q_T)$.

DEFINITION 3.1. *If $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$ and f is a nonnegative measurable function defined in Ω such that*

$$(t,x,y) \mapsto \mathbb{H}[\mu](y,t)V(x,t)f(x,t) \ \in \ L^1(Q_T \times \mathbb{R}^n;\ dxdt \otimes d\mu),$$

we set

$$\mathcal{E}(f,\mu) = \int_{Q_T} \left(\int_{\mathbb{R}^n} H(x-y,t)d\mu(y) \right) V(x,t)f(x,t)dxdt.$$

If we put

$$\breve{\mathbb{H}}[f](y) = \int_{Q_T} H(x-y,t)V(x,t)f(x,t)dxdt,$$

then by Fubini's Theorem, $\breve{\mathbb{H}}[f](y) < \infty$, $\mu-$almost everywhere in $\mathbb{R}^n$ and

$$\mathcal{E}(f,\mu) = \int_{\mathbb{R}^n} \left(\int_{Q_T} H(x-y,t)V(x,t)f(x,t)dxdt \right) d\mu(y).$$

PROPOSITION 3.2. *Let f be fixed. Then*
(a) $y \mapsto \breve{\mathbb{H}}[f](y)$ is lower semicontinuous in $\mathbb{R}^n$.
(b) $\mu \mapsto \mathcal{E}(f,\mu)$ is lower semicontinuous in $\mathfrak{M}_+(\mathbb{R}^n)$ in the weak topology.*

DEFINITION 3.3. *We denote by $\mathfrak{M}^V(\mathbb{R}^n)$ the set of all measures μ on $\mathbb{R}^n$ such that $V\mathbb{H}[|\mu|] \in L^1(Q_T)$. If μ is such a measure, we set*

$$||\mu||_{\mathfrak{M}^V} = \int_{Q_T} \left(\int_{\mathbb{R}^n} H(x-y,t)d|\mu|(y) \right) V(x,t)dxdt = ||V\mathbb{H}[|\mu|]||_{L^1(Q_T)}.$$

If $E \subset \mathbb{R}^n$ is a Borel set, we put

$$\mathfrak{M}_+(E) = \{\mu \in \mathfrak{M}_+(\mathbb{R}^n) :\ \mu(E^c) = 0\} \quad \text{and} \quad \mathfrak{M}^V_+(E) = \mathfrak{M}^V(\mathbb{R}^n) \cap \mathfrak{M}_+(E).$$

DEFINITION 3.4. *If $E \subset \mathbb{R}^n$ is any borel subset we define the set function C_V by*

$$C_V(E) := \sup\{\mu(E) :\ \mu \in \mathfrak{M}^V_+(E),\ ||\mu||_{\mathfrak{M}^V} \leq 1\};$$

this is equivalent to,

$$C_V(E) := \sup\left\{ \frac{\mu(E)}{||\mu||_{\mathfrak{M}^V}} :\ \mu \in \mathfrak{M}^V_+(E) \right\}.$$

PROPOSITION 3.5. *The set function C_V satisfies*

$$C_V(E) \leq \sup_{y\in E}\left(\int_{Q_T} H(x-y,t)V(x,t)dxdt\right)^{-1} \quad \forall\, E\subset\mathbb{R}^n,\ E \text{ Borel}.$$

Furthermore equality holds if E is compact. Finally,

$$C_V(E_1\cup E_2) = \sup\{C_V(E_1), C_V(E_2)\} \quad \forall\, E_i\subset\mathbb{R}^n,\ E_i \text{ Borel}.$$

DEFINITION 3.6. *For any Borel $E\subset\mathbb{R}^n$, we set*

$$C_V^*(E) := \inf\{||f||_{L^\infty} :\ \breve{\mathbb{H}}[f](y)\geq 1\ \forall y\in E\}.$$

PROPOSITION 3.7. *For any compact set $E\subset\mathbb{R}^n$,*

$$C_V^*(E) = C_V(E).$$

3.2. The singular set of V. In this section we assume that V satisfies (1.16), although much weaker assumption could have been possible. We define the singular set of V, Z_V by

$$Z_V = \left\{x\in\mathbb{R}^n : \iint_{Q_T} H(x-y,t)V(y,t)dydt = \infty\right\}. \tag{3.1}$$

Since the function $x\mapsto f(x) = \iint_{Q_T} H(x-y,t)V(y,t)dydt$ is lower semicontinuous, it is a Borel function and Z_V is a Borel set.

LEMMA 3.8. *If $x\in Z_V$ then for any $r>0$,*

$$\iint_{Q_T^{B_r(x)}} H(x-y,t)V(y,t)dydt = \infty.$$

Proof. We will prove it by contradiction, assuming that there exists $r>0$, such that

$$\iint_{Q_T^{B_r(x)}} H(x-y,t)V(y,t)dy \leq M.$$

Replacing H by its value, we derive

$$\begin{aligned}\iint_{Q_T} H(x-y,t)V(y,t)dydt &= \iint_{Q_T^{B_r(x)}} H(x-y,t)V(y,t)dydt + \iint_{Q_T^{B_r^c(x)}} H(x-y,t)V(y,t)dydt\\ &\leq M + C(n)\int_0^T t^{-\frac{n+2}{2}}e^{-\frac{r^2}{4t}}dt < \infty.\end{aligned}$$

Which is clearly a contradiction. □

LEMMA 3.9. *If μ is an admissible positive measure then $\mu(Z_V)=0$.*

Proof. Let $K\subset Z_V$ be a compact set. In view of the above lemma there exists a $R>0$ such that $K\subset B_R$ and for each $x\in K$, we have

$$\iint_{Q_T^{B_{2R}}} H(x-y,t)V(y)dy = \infty \tag{3.2}$$

and

$$\iint_{Q_T^{B_{2R}^c}} H(x-y,t)V(y)dy < \infty. \tag{3.3}$$

Now, $\mu_K = \chi_K \mu$ is an admissible measure and by Fubini theorem we have

$$\begin{aligned}\iint_{Q_T}\left(\int_{\mathbb{R}^n} H(x-y,t)d\mu_K(y)\right)V(x,t)dxdt &= \int_K \iint_{Q_T} H(x-y,t)V(x,t)dxdtd\mu(y)\\ &= \int_K \iint_{Q_T^{B_{2R}}} H(x-y,t)V(x)dxdtd\mu(y)\\ &\quad + \int_K \iint_{Q_T^{B^c_{2R}}} H(x-y,t)V(x)dxdtd\mu y.\end{aligned}$$

By (3.3) the second integral above is finite and by (3.2)

$$\iint_{Q_T^{B_{2R}}} H(x-y,t)V(x)dxdt = \infty \qquad \forall y \in K.$$

It follows that $\mu(K) = 0$. This implies $\mu(Z_V) = 0$ by regularity. □

THEOREM 3.10. *If $\mu \in \mathfrak{M}_T(\mathbb{R}^n)$, $\mu \geq 0$ such that $\mu(Z_V) = 0$, then μ is a good measure.*

Proof. We set $\mu_R = \chi_{B_R}\mu$. By Proposition 5.8, since $Z_V^{B_R} \subset Z_V$, μ_R is a good measure in B_R with corresponding solution u_μ^R. In view of Lemma 2.5, u_μ^R satisfies

$$\iint_{Q_T^{B_R}} |u_\mu^R| \frac{n}{4(T-t)} e^{-\frac{|x|^2}{4(T-t)}} dxdt + \iint_{Q_T^{B_R}} |u_\mu^R| V e^{-\frac{|x|^2}{4(T-t)}} dxdt \leq \int_{B_R} e^{-\frac{|x|^2}{4T}} d\mu.$$

Also $\{u_\mu^R\}$ is an increasing function, thus converges to u_μ. By the above estimate we have that u_μ belong to class $\mathcal{E}_\mathcal{V}(Q_T)$ and is a weak solution of (2.1). □

PROPOSITION 3.11. *Let $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$. Then $\mu(Z_V) = 0$ if and only if there exists an increasing sequence of positive admissible measures which converges to μ in the weak* topology.*

Proof. The proof is similar as the one of [**19**, Th 3.11] and we present it for the sake of completeness. First, we assume that $\mu(Z_V) = 0$. Then we define the set

$$K_N = \left\{x \in \mathbb{R}^n : \int_{Q_T} H(x-y,t)V(y)dydt \leq N\right\}.$$

We note that $Z_V \cap K_N = \emptyset$. We set $\mu_n = \chi_{K_N}\mu$ then we have

$$\int_{Q_T}\left(\int_{\mathbb{R}^n} H(x-y,t)d\mu_n(y)\right)V(x,t)dxdt \leq \mu(K_N).$$

Thus μ_n is admissible, increasing with respect n. By the monotone theorem it follows that $\mu_n \to \chi_{Z_V^c}\mu$. Since $\mu(Z_V) = 0$ the result follows in this direction.
For the other direction. Let $\{\mu_n\}$ be an increasing sequence of positive admissible measure. Then by Lemma 3.9 we have that $\mu_n(Z_V) = 0$, $\forall\ n \geq 1$. Since $\mu_n \leq \mu$, there exist an increasing functions h_n $\mu-$integrable such that $\mu_n = h_n\mu$. Since $0 = \mu_n(Z_V) \to \mu(Z_V)$ the result follows. □

3.3. Properties of positive solutions and representation formula. We first recall the construction of the kernel function for the operator $w \mapsto \partial_t w - \Delta w + Vw$ in Q_T, always assuming that V satisfies (1.16). For $\delta > 0$ and $\mu \in \mathfrak{M}_T$, we denote by w_δ the solution of

$$\begin{aligned}\partial_t w - \Delta w + V_\delta w &= 0, \qquad &&\text{in } Q_T\\ w(.,0) &= \mu \qquad &&\text{in } \mathbb{R}^n.\end{aligned} \tag{3.4}$$

where $V_\delta = V\chi_{Q_{\delta,T}}$ and $Q_{\delta,T} = (\delta, T) \times \mathbb{R}^n$. Then

$$w_\delta(x,t) = \int_{\mathbb{R}^n} H_{V_\delta}(x,y,t)d\mu(y) \tag{3.5}$$

LEMMA 3.12. *The mapping $\delta \mapsto H_{V_\delta}(x,y,t)$ is increasing and converges to $H_V \in C(\mathbb{R}^n \times \mathbb{R}^n \times (0,T]))$ when $\delta \to 0$. Furthermore there exists a function $H_V \in C(\mathbb{R}^n \times \mathbb{R}^n \times (0,T]))$ such that for any $\mu \in \mathfrak{M}_T(\mathbb{R}^n)$*

$$\lim_{\delta\to 0} w_\delta(x,t) = w(x,t) = \int_{\mathbb{R}^n} H_V(x,y,t)d\mu(y). \tag{3.6}$$

Proof. Without loss of generality we can assume $\mu \geq 0$. By the maximum principle $\delta \mapsto H_{V_\delta}(x,y,t)$ is increasing and the result follows by the monotone convergence theorem. □

If $\mathbb{R}^n$ is replaced by a smooth bounded domain Ω, we can consider the problem

$$\begin{array}{rl} \partial_t w - \Delta w + V_\delta w = 0 & \text{in } Q_T^\Omega \\ w = 0 & \text{in } \partial_l Q_T^\Omega := \partial\Omega \times (0,T] \\ w(.,0) = \mu & \text{in } \Omega. \end{array} \tag{3.7}$$

where $V'_\delta = V\chi_{Q^\Omega_{\delta,T}}$ and $Q^\Omega_{\delta,T} = (\delta, T) \times \Omega$. Then

$$w_\delta(x,t) = \int_\Omega H^\Omega_{V_\delta}(x,y,t)d\mu(y) \tag{3.8}$$

The proof of the next result is straightforward.

LEMMA 3.13. *The mapping $\delta \mapsto H^\Omega_{V_\delta}(x,y,t)$ increases and converges to $H^\Omega_V \in C(\Omega \times \Omega \times (0,T]))$ when $\delta \to 0$. Furthermore There exists a fonction $H^\Omega_V \in C(\Omega \times \Omega \times (0,T]))$ such that for any $\mu \in \mathfrak{M}_b(\Omega)$*

$$\lim_{\delta\to 0} w_\delta(x,t) = w(x,t) = \int_\Omega H^\Omega_V(x,y,t)d\mu(y). \tag{3.9}$$

Furthermore $H^\Omega_V \leq H^{\Omega'}_V \leq H_V$ if $\Omega \subset \Omega'$.

It is important to notice that the above results do not imply that w is a weak solution of problem (1.1). This question will be considered later on with the notion of reduced measure.

LEMMA 3.14. *Assume $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$ is a good measure and let u be a positive weak solution of problem (2.1). If Ω is a smooth bounded domain, then there exists a unique positive weak solution v of problem*

$$\begin{array}{rl} \partial_t v - \Delta v + Vv = 0, & \text{in } Q_T^\Omega, \\ v = 0 & \text{on } \partial_l Q_T^\Omega \\ v(.,0) = \chi_\Omega \mu & \text{in } \Omega. \end{array} \tag{3.10}$$

Furthermore

$$v(x,t) = \int_\Omega H^\Omega_V(x,y,t)d\mu(y). \tag{3.11}$$

Proof. Let $\{t_j\}_{j=1}^{\infty}$ be a sequence decreasing to 0, such that $t_j < T$, $\forall\, j \in \mathbb{N}$. We consider the following problem

$$\begin{array}{rll} \partial_t v - \Delta v + Vv = 0, & \text{in} & \Omega \times (t_j, T], \\ v = 0 & \text{on} & \partial\Omega \times (t_j, T] \\ v(.,t_j) = u(.,t_j) & \text{in} & \Omega \times \{t_j\}. \end{array} \tag{3.12}$$

Since $u, Vu \in L^1(Q_T^{B_R})$ for any $R > 0$, $t \mapsto u(.,t)$ is continuous with value in $L^1_{loc}(\mathbb{R}^n)$, therefore $u(.,t_j) \in L^1_{loc}(\mathbb{R}^n)$ and there exists a unique solution v_j to (3.12) (notice also that $V \in L^\infty(Q_T^{B_R})$). By the maximum principle $0 \le v_j \le u$ and by standard parabolic estimates, we may assume that the sequence v_j converges locally uniformly in $\Omega \times (0,T]$ to a function $v \le u$. Also, if $\phi \in C^{1,1;1}(\overline{Q_T^\Omega})$ vanishes on $\partial_l Q_T^\Omega$ and satisfies $\phi(x,T) = 0$, we have

$$-\int_{t_j}^T \int_\Omega v_j(\partial_t\phi + \Delta\phi)dxdt + \int_{t_j}^T \int_\Omega V v_j \phi dxdt + \int_\Omega \phi(x, T-t_j)v_j(x,T-t_j)dx = \int_\Omega \phi(x,0)u(x,t_j)dx,$$

where in the above equality we have taken $\phi(.,.-t_j)$ as test function. Since $\phi(.,T-t_j) \to 0$ uniformly and $u(.,t_j) \to \mu$ in the weak sense of measures, it follows by the dominated convergence theorem that

$$-\iint_{Q_T^\Omega} v(\partial_t\phi + \Delta\phi)dxdt + \iint_{Q_T^\Omega} V v\phi dxdt = \int_\Omega \phi(y,0)d\mu(y),$$

thus v is a weak solution of problem (3.10). Uniqueness follows as in Lemma 2.4. Finally, for $\delta > 0$, we consider the solution w_δ of (3.7). Then it is expressed by (3.5). Furthermore

$$-\iint_{Q_T^\Omega} w_\delta(\partial_t\phi + \Delta\phi)dxdt + \iint_{Q_T^\Omega} V_\delta w_\delta \phi dxdt = \int_\Omega \phi(x,0)d\mu(x),$$

The sequence w_δ is decreasing, with limit w. Since $w_\delta \ge v$, then $w \ge v$. If we assume $\phi \ge 0$, it follows from dominated convergence and Fatou's lemma that

$$-\iint_{Q_T^\Omega} w(\partial_t\phi + \Delta\phi)dxdt + \iint_{Q_T^\Omega} V w\phi dxdt \le \int_\Omega \phi(x,0)d\mu(x),$$

Thus w is a subsolution for problem (3.10) for which we have comparison when existence. Finally $w = v$ and (3.11) holds. □

LEMMA 3.15. *Assume $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$ is a good measure and let u be a positive weak solution of problem (2.1). Then for any $(x,t) \in \mathbb{R}^n \times (0,T]$, we have*

$$\lim_{R\to\infty} u_R = u,$$

where $\{u_R\}$ is the increasing sequence of the weak solutions of the problem (3.10) with $\Omega = B_R$. Moreover, the convergence is uniform in any compact subset of $\mathbb{R}^n \times (0,T]$ and we have the representation formula

$$u(x,t) = \int_{\mathbb{R}^n} H_V(x,y,t)d\mu(y).$$

Proof. By the maximum principle, $u_R \le u_{R'} \le u$ for any $0 < R \le R'$. Thus $u_R \to w \le u$. Also by standard parabolic estimates, this convergence is locally uniformly. Now by dominated convergence theorem, it follows that w is a weak

solution of problem (2.1) with initial data μ. Now we set $\widetilde{w} = u - w \geq 0$. Since $\widetilde{w}$ satisfies in the weak sense

$$\begin{aligned} \widetilde{w}_t - \Delta\widetilde{w} + V\widetilde{w} &= 0 \qquad \text{in } Q_T \\ \widetilde{w}(x,t) &\geq 0 \qquad \text{in } Q_T \\ \widetilde{w}(x,0) &= 0 \qquad \text{in } \mathbb{R}^n, \end{aligned}$$

and $V \geq 0$, it clearly satisfies

$$\begin{aligned} \widetilde{w}_t - \Delta\widetilde{w} &\leq 0 \qquad \text{in } Q_T \\ \widetilde{w}(x,t) &\geq 0 \qquad \text{in } Q_T \\ \widetilde{w}(x,0) &= 0 \qquad \text{in } \mathbb{R}^n, \end{aligned}$$

which implies $\widetilde{w} = 0$. By the previous lemme u_R admits the representation

$$u^R(x,t) = \int_{B_R} H_V^{B_R}(x,y,t)d\mu(y).$$

Since $\{H_V^{B_R}\}$ is an increasing sequence and $\lim_{R\to\infty} H_V^{B_R} = H_V$, we have using again Fatou's lemma as in the proof of Lemma 3.15

$$u(x,t) = \lim_{R\to\infty} u^R(x,t) = \lim_{R\to\infty} \int_{B_R} H_V^{B_R}(x,y,t)d\mu(y) = \int_{\mathbb{R}^n} H_V(x,y,t)d\mu(y)$$

□

LEMMA 3.16. ***Harnack inequality*** *Let $C_1 > 0$ and $V(x,t)$ be a potential satisfying (1.16) If u is a positive solution of (1.17), then the Harnack inequality is valid:*

$$u(y,s) \leq u(x,t)\exp\left(C(n,C_1)\left(\frac{|x-y|^2}{t-s} + \frac{t}{s} + 1\right)\right), \qquad \forall\,(y,s),(x,t) \in Q_T\,,\ s < t.$$

Proof. We extend V for $t \geq T$ by the value $C_1 t^{-1}$. We consider the linear parabolic problem

$$\partial_t u - \Delta u + Vu = 0, \qquad \text{in} \qquad \mathbb{R}^n \times [1,\infty), \tag{3.13}$$

It is well known that, under the assumption (1.16), every positive solution $u(x,t)$ of (3.13) satisfies the Harnack inequality

$$u(y,s) \leq u(x,t)\exp\left(C(n,C_1)\left(\frac{|x-y|^2}{t-s} + \frac{t}{s} + 1\right)\right), \qquad \forall\,(x,t) \in \mathbb{R}^n \times [1,\infty).$$

Set $\tilde{u}(x,t) = u(\frac{t}{\lambda^2}\frac{x}{\lambda})$. Then $\tilde{u}$ satisfies

$$\mathrm{u}_t - \Delta \mathrm{u} + \frac{1}{\lambda^2}V(\frac{t}{\lambda^2}\frac{x}{\lambda})\tilde{u} = 0, \qquad \text{in } \mathbb{R}^n \times (0,\infty).$$

We note here that $\frac{1}{\lambda^2}V(\frac{t}{\lambda^2}\frac{x}{\lambda}) \leq C_1$, $\forall t \geq \frac{1}{\lambda^2}$, thus $\tilde{u}$ satisfies the Harnack inequality

$$\widetilde{u}(y,s) \leq \widetilde{u}(x,t)\exp\left(C(n,C_1)\left(\frac{|x-y|^2}{t-s} + \frac{t}{s} + 1\right)\right), \qquad \forall\,(x,t) \in \mathbb{R}^n \times [\frac{1}{\lambda^2},\infty).$$

By the last inequality and the definition of $\tilde{u}$ we derive the desired result. □

Next, we set

$$\mathcal{S}ing_V(\mathbb{R}^n) := \{y \in \mathbb{R}^n :\ H_V(x,y,t) = 0\} \tag{3.14}$$

If $H_V(x,y,t) = 0$ for some $(x,t) \in Q_T$, then $H_V(x',y,t') = 0$ for any $(x',t') \in Q_T$, $t' < t$ by Harnack inequality principle. We prove the **Representation formula**.

THEOREM 3.17. *Let u be a positive solution of (1.17). Then there exists a measure $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$ such that*

$$u(x,t) = \int_{\mathbb{R}^n} H_V(x,y,t)d\mu(y),$$

and μ is concentrated on $(\mathcal{S}ing_V(\mathbb{R}^n))^c$.

Proof. By Lemma 3.15 we have

$$u(x,t) = \int_{\mathbb{R}^n} H_V(x,y,t-s)u(y,s)dy \quad \text{for any } s < t \leq T.$$

We assume that $s \leq \frac{T}{2}$. By Harnack inequality on $x \mapsto H_V(x,y,\frac{T}{2})$

$$\int_{\mathbb{R}^n} H_V(0,y,\frac{T}{2})u(y,s)dy \leq c(n)\int_{\mathbb{R}^n} H_V(0,y,T-s,)u(y,s)dy = c(n)u(0,T).$$

For any Borel set E, we define the measure ρ_s by

$$\rho_s(E) := \int_E H_V(0,y,\frac{T}{2})u(y,s)dy \leq \int_{\mathbb{R}^n} H_V(0,y,\frac{T}{2})u(y,s)dy \leq c(n)u(T,0).$$

Thus there exists a decreasing sequence $\{s_j\}_{j=1}^{\infty}$ which converges to origin, such that the measure ρ_{s_j} converges in the weak* topology to a positive Radon measure ρ. Also we have the estimate $\rho(\mathbb{R}^n) \leq C(n)u(0,T)$. Now choose $(x,t) \in Q_T$ and j_0 large enough such that $t > s_{j_0}$. Let $\varepsilon > 0$, we set for any $j \geq j_0$,

$$W_j(y) = \frac{H_V(x,y,t-s_j)}{H_V(0,y,\frac{T}{2})+\varepsilon}.$$

For any $R > 0$ and $|y| > R$ we have

$$W_j(y) \leq \frac{1}{\varepsilon}H_V(x,y,t-s_j) \leq \frac{1}{\varepsilon}H(x-y,t-s_j) < \frac{1}{\varepsilon}C(x,R,t-s_j),$$

where $\lim_{R\to\infty} C(x,R,t-s_j) = 0$. We have also

$$\int_{|y|\geq R} W_j(y)d\rho_j \leq \frac{1}{\varepsilon}C(x,R,t-s_j)c(n)u(T,0).$$

For any $|y| < R$, we have by standard parabolic estimates that $W_j(y) \to \frac{H_V(x,y,t)}{H_V(\frac{T}{2},0,y)+\varepsilon}$ when $j \to \infty$, uniformly with respect to y. Thus by the above estimates it follows

$$\int_{\mathbb{R}^n} W_j(y)d\rho_j \to \int_{\mathbb{R}^n} \frac{H_V(x,y,t)}{H_V(0,y,\frac{T}{2})+\varepsilon}d\rho.$$

For sufficiently large j we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{H_V(x,y,t-s_j)}{H_V(0,y,\frac{T}{2})+\varepsilon}d\rho_{s_j} &= \int_{\mathbb{R}^n} \frac{H_V(x,y,t-s_j)}{H_V(0,y,\frac{T}{2})+\varepsilon}\left(H_V(0,y,\frac{T}{2})+\varepsilon-\varepsilon\right)u(y,s_j)dy \\ &= u(x,t) - \varepsilon\int_{\mathbb{R}^n} \frac{H_V(x,y,t-s_j,)}{H_V(0,y,\frac{T}{2})+\varepsilon}u(y,s_j)dy.\end{aligned}$$

Note that this is a consequence of the identity

$$\int_{\mathbb{R}^n} H_V(x,y,t-s_j)u(y,s_j)dy = u(x,t).$$

Thus as before, we define $d\widetilde{\rho}_j = H_V(x,y,t-s_j)u(y,s_j)dy$ and thus there exists a subsequence, say $\{\widetilde{\rho}_j\}$ converges in the weak* topology to a positive Radon measure $\widetilde{\rho}$. Thus we have

$$\begin{aligned}\varepsilon\int_{\mathbb{R}^n}\frac{H_V(x,y,t-s_j)}{H_V(0,y,\frac{T}{2})+\varepsilon}u(y,s_j)dy &= \varepsilon\int_{\mathbb{R}^n}\chi_{(\mathcal{S}ing_V(\mathbb{R}^n))^c}\frac{H_V(x,y,t-s_j)}{H_V(0,y,\frac{T}{2})+\varepsilon}u(s_j,y)dy\\ &\to \varepsilon\int_{\mathbb{R}^n}\chi_{(\mathcal{S}ing_V(\mathbb{R}^n))^c}\frac{1}{H_V(\frac{T}{2},0,y)+\varepsilon}d\widetilde{\rho}.\end{aligned}$$

Combining the above relations, we derive

$$\int_{\mathbb{R}^n}\frac{H_V(x,y,t)}{H_V(0,y,\frac{T}{2})+\varepsilon}d\rho = u(x,t)-\varepsilon\int_{\mathbb{R}^n}\chi_{(\mathcal{S}ing_V(\mathbb{R}^n))^c}\frac{1}{H_V(0,y,\frac{T}{2})+\varepsilon}d\widetilde{\rho}. \tag{3.15}$$

Now, we have

$$\lim_{\varepsilon\to 0}\chi_{(\mathcal{S}ing_V(\mathbb{R}^n))^c}\frac{\varepsilon}{H_V(\frac{T}{2},0,y)+\varepsilon}=0,$$

and by Harnack inequality on the function $x\mapsto H_V(x,y,t)$

$$\frac{H_V(x,y,t)}{H_V(0,y,\frac{T}{2})+\varepsilon}\le C(t,T),$$

thus by dominated convergence theorem, we can let ε tend to 0 in (3.15) and obtain

$$\int_{\mathbb{R}^n}\frac{H_V(x,y,t)}{H_V(0,y,\frac{T}{2})}d\rho = u(x,t).$$

And the result follows if we set

$$d\mu = \chi_{(\mathcal{S}ing_V(\mathbb{R}^n))^c}\frac{1}{H_V(0,y,\frac{T}{2})}d\rho.$$

□

In the next result we give a construction of H_V, with some estimates and a different proof of the existence of an initial measure for positive solutions of (1.16).

THEOREM 3.18. *Assume V satisfies (1.16) and u is a positive solution of (1.17) then there exists a positive Radon measure μ in $\mathbb{R}^n$ such that*

$$u(x,t)=\int_{\mathbb{R}^n}\epsilon^{\psi(x,t)}\Gamma(x,y,t,0)d\mu(y) \tag{3.16}$$

where

$$\psi(x,t)=\int_t^T\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-y|^2}{4(s-t)}}}{4\pi(t-s)}V(y,s)dyds \tag{3.17}$$

and

$$c_1\frac{e^{-\gamma_1\frac{|x-y|^2}{s-t}}}{(t-s)^{\frac{n}{2}}}\le\Gamma(x,y,t,s)\le c_2\frac{e^{-\gamma_2\frac{|x-y|^2}{s-t}}}{(t-s)^{\frac{n}{2}}} \tag{3.18}$$

for some positive constants c_i and γ_i, $i=1,2$.

Proof. Assuming that u is a positive solution of (1.17), we set $u(x,t)=e^{\psi(x,t)}v(x,t)$. Then

$$\partial_t v-\Delta v-2\nabla\psi.\nabla v-|\nabla\psi|^2v+(\partial_t\psi-\Delta\psi+V)v=0. \tag{3.19}$$

We choose ψ as the solution of problem

$$\begin{aligned} -\partial_t\psi - \Delta\psi + V\psi = 0 \qquad &\text{in } \ Q_T \\ \psi(.,T) = 0 \qquad &\text{in } \ \mathbb{R}^n. \end{aligned} \tag{3.20}$$

Then ψ is expressed by (3.17). Furthermore, by standard computations,

$$\begin{aligned} &(i) \qquad 0 \le \psi(x,t) \le c \ln \tfrac{T}{t} \\ &(ii) \qquad |\nabla\psi(x,t)| \le c_1(T) + c_2(T) \ln \tfrac{T}{t} \end{aligned} \tag{3.21}$$

The function v satisfies

$$\partial_t v - \Delta v - 2\nabla\psi.\nabla v - |\nabla\psi|^2 v = 0. \tag{3.22}$$

Then, by (3.21),

$$\begin{aligned} &(i) \qquad 0 \le \int_{\mathbb{R}^n} \sup\{|\psi(x,s)|^q : x \in \mathbb{R}^n\} ds \le M_1 \\ &(ii) \qquad 0 \le \int_{\mathbb{R}^n} \sup\{|\nabla\psi(x,s)|^q : x \in \mathbb{R}^n\} ds \le M_2 \end{aligned} \tag{3.23}$$

for any $1 \le q < \infty$ for some $M_i \in \mathbb{R}_+$. This is the condition H in [**4**] with $R_0 = \infty$ and $p = \infty$. Therefore there exists a kernel function $\Gamma \in C(\mathbb{R}^n \times \mathbb{R}^n \times (0,T) \times (0,T))$ which satisfies (3.18) and there exists also a positive Radon measure μ in $\mathbb{R}^n$ such that

$$v(x,t) = \int_{\mathbb{R}^n} \Gamma(x,y,t,0) d\mu(y). \tag{3.24}$$

Finally u verifies

$$u(x,t) = e^{\psi(x,t)} \int_{\mathbb{R}^n} \Gamma(x,y,t,0) d\mu(y). \tag{3.25}$$

□

We recall that $\mathcal{S}ing_V(\mathbb{R}^n) := \{y \in \mathbb{R}^n : \ H_V(x,y,t) = 0\}$.

THEOREM 3.19. *Let δ_ξ be the Dirac measure concentrated at y and let V satisfies (1.16). Then*

$$H_V(x,\xi,t) = \int_{\mathbb{R}^n} e^{\psi(x,t)} \Gamma(x,y,t) d\mu_\xi(y),$$

where μ_ξ is a positive Radon measure such that

$$\delta_\xi \ge \mu_\xi,$$

and ψ, Γ are the functions in (3.17) and (3.18) respectively.
Furthermore, if

$$\limsup_{t\to 0} \psi(\xi,t) = \limsup_{t\to 0} \int_t^T \int_{\mathbb{R}^n} \left(\frac{1}{4\pi(s-t)}\right)^{\frac{n}{2}} e^{-\frac{|\xi-y|^2}{4(s-t)}} V(y,s) dy ds = \infty$$

then

$$\xi \in \mathcal{S}ing_V, \text{ i.e. } H_V(x,\xi,t) = 0, \ \forall (x,t) \in \mathbb{R}^n \times (0,\infty).$$

proof. First we note that $H_{V_k}(x,\xi,t)$ is the solution of problem (1.24) with δ_ξ as initial data. Since $H_{V_k}(x,\xi,t) \downarrow H_V(x,\xi,t)$, we have by maximum principle , $H(x,\xi,t) \geq H_V(x,\xi,t)$. Now by Theorem 3.18, there exists a positive Radon measure μ_ξ in $\mathbb{R}^n$ such that

$$H_V(x,,\xi,t) = \int_{\mathbb{R}^n} \epsilon^{\psi(x,t)}\Gamma(x,y,t,0)d\mu_\xi(y) \tag{3.26}$$

Let $\phi \in C_0(\mathbb{R}^n)$ then we have by the properties of $\Gamma(x,\xi,t)$ (see [**4**]) and (3.26)

$$\lim_{t\to 0}\int_{\mathbb{R}^n} H_V(x,\xi,t)\phi(x)dx \geq \lim_{t\to 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Gamma(x,y,t)\phi(x)dxd\mu_\xi(y) = \int_{\mathbb{R}^n}\phi(y)\mu_\xi(y),$$

That is

$$\int_{\mathbb{R}^n}\phi(x)d\delta_\xi(x) \geq \int_{\mathbb{R}^n}\phi(x)d\mu_\xi(x) \Rightarrow \delta_\xi \geq \mu_\xi, \tag{3.27}$$

since ϕ is an abstract function in space $C_0(\mathbb{R}^n)$.
Also we have that there exist positive constants $C_1,\ C_2$ such that

$$\Gamma(x,y,t) \geq C_1 H(x,y,C_2 t). \tag{3.28}$$

Also we have

$$\begin{aligned}
H(\xi,\xi,t) \geq H_V(\xi,\xi,t) &= \int_{\mathbb{R}^n} H_V(\xi,y,t)d\mu_\xi(y) = \int_{\mathbb{R}^n} e^{\psi(\xi,t)}\Gamma(\xi,y,t)d\mu_\xi(y)\\
\text{(by (3.28))} &\geq C_1\int_{B(\xi,\sqrt{C_2 t})} e^{\psi(t,\xi)}H(\xi,y,C_2 t)d\mu_\xi(y)\\
\text{(By Harnack inequality)} &\geq C(T,n,C_1,C_2)\int_{B(\xi,\sqrt{C_2 t})} e^{\psi(\xi,t)}H(\xi,\xi,\frac{C_2 t}{2})d\mu_\xi(y)\\
&= C(T,n,C_1,C_2)e^{\psi(\xi,t)}H(\xi,\xi,\frac{C_2 t}{2})\mu_\xi(B(\xi,\sqrt{C_2 t}))
\end{aligned}$$

Thus by the last inequality and the fact that

$$\frac{H(\xi,\xi,t)}{H(\xi,\xi,\frac{C_2 t}{2})} = C(C_2,n) > 0,$$

we have

$$C(T,n,C_1,C_2) \geq e^{\psi(t,\xi)}\mu_\xi(B(\xi,\sqrt{C_2 t})).$$

But $limsup_{t\to 0}\psi(\xi,t) = \infty$ which implies

$$\lim_{t\to 0}\mu_\xi(B(\xi,\sqrt{C_2 t}) = \mu_\xi(\{\xi\}) = 0.$$

Thus by (3.27) we have $\mu_\xi \equiv 0$, i.e. $H_V(x,\xi,t) = 0,\ \forall (x,t) \in \mathbb{R}^n \times (0,\infty)$. □

3.4. Reduced measures. In this section we assume that V is nonnegative, but not necessarily satisfies (1.16), therefore we can construct $H_V[\mu]$ for $\mu \in \mathfrak{M}_T(\mathbb{R}^n)$. Furthermore, if μ is nonnegative we can consider the solution u_k of the problem

$$\begin{aligned}
\partial_t u - \Delta u + V^k u = 0, \qquad &\text{in } Q_T\\
u(.,0) = \mu \qquad &\text{in } \mathbb{R}^n,
\end{aligned} \tag{3.29}$$

where $V^k = \min\{V,k\}$. Then there holds

$$u_k(x,t) = \int_{\mathbb{R}^n} H_{V^k}(t,x,y)d\mu(y) = \mathbb{H}_{V^k}[\mu](x,t),$$

and

$$u_k + \int_0^t \int_{\mathbb{R}^n} H(t-s,x,y)V^k u_k dyds = \mathbb{H}[\mu].$$

Since $k \mapsto H_{V^k}$ is decreasing and converges to H_V, we derive

$$\lim_{k\to\infty} u_k = u = \int_{\mathbb{R}^n} H_V(t,x,y)d\mu(y).$$

By Fatou's lemma

$$\int_0^t \int_{\mathbb{R}^n} H(t-s,x,y)Vudyds \le \liminf_{k\to\infty} \int_0^t \int_{\mathbb{R}^n} H(t-s,x,y)V^k u_k dyds.$$

It follows

$$u(x,t) + \int_0^t \int_{\mathbb{R}^n} H(t-s,x,y)Vudyds \le \int_{\mathbb{R}^n} H_V(t,x,y)d\mu(y), \quad \forall (x,t) \in Q_T.$$

Now since $Vu \in L^1_{loc}(\overline{Q}_T)$ and

$$\partial_t u - \Delta u + Vu = 0, \qquad \text{in} \qquad Q_T,$$

the function

$$u(x,t) + \int_0^t \int_{\mathbb{R}^n} H(t-s,x,y)Vudyds$$

is nonnegative and satisfies the heat equation in Q_T. Therefore it admits an initial trace $\mu^* \in \mathfrak{M}_+(\mathbb{R}^n)$ and actually $\mu^* \in \mathfrak{M}_T(\mathbb{R}^n)$. Furthermore, we have

$$u(x,t) + \int_0^t \int_{\mathbb{R}^n} H(t-s,x,y)Vudyds = \int_{\mathbb{R}^n} H(x-y,t)d\mu^*(y), \ \forall (x,t) \in Q_T.,$$

or equivalently, u is a positive weak solution of the problem

$$\begin{aligned} \partial_t u - \Delta u + Vu = 0 \qquad & \text{in } \ Q_T \\ u(.,0) = \mu^* \qquad & \text{in } \ \mathbb{R}^n. \end{aligned}$$

Note that $\mu^* \le \mu$ and the mapping $\mu \mapsto \mu^*$ is nondecreasing.

Definition 3.20. *The measure μ^* is the reduced measure associated to μ*

The proofs of the next two Propositions are similar to the ones of [**19**, Section 5].

Proposition 3.21. *There holds $\mathbb{H}_V[\mu] = \mathbb{H}_V[\mu^*]$. Furthermore the reduced measure μ^* is the largest measure for which the following problem*

$$\begin{aligned} \partial_t v - \Delta v + Vv = 0 \qquad & \text{in } \ Q_T \\ \lambda \in \mathfrak{M}_+(\mathbb{R}^n), \ \lambda \le \mu & \\ v(.,0) = \lambda \qquad & \text{in } \mathbb{R}^n, \end{aligned} \tag{3.30}$$

admits a solution.

Proposition 3.22. *Let W_k be an increasing sequence of nonnegative bounded measurable functions converging to V a.e. in Q_T. Then the solution u_k of*

$$\begin{aligned} \partial_t v - \Delta v + W_k v = 0 \qquad & \text{in } \ Q_T \\ v(.,0) = \mu \qquad & \text{in } \ \mathbb{R}^n, \end{aligned}$$

converges to u_{μ^*}.

We recall that $\mathcal{S}ing_V(\mathbb{R}^n) := \{y \in \mathbb{R}^n : \ H_V(x,y,t) = 0\}$.

PROPOSITION 3.23. *Let μ be a nonnegative measure in $\mathcal{M}_{\mathcal{T}}(\mathbb{R}^n)$. Then*
(i) $(\mu-\mu^*)\left((\mathcal{S}ing_V(\mathbb{R}^n))^c\right)=0$
(ii) If $\mu\left((\mathcal{S}ing_V(\mathbb{R}^n))^c\right)=0$, *then* $\mu^*=0$.
(iii) There always holds $\mathcal{S}ing_V(\mathbb{R}^n)=Z_V$.

proof. The proofs of (i), (ii) and the fact that $\mathcal{S}ing_V(\mathbb{R}^n)\subset Z_V$ are similar as in [**19**, Section 5], and we omit them.

The proof of $Z_V\subset\mathcal{S}ing_V(\mathbb{R}^n)$ is a immediately consequence of Theorem 3.19. Indeed, if $\xi\in Z_V$ then

$$\limsup_{t\to 0}\int_t^T\int_{\mathbb{R}^n}\left(\frac{1}{4\pi(s-t)}\right)^{\frac{n}{2}}e^{-\frac{|\xi-y|^2}{4(s-t)}}V(y,s)dyds=\infty,$$

thus $\xi\in\mathcal{S}ing_V(\mathbb{R}^n)$. □

4. Initial trace

4.1. The direct method. The initial trace that we developed in this section is an adaptation to the parabolic case of the notion of boundary trace for elliptic equations (see [**14**], [**15**], [**19**]). If $G\subset\overline{Q_T}$ is a relatively open set, we denote

$$W(G)=\bigcap_{1\le p<\infty}W_p^{2,1}(G)\quad\text{and}\quad W_{loc}(G)=\bigcap_{1\le p<\infty}W_{p\,loc}^{2,1}(G).$$

Since $V\in L^\infty_{loc}(Q_T)$, any solution of (1.17) belongs to $W_{loc}(Q_T)$.

PROPOSITION 4.1. *Let $u\in W_{loc}(Q_T)$ be a positive solution (1.17). Assume that, for some $x\in\mathbb{R}^n$, there exists an open bounded neighborhood U of x such that*

$$\iint_{Q_T^U}u(y,t)V(y,t)dxdt<\infty \tag{4.1}$$

Then $u\in L^1(U\times(0,T))$ and there exists a unique positive Radon measure μ in U such that

$$\lim_{t\to 0}\int_U u(y,t)\phi(x)dx=\int_U\phi(x)d\mu,\quad\forall\phi\in C_0^\infty(U).$$

Proof. Since $Vu\in L^1(U\times(0,T))$ the following problem has a weak solution v (see [**14**]).

$$\begin{aligned}\partial_t v-\Delta v &= Vu, \qquad \text{in} \qquad U\times(0,T],\\ v(x,t) &= 0 \qquad \text{on } \partial U\times(0,T]\\ v(x,0) &= 0 \qquad \text{in } U.\end{aligned}$$

Thus the function $w=u+v$ satisfies the heat equation. Thus there exists a unique Radon measure μ such that

$$\lim_{t\to 0}\int_U w(y,t)\phi(x)dx=\int_U\phi(x)d\mu,\quad\forall\phi\in C_0^\infty(U).$$

And the result follows since the initial data of v is zero. □

We set

$$\mathcal{R}(u)=\left\{y\in\mathbb{R}^n:\ \exists\text{ bounded neighborhood } U \text{ of } y,\ \iint_{Q_T^U}u(y,t)V(y,t)dxdt<\infty\right\}. \tag{4.2}$$

Then $\mathcal{R}(u)$ is open and there exists a unique positive Radon measure μ on $\mathcal{R}(u)$ such that

$$\lim_{t\to 0}\int_{\mathcal{R}} u(y,t)\phi(x)dx = \int_{\mathcal{R}}\phi(x)d\mu, \quad \forall \phi\in C_0^\infty(\mathcal{R}). \tag{4.3}$$

PROPOSITION 4.2. *Let $u\in W_{loc}(\mathbb{R}^n\times(0,T])$ be a positive solution of (1.17). Assume that, for some $x\in\mathbb{R}^n$, there holds*

$$\iint_{Q_T^U} u(y,t)V(y,t)dydt = \infty \tag{4.4}$$

for any bounded open neighborhood U of x. Then

$$\limsup_{t\to 0}\int_U u(y,t)dy = \infty. \tag{4.5}$$

Proof. We will prove it by contradiction. We assume that there exists an open neighborhood of x such that

$$\int_U u(y,t)dy \le M<\infty \qquad \forall t\in(0,T).$$

Then $\|u\|_{L^1(Q_T^U)}\le MT$. Let $B_r(x)\subset\subset U$ for some $r>0$, and $\zeta\in C_0^\infty(B_r(x))$, such that $\zeta=1$ in $B_{\frac{r}{2}}(x)$, $\zeta=0$ in $B_r^c(x)$ and $0\le\zeta\le 1$. Then since u is a positive solution we have

$$\int_U \partial_t u\zeta dx - \int_U u\Delta\zeta dx + \int_U Vu\zeta dx = 0 \Rightarrow \int_{B_{\frac{r}{2}}} Vudx \le \int_U \partial_t u\zeta dx - \int_U u\Delta\zeta dx \Rightarrow$$

$$\int_U \partial_t u\zeta dx - \int_U u\Delta\zeta dx + \int_U Vu\zeta dx = 0 \Rightarrow \int_{B_{\frac{r}{2}}} Vudx \le -\int_U \partial_t udx + M\left\|\Delta\zeta\right\|_{L^\infty}.$$

Integrating the above inequality on (s,T), we get

$$\int_s^T\int_{B_{\frac{r}{2}}} Vudxdr \le -\int_U u(x,T)dx + \int_U u(x,s)dx + \|u\|_{L^1(Q_T^U)}\left\|\Delta\zeta\right\|_{L^\infty}. \tag{4.6}$$

Letting $s\to 0$, we reach a contradiction. □

Remark. It is not clear wether there holds

$$\liminf_{t\to 0}\int_U u(y,t)dy = \infty. \tag{4.7}$$

However, it follows from (4.6) that if $u\in L^1(Q_T^U)$, the above equality holds.

DEFINITION 4.3. *If u is a positive solution of (1.17), we set $\mathcal{S}(u)=\mathbb{R}^n\setminus\mathcal{R}(u)$. The couple $(\mathcal{S}(u),\mu)$ is called the initial trace of u, denoted by $tr_{\{t=0\}}(u)$. The sets $\mathcal{R}(u)$ and $\mathcal{S}(u)$ are respectively the regular and the singular sets of $tr_{\{t=0\}}(u)$ and $\mu\in\mathfrak{M}_+(\mathcal{R}(u))$ is its regular part.*

Example Take $V(x,t)=ct^{-1}$, $c>0$. If u satisfies

$$\partial_t u - \Delta u + \frac{c}{t}u = 0 \tag{4.8}$$

then $v(x,t)=t^c u(x,t)$ satisfies the heat equation. Thus, if $u\ge 0$, there exists $\mu\in\mathfrak{M}_+(\mathbb{R}^n)$ such that

$$u(x,t) = t^{-c}\mathbb{H}[\mu](x,t). \tag{4.9}$$

This is a representation formula. Notice that $Vu(x,t) = ct^{-c-1}\mathbb{H}[\mu](x,t)$, therefore the regular set of $tr_{\{t=0\}}(u)$ may be empty.

PROPOSITION 4.4. *Assume V satisfies (1.16) and let $u \in W_{loc}(Q_T)$ be a positive solution of (1.17) with initial trace $(\mathcal{S}(u), \mu_u)$. Then $u \geq u_{\mu_u}$.*

Proof. We assume $\mathcal{S}(u) \neq \mathbb{R}^n$ otherwise the result is proved. Let G and E be open bounded domains such that $G \subset\subset E \subset\subset \mathcal{R}(u)$. Let $0 < \delta = \inf\{|x-y| : x \in G,\ y \in E^c\}$. Choose $R > 0$ such that $E \subset\subset B_R$. Let $\{t_j\}_{j=1}^{\infty}$ be a decreasing sequence converging to 0. We denote by u_j the weak solution of the problem

$$\begin{array}{ll} \partial_t v - \Delta v + Vv = 0 & \text{in } B_R \times (t_j, T] \\ v(x,t) = 0 & \text{on } \partial B_R \times (t_j, T] \\ v(.,t_j) = \chi_G u(.,t_j) & \text{in } B_R \times \{t_j\}, \end{array}$$

where χ is the characteristic function on G. Let v_j^R, be the solution

$$\begin{array}{ll} \partial_t v - \Delta v = 0 & \text{in } \mathbb{R}^n \times (t_j, \infty] \\ v(.,t_j) = \chi_G u(.,t_j) & \text{in } \mathbb{R}^n \times \{t_j\}. \end{array}$$

Then by maximum principle we have $u_j^R \leq u$ and $u_j^R \leq v_j$ in $B_R \times (t_j, T]$, for any $j \in \mathbb{N}$. By standard parabolic estimates, we may assume that the sequence u_j^R converges locally uniformly in $Q_T^{B_R}$ to a function $u^R \leq u$. Moreover, since $\chi_G \mu_u(.,t_j) \rightharpoonup \chi_G \mu_u$ in the weak* topology, we derive from the representation formula that $v_j \to \mathbb{H}[\chi_G \mu_u]$. Furthermore $u^R \leq v$, which implies $\chi_{(t_j,T)} u_j^R \to u^R$ in $L^1(Q_T^{B_R})$. There also holds

$$\int_{t_j}^T \int_{B_R} u_j^R V dxdt = \int_{t_j}^T \int_E u_j^R V dxdt + \int_{t_j}^T \int_{B_R \setminus E} u_j^R V dxdt,$$

and, by the choice of E and dominated convergence theorem,

$$\int_{t_j}^T \int_E u_j^R V dxdt \leq \int_0^T \int_E uV dxdt < \infty \Rightarrow \lim_{j\to\infty} \int_{t_j}^T \int_E u_j^R V dxdt = \int_0^T \int_E u^R V dxdt.$$

Furthermore, for any $x \in B_R \setminus E$,

$$v_j(x,t) = \left(\frac{1}{4\pi(t-t_j)}\right)^{\frac{n}{2}} \int_{R^n} e^{-\frac{|x-y|^2}{4(t-t_j)}} \chi_G u(y,t_j) dy \leq \left(\frac{1}{4\pi(t-t_j)}\right)^{\frac{n}{2}} e^{-\frac{\delta^2}{4(t-t_j)}} \int_G u(y,t_j) dy.$$

Next, since $V(x,t) \leq Ct^{-1}$ and $u_j^R \leq v_j$, we obtain

$$\lim_{j\to\infty} \int_{t_j}^T \int_{B_R \setminus E} u_j^R V dxdt = \int_0^T \int_{B_R \setminus E} u^R V dxdt, \tag{4.10}$$

by using the previous estimate and the fact that $\chi_G \mu_u(x,t_j) \rightharpoonup \chi_G \mu_u$ in the weak* topology. It follows $\chi_{(t_j,T)} V u_j^R \to V u^R$ in $L^1(Q_T^{B_R})$. There holds also $u_G^R \leq u$; by the maximum principle, the mapping $R \mapsto u_G^R$ is increasing and bounded from above by u. In view of Lemma 3.15,

$$\lim_{R\to\infty} u_G^R = u_G \leq u,$$

and u_G is a positive weak solution of

$$\begin{array}{ll} \partial_t v - \Delta v + Vv = 0 & \text{in } Q_T \\ v(.,0) = \chi_G \mu_u & \text{in } \mathbb{R}^n. \end{array}$$

Consider an increasing sequence $\{G_i\}_{i=1}^{\infty}$ of bounded open subsets, $G_i \subset\subset \mathcal{R}(u)$, with the property that $\bigcup_{i=1}^{\infty} G_i = \mathcal{R}(u)$. In view of Lemma 3.15 the sequence $\{u_i = u_{G_i}\}_{i=1}^{\infty}$ is increasing and converges to $\widetilde{u} \leq u$. Also we have

$$u_i(x,t) + \int_0^t \int_{\mathbb{R}^n} H(t-s,x,y) V u_i dy ds = \int_{\mathbb{R}^n} H(x-y,t) d\mu_i, \ \forall (x,t) \in Q_T,$$

where $\mu_i = \chi_{G_i}\mu$. Now since $\mu_i \rightharpoonup \mu_u$, by the monotone convergence theorem we have

$$\tilde{u}(x,t) + \int_0^t \int_{\mathbb{R}^n} H(t-s,x,y) V \tilde{u} dy ds = \int_{\mathbb{R}^n} H(x-y,t) d\mu_u, \ \forall (x,t) \in Q_T,$$

and $\tilde{u} \leq u$. this implies $\tilde{u} = u_{\mu_u}$, which ends the proof. □

Remark. Assumption (1.17) is too strong and has only been used in (4.10). It could have been replaced by the following much weaker one: for any $R > 0$ there exists a positive increasing function ϵ_R such that $\lim_{t\to 0} \epsilon(t) = 0$ satisfying

$$V(x,t) \leq e^{t^{-1}\epsilon_R(t)} \qquad \forall (x,t) \in Q_T^{B_R}. \tag{4.11}$$

We end this section with a result which shows that the stability of the initial value problem with respect to convergence the initial data in the weak* topology implies that the initial of positive solution has no singular part.

THEOREM 4.5. *Assume V satisfies, for some $\tau_0 > 0$,*

$$\lim_{|E|\to 0} \iint_E H(x-y,t) V(x,t+\tau) dx dt = 0, \qquad E \text{ Borel subset of } Q_T^{B_R} \tag{4.12}$$

for any $R > 0$, uniformly with respect to y is a compact set and $\tau \in [0,\tau_0]$. If u is a positive solution of (1.17), then $\mathcal{R}(u) = \mathbb{R}^n$

Proof. We assume that $\mathcal{S}(u) \neq \emptyset$ and if $z \in \mathcal{S}(u)$ there holds

$$\iint_{Q_T^{B_{r(z)}}} V u dx dt = \infty \qquad \forall r > 0.$$

In view of Proposition 4.1, there exist two sequences $\{r_k\}$ and $\{t_j\}$ decreasing to 0 such that

$$\lim_{j\to\infty} \int_{B_{r_k}(z)} u(x,t_j) dx = \infty \qquad \forall k \in \mathbb{N}.$$

For $k \in \mathbb{N}$ and $m > 0$ fixed, there exists $j(k)$ such that

$$\int_{B_{r_k}(z)} u(x,t_j) dx \geq m \qquad \forall j \geq j(k) \in \mathbb{N},$$

and there exists $\ell_k > 0$ such that

$$\int_{B_{r_k}(z)} \min\{u(x,t_{j(k)}), \ell_k\} dx = m$$

Furthermore $j(k) \to \infty$ when $k \to \infty$. Let $R > \max\{r_j : j = 1,2,...\}$ and u_k be the solution of

$$\begin{aligned} \partial_t v - \Delta v + V v &= 0 && \text{in } \mathbb{R}^n \times (t_{j(k)}, T] \\ v(.,t_j) &= \chi_{B_{r_k}(z)} \min\{u(.,t_{j(k)}), \ell_k\} && \text{in } \mathbb{R}^n \times \{t_{j(k)}\}, \end{aligned}$$

Then $\chi_{B_{r_k}(z)} \min\{u(.,t_{j(k)}), \ell_k\} \to m\delta_z$ in the weak sense of measures. By Proposition 5.5 we obtain that $u \geq u_k$ on $B_R(z) \times (t_{j(k)}, T]$. Applying Proposition 2.8, and

the remark here after, we conclude that $u_k(.,.+t_{j(k)}) \to u_{m\delta_z} = mu_{\delta_z}$ in $L^1_{loc}(\overline{Q}^T_R)$ This implies $u \geq mu_{\delta_z}$, and as m is arbitrary, $u = \infty$, contradiction. □

4.2. The sweeping method. In this subsection we adapt to equation (1.17) the sweeping method developed in [**19**] for constructing the boundary trace of solutions of stationnary Shrödinger equations. If $A \subset \mathbb{R}^n$ is a Borel set, we denote by

$$\mathfrak{M}_{T\,+}(A) = \{\mu \in \mathfrak{M}_+(\mathbb{R}^n) :\ \mu(A^c) = 0,\ \int_A e^{-\frac{|x|^2}{4T}} d\mu < \infty\}.$$

We recall that μ^* denotes the reduced measure associated to μ.

PROPOSITION 4.6. *Let $u \in W_{loc}(Q_T)$ be a positive solution of (1.17) with singular set $\mathcal{S}(u) \subsetneq \mathbb{R}^n$. If $\mu \in \mathfrak{M}_{T\,+}(\mathcal{S}(u))$, we set $v_\mu = \inf\{u, u_{\mu^*}\}$. Then*

$$\partial_t v_\mu - \Delta v_\mu + V v_\mu \geq 0 \qquad \text{in } Q_T,$$

and v_μ admits a boundary trace $\gamma_u(\mu) \in \tilde{\mathfrak{M}}_+(\mathcal{S}(u))$. The mapping $\mu \mapsto \gamma_u(\mu)$ is nondecreasing and $\gamma_u(\mu) \leq \mu$.

Proof. It is classical that $v_\mu := \inf\{u, u_{\mu^*}\}$ is a supersolution of (1.17) and $v_\mu \in \mathcal{E}_\nu(Q_T)$ as it holds with u_{μ^*} . The function

$$(x,t) \mapsto w(x,t) = \int_0^t \int_{\mathbb{R}^n} H(t-s,x,y)V(y,s)v_\mu(y,s)dyds$$

satisfies

$$\begin{aligned} \partial_t w - \Delta w - Vw = 0 \quad &\text{in } Q_T \\ w(.,0) = 0 \quad &\text{in } \mathbb{R}^n \times \{0\}. \end{aligned}$$

Thus $v_\mu + w$ is a nonnegative supersolution of the heat equation in Q_T. It admits an initial trace in $\mathfrak{M}_{T\,+}(\mathcal{S}(u))$ that we denote by $\gamma_u(\mu)$. Clearly $\gamma_u(\mu) \leq \mu^* \leq \mu$ since $v_\mu \leq u_{\mu^*}$ and $\gamma_u(\mu)$ is nondecreasing with respect to μ as it is the case with $\mu \mapsto u_{\mu^*}$ is. Finally, since v_μ is a positive supersolution, it is larger that the solution of 2.1 where the initial data μ is replaced by $\gamma_u(\mu)$, that is $u_{\gamma_u(\mu)} \leq v_\mu$. □

The proofs of the next four propositions are mere adaptations to the parabolic case of similar results dealing with elliptic equations and proved in [**19**]; we omit them.

PROPOSITION 4.7. *Let*

$$\nu_S(u) := \sup\{\gamma_u(\mu) :\ \mu \in \mathfrak{M}_{T\,+}(\mathcal{S}(u))\}.$$

Then $\nu_S(u)$ is a Borel measure on $\mathcal{S}(u)$.

DEFINITION 4.8. *The Borel measure $\nu(u)$ defined by*

$$\nu(u)(A) := \nu_S(u)(A \cap \mathcal{S}(u)) + \mu_u(A \cap \mathcal{R}(u)), \qquad \forall\ A \subset \mathbb{R}^n,\ A \text{ Borel},$$

is called the extended initial trace of u, denoted by $tr^e_{\{t=0\}}(u)$.

PROPOSITION 4.9. *If $A \subset \mathcal{S}(u)$ is a Borel set, then*

$$\nu_S(A) := \sup\{\gamma_u(\mu)(A) :\ \mu \in \mathfrak{M}_{T\,+}(A)\}.$$

PROPOSITION 4.10. *There always holds $\nu(\mathcal{S}ing_V(\mathbb{R}^n)) = 0$, where $\mathcal{S}ing_V(\mathbb{R}^n)$ is defined in (3.14).*

PROPOSITION 4.11. *Assume V satisfies condition (4.12). If u is a positive solution of (1.17), then $tr^e_{\{t=0\}}(u) = \mu_u \in \mathfrak{M}_{T\,+}(\mathbb{R}^n)$.*

5. Appendix: the case of a bounded domain

5.1. The subcritical case. Let Ω be a bounded domain with a C^2 boundary. We denote by $\mathfrak{M}(\Omega)$ the space of Radon measures in Ω, by $\mathfrak{M}_+(\Omega)$ its positive cone and by $\mathfrak{M}_\rho(\Omega)$ the space of Radon measures in Ω which satisfy

$$\int_\Omega \rho d|\mu| < \infty, \tag{5.1}$$

for some weight function $\rho : \Omega \mapsto \mathbb{R}_+$. As an important particular case $\rho(x) = d^\alpha(x)$, where $d(x) = \text{dist}\,(x, \partial\Omega)$ and $\alpha \geq 0$. We consider the linear parabolic problem

$$\begin{array}{rl} \partial_t u - \Delta u + Vu = 0, & \text{in } \ Q_T^\Omega = \Omega \times (0,T] \\ u = 0 & \text{on } \ \partial_l Q_T^\Omega = \partial\Omega \times (0,T] \\ u(.,0) = \mu & \text{in } \ \Omega. \end{array} \tag{5.2}$$

Definition 5.1. *We say that $\mu \in \mathfrak{M}_d(\Omega)$ is a good measure if the above problem has a weak solution u, i.e. there exists a function $u \in L^1(Q_T^\Omega)$, such that $Vu \in L^1_d(Q_T^\Omega)$ which satisfies*

$$-\int_0^T \int_\Omega u(\partial_t\phi + \Delta\phi)dxdt + \int_0^T \int_\Omega Vu\phi dxdt = \int_\Omega \phi(x,0)d\mu, \tag{5.3}$$

$\forall \phi \in C^{1,1;1}(\overline{Q_T^\Omega})$ which vanishes on $\partial_l Q_T^\Omega$ and satisfies $\phi(x,T) = 0$.

Definition 5.2. *Let $H^\Omega(x,y,t)$ be the heat kernel in Ω. Then we say that $\mu \in \mathfrak{M}_d(\Omega)$ is a admissible measure if*

$$||\mathbb{H}^\Omega[|\mu|]||_{L^1(Q_T^\Omega)} = \int_{Q_T^\Omega} \left(\int_\Omega H^\Omega(x-y,t)d|\mu(y)| \right) V(x,t)\psi(x)dxdt < \infty.$$

The next a proposition is direct consequence of [**14**, Lemma 2.4].

Proposition 5.3. *Assume $\mu \in \mathfrak{M}_d(\Omega)$ and let u be a weak solution of problem (5.2), then the following inequalities are valid*
(i)

$$||u||_{L^1(Q_T^\Omega)} + ||Vu||_{L^1_\psi(Q_T^\Omega)} \leq C(n,\Omega) \int_\Omega dd|\mu|,$$

(ii)

$$-\int_0^T \int_\Omega |u|(\partial_t\phi + \Delta\phi)dxdt + \int_0^T \int_\Omega |u|V\phi dxdt \leq \int_\Omega \phi(x,0)d|\mu|,$$

$\forall \phi \in C^{1,1;1}(\overline{Q_T^\Omega})$, $\phi \geq 0$.
(iii)

$$\lambda_\Omega \int_0^T \int_\Omega (x)u^+ dxdt + \int_0^T \int_\Omega Vu^+\psi dxdt \leq \int_\Omega \psi(x)d\mu^+.,$$

where ψ is the solution of

$$\begin{array}{rl} -\Delta\psi = 1, & \text{in } \ \Omega \\ \psi = 0 & \text{on } \ \partial\Omega. \end{array} \tag{5.4}$$

Proof. For (ii), in [**14**, Lemma 2.4, p 1456], above from the relation (2.39), we can take $\tilde{\zeta} = \gamma(u)\zeta$ for some $0 \leq \zeta \in C^{1,1;1}(\overline{Q_T^\Omega})$, since $u = 0$ on $\partial_l Q_T^\Omega$. For (iii) we consider (as in [**14**, Remark 2.5]) $\phi(x,t) = t\psi(x)$. The inequality holds by the same type of calculations as in [**19**]. □

PROPOSITION 5.4. *The problem (5.2) admits at most one solution. Furthermore, if μ is admissible, then there exists a unique solution; we denote it u_μ.*

Similarly as Theorem 2.7 and Proposition 2.7, we have the following stability results

PROPOSITION 5.5. *(i) Assume that V satisfies the stability condition*

$$\lim_{|E|\to 0}\iint_E H^\Omega(x,y,t)V(y,t)d(x)dydt = 0, \quad \forall E \subset Q_T^\Omega,\ E \text{ Borel.} \tag{5.5}$$

uniformly with respect to $y \in \Omega$. If $\{\mu_k\}$ is a bounded sequence in $\mathfrak{M}_d(\Omega)$ converging to μ in the dual sense of $\mathfrak{M}_d(\Omega)$, then (u_{μ_k}, Vu_{μ_k}) converges to (u_μ, Vu_μ) in $L^1(Q_T^\Omega) \times L^1_d(Q_T^\Omega)$. (ii) Furthermore if

$$\lim_{|E|\to 0}\iint_E H^\Omega(x,y,t+\tau_n)V(y,t)d(x)dydt = 0, \quad \forall E \subset Q_T^\Omega,\ E \text{ Borel.} \tag{5.6}$$

uniformly with respect to $y \in \Omega$ and $\tau_k \in [0,\tau_0]$ converges to 0 and $\{\mu_k\}$ is in (i), then the solutions u_{τ_k,μ_k} of the shifted problem

$$\begin{array}{rll} \partial_t u - \Delta u + Vu = 0 & on & \Omega \times (\tau_k, T) \\ u = 0 & on & \partial\Omega \times (\tau_k, T) \\ u(.,\tau_k) = \mu_k & on & \Omega \times \{\tau_k\} \end{array} \tag{5.7}$$

(extended by 0 on $(0,\tau_k)$) converge to u_μ in $L^1_d(Q_T^\Omega)$, and $\{Vu_{\mu_k}\}$ converges to Vu_μ in $L^1_d(Q_T^\Omega)$.

Proof. We can easily see that the measure μ_n is admissible and uniqueness holds; furthermore any admissible measure is a good measure is a good measure as in Theorem 2.5, and

$$\iint_{Q_T^\Omega} u_{\mu_n} dxds + \iint_{Q_T^\Omega} u_{\mu_n} V\psi dxds \le C\int_\Omega d\mu_n < C.$$

The remaining of the proof is similar to the one of Theorem 2.7. □

5.2. The supercritical case.

LEMMA 5.6. *Let $\{\mu_n\}_{n=1}^\infty$ be an increasing sequence of good measures converging to some measure μ in the weak* topology, then μ is good.*

Proof. Let u_{μ_n} be the weak solution of (5.2) with initial data μ_n. Then by Proposition 5.5 -(iii), $\{u_{\mu_n}\}$ is an increasing sequence. By 5.5 -(i) the sequence $\{u_{\mu_n}\}$ is bounded in $L^1(Q_T^\Omega)$. Thus $u_{\mu_n} \to u \in L^1(Q_T^\Omega)$. Also by (iii) of Proposition 5.5, we have that $Vu_{\mu_n} \to Vu$ in $L^1_\psi(Q_T^\Omega)$. Thus we can easily prove that u is a weak solution of (5.2) with μ as initial data. □

Let

$$Z_V^\Omega = \{x \in \Omega : \int_{Q_T^\Omega} H^\Omega(t,x,y)V(y)\psi(y)dy = \infty\}. \tag{5.8}$$

We note that, since $H^\Omega(t,x,y) \le H(x-y,t)$ for any bounded Ω with smooth boundary, it holds $Z_V^\Omega \subset Z_V$. By the same arguments as in [**19**] we can prove the following results

PROPOSITION 5.7. *Let μ be an admissible positive measure. Then $\mu(Z_V^\Omega) = 0$*

PROPOSITION 5.8. *Let $\mu \in \mathfrak{M}_{d+}(\Omega)$ such that $\mu(Z_V^\Omega) = 0$, then μ is good.*

PROPOSITION 5.9. *Let $\mu \in \mathfrak{M}_{d+}(\Omega)$ be a good measure. Then the following assertions are equivalent:*
(i) $\mu(Z_V^\Omega) = 0$.
(ii) There exists an increasing sequence of admissible measures $\{\mu_n\}$ which converges to μ in the weak-topology*

References

[1] Adams D. R. and Hedberg L. I., **Function spaces and potential theory**, Grundlehren Math. Wissen. **314**, Springer (1996).
[2] Ancona A. *Positive solutions of Schrdinger equations and fine regularity of boundary points*, **Math. Z. 272**, 429 (2012).
[3] Ancona A. *Positive solutions of Schrdinger equations and fine regularity of boundary points*, **Math. Z. 272**, 429 (2012).
[4] Aronson D., *Non-negative solutions of linear parabolic equations*, **Ann. Sc. Norm. Sup. Pisa Cl. Sc. 22**, 607-694 (1968).
[5] Baras P. and Goldstein J., *The heat equation with a singular potential*, **Trans. Amer. Math. Soc. 284**, 121-139 (1984).
[6] Choquet G., *Theory of capacities*, **Ann. Inst. Fourier 5**, 131-295 (1953-54).
[7] Doob J. L., **Classical potential theory and its probabilistic counterpart**, Reprint of the 1984 edition, Classics in Mathematics, Springer-Verlag, Berlin (2001).
[8] Dynkin E. B., *Diffusions, Superdiffusions and Partial Differential Equations*, American Math. Soc., Providence, Rhode Island, **Colloquium Publications 50** (2002).
[9] Fuglede B., *Le théorème du minimax et la théorie fine du potentiel*, **Ann. Inst. Fourier 15**, 65-87 (1965).
[10] Fuglede B., *Application du théorème minimax l'étude de diverses capacités*, **C.R. Acad. Sci. Paris 266**, 921-923 (1968).
[11] Gkikas K. and Véron L., *Classification of positive solutions of heat equation with supercritical absorption*, in preparation.
[12] Gulisashvili A. *On the heat equation with a time-dependent singular potential*, **J. Funct. Anal. 194**, 17-52 (2002).
[13] Kato T., *Shrödinger operators with singular potentials*, **Israel J. Math. 13**, 135-148 (1972).
[14] Marcus M. and Véron L., *Initial trace of positive solutions of some nonlinear parabolic equations*, **Comm. Part. Diff. Equ. 24**, 1445-1499 (1999).
[15] Marcus M. and Véron L., *Initial trace of positive solutions to semilinear parabolic inequalities*, **Adv. Nonlinear Studies 2**, 395-436 (2002)
[16] Pierfelice V., *Strichartz estimates for the Schrödinger and heat equations perturbed with singular and time dependent potentials*, **Asymptot. Anal. 47**, 1-18 (2004).
[17] Richard Y. and Véron L., *Isotropic singularities of solutions of nonlinear elliptic inequalities*, **Ann. Inst. H. Poincaré-Anal. Non Linéaire 6**, 37-72 (1989).
[18] Schnaubelt R. and Voigt J. *The non-autonomous Kato class*, **Arch. Math. 72**, 454-460 (1999).
[19] Véron L. and Yarur C., *Boundary value problems with measures for elliptic equations with singular potentials*, **J. Funct. Anal 262**, 733-772 (2012).

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