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# Initial value problems for diffusion equations with singular potential

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## Abstract

Let  $V$  be a nonnegative locally bounded function defined in  $Q_\infty := \mathbb{R}^n \times (0, \infty)$ . We study under what conditions on  $V$  and on a Radon measure  $\mu$  in  $\mathbb{R}^d$  does it exist a function which satisfies  $\partial_t u - \Delta u + Vu = 0$  in  $Q_\infty$  and  $u(\cdot, 0) = \mu$ . We prove the existence of a subcritical case in which any measure is admissible and a supercritical case where capacity conditions are needed. We obtain a general representation theorem of positive solutions when  $tV(x, t)$  is bounded and we prove the existence of an initial trace in the class of outer regular Borel measures.

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## 1 Introduction

In this article we study the initial value problem for the heat equation

$$\begin{aligned} \partial_t u - \Delta u + V(x, t)u &= 0 && \text{in } Q_T := \mathbb{R}^n \times (0, T) \\ u(\cdot, 0) &= \mu && \text{in } \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where  $V \in L_{loc}^\infty(Q_T)$  is a nonnegative function and  $\mu$  a Radon measure in  $\mathbb{R}^n$ . By a (weak) solution of (1.1) we mean a function  $u \in L_{loc}^1(\overline{Q}_T)$  such that  $Vu \in L_{loc}^1(\overline{Q}_T)$ , satisfying

$$-\iint_{Q_T} (\partial_t \phi + \Delta \phi) u dxdt + \iint_{Q_T} Vu \phi dxdt = \int_{\Omega} \zeta d\mu \tag{1.2}$$

for every function  $\zeta \in C_c^{1,1}(\overline{Q}_T)$  which vanishes for  $t = T$ . Besides the singularity of the potential at  $t = 0$ , there are two main difficulties which appear for constructing weak solutions : the growth of the measure at infinity and the concentration of the measure near some points in  $\mathbb{R}^n$ .

We denote by  $H(x, t) = \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$  the Gaussian kernel in  $\mathbb{R}^n$  and by  $\mathbb{H}[\mu]$  the corresponding heat potential of a measure  $\mu \in \mathfrak{M}(\mathbb{R}^n)$ . Thus

$$\mathbb{H}[\mu](x, t) = \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \int e^{-\frac{|x-y|^2}{4t}} d\mu(y), \tag{1.3}$$

whenever this expression have a meaning: for example it is straightforward that if  $\mu \in \mathfrak{M}(\mathbb{R}^n)$  satisfies

$$\|\mu\|_{\mathfrak{M}_T} := \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4T}} d|\mu|(y) < \infty, \tag{1.4}$$

then (1.3) has a meaning as long as  $t < T$ , and let be  $\mathfrak{M}_T(\mathbb{R}^n)$  the set of Radon measures in  $\mathbb{R}^n$  satisfying (1.4). If  $G \subset \mathbb{R}^n$ , let  $Q_T^G$  be the cylinder  $G \times (0, T)$ ,  $B_R(x)$  the ball of center  $x$  and radius  $R$  and  $B_R = B_R(0)$ . We prove

**Theorem A** *Let the measure  $\mu$  verifies*

$$\iint_{Q_T^{B_R}} \mathbb{H}[|\mu|](x, t) V(x, t) dxdt \leq M_R \quad \forall R > 0. \tag{1.5}$$

*Then (1.1) admits a solution in  $Q_T$ .*

A measure which satisfies (1.5) is called an *admissible measure* and a measure for which there exists a solution to problem (1.1) is called a *good measure*. Notice that even when  $V = 0$ , uniqueness without any restriction on  $u$  is not true, however the next uniqueness result holds:

**Theorem B** *Let  $u$  be a weak solution of (1.1) with  $\mu = 0$ . If  $u$  satisfies*

$$\iint_{Q_T} (1 + V(x, t)) e^{-\lambda|x|^2} |u(x, t)| dxdt < \infty \tag{1.6}$$

for some  $\lambda > 0$ , then  $u = 0$ .

We denote by  $\mathcal{E}_\nu(Q_T)$  the set of functions  $u \in L^1_{loc}(Q_T)$  for which (1.6) holds for some  $\lambda > 0$ . The general result we prove is the following.

**Theorem C** *Let  $\mu \in \mathfrak{M}(\mathbb{R}^n)$  be an admissible measure satisfying (1.4). Then there exists a unique solution  $u_\mu \in \mathcal{E}_\nu(Q_T)$  to problem (1.1). Furthermore*

$$\iint_{Q_T} \left( \frac{n}{2T} + V \right) |u| e^{-\frac{|x|^2}{4(T-t)}} dx dt \leq \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4T}} d|\mu|(y). \quad (1.7)$$

We consider first the *subcritical case*, which means that any positive measure satisfying (1.4) is a good measure and we prove that such is the case if for any  $R > 0$  there exist  $m_R > 0$  such that

$$\iint_{Q_T^{B_R}} H(x-y, t) V(x, t) dx dt \leq m_R e^{-\frac{|y|^2}{4T}}. \quad (1.8)$$

Moreover we prove a stability result among the measures satisfying (1.4): if  $V$  verifies for all  $R > 0$

$$\sup_{y \in \mathbb{R}^n} e^{\frac{|y|^2}{4T}} \iint_E H(x-y, t) V(x, t) dx dt \rightarrow 0 \quad \text{when } |E| \rightarrow 0, \quad E \text{ Borel subset of } Q_T^{B_R}, \quad (1.9)$$

then if  $\{\mu_k\}$  is a sequence of Radon measures bounded in  $\mathfrak{M}_T(\mathbb{R}^n)$  which converges in the weak sense of measures to  $\mu$ , then  $\{(u_{\mu_k}, V u_{\mu_k})\}$  converges to  $(u_\mu, V u_\mu)$  in  $L^1_{loc}(\overline{Q_T})$ .

In the *supercritical case*, that is when not any measure in  $\mathfrak{M}_T(\mathbb{R}^n)$  is a good measure, we develop a capacity framework in order to characterize the good measures. We denote by  $\mathfrak{M}^V(\mathbb{R}^n)$  the set of Radon measures such that  $V\mathbb{H}[\mu] \in L^1(Q_T)$  and  $\|\mu\|_{\mathfrak{M}^V} := \|V\mathbb{H}[\mu]\|_{L^1}$ . If  $E \subset Q_T$  is a Borel set, we set

$$C_V(E) = \sup\{\mu(E) : \mu \in \mathfrak{M}_+^V(\mathbb{R}^n), \mu(E^c) = 0, \|\mu\|_{\mathfrak{M}^V} \leq 1\}. \quad (1.10)$$

This defines a capacity. If

$$C_V^*(E) = \inf\{\|f\|_{L^\infty} : \check{H}[f](y) \geq 1 \quad \forall y \in E\}, \quad (1.11)$$

where

$$\check{H}[f](y) = \iint_{Q_T} H(x-y, t) V(x, t) f(x, t) dx dt = \int_0^T \mathbb{H}[Vf](y, t) dt \quad \forall y \in \mathbb{R}^n, \quad (1.12)$$

then  $C_V^*(E) = C_V(E)$  for any compact set. Denote by  $Z_V$  the *singular set of  $V$* , that is the largest set with zero  $C_V$  capacity. Then

$$Z_V = \{x \in \mathbb{R}^n : \iint_{Q_T} H(x-y, t) V(y, t) dx dt = \infty\}, \quad (1.13)$$

and the following result characterizes the good measures.

**Theorem D** *If  $\mu$  is an admissible measure then  $\mu(Z_V) = 0$ . If  $\mu \in \mathfrak{M}_T(\mathbb{R}^n)$  satisfies  $\mu(Z_V) = 0$ , then it is a good measure. Furthermore  $\mu$  is a positive good measure if and only if there exists*

an increasing sequence of positive admissible measures  $\{\mu_k\}$  which converges to  $\mu$  in the weak \* topology.

Since many important applications deal with the nonlinear equation

$$\partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } Q_\infty := \mathbb{R}^n \times (0, \infty), \quad (1.14)$$

where  $q > 1$  and due to the fact that *any* solution defined in  $Q_\infty$  satisfies

$$|u(x, t)|^{q-1} \leq \frac{1}{t(q-1)} \quad \forall (x, t) \in Q_\infty, \quad (1.15)$$

we shall concentrate on potentials  $V$  which satisfy

$$0 \leq V(x, t) \leq \frac{C_1}{t} \quad \forall (x, t) \in Q_T, \quad (1.16)$$

for some  $C_1 > 0$ . For such potentials we prove the existence of a representation theorem for positive solutions of

$$\partial_t u - \Delta u + V(x, t)u = 0 \quad \text{in } Q_T. \quad (1.17)$$

If  $u$  is a positive solution of (1.1) in  $Q_T$  with  $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$ , it is the increasing limit of the solutions  $u = u_R$  of

$$\begin{aligned} \partial_t u - \Delta u + V(x, t)u &= 0 && \text{in } Q_T^{B_R} \\ u &= 0 && \text{in } \partial B_R \times (0, T) \\ u(\cdot, 0) &= \chi_{B_R} \mu && \text{in } B_R, \end{aligned} \quad (1.18)$$

when  $R \rightarrow \infty$ , thus there exists a positive function  $H_V \in C(\mathbb{R}^n \times \mathbb{R}^n \times (0, T))$  such that

$$u(x, t) = \int_{\mathbb{R}^N} H_V(x, y, t) d\mu(y). \quad (1.19)$$

Furthermore we show how to construct  $H_V$  from  $V$  and we prove the following formula

$$H_V(x, y, t) = \int_{\mathbb{R}^N} e^{\psi(x, t)} \Gamma(x, \xi, t) d\mu_y(\xi), \quad (1.20)$$

where  $\mu_y$  is a Radon measure such that

$$\delta_y \geq \mu_y, \quad (1.21)$$

( $\delta_y$  is the Dirac measure concentrated at  $y$ ),

$$\psi(x, t) = \int_t^T \int_{\mathbb{R}^n} \left( \frac{1}{4\pi(s-t)} \right)^{\frac{n}{2}} e^{-\frac{|x-y|^2}{4(s-t)}} V(y, s) dy ds \quad (1.22)$$

and  $\Gamma$  satisfies the following estimate

$$c_1 t^{-\frac{n}{2}} e^{-\gamma_1 \frac{|x-y|^2}{t}} \leq \Gamma(x, y, t) \leq c_2 t^{-\frac{n}{2}} e^{-\gamma_2 \frac{|x-y|^2}{t}} \quad (1.23)$$

where  $A_i, c_i$  depends on  $T, d$  and  $V$ . Conversely, we first prove the following representation result

**Theorem E** Assume  $V$  satisfies (1.16). If  $u$  is a positive solution of (1.1) in  $Q_T$ , there exists a positive Radon measure  $\mu$  in  $\mathbb{R}^n$  such that (1.19) holds.

If  $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$  is positive, we can define for any  $k > 0$  the solution  $u_k$  of

$$\begin{aligned} \partial_t u - \Delta u + V_k(x, t)u &= 0 && \text{in } Q_T \\ u(\cdot, 0) &= \mu && \text{in } \mathbb{R}^n, \end{aligned} \quad (1.24)$$

where  $V_k(x, t) = \min\{k, V(x, t)\}$ , and

$$u_k(x, t) = \int_{\mathbb{R}^n} H_{V_k}(x, y, t) d\mu(y). \quad (1.25)$$

Moreover  $\{H_{V_k}\}$  and  $\{u_k\}$  decrease respectively to  $H_V$  and  $u^*$  there holds

$$u^*(x, t) = \int_{\mathbb{R}^n} H_V(x, y, t) d\mu(y). \quad (1.26)$$

However  $u^*$  is not a solution of (1.1), but of a relaxed problem where  $\mu$  is replaced by a smaller measure  $\mu^*$  called *the reduced measure associated to  $\mu$* . If we define the *zero set of  $V$*  by

$$\text{Sing}_V := \{y \in \mathbb{R}^n : H_V(x, y, t) = 0\}, \quad (1.27)$$

we prove

**Theorem F** If

$$\limsup_{t \rightarrow 0} \int_t^T \int_{\mathbb{R}^n} \left( \frac{1}{4\pi(s-t)} \right)^{\frac{n}{2}} e^{-\frac{|\xi-y|^2}{4(s-t)}} V(y, s) dy ds = \infty, \quad (1.28)$$

then

$$\xi \in \text{Sing}_V, \text{ i.e. } H_V(x, \xi, t) = 0, \forall (x, t) \in \mathbb{R}^n \times (0, \infty).$$

We note here that if  $V$  satisfies (1.28) then  $\delta_\xi$  is not admissible measure and the reduced measure  $(\delta_\xi)^* = \mu_\xi$  associated to  $\delta_\xi$  is zero.

**Theorem G** Assume  $V$  satisfies (1.15) and  $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$ . Then

- (i)  $\text{supp}(\mu - \mu^*) \subset \text{Sing}_V$ .
- (ii) If  $\mu(\text{Sing}_V) = 0$ , then  $\mu^* = 0$ .
- (iii)  $\text{Sing}_V = Z_V$ .

The last section is devoted to the initial trace problem: to any positive solution  $u$  of (1.1) we can associate an open subset  $\mathcal{R}(u) \subset \mathbb{R}^n$  which is the set of points  $y$  which possesses a neighborhood  $U$  such that

$$\iint_{Q_T^U} V(x, t)u(x, t) dx dt < \infty. \quad (1.29)$$

There exists a positive Radon measure  $\mu_u$  on  $\mathcal{R}(u)$  such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} u(x, t)\zeta(x) dx = \int_{\mathbb{R}^n} \zeta d\mu \quad \forall \zeta \in C_c(\mathcal{R}(u)). \quad (1.30)$$

The set  $\mathcal{S}(u) = \mathbb{R}^n \setminus \mathcal{R}(u)$  is the set of points  $y$  such that for any open set  $U$  containing  $y$ , there holds

$$\iint_{Q_T^U} V(x, t)u(x, t)dxdt = \infty. \quad (1.31)$$

If  $V$  satisfies (1.17),  $\mathcal{S}(u)$  it has the property that

$$\limsup_{t \rightarrow 0} \int_U u(x, t)dx = \infty. \quad (1.32)$$

Furthermore, if it satisfies (1.9), then  $\mathcal{S}(u) = \emptyset$ .

An alternative construction of the initial trace based on the sweeping method is also developed.

## 2 The subcritical case

Let  $Q_T = \mathbb{R}^n \times (0, T]$ . In this section we consider the linear parabolic problem

$$\begin{aligned} \partial_t u - \Delta u + Vu &= 0 & \text{in } Q_T \\ u(\cdot, 0) &= \mu & \text{in } \mathbb{R}^n \times \{0\}, \end{aligned} \quad (2.1)$$

where  $V \in L_{loc}^1(Q_T)$  is nonnegative and  $\mu$  is a Radon measure.

**Definition 2.1** We say that  $\mu \in \mathfrak{M}(\mathbb{R}^n)$  is a good measure if problem (2.1) has a weak solution  $u$  i.e. there exists a function  $u \in L_{loc}^1(\overline{Q}_T)$ , such that  $Vu \in L_{loc}^1(\overline{Q}_T)$  which satisfies

$$- \iint_{Q_T} u(\partial_t \phi + \Delta \phi)dxdt + \iint_{Q_T} Vu\phi dxdt = \int_{\mathbb{R}^n} \phi(x, 0)d\mu \quad \forall \phi \in X(Q_T), \quad (2.2)$$

where  $X(Q_T)$  is the space of test functions defined by

$$X(Q_T) = \{\phi \in C_c(\overline{Q}_T), \partial_t \phi + \Delta \phi \in L_{loc}^\infty(\overline{Q}_\infty), \phi(x, T) = 0\}$$

**Definition 2.2** Let  $H(x, t)$  be the heat kernel of heat equation in  $\mathbb{R}^n$ , we say that  $\mu \in \mathfrak{M}(\mathbb{R}^n)$  is an admissible measure if

(i)

$$\|V\mathbb{H}[|\mu|]\|_{L^1(Q_T^{B_R})} = \iint_{Q_T^{B_R}} \left( \int_{\mathbb{R}^n} H(x-y, t)d|\mu(y)| \right) V(x, t)dxdt < M_{R,T}$$

where  $M_{R,T}$  is a positive constant.

**Definition 2.3** A function  $u(x, t)$  will be said to belong to the class  $\mathcal{E}_V(Q_T)$  if there exists  $\lambda > 0$  such that

$$\iint_{Q_T} e^{-\lambda|x|^2} |u(x, t)|(1 + V(x, t))dxdt < \infty.$$

A measure in  $\mathbb{R}^n$  belongs to the class  $\mathfrak{M}_T(\mathbb{R}^n)$  if

$$\|\mu\|_{\mathfrak{M}_T} := \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4T}} d|\mu| < \infty.$$

**Lemma 2.4** *There exists at most one weak solution of problem (2.1) in the class  $\mathcal{E}_V(Q_T)$ .*

*Proof.* Let  $u_1$  and  $u_2$  be two solutions in the class  $\mathcal{E}_V(Q_T)$  then  $w = u_1 - u_2$  is a solution with initial data 0. Choose a standard mollifier  $\rho : B(0, 1) \mapsto [0, 1]$  and define

$$w_j(x, t) = j^n \int_{B_{\frac{1}{j}}(x)} \rho(j(x-y))w(y, t)dy \equiv \int_{B_{\frac{1}{j}}(x)} \rho_j(x-y)w(y, t)dy.$$

Then  $w_j(\cdot, t)$  is  $C^\infty$  and from the equation satisfied by  $w$ , it holds

$$\partial_t w_j - \Delta w_j + \int_{B_{\frac{1}{j}}(x)} V(y, t)\rho_j(x-y)w(y)dy = 0,$$

where  $\partial_t w_j$  is taken in the weak sense.

First we consider the case  $\lambda > 0$  and  $t \leq \min\{\frac{1}{16\lambda}, T\} = T'$ .

Set  $\phi(x, t) = \xi(x, t)\zeta(x)$ , where  $\xi(x, t) = e^{-\frac{|x|^2}{4(\lambda x - t)}}$  and  $\zeta \in C_c^\infty(\mathbb{R}^n)$ . Given  $\varepsilon > 0$  we define

$$g_j = \sqrt{w_j^2 + \varepsilon}.$$

Because  $\partial_t(g_j\phi) = \frac{\partial_t w_j}{\sqrt{w_j^2 + \varepsilon}}\phi + g_j\partial_t\phi$ , by a straightforward calculation we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left[ g_j\phi(\cdot, s) \right]_{s=t}^{s=0} dx &= \iint_{Q_t} \frac{w_j}{\sqrt{w_j^2 + \varepsilon}} \phi \Delta w_j dx ds \\ &\quad - \iint_{Q_t} \frac{w_j(x, s)}{\sqrt{w_j^2(x, s) + \varepsilon}} \phi(x, s) \left( \int_{B_{\frac{1}{j}}(x)} V(y, t)\rho_j(x-y)w(y, s)dy \right) dx ds \\ &\quad + \iint_{Q_t} g_j \phi_s dx ds \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By integration by parts, we obtain

$$\begin{aligned} I_1 &= - \iint_{Q_t} \frac{|\nabla w_j|^2}{\sqrt{w_j^2(x, s) + \varepsilon}} \phi dx ds + \iint_{Q_t} \frac{|\nabla w_j|^2 w_j^2}{(w_j^2(x, s) + \varepsilon)^{\frac{3}{2}}} \phi dx ds - \iint_{Q_t} \frac{w_j}{\sqrt{w_j^2 + \varepsilon}} \nabla w_j \cdot \nabla \phi dx ds \\ &\leq - \iint_{Q_t} \frac{w_j}{\sqrt{w_j^2 + \varepsilon}} \nabla w_j \cdot \nabla \phi dx ds \\ &\leq - \iint_{Q_t} \nabla g_j \cdot \nabla \phi dx ds \\ &= - \iint_{Q_t} \zeta \nabla g_j \cdot \nabla \xi dx ds - \iint_{Q_t} \xi \nabla g_j \cdot \nabla \zeta dx ds \\ &= \iint_{Q_t} \zeta g_j \Delta \xi dx ds + \iint_{Q_t} g_j \nabla \zeta \cdot \nabla \xi dx ds. \end{aligned}$$



Since  $t \leq T$ , there holds  $\xi|\nabla g_j| \in L^1(Q_{T'})$ ,  $\xi g_j \in L^1(Q_{T'})$ ,  $|\Delta \xi|g_j \in L^1(Q_{T'})$ ,  $\partial_s \xi g_j \in L^1(Q_{T'})$  and

$$\iint_{Q_t} \frac{w_j(x, s)}{\sqrt{w_j^2(x, s) + \varepsilon}} \left( \int_{B_{\frac{1}{j}}(x)} V(y, t) \rho_j(x - y) w(y, s) dy \right) \xi dx ds < \infty.$$

The reason for which  $\xi|\nabla g_j| \in L^1(Q_{T'})$  follows from the next inequality

$$\begin{aligned} \iint_{Q_{T'}} |\nabla g_j| \xi dx ds &= \iint_{Q_{T'}} \frac{|\nabla w_j|}{\sqrt{\varepsilon + w_j^2}} \xi dx ds \\ &\leq \iint_{Q_{T'}} e^{-\frac{|x|^2}{4(\frac{1}{8\lambda} - t)}} \left( \int_{B_{\frac{1}{j}}(x)} |\nabla \rho_j(x - y)| w(y, s) dy \right) dx ds. \end{aligned}$$

Since  $\forall y \in B_{\frac{1}{j}}(x)$ , we have  $|x|^2 \geq (|y|^2 - \frac{1}{j^2})^2 = |y|^2 + \frac{1}{j^2} - 2\frac{|y|}{j} \geq \frac{|y|^2}{2} - (C - 1)\frac{1}{j^2}$ , for some positive constant  $C > 0$  independent on  $j, y$  and  $x$ . Thus we have, using the fact that  $e^{-\lambda|y|^2} w \in L^1(Q_T)$ ,

$$\iint_{Q_{T'}} |\nabla g_j| \xi dx ds \leq C(j, \lambda) \iint_{Q_{T'}} \int_{B_{\frac{1}{j}}(x)} e^{-\frac{|y|^2}{8(\frac{1}{8\lambda} - t)}} |\nabla \rho_j(x - y)| w(y, s) dy dx ds < \infty.$$

Also

$$\iint_{Q_t} \frac{w_j(x, s)}{\sqrt{w_j^2(x, s) + \varepsilon}} \xi \left( \int_{B_{\frac{1}{j}}(x)} V(y, t) \rho_j(x - y) w(y, s) dy \right) dx ds \xrightarrow{j \rightarrow \infty} \iint_{Q_t} \frac{w^2(x, s)}{\sqrt{w^2(x, s) + \varepsilon}} \xi V(y, t) dx ds$$

and

$$\int_{\mathbb{R}^n} \sqrt{w_j^2(x, s) + \varepsilon} (\xi_s + \Delta \xi) dx ds \xrightarrow{j \rightarrow \infty} \int_{\mathbb{R}^n} \sqrt{w^2(x, s) + \varepsilon} (\xi_s + \Delta \xi) dx ds.$$

We choose  $\zeta_R = 1$  in  $B_R$ ,  $0 \leq \zeta_R \leq 1$  in  $B_{R+1} \setminus B_R$  and 0 otherwise. Letting successively  $j \rightarrow \infty$ ,  $R \rightarrow \infty$  and finally  $\varepsilon \rightarrow 0$ , we derive

$$\int_{\mathbb{R}^n} |w(x, t)| \xi(x, t) dx \leq \iint_{Q_t} |w| (\xi_s + \Delta \xi) dx ds - \iint_{Q_t} w(x, s) \xi V(y, t) dx ds.$$

Since

$$\xi_s + \Delta \xi = -\frac{n}{2(\frac{1}{8\lambda} - s)},$$

and  $V \geq 0$ , we have  $w(x, t) = 0 \forall (x, t) \in Q_{T'}$ . If  $T' = T$  this complete the proof for  $\lambda \geq 0$ , otherwise the proof can be completed by a finite number of iterations of the same argument on  $\mathbb{R}^n \times (T', 2T')$ ,  $\mathbb{R}^n \times (2T', 3T')$ , etc. If  $\lambda = 0$  we set  $\xi = 1$  and the result follows by similar argument  $\square$

**Theorem 2.5** *If  $\mu \in \mathfrak{M}_T(\mathbb{R}^n)$  is an admissible measure, there exists a unique  $u = u_\mu \in \mathcal{E}_V(Q_T)$  solution of (2.1). Furthermore the following estimate holds*

$$\frac{n}{2T} \iint_{Q_T} |u| e^{-\frac{|x|^2}{4(T-t)}} dx ds + \iint_{Q_T} |u| V e^{-\frac{|x|^2}{4(T-t)}} dx ds \leq \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4T}} d|\mu|. \quad (2.3)$$

*Proof.* First we assume that  $\mu \geq 0$ . Let  $\mu_R = \chi_{B_R}\mu$ . It is well known that the heat kernel  $H^{B_R}(x, y, t)$  in  $\Omega = B_R$  is increasing with respect to  $R$  and  $H^{B_R} \rightarrow H$ , as  $R \rightarrow \infty$  in  $L^1(Q_T)$  for any  $T > 0$ . Thus  $\mu_R$  is an admissible measure in  $B_R$  and by Proposition 5.4, there exists a unique weak solution  $u_R$  of problem 5.2 on  $\Omega = B_R$ . By (ii) of Proposition 5.5 we have

$$-\iint_{Q_T} |u_R|(\partial_t \phi + \Delta \phi) dxdt + \iint_{Q_T} |u_R|V\phi dxdt \leq \int_{B_R} \phi(x, 0) d|\mu_R|.$$

If we set  $\phi_\varepsilon(x, t) = e^{-\frac{|x|^2}{4(T+\varepsilon-t)}}$ ;  $\varepsilon > 0$ , then

$$\partial_t \phi + \Delta \phi = -\frac{n}{2(T+\varepsilon-t)} e^{-\frac{|x|^2}{4(T+\varepsilon-t)}},$$

thus we have

$$\iint_{Q_T} |u_R| \frac{n}{2(T+\varepsilon-t)} e^{-\frac{|x|^2}{4(T+\varepsilon-t)}} dxdt + \iint_{Q_T} |u_R|V e^{-\frac{|x|^2}{4(T+\varepsilon-t)}} dxdt \leq \int_{B_R} e^{-\frac{|x|^2}{4T+4\varepsilon}} d\mu_R,$$

which implies

$$\frac{n}{2T+\varepsilon} \int_0^T \int_{B_R} |u_R| e^{-\frac{|x|^2}{4(T+\varepsilon-t)}} dxdt + \int_0^T \int_{B_R} |u_R|V e^{-\frac{|x|^2}{4(T+\varepsilon-t)}} dxdt \leq \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4T+4\varepsilon}} d\mu_R.$$

Letting  $\varepsilon \rightarrow 0$ , we derive

$$\frac{n}{2T} \iint_{Q_T} |u_R| e^{-\frac{|x|^2}{4(T-t)}} dxdt + \iint_{Q_T} |u_R|V e^{-\frac{|x|^2}{4(T-t)}} dxdt \leq \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4T}} d\mu_R \leq \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4T}} d\mu.$$

Now by the maximum principle  $\{u_R\}$  is increasing with respect to  $R$  and converges to some function  $u$ . By the above inequality  $u \in \mathcal{E}_V(Q_T)$  satisfies the estimate (2.5) and  $u$  is a weak solution of problem (2.1). By Lemma 2.4 it is unique. In the general case we write  $\mu = \mu^+ - \mu^-$  and the result follows by the above arguments and Lemma 2.4. *In the sequel we shall denote by  $u_\mu$  this unique solution.*  $\square$

**Definition 2.6** A potential  $V$  is called subcritical in  $Q_T$  if for any  $R > 0$  there exists  $m_R > 0$  such that

$$\iint_{Q_T^{B_R}} H(x-y, t)V(x, t) dxdt \leq m_R e^{-\frac{|y|^2}{4T}} \quad \forall y \in \mathbb{R}^n. \quad (2.4)$$

It is called strongly subcritical if moreover

$$e^{\frac{|y|^2}{4T}} \iint_E H(x-y, t)V(x, t) dxdt \rightarrow 0 \quad \text{when } |E| \rightarrow 0, \quad E \text{ Borel subset of } Q_T^{B_R}, \quad (2.5)$$

uniformly with respect to  $y \in \mathbb{R}^n$

**Theorem 2.7** Assume  $V$  is subcritical. Then any measure in  $\mathfrak{M}_T(\mathbb{R}^n)$  is admissible. Furthermore, if  $V$  is strongly subcritical and  $\{\mu_k\}$  is a sequence of measures uniformly bounded in  $\mathfrak{M}_T(\mathbb{R}^n)$  which converges weakly to  $\mu$ , then the corresponding solutions  $\{u_{\mu_k}\}$  converge to  $u_\mu$  in  $L_{loc}^1(\overline{Q_T})$ , and  $\{Vu_{\mu_k}\}$  converges to  $Vu_\mu$  in  $L_{loc}^1(\overline{Q_T})$ .

*Proof.* For the first statement we can assume  $\mu \geq 0$  and there holds

$$\begin{aligned} \iint_{Q_T^{B_R}} H(x-y, t) d\mu(y) V(x, t) dx dt &= \int_{\mathbb{R}^n} \left( \iint_{Q_T^{B_R}} H(t, x-y) V(x, t) dx dt \right) d\mu(y) \\ &\leq m_R \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4T}} d\mu(y) \\ &\leq m_R \|\mu\|_{\mathfrak{M}_T}. \end{aligned}$$

Thus  $\mu$  is admissible. For the second statement, we assume first that  $\mu_k \geq 0$ . By lower semi-continuity  $\mu \in \mathfrak{M}_T(\mathbb{R}^N)$  and  $\|V\mathbb{H}[\mu]\|_{L^1(Q_T^{B_R})} \leq M_{R,T}$  for any  $k$ . Since  $0 \leq u_{\mu_k} \leq \mathbb{H}[\mu_k]$  and  $\mathbb{H}[\mu_k] \rightarrow \mathbb{H}[\mu]$  in  $L^1_{loc}(\overline{Q}_T)$ , the sequence  $\{u_{\mu_k}\}$  is uniformly integrable and thus relatively compact in  $L^1_{loc}(\overline{Q}_T)$ . Furthermore  $0 \leq Vu_{\mu_k} \leq V\mathbb{H}[\mu_k]$ . Let  $E \subset Q_T^{B_R}$  be a Borel subset, then

$$\begin{aligned} \iint_E V\mathbb{H}[\mu_k] dx dt &= \int_{\mathbb{R}^n} \left( \iint_E V H(x-y, t) dx dt \right) d\mu_k(y) \\ &= \int_{\mathbb{R}^n} \left( e^{-\frac{|y|^2}{4T}} \iint_E V(x) H(x-y, t) dx dt \right) e^{-\frac{|y|^2}{4T}} d\mu_k(y) \\ &\leq \epsilon(|E|) \|\mu_k\|_{\mathfrak{M}_T}, \end{aligned}$$

where  $\epsilon(r) \rightarrow 0$  as  $r \rightarrow 0$ . Thus  $\{(u_{\mu_k}, Vu_{\mu_k})\}$  is locally compact in  $L^1_{loc}(\overline{Q}_T)$  and, using a diagonal sequence, there exist  $u \in L^1_{loc}(\overline{Q}_T)$  with  $Vu \in L^1_{loc}(\overline{Q}_T)$  and a subsequence  $\{k_j\}$  such that  $\{(u_{\mu_{k_j}}, Vu_{\mu_{k_j}})\}$  converges to  $(u, Vu)$  a.e. and in  $L^1_{loc}(\overline{Q}_T)$ . From the integral expression (2.2) satisfied by the  $u_{\mu_k}$ ,  $u$  is a weak solution of problem (2.1). Since the  $u_{\mu_k}$  satisfy (2.3), the property holds for  $u$ , therefore  $u = u_\mu$  is the unique solution of (2.1), which ends the proof.  $\square$

As a variant of the above result which will be useful later on we have

**Proposition 2.8** *Assume  $V$  satisfies*

$$e^{-\frac{|y|^2}{4T}} \iint_E H(x-y, t) V(x, t + \tau) dx dt \rightarrow 0 \quad \text{when } |E| \rightarrow 0, \quad E \text{ Borel subset of } Q_T^{B_R}, \quad (2.6)$$

*uniformly with respect to  $y \in \mathbb{R}^n$  and  $\tau \in [0, \tau_0]$ . Let  $\tau_k > 0$  with  $\tau_k \rightarrow 0$  and  $\{\mu_k\}$  be a sequence uniformly bounded in  $\mathfrak{M}_T(\mathbb{R}^N)$  which converges weakly to  $\mu$ . Then the solutions  $\{u_{\tau_k, \mu_k}\}$  of*

$$\begin{aligned} \partial_t u - \Delta u + Vu &= 0 & \text{on } \mathbb{R}^n \times (\tau_k, T) \\ u(\cdot, \tau_k) &= \mu_k & \text{on } \mathbb{R}^n \times \{\tau_k\} \end{aligned} \quad (2.7)$$

*(extended by 0 on  $(0, \tau_k)$ ) converge to  $u_\mu$  in  $L^1_{loc}(Q_T)$ , and  $\{Vu_{\mu_k}\}$  converges to  $Vu_\mu$  in  $L^1_{loc}(Q_T)$ .*

Condition (2.5) may be very difficult to verify and we give below a sufficient condition for it to hold.

**Proposition 2.9** *Assume  $V$  satisfies*

$$\lim_{\lambda \rightarrow 0} e^{\frac{|y|^2}{4T}} \lambda^{-n} \int_0^\lambda \int_{B_{\lambda^2}(y)} V(x, t) dx dt = 0 \quad (2.8)$$

*uniformly with respect to  $y \in \mathbb{R}^n$ , then  $V$  is strongly subcritical.*

*Proof.* Let  $E \subset Q_T^{BR}$  be a Borel set. For  $\delta > 0$ , we define the weighted heat ball of amplitude  $\delta e^{-\frac{|y|^2}{4T}}$  by

$$P_\delta = P_\delta(y, T) = \left\{ (x, t) \in Q_T : H(x - y, t) \geq \delta e^{-\frac{|y|^2}{4T}} \right\}.$$

By an straightforward computation, one sees that

$$P_\delta(y, T) \subset B_{\frac{1}{a_n \delta^{-\frac{1}{n}} e^{\frac{|y|^2}{4nT}}}}(y) \times [0, b_n \delta^{-\frac{2}{n}} e^{\frac{|y|^2}{2nT}}] := R_\delta(y, T),$$

for some  $a_n, b_n > 0$ . We write

$$\iint_E H(x - y, t) V(x, t) dx dt = \iint_{E \cap P_\delta} H(x - y, t) V(x, t) dx dt + \iint_{E \cap P_\delta^c} H(x - y, t) V(x, t) dx dt.$$

Then

$$\iint_{E \cap P_\delta^c} H(x - y, t) V(x, t) dx dt \leq \delta e^{-\frac{|y|^2}{4T}} \iint_E V(x, t) dx dt,$$

and

$$\begin{aligned} \iint_{E \cap P_\delta} H(x - y, t) V(x, t) dx dt &\leq \int_0^\delta \int_{\{(x, t) \in Q_T^{BR} : H(x - y, t) = \tau e^{-\frac{|y|^2}{4T}}\}} V(x, t) dS_\tau(x, t) \tau d\tau \\ &\leq \left[ \tau \int_0^\tau \int_{\{(x, t) \in Q_T^{BR} : H(x - y, t) = \sigma e^{-\frac{|y|^2}{4T}}\}} V(x, t) dS_\sigma(x, t) d\sigma \right]_{\tau=0}^{\tau=\delta} \\ &\quad - \int_0^\delta \int_0^\tau \int_{\{(x, t) \in Q_T^{BR} : H(x - y, t) = \sigma e^{-\frac{|y|^2}{4T}}\}} V(x, t) dS_\sigma(x, t) d\sigma d\tau \\ &\leq \delta \int_0^\delta \int_{\{(x, t) \in Q_T^{BR} : H(x - y, t) = \sigma e^{-\frac{|y|^2}{4T}}\}} V(x, t) dS_\sigma(x, t) d\sigma. \end{aligned}$$

The first integration by parts is justified since  $V \in L^1(Q_T^{BR})$ . Notice that

$$\delta \int_0^\delta \int_{\{(x, t) \in Q_T^{BR} : H(x - y, t) = \sigma e^{-\frac{|y|^2}{4T}}\}} V(x, t) dS_\sigma(x, t) d\sigma = \delta \iint_{Q_T^{BR} \cap P_\delta} V(x, t) dx dt$$

and

$$\begin{aligned} \delta \iint_{Q_T^{BR} \cap P_\delta} V(x, t) dx dt &\leq \delta \iint_{Q_T^{BR} \cap R_\delta(y, T)} V(x, t) dx dt \\ &\leq \beta r^{-n} \int_0^{\alpha r} \int_{B_R \cap B_{(\alpha r)^2}(y)} V(x, t) dx dt, \end{aligned}$$

for some  $\alpha, \beta > 0$  and if we have set  $r = \delta^{-\frac{1}{n}}$ . Notice also that  $B_R \cap B_{(\alpha r)^2}(y) = \emptyset$  if  $|y| \geq R + (\alpha r)^2$ , or, equivalently, if  $|y| \geq R + \alpha^2 \delta^{-\frac{2}{n}}$ .

(i) If  $|y| \geq R + \alpha^2$ , we fix  $\delta$  such that  $1 < \delta$ , then

$$e^{\frac{|y|^2}{4T}} \iint_E H(x-y, t) V(x, t) dx dt \leq \delta \iint_E V(x, t) dx dt,$$

which can be made smaller than  $\epsilon$  provided  $|E|$  is small enough.

(ii) If  $|y| < R + \alpha^2$ , then

$$\begin{aligned} e^{\frac{|y|^2}{4T}} \iint_{E \cap P_\delta^c} H(x-y, t) V(x, t) dx dt &\leq e^{\frac{R^2 + \alpha^4}{2T}} \iint_{E \cap P_\delta^c} H(x-y, t) V(x, t) dx dt \\ &\leq \delta e^{\frac{R^2 + \alpha^4}{2T}} \iint_E V(x, t) dx dt. \end{aligned}$$

Given  $\epsilon > 0$ , we fix  $\delta = r^{-n}$  such that

$$e^{\frac{R^2 + \alpha^4}{2T}} \iint_{E \cap P_\delta} H(x-y, t) V(x, t) dx dt \leq \beta e^{\frac{R^2 + \alpha^4}{2T}} r^{-n} \int_0^{\alpha r} \int_{B_R \cap B_{(\alpha r)^2}(y)} V(x, t) dx dt \leq \frac{\epsilon}{2},$$

and then  $\eta > 0$  such that  $|E| \leq \eta$  implies

$$e^{\frac{|y|^2}{4T}} \iint_{E \cap P_\delta^c} H(x-y, t) V(x, t) dx dt \leq \delta e^{\frac{R^2 + \alpha^4}{2T}} \iint_E V(x, t) dx dt \leq \frac{\epsilon}{2}.$$

Therefore

$$e^{\frac{|y|^2}{4T}} \iint_E H(x-y, t) V(x, t) dx dt \leq \epsilon,$$

which is (2.5). □

**Remark** In Theorem 2.7 and Proposition 2.9, the assumption of uniformity with respect to  $y \in \mathbb{R}^n$  in (2.5), (2.6) and (2.8) can be replaced by uniformity with respect to  $y \in B_{R_0}$  if all the measures  $\mu_k$  have their support in  $B_{R_0}$ . A extension of these assumptions, valid when the convergent measures  $\mu_k$  have their support in a fixed compact set is to assume that  $V$  is **locally strongly subcritical**, which means that (2.5) holds uniformly with respect to  $y$  in a compact set. Similar extension holds for (2.8).

## 3 The supercritical case

### 3.1 Capacities

All the proofs in this subsection are similar to the ones of [12] and inspired by [6]; we omit them. We assume also that there exists a positive measure  $\mu_0$  such that  $\mathbb{H}[\mu_0]V \in L^1(Q_T)$ .

**Definition 3.1** If  $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$  and  $f$  is a nonnegative measurable function defined in  $\Omega$  such that

$$(t, x, y) \mapsto \mathbb{H}[\mu](y, t) V(x, t) f(x, t) \in L^1(Q_T \times \mathbb{R}^n; dx dt \otimes d\mu),$$

we set

$$\mathcal{E}(f, \mu) = \int_{Q_T} \left( \int_{\mathbb{R}^n} H(x-y, t) d\mu(y) \right) V(x, t) f(x, t) dx dt.$$

If we put

$$\check{\mathbb{H}}[f](y) = \int_{Q_T} H(x-y, t) V(x, t) f(x, t) dx dt,$$

then by Fubini's Theorem,  $\check{\mathbb{H}}[f](y) < \infty$ ,  $\mu$ -almost everywhere in  $\mathbb{R}^n$  and

$$\mathcal{E}(f, \mu) = \int_{\mathbb{R}^n} \left( \int_{Q_T} H(x-y, t) V(x, t) f(x, t) dx dt \right) d\mu(y).$$

**Proposition 3.2** *Let  $f$  be fixed. Then*

(a)  $y \mapsto \check{\mathbb{H}}[f](y)$  is lower semicontinuous in  $\mathbb{R}^n$ .

(b)  $\mu \mapsto \mathcal{E}(f, \mu)$  is lower semicontinuous in  $\mathfrak{M}_+(\mathbb{R}^n)$  in the weak\* topology.

**Definition 3.3** *We denote by  $\mathfrak{M}^V(\mathbb{R}^n)$  the set of all measures  $\mu$  on  $\mathbb{R}^n$  such that  $V\mathbb{H}[|\mu|] \in L^1(Q_T)$ . If  $\mu$  is such a measure, we set*

$$\|\mu\|_{\mathfrak{M}^V} = \int_{Q_T} \left( \int_{\mathbb{R}^n} H(x-y, t) d|\mu|(y) \right) V(x, t) dx dt = \|V\mathbb{H}[|\mu|]\|_{L^1(Q_T)}.$$

If  $E \subset \mathbb{R}^n$  is a Borel set, we put

$$\mathfrak{M}_+(E) = \{\mu \in \mathfrak{M}_+(\mathbb{R}^n) : \mu(E^c) = 0\} \quad \text{and} \quad \mathfrak{M}_+^V(E) = \mathfrak{M}^V(\mathbb{R}^n) \cap \mathfrak{M}_+(E).$$

**Definition 3.4** *If  $E \subset \mathbb{R}^n$  is any borel subset we define the set function  $C_V$  by*

$$C_V(E) := \sup\{\mu(E) : \mu \in \mathfrak{M}_+^V(E), \|\mu\|_{\mathfrak{M}^V} \leq 1\};$$

this is equivalent to,

$$C_V(E) := \sup \left\{ \frac{\mu(E)}{\|\mu\|_{\mathfrak{M}^V}} : \mu \in \mathfrak{M}_+^V(E) \right\}.$$

**Proposition 3.5** *The set function  $C_V$  satisfies*

$$C_V(E) \leq \sup_{y \in E} \left( \int_{Q_T} H(x-y, t) V(x, t) dx dt \right)^{-1} \quad \forall E \subset \mathbb{R}^n, E \text{ Borel.}$$

Furthermore equality holds if  $E$  is compact. Finally,

$$C_V(E_1 \cup E_2) = \sup\{C_V(E_1), C_V(E_2)\} \quad \forall E_i \subset \mathbb{R}^n, E_i \text{ Borel.}$$

**Definition 3.6** *For any Borel  $E \subset \mathbb{R}^n$ , we set*

$$C_V^*(E) := \inf\{\|f\|_{L^\infty} : \check{\mathbb{H}}[f](y) \geq 1 \forall y \in E\}.$$

**Proposition 3.7** *For any compact set  $E \subset \mathbb{R}^n$ ,*

$$C_V^*(E) = C_V(E).$$

### 3.2 The singular set of $V$

In this section we assume that  $V$  satisfies (1.16), although much weaker assumption could have been possible. We define the singular set of  $V$ ,  $Z_V$  by

$$Z_V = \left\{ x \in \mathbb{R}^n : \iint_{Q_T} H(x-y, t)V(y, t)dydt = \infty \right\}. \quad (3.1)$$

Since the function  $x \mapsto f(x) = \iint_{Q_T} H(x-y, t)V(y, t)dydt$  is lower semicontinuous, it is a Borel function and  $Z_V$  is a Borel set.

**Lemma 3.8** *If  $x \in Z_V$  then for any  $r > 0$ ,*

$$\iint_{Q_T^{B_r(x)}} H(x-y, t)V(y, t)dydt = \infty.$$

*Proof.* We will prove it by contradiction, assuming that there exists  $r > 0$ , such that

$$\iint_{Q_T^{B_r(x)}} H(x-y, t)V(y, t)dy \leq M.$$

Replacing  $H$  by its value, we derive

$$\begin{aligned} \iint_{Q_T} H(x-y, t)V(y, t)dydt &= \iint_{Q_T^{B_r(x)}} H(x-y, t)V(y, t)dydt + \iint_{Q_T^{B_r^c(x)}} H(x-y, t)V(y, t)dydt \\ &\leq M + C(n) \int_0^T t^{-\frac{n+2}{2}} e^{-\frac{r^2}{4t}} dt < \infty. \end{aligned}$$

Which is clearly a contradiction. □

**Lemma 3.9** *If  $\mu$  is an admissible positive measure then  $\mu(Z_V) = 0$ .*

*Proof.* Let  $K \subset Z_V$  be a compact set. In view of the above lemma there exists a  $R > 0$  such that  $K \subset B_R$  and for each  $x \in K$ , we have

$$\iint_{Q_T^{B_{2R}}} H(x-y, t)V(y)dy = \infty \quad (3.2)$$

and

$$\iint_{Q_T^{B_{2R}^c}} H(x-y, t)V(y)dy < \infty. \quad (3.3)$$

Now,  $\mu_K = \chi_K \mu$  is an admissible measure and by Fubini theorem we have

$$\begin{aligned} \iint_{Q_T} \left( \int_{\mathbb{R}^n} H(x-y, t)d\mu_K(y) \right) V(x, t)dxdt &= \int_K \iint_{Q_T} H(x-y, t)V(x, t)dxdt d\mu(y) \\ &= \int_K \iint_{Q_T^{B_{2R}}} H(x-y, t)V(x)dxdt d\mu(y) \\ &\quad + \int_K \iint_{Q_T^{B_{2R}^c}} H(x-y, t)V(x)dxdt d\mu(y). \end{aligned}$$

By (3.3) the second integral above is finite and by (3.2)

$$\iint_{Q_T^{B_{2R}}} H(x-y, t)V(x)dxdt = \infty \quad \forall y \in K.$$

It follows that  $\mu(K) = 0$ . This implies  $\mu(Z_V) = 0$  by regularity.  $\square$

**Theorem 3.10** *If  $\mu \in \mathfrak{M}_T(\mathbb{R}^n)$ ,  $\mu \geq 0$  such that  $\mu(Z_V) = 0$ , then  $\mu$  is a good measure.*

*Proof.* We set  $\mu_R = \chi_{B_R}\mu$ . By Proposition 5.8, since  $Z_V^{B_R} \subset Z_V$ ,  $\mu_R$  is a good measure in  $B_R$  with corresponding solution  $u_\mu^R$ . In view of Lemma 2.5,  $u_\mu^R$  satisfies

$$\iint_{Q_T^{B_R}} |u_\mu^R| \frac{n}{4(T-t)} e^{-\frac{|x|^2}{4(T-t)}} dxdt + \iint_{Q_T^{B_R}} |u_\mu^R| V e^{-\frac{|x|^2}{4(T-t)}} dxdt \leq \int_{B_R} e^{-\frac{|x|^2}{4T}} d\mu.$$

Also  $\{u_\mu^R\}$  is an increasing function, thus converges to  $u_\mu$ . By the above estimate we have that  $u_\mu$  belong to class  $\mathcal{E}_V(Q_T)$  and is a weak solution of (2.1).  $\square$

**Proposition 3.11** *Let  $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$ . Then  $\mu(Z_V) = 0$  if and only if there exists an increasing sequence of positive admissible measures which converges to  $\mu$  in the weak\* topology.*

*Proof.* The proof is similar as the one of [12, Th 3.11] and we present it for the sake of completeness. First, we assume that  $\mu(Z_V) = 0$ . Then we define the set

$$K_N = \left\{ x \in \mathbb{R}^n : \int_{Q_T} H(x-y, t)V(y)dydt \leq N \right\}.$$

We note that  $Z_V \cap K_N = \emptyset$ . We set  $\mu_n = \chi_{K_N}\mu$  then we have

$$\int_{Q_T} \left( \int_{\mathbb{R}^n} H(x-y, t)d\mu_n(y) \right) V(x, t)dxdt \leq \mu(K_N).$$

Thus  $\mu_n$  is admissible, increasing with respect  $n$ . By the monotone theorem it follows that  $\mu_n \rightarrow \chi_{Z_V^c}\mu$ . Since  $\mu(Z_V) = 0$  the result follows in this direction.

For the other direction. Let  $\{\mu_n\}$  be an increasing sequence of positive admissible measure. Then by Lemma 3.9 we have that  $\mu_n(Z_V) = 0$ ,  $\forall n \geq 1$ . Since  $\mu_n \leq \mu$ , there exist an increasing functions  $h_n$   $\mu$ -integrable such that  $\mu_n = h_n\mu$ . Since  $0 = \mu_n(Z_V) \rightarrow \mu(Z_V)$  the result follows.  $\square$

### 3.3 Properties of positive solutions and representation formula

We first recall the construction of the kernel function for the operator  $w \mapsto \partial_t w - \Delta w + Vw$  in  $Q_T$ , always assuming that  $V$  satisfies (1.16). For  $\delta > 0$  and  $\mu \in \mathfrak{M}_T$ , we denote by  $w_\delta$  the solution of

$$\begin{aligned} \partial_t w - \Delta w + V_\delta w &= 0, & \text{in } Q_T \\ w(\cdot, 0) &= \mu & \text{in } \mathbb{R}^n. \end{aligned} \tag{3.4}$$

where  $V_\delta = V\chi_{Q_{\delta, T}}$  and  $Q_{\delta, T} = (\delta, T) \times \mathbb{R}^n$ . Then

$$w_\delta(x, t) = \int_{\mathbb{R}^n} H_{V_\delta}(x, y, t)d\mu(y) \tag{3.5}$$



**Lemma 3.12** *The mapping  $\delta \mapsto H_{V_\delta}(x, y, t)$  is increasing and converges to  $H_V \in C(\mathbb{R}^n \times \mathbb{R}^n \times (0, T])$  when  $\delta \rightarrow 0$ . Furthermore there exists a function  $H_V \in C(\mathbb{R}^n \times \mathbb{R}^n \times (0, T])$  such that for any  $\mu \in \mathfrak{M}_T(\mathbb{R}^n)$*

$$\lim_{\delta \rightarrow 0} w_\delta(x, t) = w(x, t) = \int_{\mathbb{R}^n} H_V(x, y, t) d\mu(y). \quad (3.6)$$

*Proof.* Without loss of generality we can assume  $\mu \geq 0$ . By the maximum principle  $\delta \mapsto H_{V_\delta}(x, y, t)$  is increasing and the result follows by the monotone convergence theorem.  $\square$

If  $\mathbb{R}^n$  is replaced by a smooth bounded domain  $\Omega$ , we can consider the problem

$$\begin{aligned} \partial_t w - \Delta w + V_\delta w &= 0 && \text{in } Q_T^\Omega \\ w &= 0 && \text{in } \partial_t Q_T^\Omega := \partial\Omega \times (0, T] \\ w(\cdot, 0) &= \mu && \text{in } \Omega. \end{aligned} \quad (3.7)$$

where  $V'_\delta = V \chi_{Q_{\delta, T}^\Omega}$  and  $Q_{\delta, T}^\Omega = (\delta, T) \times \Omega$ . Then

$$w_\delta(x, t) = \int_{\Omega} H_{V'_\delta}^\Omega(x, y, t) d\mu(y) \quad (3.8)$$

The proof of the next result is straightforward.

**Lemma 3.13** *The mapping  $\delta \mapsto H_{V'_\delta}^\Omega(x, y, t)$  increases and converges to  $H_V^\Omega \in C(\Omega \times \Omega \times (0, T])$  when  $\delta \rightarrow 0$ . Furthermore There exists a function  $H_V^\Omega \in C(\Omega \times \Omega \times (0, T])$  such that for any  $\mu \in \mathfrak{M}_b(\Omega)$*

$$\lim_{\delta \rightarrow 0} w_\delta(x, t) = w(x, t) = \int_{\Omega} H_V^\Omega(x, y, t) d\mu(y). \quad (3.9)$$

Furthermore  $H_V^\Omega \leq H_V^{\Omega'} \leq H_V$  if  $\Omega \subset \Omega'$ .

It is important to notice that the above results do not imply that  $w$  is a weak solution of problem (1.1). This question will be considered later on with the notion of reduced measure.

**Lemma 3.14** *Assume  $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$  is a good measure and let  $u$  be a positive weak solution of problem (2.1). If  $\Omega$  is a smooth bounded domain, then there exists a unique positive weak solution  $v$  of problem*

$$\begin{aligned} \partial_t v - \Delta v + Vv &= 0, && \text{in } Q_T^\Omega, \\ v &= 0 && \text{on } \partial_t Q_T^\Omega \\ v(\cdot, 0) &= \chi_\Omega \mu && \text{in } \Omega. \end{aligned} \quad (3.10)$$

Furthermore

$$v(x, t) = \int_{\Omega} H_V^\Omega(x, y, t) d\mu(y). \quad (3.11)$$

*Proof.* Let  $\{t_j\}_{j=1}^\infty$  be a sequence decreasing to 0, such that  $t_j < T$ ,  $\forall j \in \mathbb{N}$ . We consider the following problem

$$\begin{aligned} \partial_t v - \Delta v + Vv &= 0, && \text{in } \Omega \times (t_j, T], \\ v &= 0 && \text{on } \partial\Omega \times (t_j, T] \\ v(\cdot, t_j) &= u(\cdot, t_j) && \text{in } \Omega \times \{t_j\}. \end{aligned} \quad (3.12)$$

Since  $u, Vu \in L^1(Q_T^{B_R})$  for any  $R > 0$ ,  $t \mapsto u(\cdot, t)$  is continuous with value in  $L^1_{loc}(\mathbb{R}^n)$ , therefore  $u(\cdot, t_j) \in L^1_{loc}(\mathbb{R}^n)$  and there exists a unique solution  $v_j$  to (3.12) (notice also that  $V \in L^\infty(Q_T^{B_R})$ ). By the maximum principle  $0 \leq v_j \leq u$  and by standard parabolic estimates, we may assume that the sequence  $v_j$  converges locally uniformly in  $\Omega \times (0, T]$  to a function  $v \leq u$ . Also, if  $\phi \in C^{1,1;1}(\overline{Q_T^\Omega})$  vanishes on  $\partial_t Q_T^\Omega$  and satisfies  $\phi(x, T) = 0$ , we have

$$- \int_{t_j}^T \int_{\Omega} v_j (\partial_t \phi + \Delta \phi) dx dt + \int_{t_j}^T \int_{\Omega} V v_j \phi dx dt + \int_{\Omega} \phi(x, T - t_j) v_j(x, T - t_j) dx = \int_{\Omega} \phi(x, 0) u(x, t_j) dx,$$

where in the above equality we have taken  $\phi(\cdot, \cdot - t_j)$  as test function. Since  $\phi(\cdot, T - t_j) \rightarrow 0$  uniformly and  $u(\cdot, t_j) \rightarrow \mu$  in the weak sense of measures, it follows by the dominated convergence theorem that

$$- \int \int_{Q_T^\Omega} v (\partial_t \phi + \Delta \phi) dx dt + \int \int_{Q_T^\Omega} V v \phi dx dt = \int_{\Omega} \phi(y, 0) d\mu(y),$$

thus  $v$  is a weak solution of problem (3.10). Uniqueness follows as in Lemma 2.4. Finally, for  $\delta > 0$ , we consider the solution  $w_\delta$  of (3.7). Then it is expressed by (3.5). Furthermore

$$- \int \int_{Q_T^\Omega} w_\delta (\partial_t \phi + \Delta \phi) dx dt + \int \int_{Q_T^\Omega} V w_\delta \phi dx dt = \int_{\Omega} \phi(x, 0) d\mu(x),$$

The sequence  $w_\delta$  is decreasing, with limit  $w$ . Since  $w_\delta \geq v$ , then  $w \geq v$ . If we assume  $\phi \geq 0$ , it follows from dominated convergence and Fatou's lemma that

$$- \int \int_{Q_T^\Omega} w (\partial_t \phi + \Delta \phi) dx dt + \int \int_{Q_T^\Omega} V w \phi dx dt \leq \int_{\Omega} \phi(x, 0) d\mu(x),$$

Thus  $w$  is a subsolution for problem (3.10) for which we have comparison when existence. Finally  $w = v$  and (3.11) holds.  $\square$

**Lemma 3.15** *Assume  $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$  is a good measure and let  $u$  be a positive weak solution of problem (2.1). Then for any  $(x, t) \in \mathbb{R}^n \times (0, T]$ , we have*

$$\lim_{R \rightarrow \infty} u_R = u,$$

where  $\{u_R\}$  is the increasing sequence of the weak solutions of the problem (3.10) with  $\Omega = B_R$ . Moreover, the convergence is uniform in any compact subset of  $\mathbb{R}^n \times (0, T]$  and we have the representation formula

$$u(x, t) = \int_{\mathbb{R}^n} H_V(x, y, t) d\mu(y).$$

*Proof.* By the maximum principle,  $u_R \leq u_{R'} \leq u$  for any  $0 < R \leq R'$ . Thus  $u_R \rightarrow w \leq u$ . Also by standard parabolic estimates, this convergence is locally uniformly. Now by dominated convergence theorem, it follows that  $w$  is a weak solution of problem (2.1) with initial data  $\mu$ . Now we set  $\tilde{w} = u - w \geq 0$ . Since  $\tilde{w}$  satisfies in the weak sense

$$\begin{aligned} \tilde{w}_t - \Delta \tilde{w} + V \tilde{w} &= 0 && \text{in } Q_T \\ \tilde{w}(x, t) &\geq 0 && \text{in } Q_T \\ \tilde{w}(x, 0) &= 0 && \text{in } \mathbb{R}^n, \end{aligned}$$

and  $V \geq 0$ , it clearly satisfies

$$\begin{aligned}\tilde{w}_t - \Delta \tilde{w} &\leq 0 && \text{in } Q_T \\ \tilde{w}(x, t) &\geq 0 && \text{in } Q_T \\ \tilde{w}(x, 0) &= 0 && \text{in } \mathbb{R}^n,\end{aligned}$$

which implies  $\tilde{w} = 0$ . By the previous lemma  $u_R$  admits the representation

$$u^R(x, t) = \int_{B_R} H_V^{B_R}(x, y, t) d\mu(y).$$

Since  $\{H_V^{B_R}\}$  is an increasing sequence and  $\lim_{R \rightarrow \infty} H_V^{B_R} = H_V$ , we have using again Fatou's lemma as in the proof of Lemma 3.15

$$u(x, t) = \lim_{R \rightarrow \infty} u^R(x, t) = \lim_{R \rightarrow \infty} \int_{B_R} H_V^{B_R}(x, y, t) d\mu(y) = \int_{\mathbb{R}^n} H_V(x, y, t) d\mu(y)$$

□

**Lemma 3.16 Harnack inequality** *Let  $C_1 > 0$  and  $V(x, t)$  be a potential satisfying (1.16) If  $u$  is a positive solution of (1.17), then the Harnack inequality is valid:*

$$u(y, s) \leq u(x, t) \exp\left(C(n, C_1) \left(\frac{|x-y|^2}{t-s} + \frac{t}{s} + 1\right)\right), \quad \forall (y, s), (x, t) \in Q_T, s < t.$$

*Proof.* We extend  $V$  for  $t \geq T$  by the value  $C_1 t^{-1}$ . We consider the linear parabolic problem

$$\partial_t u - \Delta u + V u = 0, \quad \text{in } \mathbb{R}^n \times [1, \infty), \quad (3.13)$$

It is well known that, under the assumption (1.16), every positive solution  $u(x, t)$  of (3.13) satisfies the Harnack inequality

$$u(y, s) \leq u(x, t) \exp\left(C(n, C_1) \left(\frac{|x-y|^2}{t-s} + \frac{t}{s} + 1\right)\right), \quad \forall (x, t) \in \mathbb{R}^n \times [1, \infty).$$

Set  $\tilde{u}(x, t) = u(\frac{t}{\lambda^2} \frac{x}{\lambda})$ . Then  $\tilde{u}$  satisfies

$$u_t - \Delta u + \frac{1}{\lambda^2} V(\frac{t}{\lambda^2} \frac{x}{\lambda}) \tilde{u} = 0, \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

We note here that  $\frac{1}{\lambda^2} V(\frac{t}{\lambda^2} \frac{x}{\lambda}) \leq C_1, \forall t \geq \frac{1}{\lambda^2}$ , thus  $\tilde{u}$  satisfies the Harnack inequality

$$\tilde{u}(y, s) \leq \tilde{u}(x, t) \exp\left(C(n, C_1) \left(\frac{|x-y|^2}{t-s} + \frac{t}{s} + 1\right)\right), \quad \forall (x, t) \in \mathbb{R}^n \times [\frac{1}{\lambda^2}, \infty).$$

By the last inequality and the definition of  $\tilde{u}$  we derive the desired result. □

Next, we set

$$\text{Sing}_V(\mathbb{R}^n) := \{y \in \mathbb{R}^n : H_V(x, y, t) = 0\} \quad (3.14)$$

If  $H_V(x, y, t) = 0$  for some  $(x, t) \in Q_T$ , then  $H_V(x', y, t') = 0$  for any  $(x', t') \in Q_T, t' < t$  by Harnack inequality principle. We prove the **Representation formula**.

**Theorem 3.17** *Let  $u$  be a positive solution of (1.17). Then there exists a measure  $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$  such that*

$$u(x, t) = \int_{\mathbb{R}^n} H_V(x, y, t) d\mu(y),$$

and  $\mu$  is concentrated on  $(\text{Sing}_V(\mathbb{R}^n))^c$ .

*Proof.* By Lemma 3.15 we have

$$u(x, t) = \int_{\mathbb{R}^n} H_V(x, y, t - s) u(y, s) dy \quad \text{for any } s < t \leq T.$$

We assume that  $s \leq \frac{T}{2}$ . By Harnack inequality on  $x \mapsto H_V(x, y, \frac{T}{2})$

$$\int_{\mathbb{R}^n} H_V(0, y, \frac{T}{2}) u(y, s) dy \leq c(n) \int_{\mathbb{R}^n} H_V(0, y, T - s) u(y, s) dy = c(n) u(0, T).$$

For any Borel set  $E$ , we define the measure  $\rho_s$  by

$$\rho_s(E) := \int_E H_V(0, y, \frac{T}{2}) u(y, s) dy \leq \int_{\mathbb{R}^n} H_V(0, y, \frac{T}{2}) u(y, s) dy \leq c(n) u(T, 0).$$

Thus there exists a decreasing sequence  $\{s_j\}_{j=1}^\infty$  which converges to origin, such that the measure  $\rho_{s_j}$  converges in the weak\* topology to a positive Radon measure  $\rho$ . Also we have the estimate  $\rho(\mathbb{R}^n) \leq C(n) u(0, T)$ . Now choose  $(x, t) \in Q_T$  and  $j_0$  large enough such that  $t > s_{j_0}$ . Let  $\varepsilon > 0$ , we set for any  $j \geq j_0$ ,

$$W_j(y) = \frac{H_V(x, y, t - s_j)}{H_V(0, y, \frac{T}{2}) + \varepsilon}.$$

For any  $R > 0$  and  $|y| > R$  we have

$$W_j(y) \leq \frac{1}{\varepsilon} H_V(x, y, t - s_j) \leq \frac{1}{\varepsilon} H(x - y, t - s_j) < \frac{1}{\varepsilon} C(x, R, t - s_j),$$

where  $\lim_{R \rightarrow \infty} C(x, R, t - s_j) = 0$ . We have also

$$\int_{|y| \geq R} W_j(y) d\rho_j \leq \frac{1}{\varepsilon} C(x, R, t - s_j) c(n) u(T, 0).$$

For any  $|y| < R$ , we have by standard parabolic estimates that  $W_j(y) \rightarrow \frac{H_V(x, y, t)}{H_V(\frac{T}{2}, 0, y) + \varepsilon}$  when  $j \rightarrow \infty$ , uniformly with respect to  $y$ . Thus by the above estimates it follows

$$\int_{\mathbb{R}^n} W_j(y) d\rho_j \rightarrow \int_{\mathbb{R}^n} \frac{H_V(x, y, t)}{H_V(0, y, \frac{T}{2}) + \varepsilon} d\rho.$$

For sufficiently large  $j$  we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{H_V(x, y, t - s_j)}{H_V(0, y, \frac{T}{2}) + \varepsilon} d\rho_{s_j} &= \int_{\mathbb{R}^n} \frac{H_V(x, y, t - s_j)}{H_V(0, y, \frac{T}{2}) + \varepsilon} \left( H_V(0, y, \frac{T}{2}) + \varepsilon - \varepsilon \right) u(y, s_j) dy \\ &= u(x, t) - \varepsilon \int_{\mathbb{R}^n} \frac{H_V(x, y, t - s_j)}{H_V(0, y, \frac{T}{2}) + \varepsilon} u(y, s_j) dy. \end{aligned}$$

Note that this is a consequence of the identity

$$\int_{\mathbb{R}^n} H_V(x, y, t - s_j) u(y, s_j) dy = u(x, t).$$

Thus as before, we define  $d\tilde{\rho}_j = H_V(x, y, t - s_j) u(y, s_j) dy$  and thus there exists a subsequence, say  $\{\tilde{\rho}_j\}$  converges in the weak\* topology to a positive Radon measure  $\tilde{\rho}$ . Thus we have

$$\begin{aligned} \varepsilon \int_{\mathbb{R}^n} \frac{H_V(x, y, t - s_j)}{H_V(0, y, \frac{T}{2}) + \varepsilon} u(y, s_j) dy &= \varepsilon \int_{\mathbb{R}^n} \chi_{(\text{Sing}_V(\mathbb{R}^n))^c} \frac{H_V(x, y, t - s_j)}{H_V(0, y, \frac{T}{2}) + \varepsilon} u(y, s_j) dy \\ &\rightarrow \varepsilon \int_{\mathbb{R}^n} \chi_{(\text{Sing}_V(\mathbb{R}^n))^c} \frac{1}{H_V(\frac{T}{2}, 0, y) + \varepsilon} d\tilde{\rho}. \end{aligned}$$

Combining the above relations, we derive

$$\int_{\mathbb{R}^n} \frac{H_V(x, y, t)}{H_V(0, y, \frac{T}{2}) + \varepsilon} d\rho = u(x, t) - \varepsilon \int_{\mathbb{R}^n} \chi_{(\text{Sing}_V(\mathbb{R}^n))^c} \frac{1}{H_V(0, y, \frac{T}{2}) + \varepsilon} d\tilde{\rho}. \quad (3.15)$$

Now, we have

$$\lim_{\varepsilon \rightarrow 0} \chi_{(\text{Sing}_V(\mathbb{R}^n))^c} \frac{\varepsilon}{H_V(\frac{T}{2}, 0, y) + \varepsilon} = 0,$$

and by Harnack inequality on the function  $x \mapsto H_V(x, y, t)$

$$\frac{H_V(x, y, t)}{H_V(0, y, \frac{T}{2}) + \varepsilon} \leq C(t, T),$$

thus by dominated convergence theorem, we can let  $\varepsilon$  tend to 0 in (3.15) and obtain

$$\int_{\mathbb{R}^n} \frac{H_V(x, y, t)}{H_V(0, y, \frac{T}{2})} d\rho = u(x, t).$$

And the result follows if we set

$$d\mu = \chi_{(\text{Sing}_V(\mathbb{R}^n))^c} \frac{1}{H_V(0, y, \frac{T}{2})} d\rho.$$

□

In the next result we give a construction of  $H_V$ , with some estimates and a different proof of the existence of an initial measure for positive solutions of (1.16).

**Theorem 3.18** *Assume  $V$  satisfies (1.16) and  $u$  is a positive solution of (1.17) then there exists a positive Radon measure  $\mu$  in  $\mathbb{R}^n$  such that*

$$u(x, t) = \int_{\mathbb{R}^n} \epsilon^{\psi(x, t)} \Gamma(x, y, t, 0) d\mu(y) \quad (3.16)$$

where

$$\psi(x, t) = \int_t^T \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-y|^2}{4(s-t)}}}{4\pi(t-s)} V(y, s) dy ds \quad (3.17)$$

and

$$c_1 \frac{e^{-\gamma_1 \frac{|x-y|^2}{s-t}}}{(t-s)^{\frac{n}{2}}} \leq \Gamma(x, y, t, s) \leq c_2 \frac{e^{-\gamma_2 \frac{|x-y|^2}{s-t}}}{(t-s)^{\frac{n}{2}}} \quad (3.18)$$

for some positive constants  $c_i$  and  $\gamma_i$ ,  $i = 1, 2$ .

*Proof.* Assuming that  $u$  is a positive solution of (1.17), we set  $u(x, t) = e^{\psi(x, t)}v(x, t)$ . Then

$$\partial_t v - \Delta v - 2\nabla\psi \cdot \nabla v - |\nabla\psi|^2 v + (\partial_t\psi - \Delta\psi + V)v = 0. \quad (3.19)$$

We choose  $\psi$  as the solution of problem

$$\begin{aligned} -\partial_t\psi - \Delta\psi + V\psi &= 0 & \text{in } Q_T \\ \psi(\cdot, T) &= 0 & \text{in } \mathbb{R}^n. \end{aligned} \quad (3.20)$$

Then  $\psi$  is expressed by (3.17). Furthermore, by standard computations,

$$\begin{aligned} (i) \quad & 0 \leq \psi(x, t) \leq c \ln \frac{T}{t} \\ (ii) \quad & |\nabla\psi(x, t)| \leq c_1(T) + c_2(T) \ln \frac{T}{t} \end{aligned} \quad (3.21)$$

The function  $v$  satisfies

$$\partial_t v - \Delta v - 2\nabla\psi \cdot \nabla v - |\nabla\psi|^2 v = 0. \quad (3.22)$$

Then, by (3.21),

$$\begin{aligned} (i) \quad & 0 \leq \int_{\mathbb{R}^n} \sup\{|\psi(x, s)|^q : x \in \mathbb{R}^n\} ds \leq M_1 \\ (ii) \quad & 0 \leq \int_{\mathbb{R}^n} \sup\{|\nabla\psi(x, s)|^q : x \in \mathbb{R}^n\} ds \leq M_2 \end{aligned} \quad (3.23)$$

for any  $1 \leq q < \infty$  for some  $M_i \in \mathbb{R}_+$ . This is the condition  $H$  in [2] with  $R_0 = \infty$  and  $p = \infty$ . Therefore there exists a kernel function  $\Gamma \in C(\mathbb{R}^n \times \mathbb{R}^n \times (0, T) \times (0, T))$  which satisfies (3.18) and there exists also a positive Radon measure  $\mu$  in  $\mathbb{R}^n$  such that

$$v(x, t) = \int_{\mathbb{R}^n} \Gamma(x, y, t, 0) d\mu(y). \quad (3.24)$$

Finally  $u$  verifies

$$u(x, t) = e^{\psi(x, t)} \int_{\mathbb{R}^n} \Gamma(x, y, t, 0) d\mu(y). \quad (3.25)$$

□

We recall that  $Sing_V(\mathbb{R}^n) := \{y \in \mathbb{R}^n : H_V(x, y, t) = 0\}$ .

**Theorem 3.19** *Let  $\delta_\xi$  be the Dirac measure concentrated at  $\xi$  and let  $V$  satisfies (1.16). Then*

$$H_V(x, \xi, t) = \int_{\mathbb{R}^n} e^{\psi(x, t)} \Gamma(x, y, t) d\mu_\xi(y),$$

where  $\mu_\xi$  is a positive Radon measure such that

$$\delta_\xi \geq \mu_\xi,$$

and  $\psi, \Gamma$  are the functions in (3.17) and (3.18) respectively.  
Furthermore, if

$$\limsup_{t \rightarrow 0} \psi(\xi, t) = \limsup_{t \rightarrow 0} \int_t^T \int_{\mathbb{R}^n} \left( \frac{1}{4\pi(s-t)} \right)^{\frac{n}{2}} e^{-\frac{|\xi-y|^2}{4(s-t)}} V(y, s) dy ds = \infty$$

then

$$\xi \in \text{Sing}_V, \text{ i.e. } H_V(x, \xi, t) = 0, \forall (x, t) \in \mathbb{R}^n \times (0, \infty).$$

*proof.* First we note that  $H_{V_k}(x, \xi, t)$  is the solution of problem (1.24) with  $\delta_\xi$  as initial data. Since  $H_{V_k}(x, \xi, t) \downarrow H_V(x, \xi, t)$ , we have by maximum principle,  $H(x, \xi, t) \geq H_V(x, \xi, t)$ . Now by Theorem 3.18, there exists a positive Radon measure  $\mu_\xi$  in  $\mathbb{R}^n$  such that

$$H_V(x, \xi, t) = \int_{\mathbb{R}^n} e^{\psi(x,t)} \Gamma(x, y, t, 0) d\mu_\xi(y) \quad (3.26)$$

Let  $\phi \in C_0(\mathbb{R}^n)$  then we have by the properties of  $\Gamma(x, \xi, t)$  (see [2]) and (3.26)

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} H_V(x, \xi, t) \phi(x) dx \geq \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Gamma(x, y, t) \phi(x) dx d\mu_\xi(y) = \int_{\mathbb{R}^n} \phi(y) \mu_\xi(y),$$

That is

$$\int_{\mathbb{R}^n} \phi(x) d\delta_\xi(x) \geq \int_{\mathbb{R}^n} \phi(x) d\mu_\xi(x) \Rightarrow \delta_\xi \geq \mu_\xi, \quad (3.27)$$

since  $\phi$  is an abstract function in space  $C_0(\mathbb{R}^n)$ .

Also we have that there exist positive constants  $C_1, C_2$  such that

$$\Gamma(x, y, t) \geq C_1 H(x, y, C_2 t). \quad (3.28)$$

Also we have

$$\begin{aligned} H(\xi, \xi, t) \geq H_V(\xi, \xi, t) &= \int_{\mathbb{R}^n} H_V(\xi, y, t) d\mu_\xi(y) = \int_{\mathbb{R}^n} e^{\psi(\xi,t)} \Gamma(\xi, y, t) d\mu_\xi(y) \\ \text{(by (3.28))} &\geq C_1 \int_{B(\xi, \sqrt{C_2 t})} e^{\psi(t,\xi)} H(\xi, y, C_2 t) d\mu_\xi(y) \\ \text{(By Harnack inequality)} &\geq C(T, n, C_1, C_2) \int_{B(\xi, \sqrt{C_2 t})} e^{\psi(\xi,t)} H(\xi, \xi, \frac{C_2 t}{2}) d\mu_\xi(y) \\ &= C(T, n, C_1, C_2) e^{\psi(\xi,t)} H(\xi, \xi, \frac{C_2 t}{2}) \mu_\xi(B(\xi, \sqrt{C_2 t})) \end{aligned}$$

Thus by the last inequality and the fact that

$$\frac{H(\xi, \xi, t)}{H(\xi, \xi, \frac{C_2 t}{2})} = C(C_2, n) > 0,$$

we have

$$C(T, n, C_1, C_2) \geq e^{\psi(t,\xi)} \mu_\xi(B(\xi, \sqrt{C_2 t})).$$

But  $\limsup_{t \rightarrow 0} \psi(\xi, t) = \infty$  which implies

$$\lim_{t \rightarrow 0} \mu_\xi(B(\xi, \sqrt{C_2 t})) = \mu_\xi(\{\xi\}) = 0.$$

Thus by (3.27) we have  $\mu_\xi \equiv 0$ , i.e.  $H_V(x, \xi, t) = 0, \forall (x, t) \in \mathbb{R}^n \times (0, \infty)$ .  $\square$

### 3.4 Reduced measures

In this section we assume that  $V$  is nonnegative, but not necessarily satisfies (1.16), therefore we can construct  $H_V[\mu]$  for  $\mu \in \mathfrak{M}_T(\mathbb{R}^n)$ . Furthermore, if  $\mu$  is nonnegative we can consider the solution  $u_k$  of the problem

$$\begin{aligned} \partial_t u - \Delta u + V^k u &= 0, & \text{in } Q_T \\ u(\cdot, 0) &= \mu & \text{in } \mathbb{R}^n, \end{aligned} \quad (3.29)$$

where  $V^k = \min\{V, k\}$ . Then there holds

$$u_k(x, t) = \int_{\mathbb{R}^n} H_{V^k}(t, x, y) d\mu(y) = \mathbb{H}_{V^k}[\mu](x, t),$$

and

$$u_k + \int_0^t \int_{\mathbb{R}^n} H(t-s, x, y) V^k u_k dy ds = \mathbb{H}[\mu].$$

Since  $k \mapsto H_{V^k}$  is decreasing and converges to  $H_V$ , we derive

$$\lim_{k \rightarrow \infty} u_k = u = \int_{\mathbb{R}^n} H_V(t, x, y) d\mu(y).$$

By Fatou's lemma

$$\int_0^t \int_{\mathbb{R}^n} H(t-s, x, y) V u dy ds \leq \liminf_{k \rightarrow \infty} \int_0^t \int_{\mathbb{R}^n} H(t-s, x, y) V^k u_k dy ds.$$

It follows

$$u(x, t) + \int_0^t \int_{\mathbb{R}^n} H(t-s, x, y) V u dy ds \leq \int_{\mathbb{R}^n} H_V(t, x, y) d\mu(y), \quad \forall (x, t) \in Q_T.$$

Now since  $Vu \in L^1_{loc}(\overline{Q}_T)$  and

$$\partial_t u - \Delta u + Vu = 0, \quad \text{in } Q_T,$$

the function

$$u(x, t) + \int_0^t \int_{\mathbb{R}^n} H(t-s, x, y) V u dy ds$$

is nonnegative and satisfies the heat equation in  $Q_T$ . Therefore it admits an initial trace  $\mu^* \in \mathfrak{M}_+(\mathbb{R}^n)$  and actually  $\mu^* \in \mathfrak{M}_T(\mathbb{R}^n)$ . Furthermore, we have

$$u(x, t) + \int_0^t \int_{\mathbb{R}^n} H(t-s, x, y) V u dy ds = \int_{\mathbb{R}^n} H(x-y, t) d\mu^*(y), \quad \forall (x, t) \in Q_T.,$$

or equivalently,  $u$  is a positive weak solution of the problem

$$\begin{aligned} \partial_t u - \Delta u + Vu &= 0 & \text{in } Q_T \\ u(\cdot, 0) &= \mu^* & \text{in } \mathbb{R}^n. \end{aligned}$$

Note that  $\mu^* \leq \mu$  and the mapping  $\mu \mapsto \mu^*$  is nondecreasing.



**Definition 3.20** *The measure  $\mu^*$  is the reduced measure associated to  $\mu$*

The proofs of the next two Propositions are similar to the ones of [12, Section 5].

**Proposition 3.21** *There holds  $\mathbb{H}_V[\mu] = \mathbb{H}_V[\mu^*]$ . Furthermore the reduced measure  $\mu^*$  is the largest measure for which the following problem*

$$\begin{aligned} \partial_t v - \Delta v + Vv &= 0 && \text{in } Q_T \\ \lambda &\in \mathfrak{M}_+(\mathbb{R}^n), \lambda \leq \mu && \\ v(\cdot, 0) &= \lambda && \text{in } \mathbb{R}^n, \end{aligned} \tag{3.30}$$

*admits a solution.*

**Proposition 3.22** *Let  $W_k$  be an increasing sequence of nonnegative bounded measurable functions converging to  $V$  a.e. in  $Q_T$ . Then the solution  $u_k$  of*

$$\begin{aligned} \partial_t v - \Delta v + W_k v &= 0 && \text{in } Q_T \\ v(\cdot, 0) &= \mu && \text{in } \mathbb{R}^n, \end{aligned}$$

*converges to  $u_{\mu^*}$ .*

We recall that  $Sing_V(\mathbb{R}^n) := \{y \in \mathbb{R}^n : H_V(x, y, t) = 0\}$ .

**Proposition 3.23** *Let  $\mu$  be a nonnegative measure in  $\mathcal{M}_{\mathcal{T}}(\mathbb{R}^n)$ . Then*

- (i)  $(\mu - \mu^*)((Sing_V(\mathbb{R}^n))^c) = 0$
- (ii) *If  $\mu((Sing_V(\mathbb{R}^n))^c) = 0$ , then  $\mu^* = 0$ .*
- (iii) *There always holds  $Sing_V(\mathbb{R}^n) = Z_V$ .*

*proof.* The proofs of (i), (ii) and the fact that  $Sing_V(\mathbb{R}^n) \subset Z_V$  are similar as in [12, Section 5], and we omit them.

The proof of  $Z_V \subset Sing_V(\mathbb{R}^n)$  is an immediate consequence of Theorem 3.19. Indeed, if  $\xi \in Z_V$  then

$$\limsup_{t \rightarrow 0} \int_t^T \int_{\mathbb{R}^n} \left( \frac{1}{4\pi(s-t)} \right)^{\frac{n}{2}} e^{-\frac{|\xi-y|^2}{4(s-t)}} V(y, s) dy ds = \infty,$$

thus  $\xi \in Sing_V(\mathbb{R}^n)$ . □

## 4 Initial trace

### 4.1 The direct method

The initial trace that we developed in this section is an adaptation to the parabolic case of the notion of boundary trace for elliptic equations (see [9], [10], [12]). If  $G \subset \overline{Q_T}$  is a relatively open set, we denote

$$W(G) = \bigcap_{1 \leq p < \infty} W_p^{2,1}(G) \quad \text{and} \quad W_{loc}(G) = \bigcap_{1 \leq p < \infty} W_{p,loc}^{2,1}(G).$$

Since  $V \in L_{loc}^\infty(Q_T)$ , any solution of (1.17) belongs to  $W_{loc}(Q_T)$ .

**Proposition 4.1** *Let  $u \in W_{loc}(Q_T)$  be a positive solution (1.17). Assume that, for some  $x \in \mathbb{R}^n$ , there exists an open bounded neighborhood  $U$  of  $x$  such that*

$$\iint_{Q_T^U} u(y, t)V(y, t)dxdt < \infty \quad (4.31)$$

*Then  $u \in L^1(U \times (0, T))$  and there exists a unique positive Radon measure  $\mu$  in  $U$  such that*

$$\lim_{t \rightarrow 0} \int_U u(y, t)\phi(x)dx = \int_U \phi(x)d\mu, \quad \forall \phi \in C_0^\infty(U).$$

*Proof.* Since  $Vu \in L^1(U \times (0, T))$  the following problem has a weak solution  $v$  (see [9]).

$$\begin{aligned} \partial_t v - \Delta v &= Vu, & \text{in } U \times (0, T], \\ v(x, t) &= 0 & \text{on } \partial U \times (0, T], \\ v(x, 0) &= 0 & \text{in } U. \end{aligned}$$

Thus the function  $w = u + v$  satisfies the heat equation. Thus there exists a unique Radon measure  $\mu$  such that

$$\lim_{t \rightarrow 0} \int_U w(y, t)\phi(x)dx = \int_U \phi(x)d\mu, \quad \forall \phi \in C_0^\infty(U).$$

And the result follows since the initial data of  $v$  is zero. □

We set

$$\mathcal{R}(u) = \left\{ y \in \mathbb{R}^n : \exists \text{ bounded neighborhood } U \text{ of } y, \iint_{Q_T^U} u(y, t)V(y, t)dxdt < \infty \right\}. \quad (4.32)$$

Then  $\mathcal{R}(u)$  is open and there exists a unique positive Radon measure  $\mu$  on  $\mathcal{R}(u)$  such that

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}} u(y, t)\phi(x)dx = \int_{\mathcal{R}} \phi(x)d\mu, \quad \forall \phi \in C_0^\infty(\mathcal{R}). \quad (4.33)$$

**Proposition 4.2** *Let  $u \in W_{loc}(\mathbb{R}^n \times (0, T])$  be a positive solution of (1.17). Assume that, for some  $x \in \mathbb{R}^n$ , there holds*

$$\iint_{Q_T^U} u(y, t)V(y, t)dydt = \infty \quad (4.34)$$

*for any bounded open neighborhood  $U$  of  $x$ . Then*

$$\limsup_{t \rightarrow 0} \int_U u(y, t)dy = \infty. \quad (4.35)$$

*Proof.* We will prove it by contradiction. We assume that there exists an open neighborhood of  $x$  such that

$$\int_U u(y, t)dy \leq M < \infty \quad \forall t \in (0, T).$$

Then  $\|u\|_{L^1(Q_T^U)} \leq MT$ . Let  $B_r(x) \subset\subset U$  for some  $r > 0$ , and  $\zeta \in C_0^\infty(B_r(x))$ , such that  $\zeta = 1$  in  $B_{\frac{r}{2}}(x)$ ,  $\zeta = 0$  in  $B_r^c(x)$  and  $0 \leq \zeta \leq 1$ . Then since  $u$  is a positive solution we have

$$\begin{aligned} \int_U \partial_t u \zeta dx - \int_U u \Delta \zeta dx + \int_U V u \zeta dx &= 0 \Rightarrow \int_{B_{\frac{r}{2}}} V u dx \leq \int_U \partial_t u \zeta dx - \int_U u \Delta \zeta dx \Rightarrow \\ \int_U \partial_t u \zeta dx - \int_U u \Delta \zeta dx + \int_U V u \zeta dx &= 0 \Rightarrow \int_{B_{\frac{r}{2}}} V u dx \leq - \int_U \partial_t u dx + M \|\Delta \zeta\|_{L^\infty}. \end{aligned}$$

Integrating the above inequality on  $(s, T)$ , we get

$$\int_s^T \int_{B_{\frac{r}{2}}} V u dx dr \leq - \int_U u(x, T) dx + \int_U u(x, s) dx + \|u\|_{L^1(Q_T^U)} \|\Delta \zeta\|_{L^\infty}. \quad (4.36)$$

Letting  $s \rightarrow 0$ , we reach a contradiction.  $\square$

**Remark.** It is not clear whether there holds

$$\liminf_{t \rightarrow 0} \int_U u(y, t) dy = \infty. \quad (4.37)$$

However, it follows from (4.36) that if  $u \in L^1(Q_T^U)$ , the above equality holds.

**Definition 4.3** If  $u$  is a positive solution of (1.17), we set  $\mathcal{S}(u) = \mathbb{R}^n \setminus \mathcal{R}(u)$ . The couple  $(\mathcal{S}(u), \mu)$  is called the initial trace of  $u$ , denoted by  $tr_{\{t=0\}}(u)$ . The sets  $\mathcal{R}(u)$  and  $\mathcal{S}(u)$  are respectively the regular and the singular sets of  $tr_{\{t=0\}}(u)$  and  $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$  is its regular part.

**Example** Take  $V(x, t) = ct^{-1}$ ,  $c > 0$ . If  $u$  satisfies

$$\partial_t u - \Delta u + \frac{c}{t} u = 0 \quad (4.38)$$

then  $v(x, t) = t^c u(x, t)$  satisfies the heat equation. Thus, if  $u \geq 0$ , there exists  $\mu \in \mathfrak{M}_+(\mathbb{R}^n)$  such that

$$u(x, t) = t^{-c} \mathbb{H}[\mu](x, t). \quad (4.39)$$

This is a representation formula. Notice that  $V u(x, t) = ct^{-c-1} \mathbb{H}[\mu](x, t)$ , therefore the regular set of  $tr_{\{t=0\}}(u)$  may be empty.

**Proposition 4.4** Assume  $V$  satisfies (1.16) and let  $u \in W_{loc}(Q_T)$  be a positive solution of (1.17) with initial trace  $(\mathcal{S}(u), \mu_u)$ . Then  $u \geq u_{\mu_u}$ .

*Proof.* We assume  $\mathcal{S}(u) \neq \mathbb{R}^n$  otherwise the result is proved. Let  $G$  and  $E$  be open bounded domains such that  $G \subset\subset E \subset\subset \mathcal{R}(u)$ . Let  $0 < \delta = \inf\{|x - y| : x \in G, y \in E^c\}$ . Choose  $R > 0$  such that  $E \subset\subset B_R$ . Let  $\{t_j\}_{j=1}^\infty$  be a decreasing sequence converging to 0. We denote by  $u_j$  the weak solution of the problem

$$\begin{aligned} \partial_t v - \Delta v + V v &= 0 && \text{in } B_R \times (t_j, T] \\ v(x, t) &= 0 && \text{on } \partial B_R \times (t_j, T] \\ v(\cdot, t_j) &= \chi_G u(\cdot, t_j) && \text{in } B_R \times \{t_j\}, \end{aligned}$$

where  $\chi$  is the characteristic function on  $G$ . Let  $v_j^R$ , be the solution

$$\begin{aligned} \partial_t v - \Delta v &= 0 && \text{in } \mathbb{R}^n \times (t_j, \infty) \\ v(\cdot, t_j) &= \chi_G u(\cdot, t_j) && \text{in } \mathbb{R}^n \times \{t_j\}. \end{aligned}$$

Then by maximum principle we have  $u_j^R \leq u$  and  $u_j^R \leq v_j$  in  $B_R \times (t_j, T]$ , for any  $j \in \mathbb{N}$ . By standard parabolic estimates, we may assume that the sequence  $u_j^R$  converges locally uniformly in  $Q_T^{B_R}$  to a function  $u^R \leq u$ . Moreover, since  $\chi_G \mu_u(\cdot, t_j) \rightharpoonup \chi_G \mu_u$  in the weak\* topology, we derive from the representation formula that  $v_j \rightarrow \mathbb{H}[\chi_G \mu_u]$ . Furthermore  $u^R \leq v$ , which implies  $\chi_{(t_j, T)} u_j^R \rightarrow u^R$  in  $L^1(Q_T^{B_R})$ . There also holds

$$\int_{t_j}^T \int_{B_R} u_j^R V dx dt = \int_{t_j}^T \int_E u_j^R V dx dt + \int_{t_j}^T \int_{B_R \setminus E} u_j^R V dx dt,$$

and, by the choice of  $E$  and dominated convergence theorem,

$$\int_{t_j}^T \int_E u_j^R V dx dt \leq \int_0^T \int_E u V dx dt < \infty \Rightarrow \lim_{j \rightarrow \infty} \int_{t_j}^T \int_E u_j^R V dx dt = \int_0^T \int_E u^R V dx dt.$$

Furthermore, for any  $x \in B_R \setminus E$ ,

$$v_j(x, t) = \left( \frac{1}{4\pi(t-t_j)} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-t_j)}} \chi_G u(y, t_j) dy \leq \left( \frac{1}{4\pi(t-t_j)} \right)^{\frac{n}{2}} e^{-\frac{\delta^2}{4(t-t_j)}} \int_G u(y, t_j) dy.$$

Next, since  $V(x, t) \leq Ct^{-1}$  and  $u_j^R \leq v_j$ , we obtain

$$\lim_{j \rightarrow \infty} \int_{t_j}^T \int_{B_R \setminus E} u_j^R V dx dt = \int_0^T \int_{B_R \setminus E} u^R V dx dt, \quad (4.40)$$

by using the previous estimate and the fact that  $\chi_G \mu_u(x, t_j) \rightharpoonup \chi_G \mu_u$  in the weak\* topology. It follows  $\chi_{(t_j, T)} V u_j^R \rightarrow V u^R$  in  $L^1(Q_T^{B_R})$ . There holds also  $u_G^R \leq u$ ; by the maximum principle, the mapping  $R \mapsto u_G^R$  is increasing and bounded from above by  $u$ . In view of Lemma 3.15,

$$\lim_{R \rightarrow \infty} u_G^R = u_G \leq u,$$

and  $u_G$  is a positive weak solution of

$$\begin{aligned} \partial_t v - \Delta v + Vv &= 0 && \text{in } Q_T \\ v(\cdot, 0) &= \chi_G \mu_u && \text{in } \mathbb{R}^n. \end{aligned}$$

Consider an increasing sequence  $\{G_i\}_{i=1}^{\infty}$  of bounded open subsets,  $G_i \subset \subset \mathcal{R}(u)$ , with the property that  $\bigcup_{i=1}^{\infty} G_i = \mathcal{R}(u)$ . In view of Lemma 3.15 the sequence  $\{u_i = u_{G_i}\}_{i=1}^{\infty}$  is increasing and converges to  $\tilde{u} \leq u$ . Also we have

$$u_i(x, t) + \int_0^t \int_{\mathbb{R}^n} H(t-s, x, y) V u_i dy ds = \int_{\mathbb{R}^n} H(x-y, t) d\mu_i, \quad \forall (x, t) \in Q_T,$$

where  $\mu_i = \chi_{G_i}\mu$ . Now since  $\mu_i \rightarrow \mu_u$ , by the monotone convergence theorem we have

$$\tilde{u}(x, t) + \int_0^t \int_{\mathbb{R}^n} H(t-s, x, y) V \tilde{u} dy ds = \int_{\mathbb{R}^n} H(x-y, t) d\mu_u, \quad \forall (x, t) \in Q_T,$$

and  $\tilde{u} \leq u$ . this implies  $\tilde{u} = u_{\mu_u}$ , which ends the proof.  $\square$

**Remark.** Assumption (1.17) is too strong and has only been used in (4.40). It could have been replaced by the following much weaker one: for any  $R > 0$  there exists a positive increasing function  $\epsilon_R$  such that  $\lim_{t \rightarrow 0} \epsilon(t) = 0$  satisfying

$$V(x, t) \leq e^{t^{-1}\epsilon_R(t)} \quad \forall (x, t) \in Q_T^{BR}. \quad (4.41)$$

We end this section with a result which shows that the stability of the initial value problem with respect to convergence the initial data in the weak\* topology implies that the initial of positive solution has no singular part.

**Theorem 4.5** *Assume  $V$  satisfies, for some  $\tau_0 > 0$ ,*

$$\lim_{|E| \rightarrow 0} \iint_E H(x-y, t) V(x, t + \tau) dx dt = 0, \quad E \text{ Borel subset of } Q_T^{BR} \quad (4.42)$$

for any  $R > 0$ , uniformly with respect to  $y$  is a compact set and  $\tau \in [0, \tau_0]$ . If  $u$  is a positive solution of (1.17), then  $\mathcal{R}(u) = \mathbb{R}^n$

*Proof.* We assume that  $\mathcal{S}(u) \neq \emptyset$  and if  $z \in \mathcal{S}(u)$  there holds

$$\iint_{Q_T^{Br(z)}} V u dx dt = \infty \quad \forall r > 0.$$

In view of Proposition 4.1, there exist two sequences  $\{r_k\}$  and  $\{t_j\}$  decreasing to 0 such that

$$\lim_{j \rightarrow \infty} \int_{B_{r_k}(z)} u(x, t_j) dx = \infty \quad \forall k \in \mathbb{N}.$$

For  $k \in \mathbb{N}$  and  $m > 0$  fixed, there exists  $j(k)$  such that

$$\int_{B_{r_k}(z)} u(x, t_j) dx \geq m \quad \forall j \geq j(k) \in \mathbb{N},$$

and there exists  $\ell_k > 0$  such that

$$\int_{B_{r_k}(z)} \min\{u(x, t_{j(k)}), \ell_k\} dx = m$$

Furthermore  $j(k) \rightarrow \infty$  when  $k \rightarrow \infty$ . Let  $R > \max\{r_j : j = 1, 2, \dots\}$  and  $u_k$  be the solution of

$$\begin{aligned} \partial_t v - \Delta v + Vv &= 0 && \text{in } \mathbb{R}^n \times (t_{j(k)}, T] \\ v(\cdot, t_j) &= \chi_{B_{r_k}(z)} \min\{u(\cdot, t_{j(k)}), \ell_k\} && \text{in } \mathbb{R}^n \times \{t_{j(k)}\}, \end{aligned}$$

Then  $\chi_{B_{r_k}(z)} \min\{u(\cdot, t_{j(k)}), \ell_k\} \rightarrow m\delta_z$  in the weak sense of measures. By Proposition 5.5 we obtain that  $u \geq u_k$  on  $B_R(z) \times (t_{j(k)}, T]$ . Applying Proposition 2.8, and the remark here after, we conclude that  $u_k(\cdot, \cdot + t_{j(k)}) \rightarrow u_{m\delta_z} = mu_{\delta_z}$  in  $L^1_{loc}(\overline{Q}_R^T)$ . This implies  $u \geq mu_{\delta_z}$ , and as  $m$  is arbitrary,  $u = \infty$ , contradiction.  $\square$

## 4.2 The sweeping method

In this subsection we adapt to equation (1.17) the sweeping method developed in [12] for constructing the boundary trace of solutions of stationary Shrödinger equations. If  $A \subset \mathbb{R}^n$  is a Borel set, we denote by

$$\mathfrak{M}_{T^+}(A) = \{\mu \in \mathfrak{M}_+(\mathbb{R}^n) : \mu(A^c) = 0, \int_A e^{-\frac{|x|^2}{4T}} d\mu < \infty\}.$$

We recall that  $\mu^*$  denotes the reduced measure associated to  $\mu$ .

**Proposition 4.6** *Let  $u \in W_{loc}(Q_T)$  be a positive solution of (1.17) with singular set  $\mathcal{S}(u) \subsetneq \mathbb{R}^n$ . If  $\mu \in \mathfrak{M}_{T^+}(\mathcal{S}(u))$ , we set  $v_\mu = \inf\{u, u_{\mu^*}\}$ . Then*

$$\partial_t v_\mu - \Delta v_\mu + V v_\mu \geq 0 \quad \text{in } Q_T,$$

and  $v_\mu$  admits a boundary trace  $\gamma_u(\mu) \in \tilde{\mathfrak{M}}_+(\mathcal{S}(u))$ . The mapping  $\mu \mapsto \gamma_u(\mu)$  is nondecreasing and  $\gamma_u(\mu) \leq \mu$ .

*Proof.* It is classical that  $v_\mu := \inf\{u, u_{\mu^*}\}$  is a supersolution of (1.17) and  $v_\mu \in \mathcal{E}_\nu(Q_T)$  as it holds with  $u_{\mu^*}$ . The function

$$(x, t) \mapsto w(x, t) = \int_0^t \int_{\mathbb{R}^n} H(t-s, x, y) V(y, s) v_\mu(y, s) dy ds$$

satisfies

$$\begin{aligned} \partial_t w - \Delta w - V w &= 0 & \text{in } Q_T \\ w(\cdot, 0) &= 0 & \text{in } \mathbb{R}^n \times \{0\}. \end{aligned}$$

Thus  $v_\mu + w$  is a nonnegative supersolution of the heat equation in  $Q_T$ . It admits an initial trace in  $\mathfrak{M}_{T^+}(\mathcal{S}(u))$  that we denote by  $\gamma_u(\mu)$ . Clearly  $\gamma_u(\mu) \leq \mu^* \leq \mu$  since  $v_\mu \leq u_{\mu^*}$  and  $\gamma_u(\mu)$  is nondecreasing with respect to  $\mu$  as it is the case with  $\mu \mapsto u_{\mu^*}$  is. Finally, since  $v_\mu$  is a positive supersolution, it is larger than the solution of 2.1 where the initial data  $\mu$  is replaced by  $\gamma_u(\mu)$ , that is  $u_{\gamma_u(\mu)} \leq v_\mu$ .  $\square$

The proofs of the next four propositions are mere adaptations to the parabolic case of similar results dealing with elliptic equations and proved in [12]; we omit them.

**Proposition 4.7** *Let*

$$\nu_S(u) := \sup\{\gamma_u(\mu) : \mu \in \mathfrak{M}_{T^+}(\mathcal{S}(u))\}.$$

*Then  $\nu_S(u)$  is a Borel measure on  $\mathcal{S}(u)$ .*

**Definition 4.8** *The Borel measure  $\nu(u)$  defined by*

$$\nu(u)(A) := \nu_S(u)(A \cap \mathcal{S}(u)) + \mu_u(A \cap \mathcal{R}(u)), \quad \forall A \subset \mathbb{R}^n, A \text{ Borel,}$$

*is called the extended initial trace of  $u$ , denoted by  $tr_{\{t=0\}}^e(u)$ .*

**Proposition 4.9** *If  $A \subset \mathcal{S}(u)$  is a Borel set, then*

$$\nu_S(A) := \sup\{\gamma_u(\mu)(A) : \mu \in \mathfrak{M}_{T^+}(A)\}.$$

**Proposition 4.10** *There always holds  $\nu(\text{Sing}_V(\mathbb{R}^n)) = 0$ , where  $\text{Sing}_V(\mathbb{R}^n)$  is defined in (3.14).*

**Proposition 4.11** *Assume  $V$  satisfies condition (4.42). If  $u$  is a positive solution of (1.17), then  $tr_{\{t=0\}}^e(u) = \mu_u \in \mathfrak{M}_{T^+}(\mathbb{R}^n)$ .*

## 5 Appendix: the case of a bounded domain

### 5.1 The subcritical case

Let  $\Omega$  be a bounded domain with a  $C^2$  boundary. We denote by  $\mathfrak{M}(\Omega)$  the space of Radon measures in  $\Omega$ , by  $\mathfrak{M}_+(\Omega)$  its positive cone and by  $\mathfrak{M}_\rho(\Omega)$  the space of Radon measures in  $\Omega$  which satisfy

$$\int_{\Omega} \rho d|\mu| < \infty, \quad (5.1)$$

for some weight function  $\rho : \Omega \mapsto \mathbb{R}_+$ . As an important particular case  $\rho(x) = d^\alpha(x)$ , where  $d(x) = \text{dist}(x, \partial\Omega)$  and  $\alpha \geq 0$ . We consider the linear parabolic problem

$$\begin{aligned} \partial_t u - \Delta u + Vu &= 0, & \text{in } Q_T^\Omega = \Omega \times (0, T] \\ u &= 0 & \text{on } \partial_t Q_T^\Omega = \partial\Omega \times (0, T] \\ u(\cdot, 0) &= \mu & \text{in } \Omega. \end{aligned} \quad (5.2)$$

**Definition 5.1** We say that  $\mu \in \mathfrak{M}_d(\Omega)$  is a good measure if the above problem has a weak solution  $u$ , i.e. there exists a function  $u \in L^1(Q_T^\Omega)$ , such that  $Vu \in L^1_d(Q_T^\Omega)$  which satisfies

$$-\int_0^T \int_{\Omega} u(\partial_t \phi + \Delta \phi) dxdt + \int_0^T \int_{\Omega} Vu\phi dxdt = \int_{\Omega} \phi(x, 0) d\mu, \quad (5.3)$$

$\forall \phi \in C^{1,1;1}(\overline{Q_T^\Omega})$  which vanishes on  $\partial_t Q_T^\Omega$  and satisfies  $\phi(x, T) = 0$ .

**Definition 5.2** Let  $H^\Omega(x, y, t)$  be the heat kernel in  $\Omega$ . Then we say that  $\mu \in \mathfrak{M}_d(\Omega)$  is a admissible measure if

$$\|\mathbb{H}^\Omega[|\mu|]\|_{L^1(Q_T^\Omega)} = \int_{Q_T^\Omega} \left( \int_{\Omega} H^\Omega(x-y, t) d|\mu(y)| \right) V(x, t) \psi(x) dxdt < \infty.$$

The next a proposition is direct consequence of [9, Lemma 2.4].

**Proposition 5.3** Assume  $\mu \in \mathfrak{M}_d(\Omega)$  and let  $u$  be a weak solution of problem (5.2), then the following inequalities are valid

(i)

$$\|u\|_{L^1(Q_T^\Omega)} + \|Vu\|_{L^1_\psi(Q_T^\Omega)} \leq C(n, \Omega) \int_{\Omega} dd|\mu|,$$

(ii)

$$-\int_0^T \int_{\Omega} |u|(\partial_t \phi + \Delta \phi) dxdt + \int_0^T \int_{\Omega} |u|V\phi dxdt \leq \int_{\Omega} \phi(x, 0) d|\mu|,$$

$\forall \phi \in C^{1,1;1}(\overline{Q_T^\Omega})$ ,  $\phi \geq 0$ .

(iii)

$$\lambda_\Omega \int_0^T \int_{\Omega} (x)u^+ dxdt + \int_0^T \int_{\Omega} Vu^+ \psi dxdt \leq \int_{\Omega} \psi(x) d\mu^+.,$$

where  $\psi$  is the solution of

$$\begin{aligned} -\Delta \psi &= 1, & \text{in } \Omega \\ \psi &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (5.4)$$

*Proof.* For (ii), in [9, Lemma 2.4, p 1456], above from the relation (2.39), we can take  $\tilde{\zeta} = \gamma(u)\zeta$  for some  $0 \leq \zeta \in C^{1,1;1}(\overline{Q_T^\Omega})$ , since  $u = 0$  on  $\partial_t Q_T^\Omega$ . For (iii) we consider (as in [9, Remark 2.5])  $\phi(x, t) = t\psi(x)$ . The inequality holds by the same type of calculations as in [12].  $\square$

**Proposition 5.4** *The problem (5.2) admits at most one solution. Furthermore, if  $\mu$  is admissible, then there exists a unique solution; we denote it  $u_\mu$ .*

Similarly as Theorem 2.7 and Proposition 2.7, we have the following stability results

**Proposition 5.5** (i) *Assume that  $V$  satisfies the stability condition*

$$\lim_{|E| \rightarrow 0} \iint_E H^\Omega(x, y, t) V(y, t) d(x) dy dt = 0, \quad \forall E \subset Q_T^\Omega, E \text{ Borel.} \quad (5.5)$$

*uniformly with respect to  $y \in \Omega$ . If  $\{\mu_k\}$  is a bounded sequence in  $\mathfrak{M}_d(\Omega)$  converging to  $\mu$  in the dual sense of  $\mathfrak{M}_d(\Omega)$ , then  $(u_{\mu_k}, Vu_{\mu_k})$  converges to  $(u_\mu, Vu_\mu)$  in  $L^1(Q_T^\Omega) \times L^1_d(Q_T^\Omega)$ . (ii) Furthermore if*

$$\lim_{|E| \rightarrow 0} \iint_E H^\Omega(x, y, t + \tau_n) V(y, t) d(x) dy dt = 0, \quad \forall E \subset Q_T^\Omega, E \text{ Borel.} \quad (5.6)$$

*uniformly with respect to  $y \in \Omega$  and  $\tau_k \in [0, \tau_0]$  converges to 0 and  $\{\mu_k\}$  is in (i), then the solutions  $u_{\tau_k, \mu_k}$  of the shifted problem*

$$\begin{aligned} \partial_t u - \Delta u + Vu &= 0 & \text{on } \Omega \times (\tau_k, T) \\ u &= 0 & \text{on } \partial\Omega \times (\tau_k, T) \\ u(\cdot, \tau_k) &= \mu_k & \text{on } \Omega \times \{\tau_k\} \end{aligned} \quad (5.7)$$

*(extended by 0 on  $(0, \tau_k)$ ) converge to  $u_\mu$  in  $L^1_d(Q_T^\Omega)$ , and  $\{Vu_{\mu_k}\}$  converges to  $Vu_\mu$  in  $L^1_d(Q_T^\Omega)$ .*

*Proof.* We can easily see that the measure  $\mu_n$  is admissible and uniqueness holds; furthermore any admissible measure is a good measure as in Theorem 2.5, and

$$\iint_{Q_T^\Omega} u_{\mu_n} dx ds + \iint_{Q_T^\Omega} u_{\mu_n} V\psi dx ds \leq C \int_\Omega d\mu_n < C.$$

The remaining of the proof is similar to the one of Theorem 2.7.  $\square$

## 5.2 The supercritical case

**Lemma 5.6** *Let  $\{\mu_n\}_{n=1}^\infty$  be an increasing sequence of good measures converging to some measure  $\mu$  in the weak\* topology, then  $\mu$  is good.*

*Proof.* Let  $u_{\mu_n}$  be the weak solution of (5.2) with initial data  $\mu_n$ . Then by Proposition 5.5 -(iii),  $\{u_{\mu_n}\}$  is an increasing sequence. By 5.5 -(i) the sequence  $\{u_{\mu_n}\}$  is bounded in  $L^1(Q_T^\Omega)$ . Thus  $u_{\mu_n} \rightarrow u \in L^1(Q_T^\Omega)$ . Also by (iii) of Proposition 5.5, we have that  $Vu_{\mu_n} \rightarrow Vu$  in  $L^1_\psi(Q_T^\Omega)$ . Thus we can easily prove that  $u$  is a weak solution of (5.2) with  $\mu$  as initial data.  $\square$

Let

$$Z_V^\Omega = \{x \in \Omega : \int_{Q_T^\Omega} H^\Omega(t, x, y) V(y) \psi(y) dy = \infty\}. \quad (5.8)$$

We note that, since  $H^\Omega(t, x, y) \leq H(x - y, t)$  for any bounded  $\Omega$  with smooth boundary, it holds  $Z_V^\Omega \subset Z_V$ . By the same arguments as in [12] we can prove the following results



**Proposition 5.7** *Let  $\mu$  be an admissible positive measure. Then  $\mu(Z_V^\Omega) = 0$*

**Proposition 5.8** *Let  $\mu \in \mathfrak{M}_{d+}(\Omega)$  such that  $\mu(Z_V^\Omega) = 0$ , then  $\mu$  is good.*

**Proposition 5.9** *Let  $\mu \in \mathfrak{M}_{d+}(\Omega)$  be a good measure. Then the following assertions are equivalent:*

(i)  $\mu(Z_V^\Omega) = 0$ .

(ii) *There exists an increasing sequence of admissible measures  $\{\mu_n\}$  which converges to  $\mu$  in the weak\*-topology*

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