

Singular Bohr-Sommerfeld conditions for 1D Toeplitz operators: elliptic case

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Abstract

In this article, we state the Bohr-Sommerfeld conditions around a global minimum of the principal symbol of a self-adjoint semiclassical Toeplitz operator on a compact connected Kähler surface, using an argument of normal form which is obtained thanks to Fourier integral operators. These conditions give an asymptotic expansion of the eigenvalues of the operator in a neighbourhood of fixed size of the singularity. We also recover the usual Bohr-Sommerfeld conditions away from the critical point. We end by investigating an example on the two-dimensional torus.

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1 Introduction

Let M be a compact, connected Kähler manifold of complex dimension 1, with fundamental 2-form ω . Assume M is endowed with a prequantum line bundle L . Let K be another holomorphic line bundle and define the quantum Hilbert space \mathcal{H}_k as the space of holomorphic sections of $L^k \otimes K$, for every positive integer k . The operators acting on \mathcal{H}_k that we consider are Berezin-Toeplitz operators ([5, 4, 6, 19], and many others). The semiclassical parameter is k , and the semiclassical limit is $k \rightarrow +\infty$. Formally, k is the inverse of Planck's constant \hbar .

Our aim is to understand the spectrum of a given self-adjoint Toeplitz operator, in the semiclassical limit. In the setting of (\hbar -)pseudodifferential operators, the similar study was done by Colin de Verdière in [13]. In his article [8], Charles obtained the description of the intersection of the spectrum of a self-adjoint Toeplitz operator with an interval of regular values of its principal symbol, in the semiclassical limit: the eigenvalues are selected by an integrality condition for some geometric quantities (actions) associated to the symbol of the operator (these are the Bohr-Sommerfeld conditions).

In this article, we extend these conditions around a global minimum of the principal symbol; since we work with only one degree of liberty, we expect to have a very precise description of the eigenvalues near the critical point.

1.1 Main theorem

Let A_k be a self-adjoint Toeplitz operator on M ; its normalized symbol $a_0 + \hbar a_1 + \dots$ is real-valued. Assume that its principal symbol a_0 admits a global minimum at $m_0 \in M$, with $a_0(m_0) = 0$. Denote by $\lambda_k^{(1)} \leq \lambda_k^{(2)} \leq \dots \leq \lambda_k^{(j)} \leq \dots$ the eigenvalues of A_k . Our main result is the following theorem.

Theorem (Theorem 6.2). *There exist $E_0 > 0$, a sequence $g(\cdot, k)$ of functions of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ which admits an asymptotic expansion of the form $g(\cdot, k) = \sum_{\ell \geq 0} k^{-\ell} g_\ell$ in the \mathcal{C}^∞ topology, and a positive integer k_0 such that for every integer $N \geq 1$ and for every $E \leq E^0$, there exists a constant $C_N > 0$ such that for $k \geq k_0$:*

$$\left(\lambda_k^{(j)} \leq E \text{ or } E_k^{(j)} \leq E \right) \Rightarrow \left| \lambda_k^{(j)} - E_k^{(j)} \right| \leq C_N k^{-N}$$

where

$$E_k^{(j)} = g \left(k^{-1} \left(j + \frac{1}{2} \right), k \right), \quad j \in \mathbb{N}.$$

This allows to compute asymptotic expansions to all order for eigenvalues of A_k lower than E^0 , except that so far, we do not know who are the g_ℓ , $\ell \geq 0$, or how to compute them. In fact, $g(\cdot, k)$ is constructed as the local

inverse of a sequence $f(\cdot, k)$ which also admits an asymptotic expansion $f(\cdot, k) = \sum_{\ell \geq 0} k^{-\ell} f_\ell$ in the \mathcal{C}^∞ topology, and the first terms f_0, f_1 are related to geometric quantities (actions) associated to A_k .

1.2 Link with the usual Bohr-Sommerfeld conditions

Let I be a set of regular values of the principal symbol a_0 ; for every E in I , the level set $\Gamma_E := a_0^{-1}(E)$ is diffeomorphic to S^1 . Fix an orientation on Γ_E depending continuously on E . Define the *principal action* $c_0 \in \mathcal{C}^\infty(I)$ in such a way that the parallel transport in L along Γ_E is the multiplication by $\exp(ic_0(E))$. Of course, $c_0(E)$ is defined up to an integer multiple of 2π , but we can always choose a determination of c_0 that is smooth on I .

Let (δ, φ) be a *half-form bundle*, that is a line bundle $\delta \rightarrow M$ together with an isomorphism of line bundles $\varphi : \delta^2 \rightarrow \Lambda^{2,0}T^*M$. It is known that for any connected compact Kähler manifold of complex dimension 1, such a couple exists. Introduce the hermitian holomorphic line bundle L_1 such that $K = L_1 \otimes \delta$. For E in I , define the *subprincipal form* κ_E as the 1-form on Γ_E such that

$$\kappa_E(X_{a_0}) = -a_1$$

where X_{a_0} stands for the Hamiltonian vector field associated to a_0 . Denote by j_E the embedding $\Gamma_E \rightarrow M$, and introduce the connection ∇^E on $j_E^*L_1 \rightarrow \Gamma_E$ defined by

$$\nabla^E = \nabla^{j_E^*L_1} + \frac{1}{i}\kappa_E,$$

with $\nabla^{j_E^*L_1}$ the connection induced by the Chern connection of L_1 on $j_E^*L_1$. Define the *subprincipal action* $c_1 \in \mathcal{C}^\infty(I)$ like the principal action, replacing L by L_1 endowed with this connection.

Finally, define an index ϵ from the half-form bundle δ as follows: the map

$$\varphi_E : \delta_E^2 \rightarrow T^*\Gamma_E \otimes \mathbb{C}, \quad u \mapsto j_E^*\varphi(u)$$

is an isomorphism of line bundles. The set

$$\left\{ u \in \delta_E; \varphi_E(u^{\otimes 2}) > 0 \right\}$$

has one or two connected components. In the first case, we set $\epsilon_E = 1$, and in the second case $\epsilon_E = 0$. In fact, ϵ_E is a constant $\epsilon_E = \epsilon$ for E in I .

If we define carefully c_0 and c_1 , the following result holds.

Proposition (Proposition 6.4). *Set $I =]0, E^0[$. Then*

$$f_0 = \frac{1}{2\pi}c_0, \quad f_1 = \frac{1}{2\pi}c_1 \tag{1}$$

on I .

Thus, we recover the regular Bohr-Sommerfeld conditions away from the minimum.

1.3 Structure of the article

The paper is organized as follows: we start by recalling some properties of Toeplitz operators on a compact manifold. Then, we briefly explain how to adapt the theory in the case where the phase space is the whole complex plane. The fourth section is devoted to the construction of Fourier integral operators that we use to construct our microlocal normal form in the following part. In section 6, we state the Bohr-Sommerfeld conditions and some consequences. In the last section, we investigate an example to give some numerical evidence of our results.

2 Preliminaries and notations

First, we introduce the notations and conventions that we will adopt through this whole article. They are already written in [8] for instance, but we recall them here for the sake of completeness.

2.1 Quantum spaces

Let M be a connected compact Kähler manifold, with fundamental 2-form $\omega \in \Omega^2(M, \mathbb{R})$. Assume M is endowed with a prequantum bundle $L \rightarrow M$, that is a Hermitian holomorphic line bundle whose Chern connection ∇ has curvature $\frac{1}{i}\omega$. Let $K \rightarrow M$ be a Hermitian holomorphic line bundle. For every positive integer k , define the quantum space \mathcal{H}_k as:

$$\mathcal{H}_k = H^0(M, L^k \otimes K) = \left\{ \text{holomorphic sections of } L^k \otimes K \right\}.$$

The space \mathcal{H}_k is a subspace of the space $L^2(M, L^k \otimes K)$ of sections of finite L^2 -norm, where the scalar product is given by

$$\langle \varphi, \psi \rangle = \int_M h_k(\varphi, \psi) \mu_M$$

with h_k the hermitian product on $L^k \otimes K$ induced by those of L and K , and μ_M the Liouville measure on M . Since M is compact, \mathcal{H}_k is finite dimensional, and is thus given a Hilbert space structure with this scalar product.

2.2 Geometric notations

Unless otherwise mentioned, “smooth” will always mean \mathcal{C}^∞ , and a section of a line bundle will always be assumed to be smooth. The space of sections of a bundle $E \rightarrow M$ will be denoted by $\Gamma(M, E)$.

Let $L_P \rightarrow P$ and $L_N \rightarrow N$ be two prequantum bundles over Kähler manifolds, whose fundamental 2-forms are denoted by ω_P and ω_N . Denote by p_1 and p_2 the projections of $P \times N$ on each factor, and $L_P \boxtimes L_N = p_1^* L_P \otimes$

$p_2^*L_N$; then, if $P \times N$ is endowed with the symplectic form $p_1^*\omega_P + p_2^*\omega_N$, $L_P \boxtimes L_N \rightarrow P \times N$ is a prequantum bundle.

Let P^{op} be the manifold P endowed with the symplectic form $-\omega_P$ and the (almost) complex structure opposed to the one of P , and let L_P^{-1} be the inverse (dual) bundle of L_P with induced hermitian and holomorphic structure and connection; then $L_P^{-1} \rightarrow P^{op}$ is a prequantum bundle. If k is a positive integer, we can identify the Schwartz kernel of an operator $T : \Gamma(P, L_P^k) \rightarrow \Gamma(N, L_N^k)$ to a section of $L_N^k \boxtimes L_P^{-k} \rightarrow N \times P^{op}$ via the following formula:

$$Ts(x) = \int_P T(x, y).s(y)\mu_P(y),$$

where μ_P is the Liouville measure on the manifold P .

2.3 Admissible and negligible sequences

Let M be a compact connected Kähler manifold. Let $(s_k)_{k \geq 1}$ be a sequence such that for each k , s_k belongs to $\Gamma(M, L^k \otimes K)$. We say that $(s_k)_{k \geq 1}$ is

- *admissible* if for every positive integer ℓ , for every vector fields X_1, \dots, X_ℓ on M , and for every compact set $C \subset M$, there exist a constant $c > 0$ and an integer N such that

$$\forall m \in C \quad \|\nabla_{X_1} \dots \nabla_{X_\ell} s_k(m)\| \leq ck^N,$$

- *negligible* if for every positive integers ℓ and N , for every vector fields X_1, \dots, X_ℓ on M , and for every compact set $C \subset M$, there exists a constant $c > 0$ such that

$$\forall m \in C \quad \|\nabla_{X_1} \dots \nabla_{X_\ell} s_k(m)\| \leq ck^{-N}.$$

We say that $(s_k)_{k \geq 1}$ is *negligible* over an open set $U \subset M$ if the previous estimates hold for every compact subset of U . We denote by $O(k^{-\infty})$ any negligible sequence or the set of negligible sequences. The *microsupport* $\text{MS}(s_k)$ of an admissible sequence $(s_k)_{k \geq 1}$ is the complement of the set of points of M which admit a neighbourhood where $(s_k)_{k \geq 1}$ is negligible. Finally, we say that two admissible sequences $(t_k)_{k \geq 1}$ and $(s_k)_{k \geq 1}$ are microlocally equal on an open set U if $\text{MS}(t_k - s_k) \cap U = \emptyset$; the symbol \sim will indicate microlocal equivalence. We can then define, *via* the sequences of their Schwartz kernels, *admissible* and *smoothing* operators, and the microsupport and microlocal equality of operators.

2.4 Toeplitz operators

Let Π_k be the orthogonal projector of $L^2(M, L^k \otimes K)$ onto \mathcal{H}_k . A *Toeplitz operator* is any sequence $(T_k : \mathcal{H}_k \rightarrow \mathcal{H}_k)_{k \geq 1}$ of operators of the form

$$T_k = \Pi_k M_{f(\cdot, k)} + R_k \tag{2}$$

where $f(\cdot, k)$ is a sequence of $\mathcal{C}^\infty(M)$ with an asymptotic expansion $f(\cdot, k) = \sum_{\ell \geq 0} k^{-\ell} f_\ell$ for the \mathcal{C}^∞ topology, $M_{f(\cdot, k)}$ is the operator of multiplication by $f(\cdot, k)$ and R_k is a smoothing operator. They are the semiclassical analogue of the Toeplitz operators studied by Boutet de Monvel and Guillemin in [5].

We recall the following essential theorem about Toeplitz operators, which is a consequence of the works of Boutet de Monvel and Guillemin [5] (see also [3, 6, 17]).

Theorem 2.1. *The set \mathcal{T} of Toeplitz operators is a star algebra whose identity is $(\Pi_k)_{k \geq 1}$. The contravariant symbol map*

$$\sigma_{\text{cont}} : \mathcal{T} \rightarrow \mathcal{C}^\infty(M)[[\hbar]]$$

sending T_k into the formal series $\sum_{\ell \geq 0} \hbar^\ell f_\ell$ is well defined, onto, and its kernel is the ideal consisting of $O(k^{-\infty})$ Toeplitz operators. More precisely, for any integer ℓ ,

$$\|T_k\| = O(k^{-\ell}) \text{ if and only if } \sigma_{\text{cont}}(T_k) = O(\hbar^\ell).$$

We will mainly work with the *normalized symbol*

$$\sigma_{\text{norm}} = \left(\text{Id} + \frac{\hbar}{2} \Delta \right) \sigma_{\text{cont}}$$

where Δ is the holomorphic Laplacian acting on $\mathcal{C}^\infty(M)$; unless otherwise mentioned, when we talk about a subprincipal symbol, this refers to the normalized symbol. This symbol has the good property that, if T_k and S_k are Toeplitz operators with respective principal symbols t_0 and s_0 , then

$$\sigma_{\text{norm}}(T_k S_k) = t_0 s_0 + \frac{\hbar}{2i} \{t_0, s_0\} + O(\hbar^2).$$

Finally, we will need to apply functional calculus to Toeplitz operators.

Proposition 2.2 ([6]). *Let T_k be a self-adjoint Toeplitz operator with symbol $\sum_{\ell \geq 0} \hbar^\ell t_\ell$ and g be a function of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$. Then $g(T_k)$ is a Toeplitz operator with principal symbol $g(t_0)$.*

3 Toeplitz operators on the complex plane

3.1 Bargmann spaces

We consider the Kähler manifold $\mathbb{C} \simeq \mathbb{R}^2$ with coordinates (x, ξ) , standard complex structure and symplectic form $\omega_0 = d\xi \wedge dx$. Let $L_0 = \mathbb{R}^2 \times \mathbb{C} \rightarrow \mathbb{R}^2$ be the trivial fiber bundle with standard hermitian metric h_0 and connection ∇^0 with 1-form $\frac{1}{i}\alpha$, where $\alpha_u(v) = \frac{1}{2}\omega_0(u, v)$; endow L_0 with the unique

holomorphic structure compatible with h_0 and ∇^0 . For every positive integer k , the quantum space that we consider is

$$\mathcal{H}_k^0 = H^0(\mathbb{R}^2, L_0^k) \cap L^2(\mathbb{R}^2, L_0^k);$$

this means that in this case, we make the arbitrary choice that the auxiliary line bundle K is the trivial bundle with flat connection. These spaces coincide with Bargmann spaces [1, 2], which are spaces of square integrable functions with respect to a Gaussian weight. More precisely, we choose the holomorphic coordinate $z = \frac{x-i\xi}{\sqrt{2}}$ and note

$$\mathcal{B}_k = \left\{ f\psi^k; f : \mathbb{C} \mapsto \mathbb{C} \text{ holomorphic, } \int_{\mathbb{R}^2} |f(z)|^2 \exp(-k|z|^2) d\lambda(z) < +\infty \right\}$$

with $\psi : \mathbb{C} \rightarrow \mathbb{C}, z \mapsto \exp\left(-\frac{1}{2}|z|^2\right)$, ψ^k its k -th tensor power, and λ the Lebesgue measure on \mathbb{R}^2 . It is easily shown that for $k \geq 1$, \mathcal{H}_k^0 is precisely \mathcal{B}_k . Sometimes, we will use the identification of the section $f\psi$ to the function f in abusive notations, such as talking about the operator $\frac{\partial}{\partial z}$ action on \mathcal{B}_k , etc. It is standard that \mathcal{B}_k is closed in $L^2(\mathbb{R}^2, \exp(-k|z|^2)d\lambda(z))$, and is thus a Hilbert space; moreover, we know an orthonormal basis of \mathcal{B}_k .

Proposition 3.1. *The family $(\varphi_{n,k})_{n \in \mathbb{N}}$, where $\varphi_{n,k}(z) = \sqrt{\frac{k^{n+1}}{2\pi n!}} z^n \psi^k$, is an orthonormal basis of \mathcal{B}_k .*

We denote by Π_k^0 the orthogonal projector from $L^2(\mathbb{R}^2, L_0^k)$ onto \mathcal{B}_k .

3.2 Admissible and negligible sequences

Since we will only deal with C^∞ sections, we can adopt the same definitions for admissible and negligible sequences as in the previous section.

3.3 Toeplitz operators

To consider Toeplitz operators acting on Bargmann spaces without raising technical issues, we could only work with operators with compactly supported kernels. However, we would miss the simple case of the harmonic oscillator. So we need to introduce symbol classes, very similar to the ones used when dealing with \hbar -pseudodifferential operators (see for instance [15]). The proofs of the results of this part are collected in the appendix.

Let d be a positive integer. For u in \mathbb{C}^d , set $m(u) = (1 + \|u\|^2)^{\frac{1}{2}}$. For every integer j , we define the symbol class \mathcal{S}_j^d as the set of sequences of functions of $C^\infty(\mathbb{C}^d)$ which admit an asymptotic expansion of the form $a(\cdot, k) = \sum_{\ell \geq 0} k^{-\ell} a_\ell$ in the sense that

- $\forall \ell \in \mathbb{N} \quad \forall \alpha, \beta \in \mathbb{N}^{2d} \quad \exists C_{\ell, \alpha, \beta} > 0 \quad |\partial_z^\alpha \partial_{\bar{z}}^\beta a_\ell| \leq C_{\ell, \alpha, \beta} m^j,$

- $\forall L \in \mathbb{N}^* \quad \forall \alpha, \beta \in \mathbb{N}^{2d} \quad \exists C_{L,\alpha} > 0 \quad \left| \partial_z^\alpha \partial_{\bar{z}}^\beta \left(a - \sum_{\ell=0}^{L-1} k^{-\ell} a_\ell \right) \right| \leq C_{L,\alpha,\beta} k^{-L} m^j.$

Set $\mathcal{S}^d = \bigcup_{j \in \mathbb{Z}} \mathcal{S}_j^d$. Now, let $a(\cdot, k)$ be a symbol in \mathcal{S}_j^1 , and consider the operator

$$A_k = \text{Op}(a(\cdot, k)) = \Pi_k^0 M_{a(\cdot, k)} \Pi_k^0 \quad (3)$$

acting on the subspace

$$\mathfrak{S}_k = \left\{ \varphi \in \mathcal{B}_k; \forall j \in \mathbb{N} \quad \sup_{z \in \mathbb{C}} \left(|\varphi(z)| (1 + |z|^2)^{j/2} \right) < +\infty \right\}$$

of \mathcal{B}_k . As shown in [2], \mathfrak{S}_k corresponds to the Schwartz space *via* a particular unitary mapping between $L^2(\mathbb{R})$ and \mathcal{B}_k , the Bargmann transform. It is easily seen that A_k sends \mathfrak{S}_k into \mathfrak{S}_k ; it is even continuous $\mathfrak{S}_k \rightarrow \mathfrak{S}_k$. Note that if $j = 0$, then A_k is bounded $\mathcal{B}_k \rightarrow \mathcal{B}_k$, and its norm is lower than $\sup |a(\cdot, k)|$.

Let t be the section of $L_0 \rightarrow \mathbb{R}^2$ with constant value 1. Let F_0 be the section of $L_0 \boxtimes L_0^{-1}$ given by

$$F_0(z_1, z_2) = \exp \left(-\frac{1}{2} \left(|z_1|^2 + |z_2|^2 - 2z_1 \bar{z}_2 \right) \right) t(z_1) \otimes t^{-1}(z_2),$$

or equivalently, if $u = (x, \xi)$ where $z = \frac{1}{\sqrt{2}}(x - i\xi)$,

$$F_0(u, v) = \exp \left(-\frac{1}{4} \|u - v\|^2 - \frac{i}{2} \omega_0(u, v) \right) t(u) \otimes t^{-1}(v).$$

Adapting the result of section 1.c of [1], with the good normalization for the weight defining our Bargmann spaces, we have the following:

Proposition 3.2. Π_k^0 admits a Schwartz kernel given by $\frac{k}{2\pi} F_0^k$.

In the rest of the paper, we will use the same letter to designate an operator and its Schwartz kernel. This proposition allows us to compute the Schwartz kernel of any Toeplitz operator.

Lemma 3.3. Let $a(\cdot, k)$ be a symbol in \mathcal{S}_j^1 ; then $A_k = \text{Op}(a(\cdot, k))$ admits a Schwartz kernel given by

$$A_k(z_1, z_2) = \frac{k}{2\pi} \exp \left(-\frac{k}{2} \left(|z_1|^2 + |z_2|^2 - 2z_1 \bar{z}_2 \right) \right) \tilde{a}(z_1, z_2, k) + R_k \exp \left(-Ck |z_1 - z_2|^2 \right), \quad (4)$$

where $\tilde{a}(\cdot, \cdot, k)$ belongs to \mathcal{S}_j^2 , R_k is negligible and C is some positive constant. Moreover, one has

$$\tilde{a}(z, z, k) = \left(\exp \left(k^{-1} \Delta \right) a \right) (z, k) \quad (5)$$

where $\Delta = \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$ is the holomorphic Laplacian acting on $\mathcal{C}^\infty(\mathbb{C}^2)$, in the sense that the asymptotic expansion of $\tilde{a}(\cdot, \cdot, k)$ is obtained by applying the formal asymptotic expansion of the operator $\exp(k^{-1}\Delta)$ to the asymptotic expansion of $a(\cdot, k)$.

This leads us to the following definition.

Definition 3.4. A *Toeplitz operator* is an operator from \mathfrak{S}_k to \mathfrak{S}_k of the form

$$\Pi_k^0 M_{a(\cdot, k)} \Pi_k^0 + S_k, \quad (6)$$

where $a(\cdot, k)$ is a symbol in \mathcal{S}^1 and the kernel of S_k satisfies

$$S_k(z_1, z_2) = R_k(z_1, z_2) \exp\left(-Ck|z_1 - z_2|^2\right) \quad (7)$$

with R_k negligible and C some positive constant. As in the compact case, $\sigma_{\text{cont}}(A_k) = \sum_{\ell \geq 0} \hbar^\ell a_\ell$ is called the *contravariant symbol* of A_k . We denote by \mathcal{T}_j the set of Toeplitz operators with contravariant symbol belonging to \mathcal{S}_j^1 .

In fact, lemma 3.3 defines the *covariant symbol* of A_k :

$$\sigma_{\text{cov}}(A_k)(z) = \sum_{\ell \geq 0} \hbar^\ell \tilde{a}_\ell(z, z).$$

The following lemma gives an important property of the latter.

Lemma 3.5. *If the covariant symbol of A_k vanishes, then the Schwartz kernel of A_k is of the form (7).*

As a corollary of lemmas 3.3 and 3.5, we obtain the stability under composition of the set of Toeplitz operators.

Corollary 3.6. *Let $A_k \in \mathcal{T}_j$ and $B_k \in \mathcal{T}_{j'}$ be two Toeplitz operators. Then $C_k = A_k B_k$ belongs to $\mathcal{T}_{j+j'}$; more precisely, its contravariant symbol is given by*

$$\sigma_{\text{cont}}(C_k)(z) = \left(\exp\left(-\hbar \frac{\partial}{\partial z_1} \frac{\partial}{\partial \bar{z}_2}\right) \sigma_{\text{cont}}(A_k)(z_1) \sigma_{\text{cont}}(B_k)(z_2) \right)_{|z_1=z_2=z}, \quad (8)$$

in the same sense as in lemma 3.3.

Define the *normalized symbol* as in the compact case:

$$\sigma_{\text{norm}} = \left(Id + \frac{\hbar}{2} \Delta \right) \sigma_{\text{cont}}.$$

From formula (8), we find that $\sigma_{\text{norm}}(A_k B_k) = a_0 b_0 + \frac{\hbar}{2i} \{a_0, b_0\} + O(\hbar^2)$, as expected.

Definition 3.7. A Toeplitz operator $A_k \in \mathcal{T}_j$ is said to be *elliptic at infinity* if there exists some $c > 0$ such that for z in \mathbb{C} , $|\sigma_{\text{cont}}(A_k)(z)| \geq c(1 + |z|^2)^{\frac{j}{2}}$.

Adapting proposition 12 of [6] and theorem 39 of [12], one can show the following:

Proposition 3.8 (Functional calculus). *If A_k belongs to \mathcal{T}_j for some $j \geq 1$, is essentially self-adjoint and elliptic at infinity and if $\eta : \mathbb{R} \rightarrow \mathbb{R}$ is a compactly supported C^∞ function, then $\eta(A_k)$ belongs to $\mathcal{T}_{j'}$ for every $j' < 0$.*

4 Fourier integral operators

The aim of this section is to construct microlocally unitary operators between \mathcal{H}_k and \mathcal{B}_k , given a local symplectomorphism χ from M to \mathbb{R}^2 . In [5], Boutet de Monvel and Guillemin introduced Fourier integral operators in the homogeneous Toeplitz setting. In the semiclassical Toeplitz theory, such operators between compact manifolds have been used by Charles [7, 9], but some difficulties arise when dealing with a non compact manifold. Nevertheless, the ideas, based on Lagrangian sections, are very similar.

Let $\chi : \Omega_1 \subset M \rightarrow \Omega_2 \subset \mathbb{R}^2$ be a symplectomorphism between the open sets Ω_1 and Ω_2 . Then the graph

$$\Lambda_\chi = \{(u, \chi(u)); u \in \Omega_1\} \subset \Omega_1 \times \Omega_2$$

of χ is a Lagrangian submanifold of the product $M \times \mathbb{C}^{op}$. As in the previous section, let t be the section of $L_0 \rightarrow \mathbb{R}^2$ with constant value 1. By definition of the connection on L_0 , we have $\nabla^0 t = \frac{1}{i}\alpha \otimes t$, where α is the primitive of ω_0 given by $\alpha_u(v) = \frac{1}{2}\omega_0(u, v)$. The following lemma is elementary.

Lemma 4.1. *Taking Ω_1 smaller if necessary, we can find a local gauge s of $L \rightarrow \Omega_1$ such that $\nabla s = \frac{1}{i}\chi^*\alpha \otimes s$.*

We consider the section t_{Λ_χ} of $L \boxtimes L_0^{-1}$ over Λ_χ given by

$$t_{\Lambda_\chi}(u, \chi(u)) = s(u) \otimes t^{-1}(\chi(u)).$$

Thanks to proposition 2.1 of [7], we can build a local section E of $L \boxtimes L_0^{-1} \rightarrow \Omega_1 \times \Omega_2^{op}$ such that

- E is equal to t_{Λ_χ} on Λ_χ ,
- for every holomorphic vector field Z on $\Omega_1 \times \Omega_2^{op}$, the covariant derivative of E with respect to \bar{Z} is zero modulo a section vanishing to infinite order along Λ_χ ,

and this section is unique modulo a section vanishing to infinite order along Λ_χ . Furthermore, we can always assume that $\|E\| < 1$ outside Λ_χ , and we will make this assumption until the end of this article.

We consider a sequence of functions of $\mathcal{C}^\infty(\Omega_1 \times \Omega_2^{op})$, which admits an asymptotic expansion $\sum_{\ell \geq 0} k^{-\ell} a_\ell$ for the \mathcal{C}^∞ topology whose coefficients are all supported in a fixed (independent of k) compact set $C \subset \Omega_1 \times \Omega_2$. Let S_k be the local section of $(L^k \otimes K) \boxtimes L_0^{-k}$ given by

$$S_k(u, v) = \frac{k}{2\pi} E^k(u, v) a(u, v, k),$$

and consider the operator S_k defined on $\Gamma(\mathbb{C}, L_0^k)$ by

$$(S_k \phi)(u) = \int_{\mathbb{R}^2} S_k(u, v) \cdot \phi(v) d\lambda(v),$$

which makes sense since $S_k(\cdot, \cdot)$ vanishes outside $\Omega_1 \times \Omega_2$.

Proposition 4.2. *The operator $R_k = S_k \Pi_k^0$ maps \mathcal{B}_k into $\Gamma(M, L^k \otimes K)$.*

Proof. We must show that if φ_k is a smooth square integrable section of $L_0^k \rightarrow \mathbb{C}$, then $S_k \Pi_k^0 \varphi_k$ is a smooth section of $L^k \otimes K \rightarrow M$. It is enough to show that the Schwartz kernel of $S_k \Pi_k^0$ and its derivatives with respect to the first variable are rapidly decreasing in the second variable. Let R_k be this kernel; one has

$$R_k(u, v) = \int_{p_2(C)} S_k(u, w) \cdot \Pi_k^0(w, v) dw,$$

with p_2 the projection from $M \times \mathbb{C}$ to \mathbb{C} . So

$$R_k(u, v) = \left(\frac{k}{2\pi}\right)^2 \int_{p_2(C)} f(u, v, w, k) E^k(u, w) \cdot t^k(w) \otimes t^{-k}(v) dw$$

with $f(u, v, w, k) = a(u, w, k) \exp\left(-\frac{k}{4}\|w - v\|^2 - \frac{ik}{2}\omega_0(w, v)\right)$. This implies the estimate

$$\|R_k(u, v)\| \leq \left(\frac{k}{2\pi}\right)^2 \int_{p_2(C)} |a(u, w, k)| \exp\left(-\frac{k}{4}\|w - v\|^2\right) \|E^k(u, w)\| dw.$$

Since $\|E\| \leq 1$ and $a(\cdot, \cdot, k)$ is bounded by some constant $c_k > 0$, this yields

$$\|R_k(u, v)\| \leq \left(\frac{k}{2\pi}\right)^2 c_k \int_{p_2(C)} \exp\left(-\frac{k}{4}\|w - v\|^2\right) dw.$$

Using $\|w - v\|^2 \geq \|v\|^2 - 2\|w\|\|v\|$, we obtain

$$\|R_k(u, v)\| \leq \left(\frac{k}{2\pi}\right)^2 c_k \exp\left(-\frac{k}{4}\|v\|^2\right) \int_{p_2(C)} \exp\left(\frac{k}{2}\|w\|\|v\|\right) dw.$$

Finally, an upper bound for the integral that appears in this inequality is $\pi r^2 \exp\left(\frac{k}{2}r\|v\|\right)$ where $p_2(C)$ is included in the closed ball of radius r and centered at the origin. This allows us to conclude that

$$\|R_k(u, v)\| \leq \left(\frac{k}{2\pi}\right)^2 c_k \pi r^2 \exp\left(-\frac{k}{4}\|v\|^2 + \frac{kr}{2}\|v\|\right).$$

The same kind of estimates hold for the successive derivatives of R_k with respect to u ; we prove them by differentiating under the integral sign. \square

Unfortunately, R_k has no reason to map holomorphic sections to holomorphic sections; to fix this problem, we set $T_k = \Pi_k R_k$; defined in this way, T_k is an operator from \mathcal{B}_k to \mathcal{H}_k .

Proposition 4.3. *The Schwartz kernel of T_k reads*

$$T_k(u, v) = \frac{k}{2\pi} E^k(u, v) b(u, v, k) + O(k^{-\infty}) \quad (9)$$

with $b(\cdot, \cdot, k)$ a sequence of smooth functions which admits an asymptotic expansion $b(\cdot, \cdot, k) = \sum_{\ell \geq 0} k^{-\ell} b_\ell$ for the \mathcal{C}^∞ topology satisfying

$$b_0(u, \chi(u), k) = \mu(u) a_0(u, \chi(u), k)$$

where μ is a smooth, nowhere vanishing function which depends only on the section E .

The proof is the same as the proof of proposition 4.2 of [7]; it is based on an application of the stationary phase lemma.

An operator $V_k : \mathcal{B}_k \rightarrow \mathcal{H}_k$ admitting a Schwartz kernel of the form of equation (9) and satisfying $\Pi_k V_k \Pi_k^0 = V_k$ will be called a *Fourier integral operator* associated to the sequence $b(\cdot, \cdot, k)$; let us denote by $\mathcal{FI}(\chi)$ the set of such operators. We define the *full symbol map*

$$\sigma : \mathcal{FI}(\chi) \rightarrow \mathcal{C}^\infty(M)[[\hbar]], \quad V_k \mapsto \sum_{\ell \geq 0} \hbar^\ell b_\ell(u, \chi(u)).$$

One can show that its kernel consists of smoothing operators. In the same way, we define $\mathcal{FI}(\chi^{-1}) : \mathcal{H}_k \rightarrow \mathcal{B}_k$. The following property is another application of the stationary phase lemma.

Proposition 4.4. *Let $R_k \in \mathcal{FI}(\chi)$ and $S_k \in \mathcal{FI}(\chi^{-1})$ with respective principal symbols $r_0(u, \chi(u))$ and $s_0(v, \chi^{-1}(v))$. Then there exists a smooth nowhere vanishing function $\nu : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for every Toeplitz operator T_k on M with principal symbol t_0 , $S_k T_k R_k$ is a Toeplitz operator on \mathbb{R}^2 with principal symbol equal to*

$$\nu(v) s_0(v, \chi^{-1}(v)) t_0(\chi^{-1}(v)) r_0(\chi^{-1}(v), v)$$

on Ω_2 .

To conclude this section, we prove that we can find *microlocally unitary* operators mapping \mathcal{B}_k to \mathcal{H}_k in the following sense.

Proposition 4.5. *There exists a Fourier integral operator $U_k : \mathcal{B}_k \rightarrow \mathcal{H}_k$ such that*

- $U_k^* U_k \sim \Pi_k^0$ on Ω_2 ,
- $U_k U_k^* \sim \Pi_k$ on Ω_1 ,

reducing Ω_1 and Ω_2 if necessary, where we recall that the symbol \sim stands for microlocal equality.

Proof. Start from a Fourier integral operator S_k associated to $s(\cdot, \cdot, k)$ with principal symbol $s_0(v, \chi^{-1}(v))$ never vanishing on $\Omega_1 \times \Omega_2$. The first step is to construct an operator with the first property; we do it by induction, correcting S_k by Toeplitz operators. More precisely, let P_k be a Toeplitz operator on \mathbb{R}^2 , and denote by p_0 its principal symbol. Set $U_k^{(0)} = S_k P_k$; then $U_k^{(0)*} U_k^{(0)}$ is a Toeplitz operator on \mathbb{R}^2 , with principal symbol $\nu(v) |p_0(v)|^2 |s_0(\chi^{-1}(v), v)|^2$. Since $\nu(v)$ and $s_0(s, \chi^{-1}(v))$ vanish in no point v , one can choose p_0 such that this principal symbol is equal to 1. Doing so, $U_k^{(0)*} U_k^{(0)}$ has the same principal symbol as Π_k^0 , so there exists a Toeplitz operator $R_k^{(0)}$ such that

$$U_k^{(0)*} U_k^{(0)} \sim \Pi_k^0 + k^{-1} R_k^{(0)}$$

on Ω_2 . From now on, when there is no ambiguity, the equality between operators will mean microlocal equality on Ω_2 . Let $n \in \mathbb{N}$ and assume that there exists an operator $U_k^{(n)} : \mathcal{B}_k \rightarrow \mathcal{H}_k$ and a Toeplitz operator $R_k^{(n)}$ (with principal symbol r_n) such that

$$U_k^{(n)*} U_k^{(n)} = \Pi_k^0 + k^{-(n+1)} R_k^{(n)}.$$

Let T_k be a Toeplitz operator on \mathbb{R}^2 with principal symbol t_0 , and set $U_k^{(n+1)} = U_k^{(n)} (\Pi_k^0 + k^{-(n+1)} T_k)$. One has

$$U_k^{(n+1)*} U_k^{(n+1)} = \Pi_k^0 + k^{-(n+1)} (T_k^* + R_k^{(n)} + T_k) + k^{-(n+2)} R_k^{(n+1)}$$

with $R_k^{(n+1)}$ a Toeplitz operator. This implies that if we choose t_0 such that $2\Re(t_0) = -r_n$, then

$$U_k^{(n+1)*} U_k^{(n+1)} = \Pi_k^0 + k^{-(n+2)} R_k^{(n+1)}.$$

So we can construct the operators $U_k^{(n)}$ by induction; it remains to apply Borel's summation lemma to find the desired operator U_k .

Composing such a U_k by U_k on the left and U_k^* on the right gives:

$$(U_k U_k^*)^2 \sim U_k U_k^* \quad \text{on } \Omega_1.$$

Since $U_k U_k^*$ is an elliptic Toeplitz operator on Ω_1 (its principal symbol vanishes nowhere), it has a microlocal inverse at each point of Ω_1 ; so the preceding equation yields $U_k U_k^* \sim \Pi_k$ on Ω_1 . \square

Of course, such operators satisfy the analogue of Egorov's theorem:

Proposition 4.6. *If U_k is as above, then, for every Toeplitz operator T_k on M with principal symbol t_0 , $S_k = U_k^* T_k U_k$ is a Toeplitz operator on \mathbb{R}^2 with principal symbol equal to $t_0 \circ \chi^{-1}$ on Ω_2 .*

For a proof, see [12, theorem 47]. The action of a Fourier integral operator at the subprincipal level is much more complicated to compute. Denote by $\sigma_0(U_k)$ the principal symbol of U_k , and by γ the 1-form on Λ_χ such that

$$\nabla^{\text{Hom}(\mathbb{C}, K)} \sigma_0(U_k) = -\frac{1}{i} \gamma \otimes \sigma_0(U_k)$$

endowing \mathbb{C} with the trivial connection and K with the one inherited from L_1 and δ . Now, notice that the symplectomorphism χ brings the complex structure of \mathbb{C}^{op} to a positive complex structure j on M . Introduce the section Ψ of $\text{Hom}(\Omega^{1,0}(\mathbb{C}), \Omega_j^{1,0}(M))|_{\Lambda_\chi} \rightarrow \Lambda_\chi$ such that for all $\alpha \in \Lambda^{1,0}(T\Omega_2^{op})^*$ and $\beta \in \Lambda^{1,0}(T\Omega_1)^*$,

$$\Psi(\alpha) \wedge \bar{\beta} = (\chi^* \alpha) \wedge \bar{\beta}.$$

This map is well-defined because the sesquilinear pairing $\Lambda_j^{1,0}(T_m \Omega_1)^* \times \Lambda^{1,0}(T_m \Omega_1)^* \rightarrow \mathbb{C}$, $(\alpha, \beta) \mapsto (\alpha \wedge \bar{\beta})/\omega_m$ is non-degenerate. Let δ be the 1-form on Λ_χ such that

$$\nabla \Psi = \delta \otimes \Psi$$

where ∇ is the connection induced by the Chern connections of $\Omega^{2,0}(\mathbb{C})$ and $\Omega^{2,0}(M)$. In [9, Theorem 3.3], Charles derived the following formula.

Theorem 4.7. *With the same notations as in the previous proposition and denoting by t_1 the subprincipal symbol of T_k , the subprincipal symbol s_1 of S_k is given by:*

$$s_1(u) = t_1(m) + \left\langle \gamma_{(m,u)} - \frac{1}{2} \delta_{(m,u)}, \left(X_{t_0}(m), ((\chi^{-1})^* X_{t_0})(u) \right) \right\rangle$$

for $u \in \mathbb{R}^2$ and $m = \chi^{-1}(u)$.

5 Microlocal normal form

5.1 The local model

Our local model will be the realization of the quantum harmonic oscillator in the Bargmann representation: $Q_k = \frac{1}{k} \left(z \frac{\partial}{\partial z} + \frac{1}{2} \right)$, with domain $\mathbb{C}[z]$ which is dense in \mathcal{B}_k . The following lemma is easily shown.

Lemma 5.1. *Q_k is an essentially self-adjoint Toeplitz operator with normalized symbol q_0 . Moreover, the spectrum of Q_k is*

$$Sp(Q_k) = \left\{ k^{-1} \left(n + \frac{1}{2} \right), n \in \mathbb{N} \right\}.$$

5.2 A symplectic Morse lemma

Let (N, ω) be a two-dimensional symplectic manifold and f a function of $\mathcal{C}^\infty(N, \mathbb{R})$. Assume f admits an elliptic critical point at $n_0 \in N$, with $f(n_0) = 0$. Replacing f by $-f$ if necessary, we can assume that this critical point is a local minimum for f . Define

$$q_0 : \mathbb{R} \rightarrow \mathbb{R}^2, \quad (x, \xi) \mapsto \frac{1}{2}(x^2 + \xi^2).$$

The following theorem is well-known.

Theorem 5.2. *There exist a local symplectomorphism $\chi : (N, n_0) \rightarrow (\mathbb{R}^2, 0)$ and a function g in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ satisfying $g(0) = 0$ and $g'(0) > 0$, such that*

$$f \circ \chi^{-1} = g \circ q_0$$

where χ^{-1} is defined.

It can be viewed as a consequence of the isochore Morse lemma [14] or of Eliasson's symplectic normal form theorem [16], but this case is in fact easier than these two results.

5.3 Semiclassical normal form

We consider a self-adjoint Toeplitz operator A_k on M ; its normalized symbol

$$a(\cdot, \hbar) = a_0 + \hbar a_1 + \dots$$

is real-valued. Assume that the principal symbol a_0 admits a non-degenerate local minimum at $m_0 \in M$. Assume also that $a_0(m_0) = 0$, so that a_0 takes positive values on a neighbourhood of m_0 . Hence, thanks to theorem 5.2, we get a neighbourhood Ω_1 of m_0 in M , a neighbourhood Ω_2 of 0 in \mathbb{R}^2 , a local symplectomorphism $\chi : \Omega_1 \rightarrow \Omega_2$ and a function g_0 of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ with $g_0(0) = 0$ and $g_0'(0) > 0$, such that:

$$a_0 \circ \chi^{-1} = g_0 \circ q_0$$

on Ω_2 . We denote by f_0 the local inverse of g_0 . Our goal is to show:

Theorem 5.3. *There exist a Fourier integral operator $U_k : \mathcal{B}_k \rightarrow \mathcal{H}_k$ and a sequence $f(\cdot, k)$ of functions of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ which admits an asymptotic expansion in the \mathcal{C}^∞ topology of the form $f(\cdot, k) = \sum_{\ell \geq 0} k^{-\ell} f_\ell$, such that:*

- $U_k^* U_k \sim \Pi_k^0$ on Ω_2 ,
- $U_k U_k^* \sim \Pi_k$ on Ω_1 ,
- $U_k^* f(A_k, k) U_k \sim Q_k$ on Ω_2 .

Proof. We consider an operator $U_k^{(0)}$ satisfying the two first points (see the previous section). We will construct the operator that we seek by successive perturbations by unitary Toeplitz operators on \mathcal{B}_k . More precisely, we show by induction that for every positive integer n , there exist an operator $U_k^{(n)} : \mathcal{B}_k \rightarrow \mathcal{H}_k$ satisfying the two first points, a sequence $f^{(n)}(\cdot, k)$ of functions of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ of the form $f^{(n)}(\cdot, k) = \sum_{\ell=0}^n k^{-\ell} f_\ell$, with f_ℓ smooth, and a Toeplitz operator $R_k^{(n)}$ acting on \mathcal{B}_k such that

$$U_k^{(n)*} f^{(n)}(A_k, k) U_k^{(n)} = Q_k + k^{-(n+1)} R_k^{(n)} \quad \text{on } \Omega_2.$$

The first step is as follows: by the results of the previous section, the operator $U_k^{(0)*} f_0(A_k) U_k^{(0)}$ is a Toeplitz operator on \mathcal{B}_k , whose principal symbol is equal to $f_0 \circ a_0 \circ \chi^{-1} = q_0$ on Ω_2 . Hence, there exists a Toeplitz operator $R_k^{(0)}$ on \mathcal{B}_k such that

$$U_k^{(0)*} f_0(A_k) U_k^{(0)} = Q_k + k^{-1} R_k^{(0)}.$$

We look for $U_k^{(1)}$ of the form $U_k^{(0)} P_k$ with P_k a unitary Toeplitz operator on \mathcal{B}_k . Moreover, we choose $f^{(1)}(\cdot, k) = f_0 + k^{-1} \theta_1 \circ f_0$ with θ_1 a smooth function that remains to determine. Expanding, we get

$$U_k^{(1)*} f^{(1)}(A_k, k) U_k^{(1)} = P_k^* U_k^{(0)*} f_0(A_k) U_k^{(0)} P_k + k^{-1} P_k^* U_k^{(0)*} (\theta_1 \circ f_0)(A_k) U_k^{(0)} P_k$$

which yields

$$U_k^{(1)*} f^{(1)}(A_k, k) U_k^{(1)} = P_k^* \left(Q_k + k^{-1} R_k^{(0)} \right) P_k + k^{-1} P_k^* U_k^{(0)*} (\theta_1 \circ f_0)(A_k) U_k^{(0)} P_k.$$

Consequently, we wish to have

$$P_k^* \left(Q_k + k^{-1} R_k^{(0)} \right) P_k + k^{-1} P_k^* U_k^{(0)*} (\theta_1 \circ f_0)(A_k) U_k^{(0)} P_k = Q_k + k^{-2} R_k^{(1)}$$

where $R_k^{(1)}$ is a Toeplitz operator; this amounts, remembering that P_k is unitary, to

$$Q_k P_k + k^{-1} \left(R_k^{(0)} P_k + U_k^{(0)*} (\theta_1 \circ f_0)(A_k) U_k^{(0)} P_k \right) = P_k Q_k + k^{-2} P_k R_k^{(1)}$$

which we can rewrite

$$[Q_k, P_k] + k^{-1} \left(R_k^{(0)} P_k + U_k^{(0)*} (\theta_1 \circ f_0) (A_k) U_k^{(0)} P_k \right) = k^{-2} P_k R_k^{(1)}.$$

This will be true if and only if the subprincipal symbol of the operator on the left of the equality vanishes, that is to say

$$\frac{1}{i} \{q_0, p_0\} + p_0(r_0 + \theta_1 \circ q_0) = 0$$

where p_0 and r_0 stand for the respective principal symbols of P_k and $R_k^{(0)}$. Set $p_0 = \exp(i\varphi_0)$ with φ_0 a smooth, real-valued function (since P_k is unitary). The previous equation then becomes

$$\{\varphi_0, q_0\} = r_0 + \theta_1 \circ q_0.$$

This equation is standard and it is well-known that it can be solved. We recall a method from [16] to find θ_1 and φ_0 smooth such that it is satisfied, since we will need to know how to construct these in part 6.2. Consider the functions:

$$F(x, \xi) = -\frac{1}{2\pi} \int_0^{2\pi} r_0(\phi_{q_0}^t(x, \xi)) dt$$

and

$$\varphi_0(x, \xi) = -\frac{1}{2\pi} \int_0^{2\pi} t r_0(\phi_{q_0}^t(x, \xi)) dt,$$

where $\phi_{q_0}^t$ stands for the Hamiltonian flow of q_0 taken at time t :

$$\phi_{q_0}^t(x, \xi) = (x \cos t + \xi \sin t, -x \sin t + \xi \cos t).$$

Since we integrate on a compact set and the flow ϕ^t is smooth with respect to (x, ξ) , both F and φ_0 are smooth. By construction, we have $\{F, q_0\} = 0$. But we have the easy lemma

Lemma 5.4. *Let f be a function of $\mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ such that $\{f, q_0\} = 0$. Then the function g such that*

$$f = g \circ q_0$$

belongs to $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$.

So there exists a function θ_1 of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that $F = \theta_1 \circ q_0$. Integrating by parts, it is easy to show that

$$\{\varphi_0, q_0\} = \theta_1 \circ q_0 + r_0$$

The next steps are practically the same; indeed, let $n \geq 1$ and assume that we have found $U_k^{(n)}$ and $f^{(n)}(\cdot, k)$ satisfying the desired properties. We now look for $U_k^{(n+1)}$ of the form $U_k^{(n)}(\Pi_k^0 + k^{-n}V_k)$ with V_k a Toeplitz operator on \mathcal{B}_k such that $\Pi_k^0 + k^{-n}V_k$ is unitary. Furthermore, we write

$f^{(n+1)}(\cdot, k) = f^{(n)}(\cdot, k) + k^{-(n+1)}\theta_{n+1} \circ f_0$ with θ_{n+1} an unknown smooth function. We want the existence of a Toeplitz operator $R_k^{(n+1)}$ such that

$$U_k^{(n)*} f^{(n+1)}(A_k, k) U_k^{(n)} \left(\Pi_k^0 + k^{-n} V_k \right) = \left(\Pi_k^0 + k^{-n} V_k \right) \left(Q_k + k^{-(n+2)} R_k^{(n+1)} \right)$$

which gives, expanding,

$$\begin{aligned} Q_k + k^{-n} Q_k V_k + k^{-(n+1)} \left(R_k^{(n)} + U_k^{(n)*} (\theta_{n+1} \circ f_0) (A_k) U_k^{(n)} \right) &= Q_k + k^{-n} V_k Q_k \\ &+ k^{-(n+2)} R_k^{(n+1)} + k^{-(2n+1)} V_k R_k^{(n+1)}. \end{aligned}$$

Thus, we wish that

$$[Q_k, V_k] + k^{-1} \left(R_k^{(n)} + U_k^{(n)*} (\theta_{n+1} \circ f_0) (A_k) U_k^{(n)} \right) = 0;$$

this will be verified if and only if

$$\frac{1}{i} \{q_0, v_0\} + r_n + \theta_{n+1} \circ q_0 = 0$$

which is treated as before.

We conclude thanks to Borel's summation lemma (applied to both $f(\cdot, k)$ and U_k). \square

6 Bohr-Sommerfeld conditions

Let A_k be a self-adjoint Toeplitz operator on M as in the previous section. Moreover, assume that $a_0(m_0)$ is a global minimum of the principal symbol a_0 , and that m_0 is the unique point of M with this property. This implies that there exists $E^0 > 0$ such that for every $E \leq E^0$, the level set $a_0^{-1}(E)$ is connected and contained in Ω_1 .

The maximum norm $\|A_k\|_\infty$ of A_k tends to the L^∞ -norm $\|a_0\|_\infty$ of a_0 as k goes to infinity [3]; hence, for k large enough, the spectrum of A_k is included in the set $[-E^1, E^1]$, where $E^1 = \|a_0\|_\infty + 1$.

6.1 Statement of the result

Before stating the Bohr-Sommerfeld conditions, it is convenient to show that the sequence $f(\cdot, k)$ can be inverted, and that its inverse still has a good asymptotic expansion.

Lemma 6.1. *For k large enough, the function $f(\cdot, k)$ that appears in theorem 6.2 is a bijection from $[-E^1, E^0]$ to its image; more precisely, it is strictly increasing. Moreover, the inverse sequence $g(\cdot, k)$ admits an asymptotic expansion in the \mathcal{C}^∞ topology of the form $g(\cdot, k) = \sum_{\ell \geq 0} k^{-\ell} g_\ell + O(k^{-\infty})$, uniformly on $[-E^1, E^0]$.*

Proof. The first assertion follows from the mean value inequality

$$\forall k \geq 1 \quad \forall E, \tilde{E} \in [-E^1, E^0] \quad |f(E, k) - f(\tilde{E}, k)| \geq \inf_{[-E^1, E^0]} |f'(\cdot, k)| |E - \tilde{E}|$$

and the fact that $f'(\cdot, k)$ is bounded below by some positive constant. This implies that for k sufficiently large, $f(\cdot, k)$ is strictly monotone on $[-E^1, E^0]$; since $f'_0(0) > 0$, $f(\cdot, k)$ is in fact strictly increasing. For the second part, the proof is once again based on Borel's summation lemma; it is done by induction thanks to Taylor's formula with integral remainder. \square

We can therefore introduce the sequences

$$E_k^{(j)} = g\left(k^{-1}\left(j + \frac{1}{2}\right), k\right), \quad j \in \mathbb{N} \quad (10)$$

for k large enough and for j such that $k^{-1}\left(j + \frac{1}{2}\right)$ belongs to the set $[f(-E^1, k), f(E^0, k)]$. Since $g(\cdot, k)$ is also strictly increasing, the $E_k^{(j)}$ are ordered:

$$\forall j \in \mathbb{N}, \quad E_k^{(j)} < E_k^{(j+1)}.$$

We can be more precise; fix $j \in \mathbb{N}$ and write

$$E_k^{(j)} = g_0\left(k^{-1}\left(j + \frac{1}{2}\right)\right) + k^{-1}g_1\left(k^{-1}\left(j + \frac{1}{2}\right)\right) + O(k^{-2}).$$

Then, applying Taylor's formula with integral remainder, we get

$$E_k^{(j)} = \underbrace{g_0(0)}_{=0} + k^{-1}\left(g_1(0) + \left(j + \frac{1}{2}\right)g'_0(0)\right) + O(k^{-2}). \quad (11)$$

One must be careful with this estimate: the $O(k^{-2})$ remainder is no longer uniform with respect to j . Denote by $\lambda_k^{(1)} \leq \lambda_k^{(2)} \leq \dots \leq \lambda_k^{(j)} \leq \dots$ the eigenvalues of A_k . The main result of this section is the following theorem.

Theorem 6.2. *There exists a positive integer $k_0 \geq 1$ such that for every integer $N \geq 1$ and for every $E \leq E^0$, there exist a constant $C_N > 0$ such that for $k \geq k_0$:*

$$\left(\lambda_k^{(j)} \leq E \text{ or } E_k^{(j)} \leq E\right) \Rightarrow \left|\lambda_k^{(j)} - E_k^{(j)}\right| \leq C_N k^{-N}. \quad (12)$$

Moreover, for k large enough, all the eigenvalues of A_k smaller than E^0 are simple. In particular, we obtain an asymptotic expansion to every order for the eigenvalues of A_k smaller than E^0 .

We will need the following lemma, based on the min-max principle.

Lemma 6.3 ([11, lemma 3.3]). *Let A and B be two self-adjoint operators acting, respectively, on the Hilbert spaces \mathcal{H}' and \mathcal{H} , both bounded from below. Denote by Π_I^A the spectral projection of A on I and by $\lambda_1^A \leq \lambda_2^A \leq \dots \leq \lambda_j^A \leq \dots$ the increasing sequence of eigenvalues below the essential spectrum of A ; if there is a finite number j_{\max} of such eigenvalues, extend the sequence for $j > j_{\max}$ by setting $\lambda_j^A = \lambda_{\text{ess}}^A$, where λ_{ess}^A is the infimum of the essential spectrum of A . Introduce the same notations for B . Suppose that there exist a bounded operator $U : \mathcal{H} \rightarrow \mathcal{H}'$, an interval $I = (-\infty, E]$, and constants $C > 0$, $c \in (0, 1)$ such that $U\Pi_I^B(\mathcal{H}) \subset \text{Dom}(A)$ and*

$$\|(U^*AU - B)\Pi_I^B\| \leq C$$

and

$$\|U^*U\Pi_I^B - \Pi_I^B\| \leq c.$$

Then, for all j such that $\lambda_j^B \leq E$, one has

$$\lambda_j^A \leq (\lambda_j^B + C) \left(1 + \frac{c}{1-c}\right).$$

Proof of theorem 6.2. Fix E in $[-E^1, E^0]$. Let \mathcal{J} be an open neighbourhood of $[-E^1, E]$ such that the open set $a_0^{-1}(\mathcal{J})$ is contained in Ω_1 , and let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function equal to 1 on $[-E^1, E]$ and 0 outside \mathcal{J} . Consider the Toeplitz operator $R_k = \eta(A_k)$ and set $B_k = (f(A_k, k) - U_k Q_k U_k^*) R_k$. By the choice of R_k , the microsupport of B_k is a subset of Ω_1 . Moreover, $f(A_k, k)$ is microlocally equal to $U_k Q_k U_k^*$ on Ω_1 . These two facts imply that B_k is negligible; since M is compact, this yields that for every $N \geq 1$, there exists a positive constant C_N such that

$$\|B_k\| \leq C_N k^{-N}.$$

Now, let $\Pi_{\leq f(E, k)}^{f(A_k, k)}$ be the spectral projection associated to $f(A_k, k)$ and corresponding to the eigenvalues smaller than $f(E, k)$. If (λ, φ) is an eigencouple for A_k with $\lambda \leq E$, then $R_k \varphi = \eta(\lambda) \varphi = \varphi$. This implies that for every ϕ in $\Pi_{\leq f(E, k)}^{f(A_k, k)}(\mathcal{H}_k)$, $R_k \phi = \phi$, and consequently

$$\|(f(A_k, k) - U_k Q_k U_k^*) \Pi_{\leq f(E, k)}^{f(A_k, k)}\| \leq C_N k^{-N}.$$

Similarly, there exists $c_N > 0$ such that

$$\|(\Pi_k - U_k U_k^*) \Pi_{\leq f(E, k)}^{f(A_k, k)}\| \leq c_N k^{-N}.$$

So lemma 6.3 shows that if $f(\lambda_k^{(j)}, k) \leq f(E, k)$, the inequality

$$k^{-1} \left(j + \frac{1}{2}\right) \leq \left(1 + \frac{c_N k^{-N}}{1 - c_N k^{-N}}\right) \left(f(\lambda_k^{(j)}, k) + C_N k^{-N}\right)$$

holds. So for k large enough (independently of E), we have $k^{-1} \left(j + \frac{1}{2} \right) \leq f(E, k)$.

Now, let ρ be a smooth function equal to 1 on $[f_0(-E^1), f_0(E)]$ and vanishing outside an open neighbourhood \mathcal{K} of $[f_0(-E^1), f_0(E)]$ such that $q_0^{-1}(\mathcal{K}) \subset \Omega_2$. Thanks to proposition 3.8, we can consider the Toeplitz operator $S_k = \rho(Q_k)$, and set $C_k = (U_k^* f(A_k, k) U_k - Q_k) S_k$. Since S_k belongs to every \mathcal{T}_j , $j < 0$, C_k belongs to \mathcal{T}_0 and is thus a bounded operator $\mathcal{B}_k \rightarrow \mathcal{B}_k$. Moreover, by construction, it is negligible. Hence there exists a positive constant \tilde{C}_N such that

$$\|C_k\| \leq \tilde{C}_N k^{-N};$$

modifying C_N if necessary, we can assume that \tilde{C}_N is equal to C_N . So, introducing the spectral projection $\Pi_{\leq f(E, k)}^{Q_k}$ corresponding to the eigenvalues of Q_k smaller than $f(E, k)$, the inequality

$$\left\| (U_k^* f(A_k, k) U_k - Q_k) \Pi_{\leq f(E, k)}^{Q_k} \right\| \leq C_N k^{-N}$$

holds. Similarly, we have

$$\left\| (U_k^* U_k - \Pi_k^0) \Pi_{\leq f(E, k)}^{Q_k} \right\| \leq c_N k^{-N}.$$

Hence, applying again lemma 6.3, we obtain that

$$f(\lambda_k^{(j)}, k) \leq \left(1 + \frac{c_N k^{-N}}{1 - c_N k^{-N}} \right) \left(k^{-1} \left(j + \frac{1}{2} \right) + C_N k^{-N} \right)$$

as soon as $k^{-1} \left(j + \frac{1}{2} \right) \leq f(E, k)$. This shows that if $f(\lambda_k^{(j)}, k) \leq f(E, k)$, then

$$\left| f(\lambda_k^{(j)}, k) - k^{-1} \left(j + \frac{1}{2} \right) \right| \leq C'_N k^{-N}$$

for some positive constant C'_N . Exchanging the roles of Q_k and A_k , and using lemma 6.1, this gives formula (12).

Using this result and the fact that there exists $c > 0$ such that for $j \in \mathbb{N}$, $E_k^{(j+1)} - E_k^{(j)}$ is equivalent to ck^{-1} , we obtain that the $\lambda_k^{(j)}$ are simple for k large enough. \square

6.2 Computation of the principal and subprincipal terms

In order to exploit these results, it remains to compute a few first terms in the asymptotic expansion of the sequence $f(\cdot, k)$. What we can do is relate the principal and subprincipal terms to the actions introduced in section 1.2.

Proposition 6.4. *Set $I =]0, E^0[$. Then*

$$f_0 = \frac{1}{2\pi}c_0, \quad f_1 = \frac{1}{2\pi}c_1 \quad (13)$$

on I .

Proof. Let us first compute f_0 . Fix a level E in I . Let $\frac{1}{i}\beta$ be the 1-form describing locally the Chern connection on L ; then $c_0(E)$ is given by

$$c_0(E) = \int_{\Gamma_E} \beta.$$

Using the relation $a_0 \circ \chi^{-1} = g_0 \circ q_0$, we can then write

$$c_0(E) = \int_{\mathcal{C}_E} (\chi^{-1})^* \beta$$

where \mathcal{C}_E is the circle centered at the origin and with radius $\sqrt{2f_0(E)}$. Using Stokes' formula and the fact that χ is a symplectomorphism, this yields that $c_0(E)$ is the area of the disk bounded by \mathcal{C}_E , that is, if the orientation that we chose is the one giving the positive area (and this is what we will assume in the rest of this section)

$$c_0(E) = 2\pi f_0(E). \quad (14)$$

Now, turn back to the proof of our normal form theorem 5.3, where f_1 is constructed from the subprincipal symbol r_0 of $U_k^{(0)*} f(A_k, k) U_k^{(0)}$. By uniqueness of f_1 , instead of starting from any operator $U_k^{(0)}$, we can choose one with symbol $u \otimes v$, where u is constant and v is a square root of Ψ . Doing so, we can compute r_0 thanks to theorem 4.7:

$$r_0 = (a_1 \circ \chi^{-1})(f'_0 \circ a_0 \circ \chi^{-1}) - \nu_{\chi^{-1}(\cdot)} \left(X_{f_0 \circ a_0} \circ \chi^{-1} \right)$$

where ν is the local connection 1-form associated to ∇^{L^1} . We have $f_1 = \theta_1 \circ f_0$ with θ_1 such that for all (x, ξ) in \mathbb{R}^2

$$(\theta_1 \circ q_0)(x, \xi) = -\frac{1}{2\pi} \int_0^{2\pi} r_0 \left(\phi_{q_0}^t(x, \xi) \right) dt$$

where $\phi_{q_0}^t$ stands for the Hamiltonian flow of q_0 . Since $q_0 = f_0 \circ a_0 \circ \chi^{-1}$, this implies that for (x, ξ) in \mathbb{R}^2

$$\begin{aligned} (f_1 \circ a_0 \circ \chi^{-1})(x, \xi) &= -\frac{1}{2\pi} \int_0^{2\pi} (a_1 f'_0 \circ a_0) \left(\chi^{-1} \left(\phi_{q_0}^t(x, \xi) \right) \right) dt \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \nu_{\chi^{-1}(\phi_{q_0}^t(x, \xi))} \left(X_{f_0 \circ a_0} \left(\chi^{-1}(\phi_{q_0}^t(x, \xi)) \right) \right) dt. \end{aligned}$$

So, for $m \neq m_0$ in Ω_1 , we have

$$(f_1 \circ a_0)(m) = -\frac{1}{2\pi} \int_0^{2\pi} (a_1 f'_0 \circ a_0) \left(\chi^{-1} \left(\phi_{q_0}^t(\chi(m)) \right) \right) dt \\ + \frac{1}{2\pi} \int_0^{2\pi} \nu_{\chi^{-1}(\phi_{q_0}^t(\chi(m)))} \left(X_{f_0 \circ a_0} \left(\chi^{-1} \left(\phi_{q_0}^t(\chi(m)) \right) \right) \right) dt;$$

thus, if $E = a_0(m)$,

$$f_1(E) = \frac{1}{2\pi} \int_0^{2\pi} \nu_{\chi^{-1}(\phi_{q_0}^t(\chi(m)))} \left(X_{f_0 \circ a_0} \left(\chi^{-1} \left(\phi_{q_0}^t(\chi(m)) \right) \right) \right) dt \\ - \frac{f'_0(E)}{2\pi} \int_0^{2\pi} a_1 \left(\chi^{-1} \left(\phi_{q_0}^t(\chi(m)) \right) \right) dt.$$

But $\chi^{-1} \circ \phi_{q_0}^t \circ \chi$ is the Hamiltonian flow of $q_0 \circ \chi = f_0 \circ a_0$, so

$$f_1(E) = \frac{1}{2\pi} \int_0^{2\pi} \nu_{\phi_{f_0 \circ a_0}^t(m)} \left(X_{f_0 \circ a_0} \left(\phi_{f_0 \circ a_0}^t(m) \right) \right) dt \\ - \frac{f'_0(E)}{2\pi} \int_0^{2\pi} a_1 \left(\phi_{f_0 \circ a_0}^t(m) \right) dt.$$

Therefore, if T_E is the period of the flow $\phi_{a_0}^t$ along Γ_E , we have $T_E = 2\pi f'_0(E)$ for E close to 0, and a change of variable gives

$$f_1(E) = \frac{1}{2\pi f'_0(E)} \int_0^{2\pi} \nu_{\phi_{f_0 \circ a_0}^{t f'_0(E)}(m)} \left(X_{f_0 \circ a_0} \left(\phi_{f_0 \circ a_0}^{t f'_0(E)}(m) \right) \right) dt \\ - \frac{1}{2\pi} \int_0^{T_E} a_1 \left(\phi_{f_0 \circ a_0}^{t f'_0(E)}(m) \right) dt$$

which yields, since the Hamiltonian vector field associated to $f_0 \circ a_0$ is $X_{f_0 \circ a_0} = (f'_0 \circ a_0) X_{a_0}$, and hence $\phi_{f_0 \circ a_0}^{t f'_0(E)}(m) = \phi_{a_0}^t(m)$:

$$f_1(E) = \frac{1}{2\pi f'_0(E)} \int_0^{2\pi} \nu_{\phi_{a_0}^t(m)} \left(f'_0(E) X_{a_0} \left(\phi_{a_0}^t(m) \right) \right) dt \\ - \frac{1}{2\pi} \underbrace{\int_0^{T_E} a_1 \left(\phi_{a_0}^t(m) \right) dt}_{= - \int_{\Gamma_E} \kappa_E}$$

and by linearity of ν

$$f_1(E) = \frac{1}{2\pi} \underbrace{\int_0^{2\pi} \nu_{\phi_{a_0}^t(m)} \left(X_{a_0} \left(\phi_{a_0}^t(m) \right) \right) dt}_{= \int_{\Gamma_E} \nu} + \frac{1}{2\pi} \int_{\Gamma_E} \kappa_E.$$

The right term of this equality is precisely equal to $c_1(E)$; so we have on I

$$c_1 = 2\pi f_1. \quad (15)$$

□

To conclude, we can show that $\epsilon = 1$ on I , so the result of theorem 6.2 matches the usual Bohr-Sommerfeld conditions on the set I of regular values.

6.3 First terms of the asymptotic expansion of the eigenvalues

Theorem 6.2 and formula (11) give an asymptotic expansion for the eigenvalues of A_k smaller than E^0 . Fix $j \in \mathbb{N}$; for k large enough, one has

$$\lambda_k^{(j)} = k^{-1} \left(g_1(0) + \left(j + \frac{1}{2} \right) g'_0(0) \right) + O(k^{-2}).$$

We can be more precise, since we know the value of $g_1(0)$: by definition of $g(\cdot, k)$, we have $g_1 = -(f_1 \circ g_0)g'_0$, and the computation made in the previous part leads to $f_1(0) = -a_1(0)f'_0(0)$; consequently, $g_1(0) = a_1(0)$ and

$$\lambda_k^{(j)} = k^{-1} \left(a_1(0) + \left(j + \frac{1}{2} \right) g'_0(0) \right) + O(k^{-2}). \quad (16)$$

But $g'_0(0) = \frac{1}{f'_0(0)}$; moreover, it is standard that the principal action c_0 is smooth even at the critical value $E = 0$. Hence, thanks to formula (14), one has

$$\lambda_k^{(j)} = k^{-1} \left(a_1(0) + \frac{2\pi \left(j + \frac{1}{2} \right)}{c'_0(0)} \right) + O(k^{-2}). \quad (17)$$

In particular, the gap between two consecutive eigenvalues is given by

$$\lambda_k^{(j+1)} - \lambda_k^{(j)} = \frac{2\pi k^{-1}}{c'_0(0)} + O(k^{-2}). \quad (18)$$

7 An example on the torus

The aim of this section is to give numerical evidence for our results by investigating the case of a particular Toeplitz operator on the torus \mathbb{T} of real dimension 2. One can find the details of the quantization of \mathbb{T} in [10], where the authors investigate some conjectures on knot states; let us briefly recall its main ingredients.

7.1 The setting

Endow \mathbb{R}^2 with the linear symplectic form ω_0 and consider a lattice Λ with symplectic volume 4π . The Heisenberg group $H = \mathbb{R}^2 \times U(1)$ with product

$$(x, u).(y, v) = \left(x + y, uv \exp \left(\frac{i}{2} \omega_0(x, y) \right) \right)$$

acts on the trivial bundle $L_0 \rightarrow \mathbb{R}^2$, with action given by the same formula. This action preserves the prequantum data, and the lattice Λ injects into H ; therefore, the fiber bundle L_0 reduces to a prequantum bundle L over $\mathbb{T} = \mathbb{R}^2/\Lambda$. The action extends to the fiber bundle L_0^k by

$$(x, u).(y, v) = \left(x + y, u^k v \exp\left(\frac{ik}{2}\omega_0(x, y)\right) \right)$$

and naturally induces an action

$$T^* : \Lambda \rightarrow \text{End}\left(\Gamma\left(\mathbb{R}^2, L_0^k\right)\right), \quad u \mapsto T_u^*.$$

The Hilbert space $\mathcal{H}_k = H^0(M, L^k)$ can naturally be identified to the space $\mathcal{H}_{\Lambda, k}$ of holomorphic sections of $L_0^k \rightarrow \mathbb{R}^2$ which are invariant under the action of Λ , endowed with the hermitian product

$$\langle \varphi, \psi \rangle = \int_D \varphi \bar{\psi} |\omega_0|$$

where D is the fundamental domain of the lattice. Furthermore, $\Lambda/2k$ acts on $\mathcal{H}_{\Lambda, k}$. Let e and f be generators of Λ satisfying $\omega_0(e, f) = 4\pi$; one can show that there exists an orthonormal basis $(\psi_\ell)_{\ell \in \mathbb{Z}/2k\mathbb{Z}}$ of $\mathcal{H}_{\Lambda, k}$ such that

$$\forall \ell \in \mathbb{Z}/2k\mathbb{Z} \quad \begin{cases} T_{e/2k}^* \psi_\ell = w^\ell \psi_\ell \\ T_{f/2k}^* \psi_\ell = \psi_{\ell+1} \end{cases} \quad (19)$$

with $w = \exp\left(\frac{i\pi}{k}\right)$. The sections ψ_ℓ can be expressed in terms of Θ functions.

Set $M_k = T_{e/2k}^*$ and $L_k = T_{f/2k}^*$. Let (q, p) be coordinates on \mathbb{R}^2 associated to the basis (e, f) and $[q, p]$ be the equivalence class of (q, p) . Both M_k and L_k are Toeplitz operators, with respective principal symbols $[q, p] \mapsto \exp(2i\pi p)$ and $[q, p] \mapsto \exp(2i\pi q)$, and vanishing subprincipal symbols. Consequently

$$A_k = M_k + M_k^{-1} + L_k + L_k^{-1}$$

is a Toeplitz operator on \mathbb{T} with principal symbol

$$a_0(q, p) = 2(\cos(2\pi p) + \cos(2\pi q)) \quad (20)$$

and vanishing subprincipal symbol. Its matrix in the basis $(\psi_\ell)_{\ell \in \mathbb{Z}/2k\mathbb{Z}}$ is

$$\begin{pmatrix} 2\alpha_0 & 1 & 0 & \dots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & 0 & 1 & 2\alpha_{2k-1} \end{pmatrix}$$

where

$$\alpha_\ell = \cos\left(\frac{\ell\pi}{k}\right).$$

The principal symbol a_0 is known as Harper's Hamiltonian. It admits a global minimum at $m_0 = [1/2, 1/2]$, with $a_0(m_0) = -4$. Figure 1 is a contour plot of this function on the fundamental domain. In figure 2, we derived numerically the spectrum of A_k . Figure 3 shows the modulus of the eigenfunction associated to the eigenvalue closest to a prescribed level E .

In this situation, we can express $c'_0(0)$, and so, by equation (17), the first eigenvalues of A_k . Indeed, the symplectic form on \mathbb{T} is $\omega = 4\pi dp \wedge dq$ and the hessian of a_0 at $m_0 = [1/2, 1/2]$ is given by:

$$d^2a_0(m_0) = 8\pi^2 I_2$$

so it is easy to obtain

$$c'_0(0) = 1.$$

Consequently, the first eigenvalues are given by:

$$\lambda_k^{(j)} = -4 + 2\pi k^{-1} \left(j + \frac{1}{2}\right) + O(k^{-2}). \quad (21)$$

This is exactly what our simulations show; we plotted the eigenvalues located in a window of length $20\pi k^{-1}$ around the minimum (so we expect to see about ten eigenvalues) and the result can be seen in figure 4.

7.2 Figures

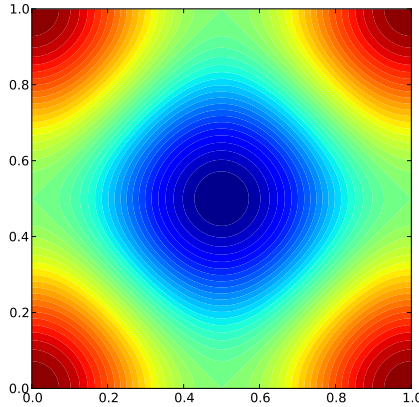


Figure 1: A few level sets of a_0

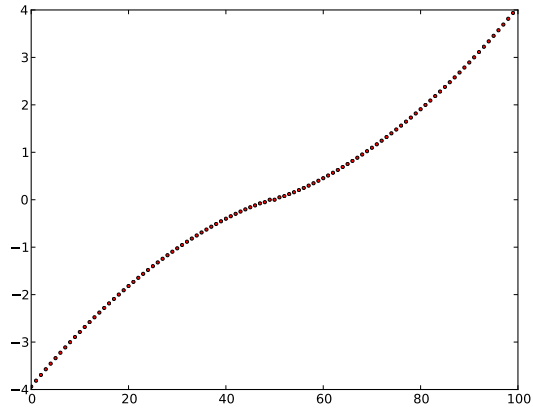
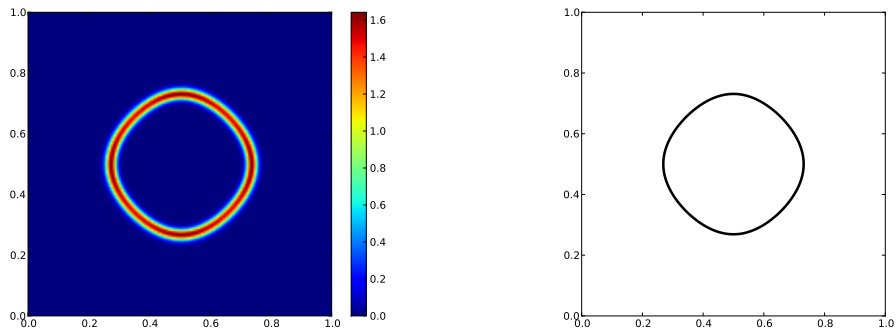


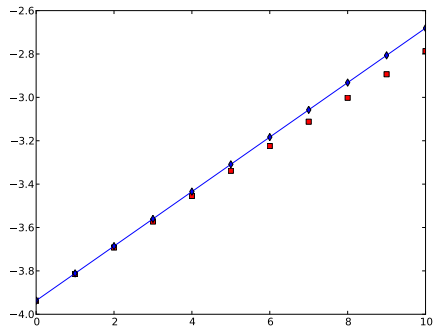
Figure 2: Ordered eigenvalues of A_k , $k = 50$



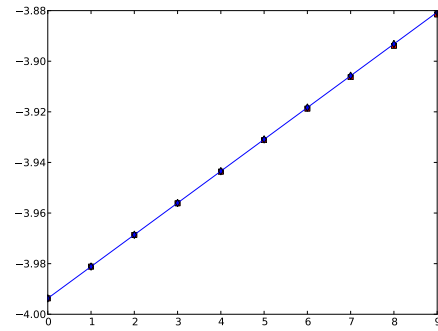
(a) Modulus of the eigenfunction associated to the eigenvalue closest to E

(b) Level set $a_0^{-1}(E)$

Figure 3: $E = a_0(0.7, 0.6)$, $k = 500$



(a) $k = 50$



(b) $k = 500$

Figure 4: Eigenvalues in $[-4, -4 + 20\pi k^{-1}]$; in red squares, the eigenvalues of A_k obtained numerically; in blue diamonds, the theoretical eigenvalues up to order k^{-2} (*i.e.* forgetting the $O(k^{-2})$ in formula (21))

A Appendix: proofs of the results of part 3.3

Proof of lemma 3.3. A first expression for this Schwartz kernel is

$$A_k(z_1, z_2) = \int_{\mathbb{C}} \Pi_k(z_1, z_3) a(z_3, k) \Pi_k(z_3, z_2) d\lambda(z_3)$$

which is equal to

$$\left(\frac{k}{2\pi}\right)^2 \int_{\mathbb{C}} \exp\left(-\frac{k}{2} \left(|z_1|^2 + |z_2|^2 + 2|z_3|^2 - 2z_1\bar{z}_3 - 2z_3\bar{z}_2\right)\right) a(z_3, k) d\lambda(z_3). \quad (22)$$

This can be written

$$A_k(z_1, z_2) = \left(\frac{k}{2\pi}\right)^2 \int_{\mathbb{C}} \exp(ik\phi(z_1, z_2, z_3)) a(z_3, k) d\lambda(z_3)$$

with the phase ϕ given by

$$\phi(z_1, z_2, z_3) = \frac{i}{2} \left(|z_1|^2 + |z_2|^2 + 2|z_3|^2 - 2z_1\bar{z}_3 - 2z_3\bar{z}_2\right).$$

It is more convenient to use real variables: let $u_j = (x_j, \xi_j)$ be the point of \mathbb{R}^2 corresponding to $z_j = \frac{1}{\sqrt{2}}(x_j - i\xi_j)$. The phase ϕ reads

$$\phi(u_1, u_2, u_3) = \frac{i}{4} \left(\|u_1 - u_3\|^2 + \|u_3 - u_2\|^2 + 2i\omega_0(u_1 - u_2, u_3)\right).$$

Using the identity

$$\|u_1 - u_3\|^2 + \|u_3 - u_2\|^2 = \frac{1}{2} \left(\|u_1 - u_2\|^2 + \|2u_3 - u_1 - u_2\|^2\right),$$

we can rewrite ϕ as

$$\phi(u_1, u_2, u_3) = \frac{i}{8} \|u_1 - u_2\|^2 + \varphi(u_1, u_2, u_3)$$

with

$$\varphi(u_1, u_2, u_3) = \frac{i}{4} \left(\frac{1}{2} \|2u_3 - u_1 - u_2\|^2 + 2i\omega_0(u_1 - u_2, u_3)\right).$$

So we have $A_k(u_1, u_2) = \exp\left(-\frac{k}{8} \|u_1 - u_2\|^2\right) I_k(u_1, u_2)$ with

$$I_k(u_1, u_2) = \left(\frac{k}{2\pi}\right)^2 \int_{\mathbb{R}^2} \exp(ik\varphi(u_1, u_2, u_3)) a(u_3, k) d\lambda(u_3);$$

a change of variable finally gives

$$I_k(u_1, u_2) = \left(\frac{k}{2\pi}\right)^2 \int_{\mathbb{R}^2} \exp(ik\psi(u_1, u_2, u_3)) a\left(u_3 + \left(\frac{u_1 + u_2}{2}\right), k\right) d\lambda(u_3),$$

with

$$\psi(u_1, u_2, u_3) = \frac{i}{2} \left(\|u_3\|^2 + i\omega_0(u_1 - u_2, u_3) + i\omega_0(u_1, u_2) \right).$$

To evaluate this integral, we cannot directly use the stationary phase lemma because $a(\cdot, k)$ may not be compactly supported; we have to adapt it. In order to do so, we start by computing the critical locus

$$C_\psi = \left\{ (u_1, u_2, u_3) \in \mathbb{C}^3; d_{u_3}\psi(u_1, u_2, u_3) = 0 \text{ and } \Im(\psi)(u_1, u_2, u_3) = 0 \right\}$$

of ψ . It is clear that $\Im(\psi)(u_1, u_2, u_3) = 0$ if and only if $u_3 = 0$; moreover, the derivative of ψ with respect to u_3 is given by

$$d_{u_3}\psi(u_1, u_2, u_3) = i\langle u_3, \cdot \rangle - \frac{1}{2}\omega_0(u_1 - u_2, \cdot),$$

hence C_ψ is the set of $(u, u, 0)$, $u \in \mathbb{R}^2$. Now, consider $\chi \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^+)$ equal to 1 in the set $\{\|x_3\| \leq \delta\}$ for some $\delta > 0$ and with compact support, and decompose I_k as $I_k = \left(\frac{k}{2\pi}\right)^2 (J_k + K_k)$, with

$$J_k(u_1, u_2) = \int_{\mathbb{R}^2} \exp(ik\psi(u_1, u_2, u_3)) a\left(u_3 + \left(\frac{u_1 + u_2}{2}\right), k\right) \chi(u_3) d\lambda(u_3)$$

and $K_k(u_1, u_2)$ equal to the same integral replacing χ by $1 - \chi$. First, we show that K_k is negligible. Choose $R > 0$ and consider the points (u_1, u_2) belonging to the ball of \mathbb{R}^4 centered at the origin and of radius R . Writing the integral in K_k in polar coordinates, we have

$$K_k(u_1, u_2) = \int_0^{2\pi} \int_\delta^{+\infty} \exp(ik\Psi(\rho, \theta)) A(\rho, \theta, k) d\rho d\theta$$

where $\Psi(\rho, \theta) = \psi(u_1, u_2, (\rho \cos \theta, \rho \sin \theta))$ and

$$A(\rho, \theta, k) = \rho a\left((\rho \cos \theta, \rho \sin \theta) + \left(\frac{u_1 + u_2}{2}\right), k\right) (1 - \chi(\rho \cos \theta, \rho \sin \theta)).$$

Performing successive integration by parts, it is easy to prove that for every positive integer N

$$K_k(u_1, u_2) = k^{-N} \int_0^{2\pi} \int_\delta^{+\infty} \exp(ik\Psi(\rho, \theta)) D^N A(\rho, \theta, k) d\rho d\theta$$

where D is the differential operator acting on $\mathcal{C}^\infty([\delta, +\infty[\times [0, 2\pi])$ given by $Df = i\frac{\partial}{\partial \rho} \left(\left(\frac{\partial \Psi}{\partial \rho}\right)^{-1} f \right)$. Furthermore, using the facts that $\frac{\partial \Psi}{\partial \rho}(\rho, \theta) = i\rho - \frac{1}{2}(x_1 - x_2) \sin \theta + \frac{1}{2}(y_1 - y_2) \cos \theta$, $D^N A$ is equal to a linear combination of terms of the form $\frac{\partial^p \Psi}{\partial \rho^p} \frac{\partial^q A}{\partial \rho^q}$ and $a(\cdot, k)$ belongs to \mathcal{S}_j^1 , we have the estimate

$$\left| \int_\delta^{+\infty} \exp(ik\Psi(\rho, \theta)) D^N A(\rho, \theta, k) d\rho \right| \leq C_N \int_\delta^{+\infty} \exp\left(-\frac{k}{2}\rho^2\right) w(\rho, \theta) d\rho$$

with

$$w(\rho, \theta) = \rho^{j_1} \left(1 + \left| \rho(\cos \theta, \sin \theta) + \frac{u_1 + u_2}{2} \right|^2 \right)^{j_2}$$

for some $C_N > 0$ and $j_1, j_2 \in \mathbb{Z}$. If $j_2 < 0$, this last integral can be bounded by $\int_\delta^{+\infty} \rho^{j_1} \exp\left(-\frac{k}{2}\rho^2\right) d\rho = O(k^{-1/2})$; if $j_2 > 0$, we have

$$\left(1 + \left| \rho(\cos \theta, \sin \theta) + \frac{u_1 + u_2}{2} \right|^2 \right)^{j_2} \leq \left(1 + |\rho|^2 + |R|^2 \right)^{j_2}$$

and hence

$$\int_\delta^{+\infty} \exp\left(-\frac{k}{2}\rho^2\right) w(\rho, \theta) d\rho \leq \tilde{C}_N \int_\delta^{+\infty} \exp\left(-\frac{k}{2}\rho^2\right) \rho^{j_3} d\rho = O(k^{-1/2}).$$

This shows that $K_k \leq c_N k^{-N}$ on the ball of radius R for some $c_N > 0$. We treat the derivatives of K_k in the same way.

It remains to estimate J_k . Since the second derivative of ψ with respect to u_3 is equal to $i\text{Id}$, and since

$$\begin{aligned} \psi(u_1, u_2, u_3) &= \frac{i}{8} \|u_1 - u_2\|^2 - \frac{1}{2} \omega_0(u_1, u_2) \\ &\quad + \frac{1}{4} (i(\xi_1 - \xi_2) + 2x_3) \partial_{x_4} \psi(u_1, u_2, u_3) \\ &\quad + \frac{1}{4} (i(x_2 - x_1) + 2\xi_3) \partial_{\xi_4} \psi(u_1, u_2, u_3), \end{aligned}$$

the stationary phase lemma [18, section 7.7] gives

$$J_k(u_1, u_2) = \frac{2\pi}{k} \exp\left(\frac{i}{8} \|u_1 - u_2\|^2 - \frac{1}{2} \omega_0(u_1, u_2)\right) \tilde{a}(u, u, k) + O(k^{-\infty}),$$

where $\tilde{a}(\cdot, \cdot, k)$ belongs to \mathcal{S}_j^2 ; the coefficients $\tilde{a}_\ell(u_1, u_2, k)$ of its asymptotic expansion, which we do not write here, are linear combinations of derivatives of the $a_m, m \geq 0$, evaluated at $\frac{u_1 + u_2}{2}$. However, the values of $a(\cdot, \cdot, k)$ along the diagonal of \mathbb{C}^2 can be easily computed, because a number of terms vanish: for z in \mathbb{C} , we have

$$\tilde{a}(z, z, k) = \left(\sum_{\ell \geq 0} \frac{k^{-\ell}}{\ell!} \Delta^\ell a \right) (z, k) = \left(\exp k^{-1} \Delta a \right) (z, k).$$

Putting this together with the fact that K_k is negligible, we obtain the result. \square

Proof of lemma 3.5. Formula (22) shows that the kernel A_k is a holomorphic section of $L_0^k \boxtimes L_0^{-k}$; differentiating equation (4), this implies that the

sequences of functions $z \in \mathbb{C} \mapsto \frac{\partial \tilde{a}}{\partial \bar{z}_1}(z, z, k)$ and $z \in \mathbb{C} \mapsto \frac{\partial \tilde{a}}{\partial z_2}(z, z, k)$ are negligible. Hence, we have for $\ell \geq 0$ and $z \in \mathbb{C}$

$$\frac{\partial \tilde{a}_\ell}{\partial \bar{z}_1}(z, z) = 0 = \frac{\partial \tilde{a}_\ell}{\partial z_2}(z, z).$$

Thanks to these holomorphy conditions, the fact that \tilde{a}_ℓ vanishes on the diagonal implies that it vanishes to all order along the diagonal. We can easily adapt lemma 1 of [6] to show that this yields the negligibility of $\exp\left(-\frac{k}{4}|z_1 - z_2|^2\right) \tilde{a}(z_1, z_2, k)$. Injecting this in formula (4) gives the result. \square

Proof of corollary 3.6. The kernel of C_k reads

$$C_k(z_1, z_2) = \int_C A_k(z_1, z_3) B_k(z_3, z_2) d\lambda(z_3).$$

Using the representations of A_k and B_k given by lemma 3.3, this yields

$$C_k(z_1, z_2) = \left(\frac{k}{2\pi}\right)^2 \int_C \exp(ik\phi(z_1, z_2, z_3)) \tilde{a}(z_1, z_3, k) \tilde{b}(z_3, z_2, k) d\lambda(z_3) \\ + R_k \exp\left(-Ck|z_1 - z_2|^2\right),$$

with $C > 0$, R_k negligible and

$$\phi(z_1, z_2, z_3) = \frac{i}{2} \left(|z_1|^2 + |z_2|^2 + 2|z_3|^2 - 2z_1\bar{z}_3 - 2z_3\bar{z}_2 \right).$$

Using the same technique as in the proof of lemma 3.3, we show that

$$C_k(z_1, z_2) = \frac{k}{2\pi} \exp\left(-\frac{k}{2} \left(|z_1|^2 + |z_2|^2 - 2z_1\bar{z}_2 \right)\right) \tilde{c}(z_1, z_2, k) \\ + R'_k \exp\left(-C'k|z_1 - z_2|^2\right)$$

with $C' > 0$, R'_k negligible, and $\tilde{c}(\cdot, \cdot, k) \in \mathcal{S}_{j+j'}^2$. Now, consider the function $\check{c}(\cdot, k)$ defined by $\check{c}(z, k) = \tilde{c}(z, z, k)$ for z in \mathbb{C} , and put $c(\cdot, k) = (\exp(-k^{-1}\Delta)\check{c})(\cdot, k)$. Then $c(\cdot, k)$ belongs to $\mathcal{S}_{j+j'}^1$ and, by lemma 3.3, the Toeplitz operator $D_k = \text{Op}(c(\cdot, k))$ admits a Schwartz kernel of the form

$$D_k(z_1, z_2) = \frac{k}{2\pi} \exp\left(-\frac{k}{2} \left(|z_1|^2 + |z_2|^2 - 2z_1\bar{z}_2 \right)\right) \tilde{d}(z_1, z_2, k) \\ + R''_k \exp\left(-C''k|z_1 - z_2|^2\right),$$

where $\tilde{d}(\cdot, \cdot, k)$ belongs to \mathcal{S}_j^2 , R''_k is negligible, C'' is a positive constant and for every z in \mathbb{C} , $\tilde{d}(z, z, k) = \check{c}(z, z, k)$. Lemma 3.5 yields that $C_k = D_k + R'''_k \exp(-C'''k|z_1 - z_2|^2)$ for some $C''' > 0$ and R'''_k negligible.

It remains to compute the contravariant symbol of C_k . For z in \mathbb{C} , put $\check{a}(z, w, k) = \tilde{a}(z, z + w, k)$ and $\check{b}(z, w, k) = \tilde{b}(z + w, z, k)$. One has

$$\check{c} = \left(\exp \left(k^{-1} \Delta_w \right) \check{a} \check{b} \right)_{|w=0}$$

with Δ_w the holomorphic laplacian with respect to w ; using lemma 3.5, we find

$$\check{c}(z, k) = \left(\exp \left(k^{-1} \frac{\partial}{\partial \bar{u}} \frac{\partial}{\partial v} \right) \check{a}(u, k) \check{b}(v, k) \right)_{|u=v=z}$$

up to a negligible term. Now, since $c(\cdot, k) = (\exp(-k^{-1} \Delta) \check{c})(\cdot, k)$, this yields

$$c(z, k) = \left(\exp \left(-k^{-1} \frac{\partial}{\partial u} \frac{\partial}{\partial \bar{v}} \right) a(u, k) b(v, k) \right)_{|u=v=z}$$

up to a negligible term, which was to be proved. \square

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