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The Linear Matching Method applied to composite materials: a micromechanical approach

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ABSTRACT

The paper considers a Direct Method for the evaluation of the maximum load corresponding to pre-assigned limits on the nonlinear behaviour of the matrix and fibres in a laminate structure. This is achieved by combining a consistent micro-macro model for linear behaviour with an extension of the Linear Matching Method (LMM), previously extensively applied to Direct Methods in plasticity. The method is developed with assumptions that allow the methodology to be displayed in its simplest form. Applications to examples of laminate elements and a laminate plate containing a hole are described, assuming a matrix with a limit on ductility.

Keywords: Composite materials; Micromechanics, Limit analysis; Linear Matching Method.

1. Introduction

This paper is concerned with methods for assessing the load bearing capacity of composite laminate structures, corresponding to appropriately defined micro failure criteria. The conventional approach to the assessment of the maximum load on a structure first requires the formulation of constitutive equations for a material element, incorporating an understanding of deformation and failure modes within the composite microstructure. This is then followed

by a conventional step by step analysis to failure. The relationship between the critical micro structural failure mechanism and the allowable maximum load is understood through the structure of the constitutive relationships and detail can become lost. The work of this paper falls within the general field of Direct Methods [1], by which we mean computational or analytic methods that allow the direct evaluation of the load or load range corresponding to a pre assigned material or design constraint, such as a yield condition or a critical strain. In the case of laminates such Direct Methods may be applied to a constitutive model (see [22, 23]) although the behaviour of laminates may not be well described by elastic-perfect plasticity. Here we explore the possibility of developing a method whereby the maximum permissible load may be related directly to a particular strain condition or failure mode at any point in the structure, in any layer of the laminate and at any point in the microstructure, taking into account nonlinear behaviour of the composite material constituents. Such a proposal requires a degree of complexity which we wish to avoid in this first attempt. We combine a kinematically consistent micromechanics model with significant elements of classical laminate theory [25] and a simple description of micro failure. This first step then allows the methodology of such a Direct Method to be displayed in its simplest form.

The history of the development of laminate failure criteria is well known. Since the work of Hill [2,3] and Tsai-Wu [4] a range of empirical and micromechanical failure criteria have been proposed which mathematically combine lamina mechanical properties into an assumed homogeneous laminate to attain idealized uniform strength and stiffness throughout the structure. Failure theories, based on a micromechanics approach, were first developed by Hashin and Rotem [5,6]. Their method can be viewed as a macro approach based on micro mechanical issues that consider a failure criterion on the basis of observed failure modes. Another step forward was made by Chang, Scott and Springer [7, 8] by introducing other possibilities of failure: matrix cracking, fibre-matrix interface shearing, fibre breakage and

material property degradation. Other researchers [9-13] treated the failure of the matrix and of the fibre separately, making some simplified assumptions. These authors employ a Mohr-Coulomb yield criterion for the matrix and assume that final failure is determined by the failure of the fibres. Mayes and Hansen [14] introduced a theory in which micro-level stresses, evaluated considering constitutive behaviour of the components, are applied to different failure theories for both the fibres and matrix. This approach includes the effect of changes in the constitutive properties of the components. For instance, similar methods were used to predict non linear stress-strain relations and failure modes for metal and polymeric matrix composites. Since then a number of developments have taken place and the subject remains one of active research. It should be emphasised that the work in this paper is not concerned with the micromechanics of failure, rather the development of the structure of computational method that is capable of taking such micro failure mechanisms into account and directly relating them to the maximum load bearing capacity of a structure. Keeping in mind the needs to first display a simple theory, we restrict attention to non-linear behaviour of a ductile matrix, assuming failure occurs at a critical micro strain. This is described in detail in Section (3). To develop a complete method, capable of comparison with experimental data, it is necessary to incorporate into the theoretical structure described here a set of possible failure modes of both matrix and fibres and this awaits further development. This is discussed further in the conclusions.

The Linear Matching Method (LMM), the method applied here, consists, essentially, of a programming method that allows a direct evaluation of the load corresponding to predefined kinematic restraints [15-21], following an iterative process. The primary area of application has been the evaluation of limits appropriate for metallic structures subjected to severe thermo-mechanical loads. Methods have been developed for the life assessment of power plant at high temperature and an entire set of such methods now form part of the UK Life

Assessment Method R5 [17,18]. The work in this paper is a first attempt to develop an appropriate method of this type for composite materials.

The elements are as follows. A structure is subjected to a load distribution λF_i where F_i is a chosen load distribution on S_F , part of the surface S , and λ is a scalar multiplier. The objective of the method is to evaluate the value of λ so that the strain field in the microstructure corresponds to a prescribed design condition, while all other conditions of the continuum problem are satisfied: equilibrium, compatibility and consistency with the material behaviour. LMM is applicable to material behaviour where a convex strain potential exists, implying that a deformation theory of inelastic behaviour must be adopted. A summary of the theoretical basis of the method, as required for this application, is given in the Appendix. For the case discussed here, the material behaviour is assumed to be either linearly elastic or elastic perfectly plastic, without unloading. The method is an iterative process where each iteration contains four stages. The process begins with the generation of a kinematically admissible strain field, conventionally by solving a linearly elastic problem for an arbitrary λ . This strain field is then scaled so that the design constraint is satisfied. The process then consists of the following stages;

Stage1: A matching linear material is defined by choosing linear moduli (which are spatially varying) so that the linear material and the actual material give the same stress state for this initial strain field. The solution of the resulting linear problem produces a new strain field which reduced the potential energy of the structure. This is discussed in detail in the Appendix.

Stage 2: The new strain field will generally not satisfy the design constraint and is scaled by a factor, to ensure that the design constraint is re-imposed.

Stage 3: The load parameter λ is now changed so that the potential energy is a minimum for the new scaled strain field, amongst all such scaled strain fields.

Stage 4: a lower static load factor is calculated that measures the degree of deviation of the equilibrium stress field in the linear solution from the actual material behaviour. The difference between the load factors evaluated in Stage 3 and Stage 4 indicates the deviation from convergence.

For the material assumptions adopted here, the specific form of this process is discussed in detail in the following sections. A consistent relationship between stresses and strains in microstructure (i.e. the fibres and the matrix) and the ply stresses and strains is described. This allows the design constraint to be assigned to the properties of the materials of the microstructure. Conventional laminate theory then relates the ply stresses to those of the laminate. Although in this application no softening behaviour is allowed, in a recent study such methods have been applied to portal frames with softening elements, with the objective of evaluating the maximum loads associated with differing design limits [20, 21]. Hence the method described here is the first step towards evaluation of a maximum load where softening effects occur in the microstructure. Such possibilities are discussed further in the Conclusions.

The outline of the paper is as follows. The constitutive assumptions of the interaction between the fibre and matrix are discussed in Section 2 and a simplified scheme of the failure modes for continuous fibre composite is reported. In Section 3 the micromechanics model for the evaluation of the stress and the strain at the micro level is presented. Section 4 concerns the application of the LMM for the evaluation of the peak load where a design failure criteria associated with the fibres and the matrix are taken into account. Numerical applications are reported in Section 5, followed by Conclusions in Section 6.

2. Constitutive assumptions of matrix and fibre

Perfect adhesion between fibre and interface is assumed. Interfacial damage (in particular, debonding) and fibre buckling are neglected. We concentrate on two possible failure modes, cracking or strain exhaustion within the matrix, characterised by a maximum effective strain, and the failure of the fibres. The methodology is, however, capable of incorporating a much wider range of possible local failure modes. The local constitutive models for the matrix and fibres are illustrated in Figure 1 and described in the following.

2.1 Fibre: elastic-brittle behaviour

In the following the superscripts f and m refer to the fibres and the matrix in a layer of a unidirectional composite laminate. The 1 direction is in the direction of the fibre. The fibre reinforcement is considered to be longitudinally continuous, isotropic, elastic and perfectly brittle, with tensile behaviour limited by a maximum principal strain failure criterion. The fibre fails when the maximum longitudinal strain reaches a critical value, i.e.

$$-\varepsilon_f^c \leq \varepsilon_1^f \leq \varepsilon_f^* \quad (1)$$

where ε_1^f denotes the axial strain in the fibre. ε_f^c and ε_f^* denote failure values in compression and tension.

2.2 Matrix: non linear behaviour.

We concentrate on non-linear properties appropriate for epoxy resins. For simplicity we ignore the pressure dependence of the yield condition and assume that the matrix satisfies a von Mises yield criterion. Hence, the matrix failure depends on the deviatoric stress and is independent of the applied hydrostatic (or mean) stress. The material is assumed to be elastic-perfectly-plastic with uniaxial yield stress σ_Y and satisfies the von Mises yield condition:

$$f(\sigma_{ij}) = \bar{\sigma}(\sigma_{ij}'^m) - \sigma_Y = 0, \quad \text{where } \sigma_{ij}'^m = \sigma_{ij}^m - \sigma_h^m \delta_{ij} \text{ and } \sigma_h^m = \frac{1}{3} \sigma_{kk}^m \quad (2)$$

where $\sigma'_{ij}{}^m$ is the deviatoric component of stress and $\bar{\sigma}(\sigma'_{ij}{}^m) = \sqrt{\frac{3}{2} \sigma'_{ij}{}^m \sigma'_{ij}{}^m}$ is the von Mises effective stress. Plastic strains at yield are given by the Prandl-Reuss associated flow rule:

$$\dot{\varepsilon}_{ij}{}^{pm} = \dot{\alpha} \sigma'_{ij}{}^m \quad \text{and} \quad \dot{\varepsilon}_{kk}{}^{pm} = 0 \quad (3a)$$

where $\dot{\alpha}$ denotes a scalar multiplier. We assume a deformation theory where total plastic strain is obtained at constant stress at yield;

$$\varepsilon_{ij}{}^{pm} = \alpha \sigma'_{ij}{}^m \quad (3b)$$

The total strain tensor may be written as the sum of a deviatoric part $\varepsilon'_{ij}{}^m$ and a volumetric part $\varepsilon_{kk}{}^m$:

$$\varepsilon'_{ij}{}^m = \frac{1}{2G_m} \sigma'_{ij}{}^m + \varepsilon_{ij}{}^{pm}, \quad \varepsilon_{kk}{}^m = \frac{\sigma_{kk}{}^m}{K_m} \quad \text{and} \quad \varepsilon_{ij}{}^m = \varepsilon'_{ij}{}^m + \varepsilon_{kk}{}^m \delta_{ij} \quad (4)$$

where G_m is the elastic shear modulus of the matrix and K_m is the elastic bulk modulus.

Hence the effective strain of the matrix $\bar{\varepsilon}(\varepsilon'_{ij}{}^m) = \sqrt{\frac{2}{3} \varepsilon'_{ij}{}^m \varepsilon'_{ij}{}^m}$ is related to the effective stress,

$$\text{by,} \quad \bar{\varepsilon} = \frac{1}{3} \left(\frac{1}{G_m} + 2\alpha \right) \bar{\sigma} \quad (5)$$

Note that for a known α the non-linear behaviour of the matrix may be represented by a linear relationship;

$$\bar{\varepsilon} = \frac{1}{3\tilde{G}_m} \bar{\sigma}, \quad \frac{1}{\tilde{G}_m} = \frac{1}{G_m} + 2\alpha \quad (6)$$

with a spatially varying shear modulus \tilde{G}_m and a constant elastic bulk modulus K_m . This form provides the basis of the application of the LMM in this case.

3. The Micromechanics model

In the application of the LMM only linear analysis is required where \tilde{G}_m (and associated Young's modulus \tilde{E}_m) replaces the non-linear properties of the matrix. In classical

theory, the relationships for overall linear ply properties are given by the Halphen-Tsai [24] equations in terms of the elastic properties of the fibres and matrix. These semi empirical relationships provide a close fit to both experimental data and the predictions of Eshelby theory, [24,25]. However they do not correspond to a consistent micromechanical model; it is not possible to find a consistent set of relations that allow the behaviour at the micro level to be evaluated from behaviour of the ply. As a result we adopt a simpler and less exact model, consistent with a set of micro mechanical assumptions. This allows the evaluation of micromechanical properties from macro properties and vice versa.

In the following, as before, quantities with a superscript, either f or m , refer to values in the fibre and matrix. Quantities with no such superscript refer to ply quantities. Consider the behaviour of a single ply, where the direction 1 corresponds to the direction of the fibres and 2 is in the plane of the ply. The most prominent assumptions are: fibre and matrix are uniformly distributed through the thickness of the ply and the ply is in a plane stress condition; strains in the fibre direction are the same in the fibres and in the matrix. Moreover, the same transverse stress, σ_2 , and shearing stresses, τ_{12} , are assumed to be applied to both the fibre and the matrix so that equilibrium of adjacent elements in the composite material must occur.

These assumptions are summarized as follow:

$$\varepsilon_1 = \varepsilon_1^m = \varepsilon_1^f, \quad \sigma_2 = \sigma_2^m = \sigma_2^f, \quad \tau_{12} = \tau_{12}^m = \tau_{12}^f \quad (7)$$

By considering Eqs. (7), standard arguments lead to the well-know relationships for ply elastic properties [25] as follows:

$$E_1 = E_f V_f + (1 - V_f) E_m, \quad E_2 = \left[\frac{V_f}{E_f} + \frac{(1 - V_f)}{E_m} \right]^{-1} \quad (8)$$

$$G_{12} = \left[\frac{V_f}{G_f} + \frac{(1 - V_f)}{G_m} \right]^{-1}, \quad \nu_{12} = V_f \nu_f + (1 - V_f) \nu_m, \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}$$

where V_f denotes the fibre volume fraction and ν denotes Poisson's ratio. Hence for ply stresses and strains,

$$\sigma_1 = \frac{E_1}{1 - \nu_{12}\nu_{21}} \{\varepsilon_1 + \nu_{21}\varepsilon_2\}, \quad \sigma_2 = \frac{E_2}{1 - \nu_{12}\nu_{21}} \{\varepsilon_2 + \nu_{12}\varepsilon_1\} \quad (9)$$

Assuming Eqs. (7), the strain ε_1^m in the matrix can be written as follows:

$$\varepsilon_1^m = \varepsilon_1^f = \varepsilon_1 = \frac{1}{E_m} (\sigma_1^m - \nu_m \sigma_2^m) = \frac{1}{E_m} (\sigma_1^m - \nu_m \sigma_2). \quad (10)$$

Hence, from (10):

$$\sigma_1^m = E_m \varepsilon_1 + \nu_m \sigma_2. \quad (11)$$

The same consideration can be done for the strain ε_2 :

$$\varepsilon_2^m = \frac{1}{E_m} (\sigma_2^m - \nu_m \sigma_1^m) \quad (12)$$

Substituting (11) into Eq. (12) and considering the assumption Eq. (7), relation (12) can be written as follows:

$$\varepsilon_2^m = \frac{(1 - \nu_m^2)}{E_m} \sigma_2 - \nu_m \varepsilon_1 \quad (13)$$

For γ_{12}^m , considering the hypothesis of Eq. (7), the following is true:

$$\gamma_{12}^m = \frac{1}{G_m} \tau_{12} \quad (14)$$

The same consideration may be applied to the fibres. The stress and strain in the matrix and fibres in terms of the ply stresses and strains are summarized in Table 1.

The equations in Table 1 allow the evaluation of stresses and strains in the fibres and matrix from the ply stresses and strains by assumptions that are consistent with the evaluation of ply elastic properties from those of the constituent parts, equations (9). These relationships are only valid for linear constitutive behaviour. The remainder of the laminate analysis follows classical laminate analysis, where the linear relationships for each ply are transformed onto a common set of axes and incorporated into the laminate kinematic conditions (see, for

example Hull and Clyne [25]). Hence the primary analysis procedure is identical to procedures already incorporated into commercial FE systems (in the case of the results of this paper, into ADINA). This is a common and important property of all LMM procedures.

STRESS IN MATRIX	STRESS IN FIBRE
$\begin{bmatrix} \sigma_1^m \\ \sigma_2^m \\ \tau_{12}^m \end{bmatrix} = \begin{bmatrix} E_m \varepsilon_1 + \nu_m \sigma_2 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$	$\begin{bmatrix} \sigma_1^f \\ \sigma_2^f \\ \tau_{12}^f \end{bmatrix} = \begin{bmatrix} E_f \varepsilon_1 + \nu_f \sigma_2 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$
STRAIN IN MATRIX	STRAIN IN FIBER
$\begin{bmatrix} \varepsilon_1^m \\ \varepsilon_2^m \\ \gamma_{12}^m \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \frac{1-\nu_m^2}{E_m} \sigma_2 - \nu_m \varepsilon_1 \\ \frac{1}{G_m} \tau_{12} \end{bmatrix}$	$\begin{bmatrix} \varepsilon_1^f \\ \varepsilon_2^f \\ \gamma_{12}^f \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \frac{1-\nu_f^2}{E_f} \sigma_2 - \nu_f \varepsilon_1 \\ \frac{1}{G_f} \tau_{12} \end{bmatrix}$

Table.1 Stress-strain relations in fibre and matrix.

4. Linear Matching Method (LMM) for the evaluation of the peak load

The LMM consists of the evaluation a sequence of linear solutions with a spatially varying shear modulus \tilde{G}_m in the matrix. At each iteration, kinematic and static bounds to the load parameter λ are evaluated which become equal at convergence. In summary, the matching process is shown in Figure (2) for the matrix material behaviour in Figure (1a). At each iteration, the modulus \tilde{G}_m^{k+1} is evaluated so that the current computed matrix stresses $\mu \bar{\sigma}_m^k$ are brought onto the yield surface at a fixed scaled strain distribution $\mu \varepsilon_m^k$ (Fig. 2). The scaling factor μ is chosen so that the design criterion is satisfied at the micro scale. The ply and laminate properties are then computed at each point of the body of volume V, in practice at each Gauss point (GP) of each finite element (FE) of the discretized domain V.

4.1 Iterative procedure

The iterative process begins with an initial arbitrary value of the load $\lambda^0 F_i$ and linear elastic material properties. A sequence of linear analyses is then conducted where the fibres retain

their elastic properties but the matrix shear modulus \tilde{G}_m^k is iteratively changed. There are three aspects to this process: the scaling of the current strain solution so that the design criterion is satisfied; the evaluation of a new load parameter and the determination of a new shear modulus \tilde{G}_m^{k+1} distribution. These are discussed in turn in the following.

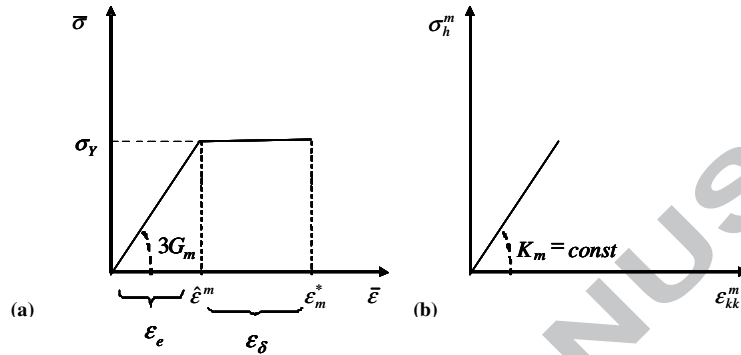


Figure 1 Constitutive properties of the matrix and fibres.

4.2 The Scaling Factor μ

The objective is to evaluate the maximum load without failure on the micro scale so that, in the matrix, $\bar{\epsilon} \leq \epsilon_m^*$ and in the fibres $-\epsilon_f^c \leq \epsilon^f \leq \epsilon_f^*$ throughout the structure. These conditions are regarded as design criteria. Hence for the current k^{th} solution, for which the strain values $\bar{\epsilon}_1^{fk}$ and $\bar{\epsilon}^{mk}$ that do not generally satisfy the design restrictions, scaling factors μ_m and μ_f may be defined as follows so that the scaled strain fields satisfy each of the

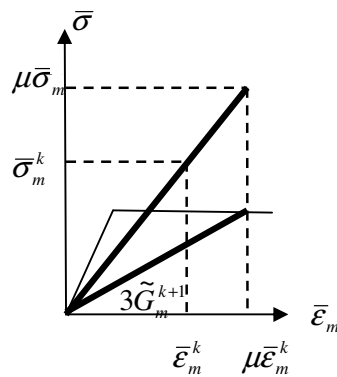


Figure 2 Definition of matching linear modulus for the matrix

$$\mu_m = \frac{\varepsilon_m^*}{\max(\bar{\varepsilon})}, \quad \mu_f^* = \frac{\varepsilon_f^*}{\max(\varepsilon_1^f)} \quad \text{and} \quad \mu_f^c = \frac{\varepsilon_f^c}{\min(\varepsilon_1^f)} \quad (15)$$

where the maximum and minimum are evaluated over all plies, and a value only accepted if it is positive. The choice of μ as the minimum of μ_m , μ_f^* and μ_f^c ensures the design condition is satisfied. In the numerical examples only yielding of the matrix is observed, hence the scaling parameter $\mu = \mu_m$.

4.3 Evaluation of the Load Parameter λ

Corresponding to this scaled solution, a corresponding load is evaluated by the following kinematic relationship:

$$\lambda_{KIN}^k = \int_V \sigma_i^c (\mu \varepsilon_i^k) \mu \varepsilon_i^k dV / \int_{S_f} F_i \mu u_i^k dS_u \quad (16)$$

where σ_i^c represents the stress components that correspond to the scaled strains $\mu \varepsilon_i^k$ via the material properties of the matrix and fibres. The integral over the volume represents summation over the matrix and fibres for all plies, and integration over the laminate. Equation (16) is derived, in general terms, in the appendix and ensures that the scaled strain field lies at the minimum of the total potential energy of the structure amongst all such scaled strain fields.

At each iteration a corresponding static approximation to the maximum load can be found by scaling the stresses $\bar{\mu} \sigma_i^k$, where σ_i^k is in equilibrium with loads $\lambda_{KIN}^{k-1} F_i$, so that for the largest possible value of $\lambda = \lambda_{LB}^k$ the scaled stresses lie on or below the stress-strain relation of Fig. (1a), i.e.

$$\bar{\mu} \sigma_i^k \leq \sigma_i^c (\mu \varepsilon_i^k) \quad (17)$$

As the stresses σ_i^k are in equilibrium with $\lambda_{KIN}^{k-1} F_i$, the scaled stresses are in equilibrium with $\lambda_{ST}^k F_i$, given by,

$$\lambda_{ST}^k = \bar{\mu} \lambda_{KIN}^{k-1} \quad (18)$$

The values of λ_{ST}^k and λ_{KIN}^k converges to the exact solution when all conditions of the problem are satisfied: equilibrium, compatibility, consistency with the material behaviour and the design criterion.

4.4 Evaluation of \tilde{G}_m^{k+1}

For the deviatoric components of the matrix stresses it is sufficient to consider the effective values. Two differing expressions correspond to the two regions as shown in Figure (1),

$$\begin{aligned}\bar{\sigma}^{mc} &= 3G_m \mu \bar{\epsilon}^{mk} & \text{for } \mu \bar{\epsilon}^{mc} < \hat{\epsilon}^m \\ \bar{\sigma}^{mc} &= \sigma_y & \text{for } \hat{\epsilon}^m < \mu \bar{\epsilon}^{mc} < \epsilon_m^*\end{aligned}\quad (19)$$

Where G_m is the matrix elastic shear modulus. At each iteration the matching shear modulus \tilde{G}_m^{k+1} is updated so that stress $\mu \bar{\sigma}$ associated with $\mu \bar{\epsilon}^k$ coincides with $\bar{\sigma}^{mc}$ as shown in Figure 2,

$$\begin{aligned}\tilde{G}_m^{k+1} &= G_m & \mu \bar{\epsilon}^{mk} \leq \hat{\epsilon}^m \\ \tilde{G}_m^{k+1} &= \frac{\sigma_y}{3\mu \bar{\epsilon}^{mk}} & \hat{\epsilon}^m < \mu \bar{\epsilon}^{mk} \leq \epsilon_m^*\end{aligned}\quad (20)$$

A new linear solution with the load $\lambda_{KIN}^k F$ and the new matching linear material with shear moduli distribution \tilde{G}_m^{k+1} (Eqs. 20) is performed. Expressions for the Young's modulus \tilde{E}_m^{k+1} and Poisson ratio $\tilde{\nu}_m^{k+1}$ consistent with \tilde{G}_m^{k+1} and the elastic bulk modulus K_m are then given by:.

$$\frac{1}{\tilde{E}_m} = \left(\frac{1}{3\tilde{G}_m} + \frac{1}{9K_m} \right), \quad \frac{\tilde{\nu}_m}{\tilde{E}_m} = \left(\frac{1}{6\tilde{G}_m} - \frac{1}{9K_m} \right)\quad (21)$$

Hence from (21) at the $k+1$ iteration:

$$\tilde{E}_m^{k+1} = 1 / \left(\frac{1}{3\tilde{G}_m^{k+1}} + \frac{1}{9K_m} \right), \quad \tilde{\nu}_m^{k+1} = \tilde{E}_m^{k+1} \left(\frac{1}{6\tilde{G}_m^{k+1}} - \frac{1}{9K_m} \right)\quad (22)$$

4.5 Summary of the iterative process

The iterative procedure has been implemented within a finite element scheme where each element contains Z ply layers, $L=1$ to Z . This consists of the following major steps.

1. *Initialization:* Specify elastic parameters of the constituents -fibre and matrix- E_m, G_m, ν_m and E_f, G_f, ν_f . The elastic parameters of each ply $E_1^L, E_2^L, G_{12}^L, \nu_{12}^L$ are evaluated through eqs. (8), leading to the elastic properties of the laminate. Here L indicates the number of the ply layer, $L=1 \dots Z$.
2. *First iteration:* Set $k=1$ and $\lambda_{KIN}^{k-1} = \lambda_{KIN}^0 = 1$. The value of the first kinematic multiplier is arbitrary as the resulting micro strains are scaled to satisfy the design criterion.
3. Perform a linear analysis with a load $\lambda_{KIN}^{k-1} F$ and the ply elastic parameters $E_{1j}^L, E_{2j}^L, G_{12j}^L, \nu_{12j}^L$ where the suffix j indicates the Gauss points (GP) in the model.
4. *kth iteration.* Results of the $(k-1)$ th linear solution are expressed in terms of ply stresses and strains $\sigma_j^{L(k-1)}$ and $\epsilon_j^{L(k-1)}$, and nodal displacements $u_i^{(k-1)}$ where the load is applied and where i denotes the nodes. Evaluate the ply stresses and strains and hence the micro stresses and strains of the matrix and the fibres using the relations summarized in Table 1 at each layer, for each element and at each GP.
5. *Scaling.* Perform linear scaling (by μ) so that constraints are satisfied at each GP and ply layer.
6. *Linear Matching* . Apply the matching process of the LMM considering the scaled strains $\mu \epsilon_j^L$ to find the new linear parameters: $(\tilde{E}_1^L)_j^k, (\tilde{E}_2^L)_j^k, (\tilde{G}_{12}^L)_j^k, (\tilde{\nu}_{12}^L)_j^k$, Eq. (8), again at each Gauss point and layer.
7. *Evaluate the kinematic and static approximations* through (16) and (18): $\lambda_{KIN}^k, \lambda_{ST}^k$.

8. Check for convergence, i.e. $|\lambda_{KIN}^k - \lambda_{ST}^k| \leq TOLERANCE$.

The process continues until convergence occurs. In the numerical examples convergence occurs after less than eight iterations.

5. Numerical applications

As a first attempt to apply this method we take the simplifying assumption that the composite fails by yielding strain exhaustion of the matrix. At point 5 of the iterative procedure in Section 4.5, the scaling factor is $\mu = \mu_m$. Hence it is assumed that the design limit is dominated by the ductility of the matrix.

The examples concern multilayer laminate plates with symmetric layups under conditions of plane stress. The constituent material properties and parameters, typical of a Glass-Epoxy system, are shown in Table 2. All the FE analyses were performed with the ADINA code [26,27], interfaced with a FORTRAN main program which controls the iterative procedure. In the first example, triangular elements with one Gauss point each are utilized. In the subsequent composite plate, model isoparametric 4 noded elements with 4 Gauss points were adopted.

Fibre	Matrix	Ply
$E_f = 74$	$E_m = 10$	$E_1 = 35.62, E_2 = 15.28$
$G_f = 30.8$	$G_m = 3.7 \quad K_m = 11.11$	$G_{12} = 5.70$
$\nu_f = 0.2$	$\nu_m = 0.35$	$\nu_{12} = 0.29$
$V_f = 0.4$	$\sigma_Y = 0.1 \quad \epsilon_m^* = 0.06$	

Table 2. Material properties GPa (Glass-Epoxy).

5.1 Example 1: Laminate Elements

Consider a 4-layered square plate of side L and area A, with material parameters given in Table 2. The first example concerns the behaviour of a four layer symmetric [+45/-45]_s laminate. Figure 3(a) shows the element structure and boundary conditions. Figure 3(b) shows

the kinematic and static multipliers as a function of the number of iterations. In a single iteration λ_{KIN} and λ_{ST} converge to the maximum load factor.

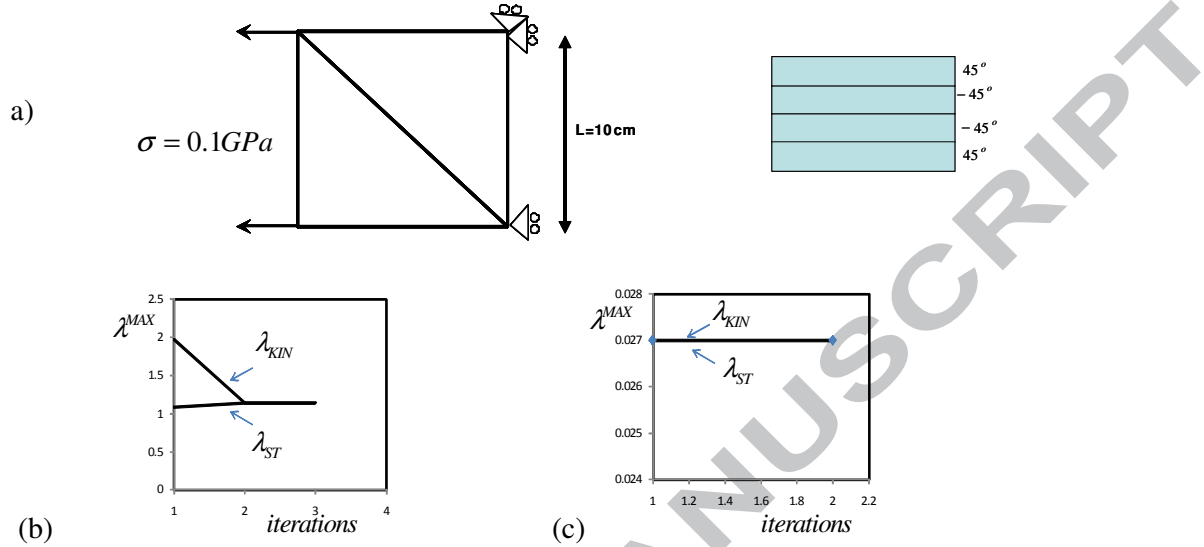


Figure 3 Laminate with $[+45/-45]_s$ lay-up, under uniform uniaxial tension. (a)Element structure, (b) convergence for properties of Table (2), (c) Convergence for stiffened fibres.

The first static bound λ_{ST}^1 (eq. 17) represents the elastic limit when yield initially occurs in the matrix; a comparison between the first static bound λ_{ST}^1 and the converged maximum load factor λ^{MAX} shows how much the load can be increased beyond first yield before the matrix exhausts its ductility. In this case the first static bound $\lambda_{ST}^1 = 1.090$ is very close to the max load $\lambda^{MAX} = 1.147$ and hence the first yield load is almost coincident with the ductility exhaustion load due the fact that the load is mainly carried by the fibres. It is useful to consider a further case where a simple analytic result exists. The fibres are made considerably stiffer than the matrix in such a way the entire load is carried by the fibre ($E_f = 740\text{GPa}$, $E_m = 0.1\text{GPa}$, $\sigma_y = 0.0239\text{GPa}$). For this case the analytic solution provides $\lambda = 0.0276$. In Fig.3(c) the results of the procedure shows that for the first iteration the kinematic and static bound coincide giving the exact value. These solutions are typical of

the behaviour of the method when applied at the laminate element level. Convergence occurs at the first iteration, with the kinematic load factor reducing and the static load factor increasing to the converged value.

5.2 Example 2: Laminate Structure

Figure 4(a) shows a plate with a hole of radius 5cm. Due to symmetry only a quarter of the plate need be studied. The material axes 1-2 coincide with y-z axes. The FE model of the plate consists of 1000 multilayer shell elements with 4 nodes and 4 Gauss points per element (Figure 8). The simplest layup $[0^\circ/0^\circ]$ is considered with, again, the material characteristics in Table.2. The initial applied load is given as a distributed load of 1 GPa. Hence values of λ correspond to applied stress levels in GPa.

The interest in this problem arises from the characteristic that the load applied in the central section of the plate edge must be transferred to the fibres at the hole edges through shear stresses transmitted through the matrix. Table 3 compares the elastic limit

Lay-up $[0,0]$	First yield λ_{ST}^I	λ^{MAX}
σ_1	0.356	1.865
τ_{12}	0.0577	0.0577

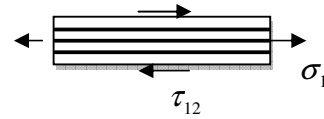


Table 3. Comparison between stresses first yield λ_{ST}^I and λ^{MAX} for material element of plate under tension and shear in GPa.

and the maximum load for a material element under stress in the direction of the fibre and subject to shear, emphasising the weakness of the composite under pure shear. If the restriction on the ductility of the matrix is removed, the ultimate load is given by shearing in the matrix along lines parallel to the plate edges tangential to the edges of the hole. Hence, the maximum possible load λ_{ult} , without constraint on the strain in the matrix, is given by

$$5\lambda_{ult} = 30\tau_y = 10\sqrt{3}\sigma_y, \text{ i.e. } \lambda_{ult} = 0.346$$

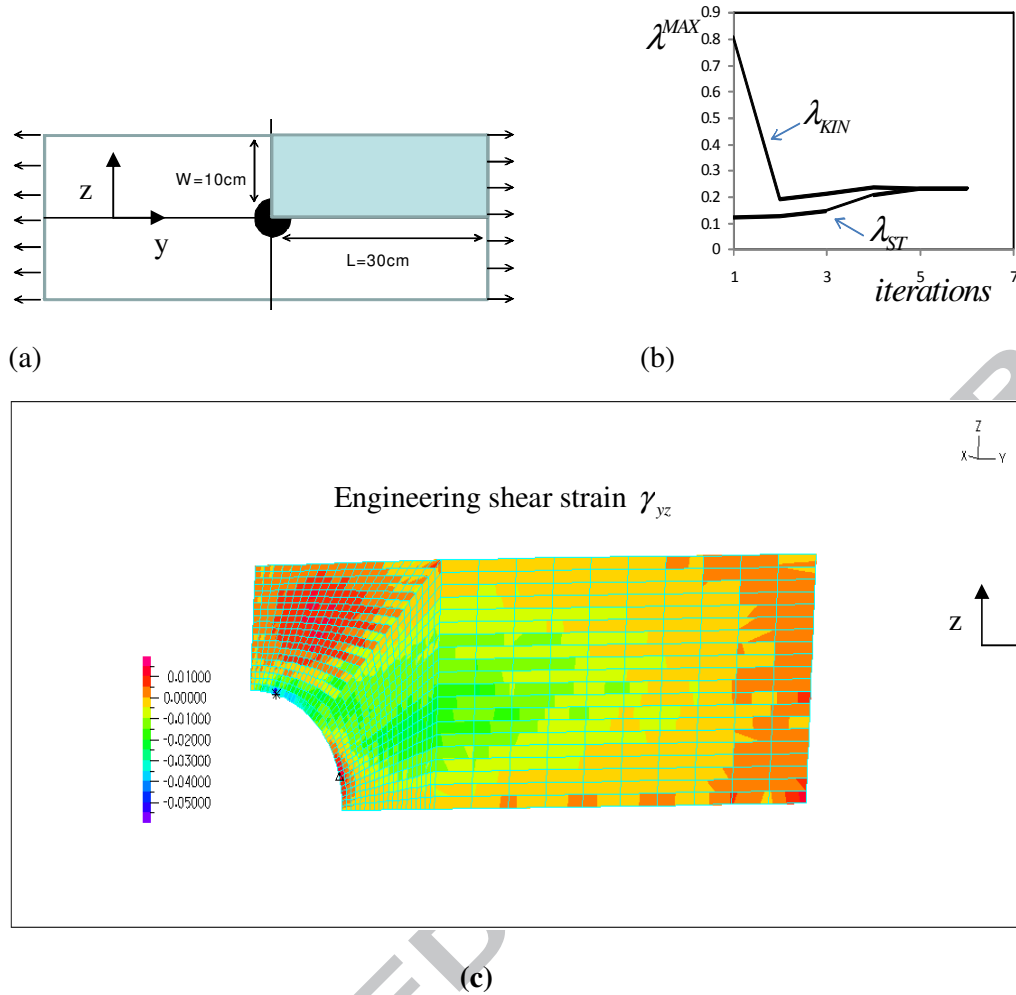


Figure 4. Plate with hole. (a) Uniform plate of [0,0] laminate with central hole and uniform tension, (b) convergence of kinematic and static load factors, (c) Distribution of engineering shear strain in laminate. The star indicates the position of maximum strain.

The results in term of kinematic and static load factors are shown in Fig. 4(b). Note that the kinematic result does not monotonically decrease but the difference between the two estimates monotonically reduces. The value of the elastic limit, the first static bound $\lambda_{ST}^l = 0.125 = 0.361 \lambda_{ult}$ is about 50% of the converged maximum load $\lambda^{MAX} = 0.233 = 0.673 \lambda_{ult}$ carried by the plate. This result means that the load can be increased significantly beyond the first yield value before the ductility exhaustion of the matrix occurs and provides a more meaningful limit than the ultimate load.

The significance of this example is its ability to show the usefulness of the procedure to deal with the effect of the ductility of the matrix on the maximum load. The total engineering shear strain γ plot in Fig.4(c) provides a visualisation of the deformation pattern. The maximum value of γ occurs at the edge of the hole. If we assume that the strain in the matrix is dominated by shear strain, then the total strain at initial yield is given by;

$$\gamma = 0.6\gamma_m + 0.4\gamma_f, \quad \gamma_m = \tau_y / 3.7, \quad \gamma_f = \tau_y / 30.8, \quad \text{hence } \gamma = 0.0101$$

Similarly the shear strain at the maximum load λ^{MAX} is given by $\gamma_m = 0.6323$. Hence the region where the strain in the matrix exceeds the elastic limit and is less than the limit value is given by elements coloured mid to dark green and blue. Over most of this region the shear strain is significantly less than the ultimate value. The limit value $\gamma_m = 0.6323$ is achieved only within a very limited region. Hence this solution demonstrates that insight into the behaviour can be gained from this method in a direct and informative manner that is distinct from that given by other analysis methods.

6. Conclusions

The method described in this paper is a first attempt to develop a Direct Method for composite materials and structures, based upon a micromechanics of failure. The objective is to seek the maximum load corresponding to a set of micromechanics material and kinematic assumptions. Here the micromechanical model is merely an adaption of classical laminate analysis where the Halphen-Tsai relationships are replaced by relationships that correspond to well defined micro mechanics assumptions, allowing the matrix and fibre stresses to be evaluated from ply stresses and strains and vice versa. Through the solution of a sequence of linear matching problems with an appropriate kinematic load factor, an iterative process converges to a solution for the maximum load where a design limit, expressed as a constraint on the matrix and ply properties, is satisfied. A static load factor gives a measure of the

deviation from the converged solution. The method is first applied at the laminate element level and convergence occurs in very few iterations. The method is then applied to a plate problem under tension with a central hole. In this case the solution converged in seven iterations. The solution to this problem demonstrates the ability of the method to define a useful and meaningful design limit, within the constraints of the mechanics model.

Despite the simplicity of both the model and the examples, the solutions here presented demonstrate the potential of such methods in design. The methodology itself is not overly constrained by the details of either a consistent micromechanical model or the design criterion. The essential theoretical structure does, however, place restrictions on the model. Deformation theory is assumed. Hence any failure criterion must be expressible in terms of the strain field and independent of precise details of the previous strain rate history. Any design constraint may be introduced provided it may be satisfied by linear scaling of the micro kinematic solutions. These properties are not regarded as overly restrictive when considering the failure modes discussed in the literature. Hence the method, here applied in very simple circumstances, is capable of being developed into a method where matrix cracking and fibre failure may be incorporated. For realistic modelling of the maximum attainable load, local softening will occur, particularly when interfacial failure between matrix and fibre is taken into account. Indeed, in the numerical example discussed in Section (5) final failure may be expected to occur because of matrix cracking emanating from the stress concentrations at the hole edge, resulting in loss of strength in the matrix. Exploration of such methods where softening behaviour is allowed has been theoretically explored and applied to portal frames, producing convergent solutions [18]. When softening is included, the search for strain fields that satisfy the design criterion by scaling needs to be augmented with tests for stability [18] as unstable solutions, corresponding to a reducing load can occur. Such solutions corresponding to a strain rate field that deviate from the class of scaled strain fields.

With this extension, stable solutions were obtained with conditions of extreme softening. An exploration of such a methodology and comparisons with experimental data for laminates await further developments, but the results of this paper demonstrate that relatively simple direct methods of this type have the potential for delivering analysis methods for direct use in design.

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Appendix – The Linear Matching Method for Strain Potential Material

Consider a material described by a convex strain potential $\Omega(\varepsilon_{ij})$:

$$\sigma_{ij} = \frac{\partial \Omega(\varepsilon_{ij})}{\partial \varepsilon_{ij}}, \quad \Omega(\varepsilon_{ij}) = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \quad (A1)$$

Consider a body with volume V and surface S , subject to a load λF_i applied over S_T , part of

S and displacement $u_i = 0$ over the remainder of S , S_u . Here F_i is a prescribed load

distribution and λ is a positive scalar multiplier. Consider a class of kinematically

admissible (KA) strains ε_{ij}^c that satisfy compatibility with corresponding displacements

$u_i^c = 0$ on S_u .

The solution to the stated problem is given by the minimum of the total potential energy

$PE(\boldsymbol{\varepsilon}_{ij}^c, \lambda)$ of the body with respect to $\boldsymbol{\varepsilon}_{ij}^c$,

$$PE(\boldsymbol{\varepsilon}_{ij}^c, \lambda) = \int_V \Omega(\boldsymbol{\varepsilon}_{ij}^c) dV - \lambda \int_{S_t} F_i u_{ij}^c dS \quad (\text{A2})$$

The objective of the Linear Matching Method is to obtain a solution $(\boldsymbol{\varepsilon}_{ij}^c, \lambda)$ so that $\boldsymbol{\varepsilon}_{ij}^c$ satisfies a design constraint, expressed as a limit to the magnitude of $\boldsymbol{\varepsilon}_{ij}^c$, and a corresponding value of λ , i.e. a solution $(\boldsymbol{\varepsilon}_{ij}^c, \lambda)$ that minimises $PE(\boldsymbol{\varepsilon}_{ij}^c, \lambda)$ subjected to the design constraint.

The objective is achieved in a four stage iterative process. In the first stage the value of $PE(\boldsymbol{\varepsilon}_{ij}^c, \lambda)$ is reduced for constant λ . In the second stage the resulting strain field is scaled so the design constraint is satisfied. In the third stage λ is adjusted so that $PE(\boldsymbol{\varepsilon}_{ij}^c, \lambda)$ is at a minimum value for the new scaled strain field amongst all such scaled strain fields. In the fourth stage a lower static value of λ is evaluate as a measure of the deviation from convergence.

Stage 1: In the first stage we assume we possess a KA strain distribution $\boldsymbol{\varepsilon}_{ij}^k$ that satisfies the design constraint and a corresponding λ^k . A linear material is now defined so that the actual material and this linear material yield the same stress for $\boldsymbol{\varepsilon}_{ij}^c = \boldsymbol{\varepsilon}_{ij}^k$, i.e.

$$\boldsymbol{\sigma}_{ij}^k = \frac{\partial \Omega(\boldsymbol{\varepsilon}_{ij}^k)}{\partial \boldsymbol{\varepsilon}_{ij}^k} = L_{ijrs} \boldsymbol{\varepsilon}_{rs}^k \quad (\text{A3})$$

where L_{ijrs} is the tensor of linear coefficients, which satisfies the usual symmetry conditions.

In addition, if $\boldsymbol{\varepsilon}_{ij}^{k+1}$ is an arbitrary strain, the following inequality must always be satisfied,

$$\Omega^L(\boldsymbol{\varepsilon}_{ij}^{k+1}) - \Omega^L(\boldsymbol{\varepsilon}_{ij}^k) \geq \Omega(\boldsymbol{\varepsilon}_{ij}^{k+1}) - \Omega(\boldsymbol{\varepsilon}_{ij}^k) \quad (\text{A4})$$

where superscript L refers to the linear material. When the potentials can be expressed in terms of effective strains, inequality (A4) implies that the effective stress-effective strain curve for the non-linear material has a reducing slope for increasing strain. This condition is satisfied for the chosen material model discussed in the text. Hence for constant λ^k , inequality (A4) gives the following relationship between the potential energies for the two materials;

$$PE^L(\boldsymbol{\varepsilon}_{ij}^{k+1}, \lambda^k) - PE^L(\boldsymbol{\varepsilon}_{ij}^k, \lambda^k) \geq PE(\boldsymbol{\varepsilon}_{ij}^{k+1}, \lambda^k) - PE(\boldsymbol{\varepsilon}_{ij}^k, \lambda^k) \quad (\text{A5})$$

If $\boldsymbol{\varepsilon}_{ij}^{k+1}$ is now chosen as the solution of the linear problem, i.e. $PE^L(\boldsymbol{\varepsilon}_{ij}^c, \lambda^k)$ is minimised by $\boldsymbol{\varepsilon}_{ij}^c = \boldsymbol{\varepsilon}_{ij}^{k+1}$ then, $PE^L(\boldsymbol{\varepsilon}_{ij}^{k+1}, \lambda^k) \leq PE^L(\boldsymbol{\varepsilon}_{ij}^k, \lambda^k)$ and hence,

$$PE(\boldsymbol{\varepsilon}_{ij}^{k+1}, \lambda^k) \leq PE(\boldsymbol{\varepsilon}_{ij}^k, \lambda^k) \quad (\text{A6})$$

Stage 2: This new KA strain field $\boldsymbol{\varepsilon}_{ij}^c = \boldsymbol{\varepsilon}_{ij}^{k+1}$ has reduced the potential energy but may not satisfy the design constraint. We assume that a value of a scalar $\mu^{k+1} = \mu^c$ may be found so that the design constraint is satisfied by $\mu^{k+1} \boldsymbol{\varepsilon}_{ij}^{k+1}$. The value of λ is now adjusted so that for $\mu^{k+1} \boldsymbol{\varepsilon}_{ij}^{k+1}$ and λ^{k+1} the potential energy is at a minimum amongst the class of strains $\mu \boldsymbol{\varepsilon}_{ij}^{k+1}$ and hence the first variation is zero;

$$\delta PE(\mu \boldsymbol{\varepsilon}_{ij}^{k+1}, \lambda) = \int_V \frac{\partial \Omega(\mu \boldsymbol{\varepsilon}_{ij}^{k+1})}{\partial \boldsymbol{\varepsilon}_{ij}^{k+1}} \delta \mu \boldsymbol{\varepsilon}_{ij}^{k+1} dV - \lambda^{k+1} = \int_{S_i} F_i \delta \mu \boldsymbol{\varepsilon}_{ij}^c dS = 0 \quad (\text{A7})$$

and hence;

$$\lambda^{k+1} = \frac{\int_V \boldsymbol{\sigma}_{ij}^c \mu \boldsymbol{\varepsilon}_{ij}^{k+1} dV}{\int_{S_i} F_i \mu \boldsymbol{\varepsilon}_{ij}^{k+1} dS} \quad (\text{A8})$$

where σ_{ij}^c denotes the stress corresponding to $\mu\epsilon_{ij}^{k+1}$ given by (A1). The new values λ^{k+1} and $\mu\epsilon_{ij}^{k+1}$ now enter into Stage 1.

The linear solution σ_{ij}^{Lk+1} satisfies equilibrium internally and with load λ^k but will generally not equal the stress σ_{ij}^c in the actual material corresponding to ϵ_{ij}^{k+1} . However it is possible to

find a lower value of $\lambda^k = \lambda_{st}^k$ so that $\lambda_{st}^k = \lambda^k \min \left\{ \frac{\sigma_{ij}^{k+1} \epsilon_{ij}^{k+1}}{\sigma_{ij}^{Lk+1} \epsilon_{ij}^{k+1}} \right\}$ where the minimum is sought

over the entire volume V . It may be shown that $\lambda_{st}^k \leq \lambda^k$ and convergence has occurred when

$$\lambda_{st}^k = \lambda^k.$$

In summary, the Linear Matching Method for this class of problems consists of a procedure for reducing the potential energy of a structure composed of a non linear material through the solution of a linear problem where the linear properties are matched to those of the non-linear material for an initial strain field. This is coupled with the assertion of a design restriction expressed in terms of the strain field and attainable through scaling. The evaluation of a load corresponding to such a scaled strain field ensures that the potential energy is a local minimum amongst the scaled strain fields. In the application to composites, the theory applies provided the relationship between laminate stresses, ply stresses and micro stresses conform to conditions of equilibrium and compatibility for the class of matching linear problems solved in the iterative process.