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**RESOLUTION OF SINGULARITIES OF THREEFOLDS  
IN MIXED CHARACTERISTIC:  
CASE OF SMALL MULTIPLICITY**

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INTRODUCTION

This article is part of the authors' program whose purpose is to prove the following conjecture on Resolution of Singularities of threefolds in mixed characteristic. The conjecture is a special case of Grothendieck's Resolution conjecture for quasi-excellent schemes.

**CONJECTURE 0.1** *Let  $C$  be an integral regular excellent curve with function field  $F$ . Let  $S/F$  be a reduced algebraic projective surface and  $\mathcal{X}$  be a flat projective  $C$ -scheme with generic fiber  $\mathcal{X}_F = S$ . There exists a birational projective  $C$ -morphism  $\pi : \mathcal{Y} \rightarrow \mathcal{X}$  such that*

- (i)  $\mathcal{Y}$  is everywhere regular.
- (ii)  $\pi^{-1}(\text{Reg}\mathcal{X}) \rightarrow \text{Reg}\mathcal{X}$  is an isomorphism.

Let us point out that the equicharacteristic techniques designed in [CP1] extend to the situation described in the above conjecture. In particular, [CP1] theorem 3.3 extends and reduces conjecture **0.1** to the following variant:

**CONJECTURE 0.2** *Let  $A$  be an excellent DVR with quotient field  $F$  and residue characteristic  $p > 0$ . Let  $(S, M, k)$  be a regular local ring of dimension three dominating  $A$ , essentially of finite type over  $A$  with  $K := QF(S)$  of transcendence degree two over  $F$ . Let finally  $L/K$  be a finite field extension and  $v$  be a valuation of  $L$ . Assume:*

- (i)  $L/K$  is cyclic Galois or purely inseparable of degree  $p$ .
- (ii)  $v$  has rank one and is centered in  $S$ .

*Then there exists a regular local ring  $T$  essentially of finite type over  $A$  with  $QF(T) = L$  such that  $v$  is centered in  $T$ .*

Applying embedded resolution techniques for surfaces, it can be assumed that such a  $v$  is centered in a local model of  $L$  of the form  $B = (S[X]/(h))_{(M, X)}$  with  $h$  monic of degree  $p$ ; more precisely,  $h = X^p - g^{p-1}X - f$ ,  $f, g \in M$  and ( $g = 0$  if  $\text{char}A = 0$ ). In particular the Local Uniformization statement of conjecture **0.2** only involves certain hypersurface singularities  $(\text{Spec}B, x)$ , of multiplicity  $m(x) \leq p = \text{char}k$ , and embedded in an excellent fourfold  $(Z = \text{Spec}S[X], x)$ . We prove here:

**MAIN THEOREM 0.3** *Let  $(R, \mathfrak{M}, k = k(x) := \frac{R}{\mathfrak{M}})$  be an excellent regular local ring of dimension four,  $(Z, x) := (\text{Spec}R, \mathfrak{M})$  and  $(X, x) := (\text{Spec}R/(h), x)$  be a reduced hypersurface. Assume that the multiplicity  $m(x)$  of  $(X, x)$  satisfies  $m(x) < p := \text{char}k(x)$ . Let  $v$  be a valuation of  $K(X)$  centered at  $x$ . Then there exists a finite sequence of local blowing ups*

$$(X, x) =: (X_0, x_0) \longleftarrow (X_1, x_1) \longleftarrow \cdots \longleftarrow (X_n, x_n),$$

where  $x_i \in X_i$ ,  $0 \leq i \leq n$  is the center of  $v$ , each blowing up center  $Y_i \subset X_i$  is permissible at  $x_i$  (in Hironaka's sense), such that  $x_n$  is regular.

The proof of theorem **0.3** builds upon classical Resolution of Singularities techniques. We use systematically the Hironaka characteristic polyhedron and Hironaka's invariants: the multiplicity  $m(x)$  and  $\tau$ -number  $\tau(x)$  for the hypersurface singularity  $(X, x) := (\text{Spec}R/(h), x)$ .

Since we are working without any ground field (at least when  $R$  is not equicharacteristic), the Tschirnhausen trick (killing the degree  $(m(x) - 1)$ -term in the equation) cannot be directly applied even though  $m(x) < p$ . Rather, we use it for the initial face of the Hironaka polyhedron (theorem **II.5**) to define well behaved invariants. The pair  $(m(x), \tau(x))$  is then further on completed to a 6-tuple  $\iota(x)$  defined in **IV.2**.

The main technical part is concentrated in section VI. We consider projections to a two dimensional space to define a refined invariant in sections **VI.3** and **VI.4.1**. Controlling the transformation law for this refined invariant under blowing up is much harder but leads essentially to the same formulæ as for the characteristic polygon of a surface singularity. The proof of the Main Theorem follows rather easily from these computations (section **VI.4.5**).

It is worth pointing out that these techniques are global in nature and it is to be expected that theorem **0.3** can be extended to a global version, i.e. without referring to a given valuation  $v$  and using global blowing up centers. We use the valuation only at a few specific places (mostly in section V) to make the argument quicker.

This article is organized as follows: section I states the reduction of the Main Theorem to the case  $\tau(x) = 1$ , immediate from [CJS]. This means that the initial form  $\text{in}_x(h)$  can be written

$$\text{in}_x(h) = \lambda Y^{m(x)}, \lambda \in k(x), \lambda \neq 0, y \text{ a regular parameter of } R, Y := \text{in}_x(y).$$

Section II first recollects known material from [H1][H2] about characteristic polyhedra and associated invariants (definition **II.2**). Special coordinates  $(z, u) := (z, u_1, u_2, u_3)$  on  $R$  are said to be *fully prepared* if they compute the Hironaka characteristic polyhedron  $\Delta(h; u)$  and if the  $\delta$ -initial  $\text{in}_\delta(h)$  of  $h$  is Tschirnhausen transformed, i.e. has no term of degree  $m(x) - 1$  (theorem **II.5** and definition **II.5.3**). The form  $\text{in}_\delta(h)$  is defined in **II.2(iii)** and is the sum of the initial forms of all those terms in  $h$  contributing to the face of minimal order at  $x$  of  $\Delta(h; u)$ . In such special coordinates, invariants  $d_1, d_2, d_3, \epsilon(x)$  (each of them is a nonnegative rational number) can be computed from the polyhedron  $\Delta(h; u)$  and the  $\delta$ -initial  $\text{in}_\delta(h)$  (definition **II.7**). Theorem **II.9** proves the Main Theorem when  $\epsilon(x) = 0$ , in which case only combinatorial blowing ups are used.

When  $\epsilon(x) > 0$ , some preparations are required in order to get the locus

$$\Sigma := \{y \in X : m(y) = m(x), \tau(y) = \tau(x), \epsilon(y) > 0\}$$

Zariski closed and of dimension at most one (theorems **II.10** and **II.11**). Section III prepares  $\text{Spec}R$  and constructs an equicharacteristic  $p$  normal crossings divisor

$$E \subseteq \text{div}(u_1 u_2 u_3) \subset \text{Spec}R$$

which contains  $\Sigma$  ( $E = E_n$  in proposition **III.1**).

Section IV then provides some further invariants build up from the ideal of coefficients of  $\text{in}_\delta(h)$  once this preparation is achieved: a refined directrix  $\mathcal{V} \subseteq \langle U_1, U_2, U_3 \rangle$  (definition **IV.1**) and associated refined numerical invariant  $\iota(x)$  (definition **IV.2**):

$$\iota(x) := (m(x), -\tau(x), \epsilon(x), -\rho(x), -t(x), -e(x)).$$

Section V introduces the notion of  $\epsilon$ -permissible blowing up centers (definition **V.1**). For curves, being  $\epsilon$ -permissible is stronger than being Hironaka permissible (proposition **V.2**); blowing up along an  $\epsilon$ -permissible center does not increase the invariant  $\iota(x)$  (proposition **V.3**). Furthermore,  $\iota(x)$  can be decreased by blowing up along  $\epsilon$ -permissible centers except possibly when  $\mathcal{V} = \langle U_3 \rangle$  and (either  $\text{div}(u_3) \subseteq E$  or  $E \subseteq \text{div}(u_1 u_2)$ ) (propositions **V.4** and **V.5**).

Section VI proves the same result in these remaining cases (theorem **VI.1**), thus concluding the proof of the Main Theorem. We now project to the  $(u_1, u_2)$ -space and define *well prepared* coordinates by minimizing the induced image of  $\Delta(h; u)$  by this projection (this requires choosing special coordinates  $(z, u_3)$ ). There are further associated invariants  $\beta(u, z), C(u, z), \gamma(u, z)$  defined in **VI.4.1**. The behaviour of these invariants by blowing up  $\epsilon$ -permissible curves and closed points are studied respectively in propositions **VI.4.2** and **VI.4.3**. Section **VI.4.5** contains the proof of theorem **VI.1** and is basically a consequence of the former computations.

The notation and assumption in the Main Theorem will be kept all along this article. The proof will be made by induction on the multiplicity  $m(x) = \text{ord}_x(h)$  of  $x \in X$ . Since it is assumed that  $m(x) < p$ ,  $(X, x)$  is already regular if  $p = 2$ , so we assume  $p \geq 3$  from now on. The formal completion of  $R$  with respect to  $\mathfrak{M}$  is denoted by  $\widehat{R}$ .

## I BASIC INVARIANTS

Two basic invariants are attached to the hypersurface singularity  $(X, x) = (\text{Spec}R/(h), x)$ . The first invariant is its multiplicity  $m(x)$  (or  $m$  for short) of  $(X, x)$ . The second invariant is  $\tau(x)$  (or  $\tau$  for short), which is the dimension of the smallest  $k(x)$ -vector subspace  $\mathcal{T}$  of  $\frac{\mathfrak{M}}{\mathfrak{M}^2}$  such that  $\text{in}_x(h) \in k(x)[\mathcal{T}]$  [H1, Ch.2, Lemma 10]. This vector space is called the *directrix* of  $\text{in}_x(h)$ .

Proving the Main Theorem in the cases  $\dim(Z) - \tau(x) \in \{0, 1, 2\}$ , i.e.  $\tau(x) \in \{2, 3, 4\}$  is done in [CJS]. So from now on, we assume that  $\tau(x) = 1$ . Equivalently:

$$\text{in}_x(h) = \lambda Y^m, \quad \lambda \in k(x), \lambda \neq 0, \quad y \text{ a regular parameter of } R, \quad Y := \text{in}_x(y).$$

## II CHARACTERISTIC POLYHEDRON

### **DEFINITION II.1**

- (i) An  $F$ -subset  $\Delta \subset \mathbb{R}_+^d$  is a closed complex subset of  $\mathbb{R}_+^d$  such that  $v \in \Delta$  implies  $v + \mathbb{R}_+^d \subset \Delta$ .
- (ii) A point  $v \in \Delta$  is called a *vertex* if there is a positive linear form  $L$  on  $\mathbb{R}^d$  (i.e. has strictly positive coefficients) such that

$$\{v\} = \Delta \cap \{A \in \mathbb{R}^d \mid L(A) = 1\}.$$

- (iii) The *essential boundary*  $\partial\Delta$  of an  $F$ -subset  $\Delta$  is the subset of  $\Delta$  consisting of those  $v \in \Delta$  such that  $v \notin v' + \mathbb{R}_+^d$  with  $v' \in \Delta$  unless  $v' = v$ . We write  $\Delta^+ = \Delta - \partial\Delta$ .

For the next definition and proposition, we will forget the hypothesis  $\dim(R) = 4$ : we will have to use the notions defined there for different regular rings of dimension at most three. Given a r.s.p.  $(y, u_1, u_2, \dots, u_d) =: (y, u)$  of a regular local ring  $R$  and  $f \in R$ , there exists a *finite* sum expansion

$$f = \sum_{A, b} C_{A, b} y^b u^A, \quad b \in \mathbb{N}, \quad A \in \mathbb{N}^d. \quad (1)$$

where each  $C_{A,b}$  is a unit in  $R$ . This follows easily from the facts that  $R$  is Noetherian and the map  $R \subseteq \widehat{R}$  faithfully flat. We regard  $u$  as “fixed” parameters and  $y$  as “varying”, which is reflected in the indexing below. Assume furthermore that

$$h \in \mathfrak{M}, \quad h \notin (u_1, \dots, u_d). \quad (2)$$

We let  $\overline{R} := R/(u_1, \dots, u_d)$ ,  $\overline{h} \in \overline{R}$  be the image of  $h$  and “ord” be the valuation of the discrete valuation ring  $\overline{R}$ . We extend our conventions by letting now

$$m := \text{ord} \overline{h} \geq 1. \quad (3)$$

Assumption (2) and notation (3) are maintained all along this article. Our original concern is for  $\tau(x) = 1$ , say  $\text{in}_x(h) = \lambda Y^m$ ,  $0 \neq \lambda \in k(x)$  which fits into these conventions provided  $Y = \text{in}_x(y)$ .

### DEFINITION II.2

(i) The polyhedron  $\Delta(h; u; y) \subset \mathbb{R}_{\geq 0}^d$  is defined as the smallest  $F$ -subset containing all points of

$$S(h) := \left\{ v = \frac{A}{m-b} \mid 0 \leq b < m \right\}.$$

The characteristic polyhedron  $\Delta(h; u) \subset \mathbb{R}_{\geq 0}^d$  is defined by the formula

$$\Delta(h; u) := \bigcap_{(\widehat{y}, u_1, \dots, u_d)} \Delta(h; u; \widehat{y}), \quad (4)$$

where the intersection runs over all r.s.p's of  $\widehat{R}$  of the form  $(\widehat{y}, u_1, \dots, u_d)$ .

(ii) For  $v \in \partial\Delta(h; u; y)$ , the  $v$ -initial of  $h$  is defined as

$$\text{in}_v(h) := \sum_{A,b} \overline{C_{A,b}} Y^b U^A \in k[U, Y] = k[U_1, U_2, \dots, U_d, Y],$$

where  $\overline{C_{A,b}} \in k$  is the residue of  $C_{A,b}$  and the sum ranges over such  $(A, b)$  that

$$C_{A,b} \neq 0, \quad (b \leq m, A = 0) \text{ or } (b < m \text{ and } v = \frac{A}{m-b}).$$

(iii) For  $A \in \mathbb{N}^d$ , let  $|A| := a_1 + \dots + a_d$ . We put

$$\delta(h, u, y) := \min \left\{ \frac{|A|}{m-b} : C_{A,b} \neq 0, b < m \right\}.$$

This is in fact an invariant of the polyhedron  $\Delta(h; u; y)$  since

$$\delta(h, u, y) = \min\{|v| : v \in \Delta(h; u; y)\}.$$

The  $\delta$ -initial of  $h$  is defined as

$$\text{in}_{\delta, u, y}(h) := \sum_{A,b} \overline{C_{A,b}} Y^b U^A \in k(x)[U, Y] = k(x)[U_1, U_2, \dots, U_d, Y],$$

where the sum ranges over such  $(A, b)$  that

$$C_{A,b} \neq 0, \quad (b \leq m, A = 0) \text{ or } (b < m \text{ and } \frac{|A|}{m-b} = \delta(h, u, y)).$$

(iv) More generally, let

$$L : (x_1, x_2, \dots, x_d) \mapsto L(x_1, x_2, \dots, x_d) = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_d x_d, \quad \lambda_1, \lambda_2, \dots, \lambda_d \in \mathbb{Q}_{\geq 0},$$

be a nonzero nonnegative linear form on  $\mathbb{R}^d$ . We define

$$l(h, u, y) := \min\{L(A) \mid A \in \Delta(h; u; y)\} \geq 0.$$

We define a monomial valuation  $v_{L,h,u,y}$  on  $R$  by setting

$$I_\lambda := (\{y^b u^A \mid l(h, u, y)b + L(A) \geq \lambda\}) \subseteq R,$$

for  $\lambda \geq 0$  and  $v_{L,h,u,y}(g) := \min\{\lambda \in \mathbb{Q} \mid g \in I_\lambda\}$  for any nonzero  $g \in R$ .

**PROPOSITION II.3** Let  $L$  be a nonzero nonnegative linear form as above, and let

$$I := \{i \mid \lambda_i > 0\}, \quad I' := \{i \mid \lambda_i = 0\} = \{1, \dots, d\} \setminus I.$$

The graded algebra  $\text{gr}_{v_{L,h,u,y}}(R)$  of  $R$  w.r.t.  $v_{L,h,u,y}$  is given by

(i) if  $l(h, u, y) \neq 0$ , then

$$\text{gr}_{v_{L,h,u,y}}(R) = \frac{R}{(y, \{u_i\}_{i \in I})} [Y, \{U_i\}_{i \in I}];$$

(ii) if  $l(h, u, y) = 0$ , then

$$\text{gr}_{v_{L,h,u,y}}(R) = \frac{R}{(\{u_i\}_{i \in I})} [\{U_i\}_{i \in I}].$$

In particular, we have  $\text{gr}_{v_{L,h,u,y}}(R) \simeq k[Y, U_1, U_2, \dots, U_d]$  whenever  $L$  is positive.

The above proposition is obvious. One also checks easily the following:

**REMARK II.3.1** Let  $v$  be a vertex of  $\Delta(h, u, y)$ . We have:

(i)  $\text{in}_v(h)$  is independent of the presentation **II.1** (1),

(ii)  $\text{in}_v(h) \neq \text{in}_x(h)$ ,

(iii)  $L$  being a positive linear form  $L$  on  $\mathbb{R}^d$  such that  $\{v\} = \Delta \cap \{A \in \mathbb{R}^d \mid L(A) = 1\}$ , (cf. **II.1** (ii)), then

$$\text{in}_v(h) = \text{in}_{v_{L,h,u,y}}(h) \in \text{gr}_{v_{L,h,u,y}}(R) = k(x)[U_1, U_2, \dots, U_d, Y].$$

When there is no ambiguity, we will write  $\text{gr}_\delta(R)$  and  $\text{in}_\delta(h) \in \text{gr}_\delta(R)$  instead of respectively  $\text{gr}_{v_{L,h,u,y}}(R)$  and  $\text{in}_{v_{L,h,u,y}}(h)$ , where  $L(x_1, x_2, \dots, x_d) = x_1 + x_2 + \dots + x_d$ .

**REMARK II.4** With notations as above, we have:

$\delta(h, u, y) \in \frac{1}{m!}\mathbb{N}$  and  $\delta(h, u, y) > 1$  if ( $m = \text{ord}_x(h)$  and  $\langle \text{in}_x(h) \rangle = \langle (\text{in}_x(y))^m \rangle$ ).

**ASSUMPTIONS II.4.1** We now apply these constructions to the case  $R := \mathcal{O}_{Z,x}$ ,  $\dim(R) = 4$ ; the element  $h \in R$  verifies assumptions **II.1**(2)(3) with  $m = \text{ord}_x(h) < p = \text{char}k(x)$  and  $\langle \text{in}_x(h) \rangle = \langle Y^m \rangle$ . In addition,  $X = \text{Spec}(R/(h))$  is reduced.

**THEOREM II.5** Given  $(y, u_1, u_2, u_3) =: (y, u)$  as above, there exists  $z \in R$ ,  $z \equiv y \pmod{(u_1, u_2, u_3)}$  such that

$$\Delta(h, u, z) = \Delta(h, u) \neq \emptyset, \quad (1)$$

$$\text{in}_{\delta, u, z}(h) = \sum_{A, b, b \neq m(x)-1} \overline{C_{A,b}} z^b U^A. \quad (2)$$

Proof. Suppose  $\Delta(h, u) = \emptyset$ , then, in  $\widehat{R}$ , we should have  $h = \gamma z^m$ ,  $\gamma$  invertible in  $\widehat{R}$  and  $z \in \widehat{R}$  a local parameter:  $h$  should be nonreduced in  $\widehat{R}$ . By excellence,  $h$  should be nonreduced in  $R$ , in contradiction with the hypothesis  $X$  reduced.

Since  $\Delta(h, u) \neq \emptyset$ ,  $\Delta(h, u)$  may be defined by a finite number  $n$  of inequalities  $L_i(x_1, x_2, x_3) \geq 1$ ,  $1 \leq i \leq n$ , with  $L_i(x_1, x_2, x_3) = a_{i,1}x_1 + a_{i,2}x_2 + a_{i,3}x_3$ ,  $a_{i,1}, a_{i,2}, a_{i,3} \in \mathbb{Q}_{\geq 0}$ ,  $L_i \neq 0$ . In a few words:

$$\Delta(h, u) = \{(x_1, x_2, x_3) | L_i(x_1, x_2, x_3) \geq 1, 1 \leq i \leq n\}.$$

We choose  $L_1(x_1, x_2, x_3) = \frac{1}{\delta(h, u)}(x_1 + x_2 + x_3)$ , with  $\delta(h, u) := \min\{|v| : v \in \Delta(h, u)\}$ .

Suppose (1) does not hold for  $(y, u)$ . Then, with notations as in **II.2**(iv), some  $L_i$ ,  $1 \leq i \leq n$  satisfies

$$L_i(\Delta(h, u, y)) = [l_i(h, u, y), \infty[ \not\subset [1, +\infty[ \Leftrightarrow l_i(h, u, y) < 1.$$

We skip the index  $i$  of  $L_i$  and of  $l_i(h, u, y)$  to simplify the notations. Following **II.2**(iv), we define the initial form of  $h$  with respect to  $L, u, y$ : in the case  $l(h, u, y) > 0$

$$\text{in}_{L, u, y}(h) := \sum_{A, b} \overline{C_{A,b}} Y^b U^A \in \text{gr}_{L, u, y}(R), \quad (3)$$

with  $bl(h, u, y) + L(A) = ml(h, u, y)$ ,  $\overline{C_{A,b}} \in \frac{R}{(u_i)_{a_i > 0 + (y)}}$ . In the case  $l(h, u, y) = 0$ , we have

$$\text{in}_{L, u, y}(h) := \sum_{A, b} \overline{C_{A,b} y^b} U^A \in \text{gr}_{L, u, y}(R), L(A) = 0, \overline{C_{A,b} y^b} \in \frac{R}{(u_i)_{a_i > 0}}. \quad (4)$$

**Claim II.5.1** In (3) (resp. (4)), there exists  $A$  with  $\overline{C_{A, m-1}} \neq 0 \in \frac{R}{(u_i)_{a_i > 0}}$  (resp.  $\overline{C_{A, m-1} y^{m-1}} \neq 0 \in \frac{R}{(u_i)_{a_i > 0}}$ ).

Indeed, in the face with equation  $L(x_1, x_2, x_3) = l(h, u, y)$  of  $\Delta(h, u, y)$  there is at least a vertex  $v$  which is solvable [H2, (3.8)]. Then  $\text{in}_v(h)$  is collinear to an  $m^{\text{th}}$ -power:  $C_{v, m-1} \notin \mathfrak{M}$  since  $m < p$  and the claim is proved. Take  $A = v$  and let

$$y_1 = y + \frac{1}{m} C_{0, m}^{-1} \sum_A C_{A, m-1} u^A \in R. \quad (5)$$

Note that, for any  $A$  with  $C_{A, m-1} \neq 0$ ,  $A \in \Delta(h, u, y)$ . So, for any  $i$ ,  $1 \leq i \leq n$ , and any  $A$  with  $C_{A, m-1} \neq 0$ ,  $L(A) \geq l(h, u, y)$ . So if in the expansion of **II.2**(1) we set

$$y = y_1 - \frac{1}{m} C_{0, m}^{-1} \sum_A C_{A, m-1} u^A,$$

we get a new expansion

$$h = \sum_{A,b} D_{A,b} y_1^b u^A, \quad D_{A,b} \in R^\times \cup \{0\}, \quad b \in \mathbb{N}, \quad A \in \mathbb{N}^3, \quad (6)$$

and  $D_{A,b} \neq 0 \Rightarrow L(A) + bl_i(h, u, y) \geq m$ . So

$$l(h, u, y_1) \geq l(h, u, y), \quad 1 \leq i \leq n.$$

Suppose  $l(h, u, y_1) = l(h, u, y)$ . Then  $v_{L,u,y_1}(y) = v_{L,u,y_1}(y_1) = l(h, u, y)$ ,

$$Y := \text{in}_{L,u,y_1}(y) = Y_1 - \text{in}_{L,u,y_1}\left(\frac{1}{m} C_{0,m}^{-1} \sum_A C_{A,m-1} u^A\right), \quad Y_1 := \text{in}_{L,u,y_1}(y_1),$$

$$\text{in}_{L,u,y_1}(h) = \sum_{A,b,L(A)+l(h,u,y)b=l(h,u,y)m} \text{in}_{L,u,y_1}(D_{A,b}) Y_1^b (\text{in}_{L,u,y_1}(u^A)), \quad (7)$$

$$\text{in}_{L,u,y_1}(h) = \sum_{A,b,L(A)+l(h,u,y)b=l(h,u,y)m} \text{in}_{L,u,y_1}(C_{A,b}) \left(Y_1 - \text{in}_{L,u,y_1}\left(\frac{1}{m} C_{0,m}^{-1} \sum_A C_{A,m-1} u^A\right)\right)^b, \quad (8)$$

where (7) is the expansion of (8). In (7), the terms with  $b = m - 1$  are all zero; in (6),  $D_{A,m-1} \neq 0$  implies  $L(A) + l(h, u, y_1)b > ml(h, u, y_1)$ . By the claim,  $l(h, u, y_1) = 1$ : a contradiction, hence

$$l(h, u, y_1) > l(h, u, y).$$

Note that (5) is independent of the linear form  $L$ , so

$$l_i(h, u, y_1) > l_i(h, u, y), \quad \text{for all } i \quad 1 \leq i \leq n.$$

By induction on the  $l_i(h, u, y)$ 's, we get (1). If (1) holds but not (2) for  $h, u, y$ , we make the change of variable (3) and get (2).

**PROPOSITION II.5.2** *With notations as above, assume furthermore that  $(z, u)$  and  $(z', u')$  satisfy (1) of the previous theorem. Then  $\delta(h, u, z) = \delta(h, u', z')$ .*

*Proof.* This is obvious if  $u_i = u'_i$ ,  $1 \leq i \leq 3$ . On the other hand, the condition

$$\text{in}_\delta(h) \neq \lambda(Z - \Phi(U_1, U_2, U_3))^m \quad \text{for every } \Phi \in k(x)[U_1, U_2, U_3] \quad (9)$$

is preserved if  $z = z'$  since  $\delta(h, u, z) > 1$  (remark II.4). In particular we have

$$\delta(h, u, z) = \delta(h, u', z) \leq \delta(h, u', z')$$

and we conclude by symmetry that  $\delta(h, u, z) = \delta(h, u', z')$ .

**DEFINITION II.5.3** *When  $z \in R$  is such that (1) (resp. (2) holds), we say that  $(z, u)$  is prepared (resp.  $\delta$ -prepared). If both of (1) and (2) hold for  $z \in R$ , we say that  $(z, u)$  is fully prepared. If there is no ambiguity on  $u = (u_1, u_2, u_3)$ , we simply say that  $z$  is prepared,  $\delta$ -prepared, or fully-prepared.*

*If  $(z, u)$  is prepared, the invariant  $\delta(h, u, z)$  will be henceforth denoted by  $\delta(x)$ .*

**THEOREM II.6** *Let  $(y, u_1, u_2, u_3) =: (y, u)$  be as before and  $E$  be a normal crossings divisor  $E \subseteq \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$ . For a component  $\text{div}(u_i)$  of  $E$ , we define*

$$d_i(u, y) := \inf\{x_i \mid (x_1, x_2, x_3) \in \Delta(h, u, y)\},$$



$$d_i(u) := \inf\{x_i | (x_1, x_2, x_3) \in \Delta(h, u)\}.$$

(i) Let  $z \in R$  be such that  $(z, u) = (z, u_1, u_2, u_3)$  is prepared. Then

$$d_i(u) \geq 1 \Leftrightarrow V(z, u_i) \text{ is a permissible blowing up center of } \text{Spec}(R/(h)). \quad (1)$$

(ii) We have  $d_i(u) > 0$  if and only if  $Y_i := V(h, u_i)_{\text{red}}$  is a regular surface.

(iii) Assume that  $d_i(u) > 0$ . Then  $d_i(u) = \delta(\eta_i)$ , where  $\eta_i \in Z$  is the generic point of  $Y_i = V(h, u_i)_{\text{red}}$  as above; in particular,  $d_i(u)$  is independent of the choice of a prepared  $(z, u) = (z, u_1, u_2, u_3)$  containing  $u_i$ .

Proof of (ii). Take  $z \in R$  such that  $\Delta(h, u) = \Delta(h, u, z)$ , so  $h = z^m \text{mod}(u_i)$  iff  $d_i(u) > 0$  iff  $Y := V(z, u_i) = V(h, u_i)_{\text{red}}$ . We get

$$d_i(u) > 0 \Leftrightarrow (V(h, u_i))_{\text{red}} \text{ is regular at } x.$$

Proof of (i). In that hypersurface case,  $Y := V(z, u_i) \in \text{Spec}R$  permissible means  $h \in (z, u_i)^m$  which is equivalent to  $d_i(u) \geq 1$ .

Proof of (iii). Let  $\eta$  be the generic point of  $Y = V(h, u_i)_{\text{red}} = V(z, u_i)$ . The following equivalence is straightforward

$$d_i(u) = 1 \Leftrightarrow V(z, u_i) \text{ is a permissible center of } \text{Spec}(R/(h)) \text{ and } \tau(\eta) = 2.$$

If  $0 < d_i(u) < 1$ , we obviously have  $d_i(u) = d_i(u, y)$  for every r.s.p.  $(y, u_1, u_2, u_3)$  (with  $\langle \text{in}_x(h) \rangle = \langle Y^m \rangle$ ).

We turn to the case:

$$d_i(u) > 1 \Leftrightarrow V(z, u_i) \text{ is a permissible center of } \text{Spec}(R/(h)) \text{ and } \tau(\eta_i) = 1.$$

We claim that  $d_i(u) = \delta(\eta_i)$ . As  $V(z, u_i) = V(h, u_i)_{\text{red}}$ , this will prove the invariance of  $d_i(u)$ . We take  $i = 1$  and write  $\eta$  for  $\eta_1$  in the following lemma.

**LEMMA II.6.1** *Let  $(y, u_1, u_2, u_3)$  be as above and assume that  $d_1(u, y) > 1$ . Then*

$$d_1(u, y) = \delta(h, u_1, y), \quad (1)$$

where the right hand side is computed w.r.t. the datum  $(h) \subset R_{(y, u_1)}$ . If furthermore  $(y, u)$  is prepared, then

$$d_1(u) = \delta(\eta). \quad (2)$$

Proof. By II.1 (1), we have a finite expansion

$$h = \sum_{A, b} C_{A, b} y^b u^A, \quad C_{A, b} \in R^\times, \quad b \in \mathbb{N}, \quad A \in \mathbb{N}^3$$

that we rewrite as:

$$h = \sum_{a_1, b} \left( \sum_{a_2, a_3} C_{A, b} u_2^{a_2} u_3^{a_3} \right) y^b u_1^{a_1}, \quad A = (a_1, a_2, a_3) \in \mathbb{N}^3.$$

As  $d_1(u, y) > 1$ , we have  $b + a_1 > m$  for all  $b < m$ . Note that every  $\sum_{a_2, a_3} C_{A, b} u_2^{a_2} u_3^{a_3}$  is invertible in  $R_{(y, u_1)} = \mathcal{O}_{Z, \eta}$ . Then  $\delta(h, u_1, y) = \inf\{\frac{a_1}{m-b} | b < m\} = d_1(u, y)$ : this is (1).

Now suppose that  $y, u_1, u_2, u_3$  is prepared. We claim that  $y, u_1$  is prepared w.r.t. the datum  $(h) \subset R_{(y, u_1)}$ . If not, then

$$\text{in}_\delta(h) = \sum_{a_1 + b d_1 = d_1 m} \left( \sum_{a_2, a_3} \overline{C_{A, b} u_2^{a_2} u_3^{a_3}} \right) Y^b u_1^{a_1} \in \text{gr}_\delta(R_{(y, u_1)}),$$

is proportional to an  $m^{\text{th}}$ -power. So there exists some  $a_1$  with

$$a_1 + (m-1)d_1 = d_1 m, \text{ and } \sum_{a_2, a_3} C_{A, b} u_2^{a_2} u_3^{a_3} \neq 0,$$

so there exists some  $A$  with  $a_1 + (m-1)d_1 = d_1 m$  and  $C_{A, m-1} \neq 0$ . Then, as in **II.5(5)**, we change the variable  $y$  by  $y_1 = y + \frac{1}{m} C_{0, m}^{-1} \sum_A C_{A, m-1} u^A \in R$ , we get a new expansion

$$h = \sum_{A, b} D_{A, b} y_1^b u^A, \quad D_{A, b} \in R^\times \cup \{0\}, \quad b \in \mathbb{N}, \quad A \in \mathbb{N}^3.$$

Now  $D_{A, b} \neq 0$  implies  $\frac{L_i(A)}{L_i(h, u, y)} + b \geq m$  for each linear form  $L_i$  such that

$$\Delta(h, u) = \{(x_1, x_2, x_3) | L_i(x_1, x_2, x_3) \geq 1, 1 \leq i \leq n\}.$$

This holds in particular for the linear form  $L(x_1, x_2, x_3) = \frac{1}{d_1} x_1$ . Since  $y$  is supposed to be nonprepared for  $u_1$ , the unique vertex  $d_1(u_1, y)$  of  $\Delta(h, u_1, y) \subset \mathbb{R}^+$  does not belong to  $\Delta(h, u_1, y_1)$ . We get  $d_1(u_1, y_1) > d_1(u, y)$ , a contradiction with the fact that  $\Delta(h, u, y)$  was minimal.

**DEFINITION and NOTATION II.7** *Given  $(y, u_1, u_2, u_3) =: (y, u)$ ,  $h$  reduced, with assumptions **II.1(2)(3)** and a normal crossings divisor  $E \subset \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$ , we let  $d_i := d_i(u)$  for each irreducible component  $\text{div}(u_i)$  of  $E$ . We let  $d_i := 0$  whenever  $\text{div}(u_i)$  is not an irreducible component of  $E$ .*

*We define  $\epsilon(x, E) \in \mathbb{Q}_{\geq 0}$  (or  $\epsilon(x)$  for short) by:*

$$\epsilon(x, E) = \delta(x) - \sum_{\text{div}(u_i) \subset E} d_i.$$

These invariants appear in [CP2] Ch.1 (II.3.3) in an equal characteristic context. The following remarks are obvious from the definitions.

**REMARK II.8** *We have*

- (i)  $\epsilon(x, E) \in \frac{1}{m!} \mathbb{N}$ ,
- (ii) if  $\epsilon(x, E) = 0$ ,  $\Delta(h, u)$  has only one vertex: the point  $v = (d_1, d_2, d_3)$ .

**THEOREM II.9** *Given  $(y, u_1, u_2, u_3) =: (y, u)$  and a divisor  $E \subset \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$  as above, assume that  $\epsilon(x, E) = 0$ . There exists a finite sequence of local blowing ups*

$$(X, x) =: (X_0, x_0) \longleftarrow (X_1, x_1) \longleftarrow \cdots \longleftarrow (X_n, x_n),$$

*where  $x_0 = x$ ,  $x_i \in X_i$ ,  $0 \leq i \leq n$  is the center of  $v$ , each blowing up center  $Y_i \subset X_i$  is permissible in Hironaka's sense, such that  $m(x_n) < m(x)$ .*

Proof. See the connection with [CP2] Ch.1 (II.4.6). Let  $z \in R$  be such that  $\Delta(h, u, z) = \Delta(h, u)$ . Then  $\delta(x) = \sum_{1 \leq i \leq 3} d_i$ . Let  $\mathcal{I} \subset \{1, 2, 3\}$  satisfy the two following conditions:

P (for permissibility):  $\sum_{i \in \mathcal{I}} d_i \geq 1$ ,

M (for maximality):  $|\mathcal{I}|$  minimal for P, i.e. the dimension of  $V(z, \langle u_i, i \in \mathcal{I} \rangle)$  is maximal for P.

Note that  $\mathcal{I} \subset \{1, 2, 3\}$  is not unique in general. Then we choose  $\mathcal{I}$  with PM and we blow up  $Z$  along  $V(z, \langle u_i, i \in \mathcal{I} \rangle)$ . Let  $e : Z' \rightarrow Z$  denote this blowing up,  $X'$  be the strict transform of  $X$ ,  $x' \in X'$  be a point above  $x$ ,  $E' \subset Z'$  be the reduced inverse image of  $E$ .

We claim that for either  $(m(x'), -\tau(x')) \prec_{lex} (m(x), -\tau(x))$  or

$$((m(x'), \tau(x')) = (m(x), \tau(x)) = (m(x), 1) \text{ and } \epsilon(x') = 0 \text{ and } \delta(x') < \delta(x)).$$

Since  $\delta(h, u, y) \in \frac{1}{m!}\mathbb{N}$ , a descending induction on  $\delta(x)$  reduces to  $m(x') < m(x)$  or  $(m(x') = m(x), \tau(x') \geq 2)$ . As stated in section I, this completes the proof.

*Proof of the claim.* We only treat the case  $\mathcal{I} = \{1, 2, 3\}$ , the other cases being similar, if somewhat simpler. By PM,  $\mathcal{I} = \{1, 2, 3\}$  means

$$d_i > 0, \quad d_i + d_j < 1 \text{ when } i \neq j, \quad 1 \leq i, j \leq 3. \quad (1)$$

By [H3, thm.3, p.331], if  $m(x') = m(x)$ , then  $x'$  lies on the strict transform of  $z = 0$ . The variables  $u_1, u_2, u_3$  play symmetric roles; so after reordering, it can be assumed that  $x'$  belongs to the affine chart  $\text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ . Let

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1).$$

Let

$$h = C_{0,m}z^m + \sum_{\substack{|A| \\ \delta(x) + b \geq m}} C_{A,b}z^b u_1^{a_1} u_2^{a_2} u_3^{a_3},$$

be an expansion **II.1**(1) of  $h$  with  $(z, u)$  fully prepared (theorem **II.5** and definition **II.5.3**),

$$h' := h/u_1^m = C_{0,m}z'^m + \sum_{\substack{|A| \\ \delta(x) + b = m}} C_{A,b}z'^b u_1^{(m-b)(\delta(x)-1)} u_2'^{a_2} u_3'^{a_3} + h'_1 \quad (2)$$

where  $h'_1 \in I_{\delta(x)}'^+ := (z'^{m+1}, z'^b u_1^{(m-b)(\delta(x)-1)+1}, 0 \leq b \leq m)$ .

Since  $\epsilon(x) = 0$ ,  $C_{A,b}$  invertible in (2) implies  $a_2 = (m-b)d_2$ ,  $a_3 = (m-b)d_3$ . Note also that  $\delta(x) - 1 = d_1 + d_2 + d_3 - 1 < d_1$  by (1). As  $d_1 + d_2 < 1$  and  $d_1 + d_3 < 1$ ,  $m(x') = m(x)$  implies  $x' = (z', u'_1, u'_2, u'_3)$ . The coordinate change  $(z, u_1, u_2, u_3) \mapsto (z', u'_1, u'_2, u'_3)$  is a monomial substitution, so  $\Delta(h', u', z')$  is again minimal. With natural notations, we get  $(d'_1, d'_2, d'_3) = (\delta(x) - 1, d_2, d_3)$ ,  $\epsilon(x') = 0$  and

$$\delta(x') = d'_1 + d'_2 + d'_3 = \delta(x) - 1 + d_2 + d_3 < \delta(x)$$

provided  $\langle \text{in}_{x'}(h') \rangle = \langle Z'^2 \rangle$ , i.e.  $\tau(x') = \tau(x) = 1$ .

**THEOREM II.10** *Given  $(y, u_1, u_2, u_3) =: (y, u)$  and a divisor  $E \subset \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$  as above, assume that  $E$  is equicharacteristic  $p = \text{chark}(x)$  and*

$$\text{Sing}_{m(x)}(X) \subset E.$$

*Then the set  $\{y \in X \mid m(y) = m(x), \epsilon(y) > 0 \text{ and } \tau(y) = 1\}$  is locally closed.*

Proof. It is well known that the set

$$\{y \in X | m(y) = m(x), \tau(y) \geq 2\} \subseteq E$$

is locally closed. Suppose  $\epsilon(x) = 0$  for some closed point  $x \in E$ . We choose a r.s.p.  $(z, u_1, u_2, u_3)$  of  $R$  at  $x$  which is fully prepared. There is a finite expansion

$$h = C_{0,m}z^m + \sum_{\frac{|A|}{\delta(x)} \geq b} C_{A,b}z^{m-b}u_1^{a_1}u_2^{a_2}u_3^{a_3} \quad (1)$$

with each  $C_{A,b}$  invertible in  $R$  and  $a_i \geq bd_i$ ,  $i = 1, 2, 3$ . Since  $\epsilon(x) = 0$ , there exists  $C_{A,b}$  such that  $a_i = bd_i$ ,  $i = 1, 2, 3$ , with  $b \geq 2$  by full preparedness. Then the locus

$$\{y \in X | m(y) = m(x), \epsilon(y) = 0 \text{ or } \tau(y) \geq 2\}$$

contains the intersection of  $\text{Sing}_{m(x)}(X)$  with the complement of the hypersurface  $V(C_{A,b})$ .

**THEOREM II.11** *With assumptions as in II.10, assume furthermore  $\epsilon(x) > 0$  and let  $F$  be an irreducible component of  $E$  with  $x \in F$ . Then*

$$\dim(\{y \in X | m(y) = m(x), \epsilon(y) > 0 \text{ and } \tau(y) = 1\} \cap F) \leq 1.$$

Proof. Say  $\text{div}(u_1)$  is the given component. If  $d_1 < 1$ , then  $\dim(\{y \in X | m(y) = m(x)\} \cap F) \leq 1$  and the result is clear.

Assume now  $d_1 > 1$  and pick a fully prepared  $(y, u_1)$  w.r.t. to datum  $h \in R_{y,u_1}$ . There exists a nonempty Zariski open set  $\Omega \subseteq F$  such that for  $y \in \Omega$  there is an expansion

$$h = \gamma_0 z^m + \sum_{1 \leq i \leq m} \gamma_i z^{m-i} u_1^{a_i},$$

with  $a_i \geq id_1$ ,  $\gamma_i \in \mathcal{O}_{X,y}$  for  $i \geq 1$ . By definition of  $d_1$ , some  $i \geq 2$  satisfies ( $a_i = id_1$  and  $\gamma_i \notin (u_1)$ ). By full preparedness, we have  $\gamma_1 \in (u_1)$  if  $a_1 = d_1$ . Let  $\Omega'$  be the intersection of  $\Omega$  with complement of the proper closed subset  $V(\gamma_i)$ , so

$$\Omega' \subseteq \{y \in X | m(y) = m(x), \epsilon(y) = 0\}.$$

Assume finally  $d_1 = 1$ . The same construction now yields

$$\Omega' \subseteq \{y \in X | m(y) = m(x), \tau(y) \geq 2\}.$$

### III CONSTRUCTION OF THE DIVISOR $E$ .

In this section, we reach the assumptions of **II.10**. We show they are stable under a class of local permissible blowing ups which we will prove ahead are sufficient to prove theorem **0.3**. We stick to assumptions **II.4.1**.

**PROPOSITION III.1** *With assumptions as above, there exists a finite sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n),$$

where  $x_0 = x$ ,  $x_i \in X_i$  ( $X_i$  denoting the strict transform of  $X$ ),  $0 \leq i \leq n$ , is the center of  $v$ , each blowing up center  $Y_i \subset X_i$  is permissible for  $X_i$  in Hironaka's sense, such that one the following properties holds:

(i)  $(m(x_n), -\tau(x_n)) <_{lex} (m(x), -\tau(x))$ , or

(ii)  $(m(x_n), -\tau(x_n)) = (m(x), -\tau(x))$  and there exists a (reduced) normal crossings divisor  $E_n \subset (Z_n, x_n)$  of equicharacteristic  $p = \text{chark}(x_n) = \text{chark}(x)$  such that

$$C_{x_n}(E_n) \perp C_{x_n}(X_n)_{red}, \quad (1)$$

$$S_n := \text{Sing}_{m(x_n)} \subset E_n, \quad (2)$$

where  $C_{x_n}$  denotes the tangent cone and  $\text{Sing}_{m(x_n)}(X_n)$  is the stalk at  $x_n$  of the set of multiplicity  $m(x)$ .

*Proof.* We begin with the following lemma.

**LEMMA III.2** *With assumptions as above, assume furthermore that there exists a normal crossings divisor  $E \subset (Z, x)$  such that*

$$C_x(E) \perp C_x(X)_{red}. \quad (1)$$

Then for any local blowing up :

$$\pi : (Z', x') \longrightarrow (Z, x)$$

of center  $Y \subset X$ , permissible for  $(X, x)$  and at normal crossing with  $E$ , we have  $(m(x'), \tau(x')) \leq (m(x), \tau(x))$ , where  $x' \in X'$  is the center of  $v$ ; if equality holds, then

$$C_{x'}(E') \perp C_{x'}(X')_{red},$$

where  $E' := \pi^{-1}(E)_{red}$ ,  $X'$  the strict transform of  $X$ .

*Proof.* The normal crossing assumption implies that we can choose a r.s.p.  $(v_1, v_2, v_3, v_4)$  of  $R := \mathcal{O}_{Z, x}$  such that  $Y = V(v_1, \dots, v_e)$  and  $E \subseteq \text{div}(v_1 \cdots v_4)$ . By permissibility, we have  $h \in (v_1, \dots, v_e)^{m(x)}$ . Assumption (1) means that  $\langle \text{in}_x(h) \rangle = \langle Z^d \rangle$ , where  $Z \notin \langle \text{in}_x(v_j), \text{div}(u_j) \subseteq E \rangle$ . Changing generators of the ideal of  $Y$ , we relabel parameters as  $(z, u_1, u_2, u_3)$  with

$$E = \text{div}(u_1 \cdots u_d) \subset \text{div}(u_1 u_2 u_3), \quad I(Y) = (z, \{u_i, i \in A\}) \text{ for some } A \subseteq \{1, 2, 3\}. \quad (2)$$

If  $m(x') = m(x)$ ,  $x'$  belongs to the strict transform of  $\text{div}(z)$  by (1). Let  $i \in A$  such that  $u_i$  generates the ideal of the exceptional divisor of  $\pi$  in a neighbourhood of  $x'$  and let  $z' = z/u_i$ . A local equation for  $(X', x')$  is  $h' := h/u_i^{m(x)}$ , where

$$h' \equiv \gamma z'^{m(x)} \pmod{(u_i, \{u_k, k \notin A\})},$$

and

$$E' \subset \text{div}(u_i \times \prod_{j \neq i, j \in A} \left(\frac{u_j}{u_i}\right) \prod_{k \notin A} u_k),$$

which proves the lemma.

*Proof of III.1.* If  $\text{Sing}_{m(x)}(X) = \{x\}$ , take  $E = \text{div}(u_1 u_2 u_3)$  with coordinates as in (2) above.

If  $\dim(\text{Sing}_{m(x)}(X)) \geq 1$ , then any regular closed set  $Y \subset S := S_0$  is permissible for  $X$ . In any case, we have  $(m(x_n), -\tau(x_n)) \leq (m(x), -\tau(x))$  since centers are permissible.

Let  $(C, x)$  be any curve contained in  $S$ . Since it is assumed that  $v$  has rank one,  $x_n$  does not belong to the strict transform  $C_n$  of  $C$  in  $X_n$  for  $n \gg 0$  if we take  $Y_i = \{x_i\}$ , the center of  $v$  in

$X_i$  for  $i \geq 0$ . In particular, it can be assumed that the strict transform  $S_n$  of  $S$  in  $X_n$  has pure dimension two. Take  $n = 0$  in what follows.

We now apply classical embedded resolution theorems for  $S$  with  $\dim(S) = 2$  ([CJS] for suitable generality). This involves blowing up closed points or regular curves on the successive strict transforms of  $S$ . By blowing up finitely closed many points as before, it can be assumed that every blown up curve is equicharacteristic  $p = \text{char}k(x)$ . We reach the following situation: the strict transform  $S_n$  of  $S$  at  $x_n$  is empty or an irreducible surface with normal crossings with the (equicharacteristic) reduced exceptional divisor  $E_n$  of  $(Z_n, x_n) \rightarrow (Z, x)$ . If  $S$  itself is equicharacteristic, enlarge  $E_n$  to  $E_n \cup S$ . Otherwise, we blow up finitely many times irreducible components of  $S \cap E_n$  (i.e. equicharacteristic curves) to get  $x_n \notin S_n$ . This is possible again because  $v$  has rank one.

#### IV REFINED DIRECTRIX, TRANSVERSENESS, ENCOMBREMENT

Assume that the conclusion of proposition **III.1** (ii) holds. We will perform local blowing ups which are permissible in Hironaka's sense, with center  $Y_n$  having normal crossing with  $E_n$ . Take  $n = 0$  in what follows,  $E = E_0$ , and consider a local blowing up :

$$\pi : (Z', x') \longrightarrow (Z, x)$$

of center  $Y \subset X$ , permissible for  $(X, x)$  and at normal crossing with  $E$ . We assume that

$$(m(x'), \tau(x')) = (m(x), \tau(x)),$$

where  $x' \in X'$  is the center of  $v$ . By lemma **III.2**, we have

$$C_{x'}(E') \perp C_{x'}(X')_{red},$$

where  $E' := \pi^{-1}(E)_{red}$ ,  $X'$  the strict transform of  $X$  in  $Z'$ .

**DEFINITION and NOTATION IV.1** *Let  $(z, u)$  be fully prepared with  $E \subset \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$  as above. Let*

$$F := \text{in}_\delta(h) = Z^m + \sum_{2 \leq j \leq m} Z^{m-j} F_j(U_1, U_2, U_3) \in \text{gr}_\delta(R) = k(x)[Z, U_1, U_2, U_3],$$

where  $Z = \text{in}_\delta(z)$ ,  $U_i = \text{in}_\delta(u_i)$ ,  $1 \leq i \leq 3$  (notations of **II.3.1**). Each  $F_j$  is zero or homogeneous of degree  $j\delta(x)$ ; we have  $F_j = 0$  if  $j\delta(x) \notin \mathbb{N}$ .

We define the refined tangent ideal of  $X$  at  $x$  as the ideal

$$I_x := (Z, \prod_{\text{div}(u_i) \subset E} U_i^{-m!d_i} F_j^{\frac{m!}{j}}, 1 \leq j \leq m) \subset k(x)[Z, U_1, U_2, U_3].$$

We define the refined directrix of  $X$  at  $x$  as the smallest vector subspace  $\mathcal{V} \subseteq \langle U_1, U_2, U_3 \rangle$  such that

$$\{U_i^{-m!d_i} F_j^{\frac{m!}{j}} \mid 1 \leq j \leq m, \text{div}(u_i) \subset E\} \subseteq k(x)[\mathcal{V}].$$

Let  $\rho(x) := \dim(\mathcal{V})$ .

**REMARK IV.1.1** *The following holds:*

- (i)  $U_i^{\lceil jd_i \rceil}$  divides  $F_j$ ,  $2 \leq j \leq m$ , for  $1 \leq i \leq 3$  such that  $\text{div}(u_i) \subset E$ .  
(ii)  $I_x$  and  $\mathcal{V}$  do not depend upon choices of  $z, u$  satisfying the assumptions.  
(iii) the polynomials

$$U_i^{-m!d_i} F_j^{\frac{m!}{j}}, \quad 1 \leq j \leq m, \quad 1 \leq i \leq 3$$

are zero or homogeneous of degree  $m!e(x)$ .

Statement (i) is a consequence of the definition of  $d_i$ . For (ii), suppose  $(z', u')$  is fully prepared, where  $E \subset \text{div}(u'_1 u'_2 u'_3)$ . Let

$$u_j = a_{1,j} u'_1 + a_{2,j} u'_2 + a_{3,j} u'_3 + b_j z, \quad a_{i,j}, b_j \in R, \quad 1 \leq i, j \leq 3,$$

for some matrix  $(a_{i,j}) \in \text{GL}(3, R)$ . Since  $\tau(x) = 1$ , we have

$$\deg U_j = \deg U'_j = \frac{1}{\delta(x)} < \deg Z = 1$$

in  $\text{gr}_\delta$ . Computing w.r.t. the r.s.p.  $(z, u')$ , we get

$$\text{in}_\delta(h) = Z^m + \sum_{2 \leq j \leq m} Z^{m-j} F'_j(U'_1, U'_2, U'_3) \in \text{gr}_\delta(R) = k(x)[Z, U'_1, U'_2, U'_3],$$

with  $F'_j(U'_1, U'_2, U'_3) = F_j(\overline{\mathcal{M}}.(U'_1, U'_2, U'_3))$ ,  $\overline{\mathcal{M}}$  being the residue of  $\mathcal{M}$  in  $\text{GL}(3, k(x))$ . Since  $(z', u')$  is fully prepared, no term in  $Z^{m-1}$  occurs in  $\text{in}_\delta(h) \in k(x)[Z', U'_1, U'_2, U'_3]$  and this implies that  $\langle \text{in}_\delta(Z) \rangle = \langle \text{in}_\delta(Z') \rangle$ .

Statement (iii) immediately follows from definition **II.7**.

**DEFINITION IV.2** Let  $E$  be a fixed normal crossings divisor and  $(z, u)$  be fully prepared (always with the condition  $E \subset \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$ ) as above.

We call “transverseness” index of  $x$ , denoted by  $t(x)$ , the maximal dimension of a subspace of  $\mathcal{V}$  which is transverse to  $\langle U_i, \text{div}(u_i) \subset E \rangle$ . This is independent of the choice of a fully prepared r.s.p.  $(z, u)$  by remark **IV.1.1(ii)**.

We call “encombrement” of  $x$ , denoted by  $e(x)$ , the minimum number of  $U_i$ ’s among all possible fully prepared  $(z, u)$  necessary to write a basis of  $\mathcal{V}$ .

We define an invariant

$$\iota(x) := (m(x), -\tau(x), \epsilon(x), -\rho(x), -t(x), -e(x)) \in \mathbb{N} \times \{-4, -3, -2, -1\} \times \mathbb{Q}_{\geq 0} \times \{-3, -2, -1, 0\}^3.$$

For convenience, we extend the definition when  $\tau(x) \geq 2$  by letting  $\iota(x) := (m(x), -\tau(x), 0, 0, 0)$ , theorem **0.3** being already proved in this special case (section I). Note that  $\epsilon(x) \in \frac{1}{m(x)!} \mathbb{N}$ , so any decreasing sequence of values (for the lexicographical ordering) taken by  $\iota$  is finite.

**EXAMPLES IV.2.1** Assume  $p \geq 5$ .

(i)  $h = z^3 + u_1^2(u_1 + u_2 + u_3)^2$ ,  $E = \text{div}(u_1)$ .

Then  $\delta(x) = \frac{4}{3}$ ,  $d_1(x) = \frac{2}{3}$ ,  $\mathcal{V} = \langle U_1 + U_2 + U_3 \rangle$ ,  $\rho(x) = 1$ ,  $t(x) = 1$ ,  $e(x) = 1$ : take the r.s.p.  $(z, u_1, u_1 + u_2 + u_3, u_3)$ .

(ii)  $h = z^3 + u_1^2(u_1 + u_2 + u_3)^2$ ,  $E = \text{div}(u_1 u_2 u_3)$ .

Then  $\delta(x) = \frac{4}{3}$ ,  $d_1(x) = \frac{2}{3}$ ,  $d_2(x) = d_3(x) = 0$ ,  $\mathcal{V} = \langle U_1 + U_2 + U_3 \rangle$ ,  $\rho(x) = 1$ ,  $t(x) = 0$ ,  $e(x) = 3$ : the only choice allowed upon  $(u_1, u_2, u_3)$  is permuting or multiplying by a unit in  $R$ .

(iii)  $h = z^3 + u_1^2(u_1^2 + u_2^2)$ ,  $E = \text{div}(u_1)$ .

Then  $\delta(x) = \frac{4}{3}$ ,  $d_1(x) = \frac{2}{3}$ ,  $\mathcal{V} = \langle U_1, U_2 \rangle$ ,  $\rho(x) = 2$ ,  $t(x) = 1$ ,  $e(x) = 2$ .

(iv)  $h = z^3 + u_1^2(u_1^2 + u_2^2)$ ,  $E = \text{div}(u_1 u_2)$ .

Then  $\delta(x) = \frac{4}{3}$ ,  $d_1(x) = \frac{2}{3}$ ,  $d_2(x) = 0$ ,  $\mathcal{V} = \langle U_1, U_2 \rangle$ ,  $\rho(x) = 2$ ,  $t(x) = 0$ ,  $e(x) = 2$ .

**REMARK IV.2.2** The French “encombrement” was proposed by J. Giraud twenty years ago (English: “cumbersomeness index” roughly).

## V PERMISSIBLE BLOWING UPS, BEHAVIOUR OF THE INVARIANTS

We stick to the assumptions of the previous section and assume furthermore that  $\epsilon(x) > 0$ .

**DEFINITION V.1** An  $\epsilon$ -permissible center (permissible center for short)  $Y$  at  $x$  is one of the following:

- (i) either  $Y := \{x\} = V(z, u_1, u_2, u_3)$ ,
- (ii) or  $Y := V(z, u_1, u_2)$  with  $(z, u)$  fully prepared,

$$d_1 + d_2 + \epsilon(x) \geq 1 \tag{1}$$

and

$$l(h, u, z) = m, \tag{2}$$

where  $L$  denotes the linear form  $L(x_1, x_2, x_3) = \frac{x_1 + x_2}{d_1 + d_2 + \epsilon(x)}$  (definition **II.2**).

**PROPOSITION V.2** An  $\epsilon$ -permissible center at  $x$  is permissible in Hironaka’s sense.

*Proof.* Indeed, we have just to look at the case of a curve  $V(z, u_1, u_2)$ . In that latter case, as  $d_1 + d_2 + \epsilon(x) \geq 1$ , we have

$$\text{ord}_\eta(h) \geq v_{L, h, u, z}(h) = m,$$

where  $\eta$  is the generic point of  $V(z, u_1, u_2)$ , so

$$\text{ord}_\eta(h) = m,$$

which means exactly that  $V(z, u_1, u_2)$  is permissible in Hironaka’s sense.

**PROPOSITION V.3** Let  $\pi : Z' \rightarrow Z$  be the blowing up along an  $\epsilon$ -permissible center  $Y$  at  $x$ ,  $X'$  be the strict transform of  $X$  (with transformed equation  $h'$  at the center  $x' \in X'$  of  $v$ ). We have:

(i)  $\iota(x') \leq \iota(x)$  (definition **IV.2**). If equality holds (in which case we say that  $x'$  is “very near”  $x$ ), then  $E' := \pi^{-1}(E)_{\text{red}}$  is transverse to the directrix  $T'$  of  $X'$  at  $x'$ .

(ii) if  $Y = \{x\}$  and  $(m(x'), -\tau(x'), \epsilon(x')) = (m(x), -\tau(x), \epsilon(x))$ , then  $x'$  lies on

$$\text{Proj}_k(x)[Z, U_1, U_2, U_3]/(Z, \mathcal{V}) \subset \text{Proj}(\text{gr}_{\mathfrak{M}}(R)) = \mathbb{P}_{k(x)}^3$$

with notations as in **IV.1**. The refined directrix  $\mathcal{V}'$  at  $x'$  satisfies  $\mathcal{V}' \equiv U^{-1}\mathcal{V} \pmod{\langle U \rangle}$  where  $U = \text{in}_{x'}(u)$ ,  $u$  an equation of the exceptional divisor of  $\pi$ .

*Proof.* First assume that  $Y = \{x\}$ . By [H3, thm.3, p.331], if  $m(x') = m(x)$ , then  $x'$  lies on the strict transform of  $z = 0$ . The variables  $u_1, u_2, u_3$  play symmetric roles; so after reordering, it can be assumed that  $x'$  belongs to the affine chart  $\text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ . Let

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_2/u_1, u_3/u_1).$$



We can choose a r.s.p. at  $x'$  in the following way: if  $x'$  is the origin, take  $(z', u'_1, u'_2, u'_3)$ ; if  $x'$  belongs to the strict transform of, say  $\text{div}(u_2)$ , we can take  $(z', u'_1, u'_2, v_3)$  with  $v_3 = \sum_a \lambda_a u'_3{}^a$ ,  $\lambda_a \in R$  a unit or zero (the sum is finite) whose residue  $\sum_a \overline{\lambda_a} U_3^a \in k(x)[U_3]$  is an irreducible polynomial; in the general case  $u'_2(x')u'_3(x') \neq 0$ , we take  $(z', u'_1, v_2, v_3)$  where  $v_c = \sum_{a,b} \lambda_{a,b,c} u'_2{}^a u'_3{}^b$ ,  $c = 2, 3$  (sums are finite),  $\lambda_{a,b,c} \in R$  a unit or zero, and

$$\langle \sum_{a,b} \overline{\lambda_{a,b,c}} U_2^a U_3^b, c = 1, 2 \rangle \subset k(x)[U_2, U_3]$$

is a maximal ideal. Let

$$F := Z^m + \sum_{2 \leq j \leq m} Z^{m-j} U_1^{a(1,j)} U_2^{a(2,j)} U_3^{a(3,j)} G_j(U_1, U_2, U_3) :=$$

$$Z^m + \sum_{2 \leq j \leq m} Z^{m-j} F_j(U_1, U_2, U_3) := \text{in}_\delta(h) \in \text{gr}_\delta(R) = k(x)[Z, U_1, U_2, U_3], \quad (1)$$

with  $a(i, j) \geq jd_i$ ,  $2 \leq j \leq m$ ,  $1 \leq i \leq 3$ ,  $G_j \in k(x)[U_1, U_2, U_3]$  homogeneous,  $G_j = 0$  or  $\deg(G_j) = j\delta(x) - (a(1, j) + a(2, j) + a(3, j))$  and  $G_j$  not divisible by  $U_i$ ,  $1 \leq i \leq 3$ .

Let  $h' := h/u_1^m$  define the strict transform of  $h$ . We define the linear form

$$L'(x'_1, x'_2, x'_3) := \frac{x'_1}{\delta(x) - 1}$$

with associated valuation  $v' := v_{L', h', u', z'}$  (definition **II.2**). We have

$$\text{in}_{v'}(h') = Z'^m + \sum_{2 \leq j \leq m} Z'^{m-j} U_1^{j(\delta(x)-1)} u'_2{}^{a(2,j)} u'_3{}^{a(3,j)} G_j(1, u'_2, u'_3) \in \text{gr}_{v'}(R'), \quad (2)$$

where  $\text{gr}_{v'}(R') = R'/(z', u'_1)[Z', U_1]$ . Here, the meaning of  $G_j(1, u'_2, u'_3)$  is given by the inclusion

$$k(x) = R/\mathfrak{M} \rightarrow R'/(u'_1) \rightarrow R'/(z', u'_1).$$

By (2),  $x'_1 = \delta(x) - 1$  is the minimum value of the first coordinate of points in  $\Delta(h', u'_1, v_2, v_3, z')$ . Since  $z$  is  $\delta$ -prepared, no vertex of  $\Delta(h', u'_1, v_2, v_3, z')$  with first coordinate equal to  $x'_1 = \delta(x) - 1$  is solvable. We get

$$d_1(x') = \delta(x) - 1,$$

and for at least one vertex  $(x'_1 = \delta(x) - 1, x'_2, x'_3)$ , we have

$$x'_2 + x'_3 \leq \min\left\{\frac{\text{ord}_{x'}(u'_2{}^{a(2,j)} u'_3{}^{a(3,j)} G_j(1, u'_2, u'_3))}{j}, 2 \leq j \leq m\right\}.$$

In case  $x'$  belongs to the strict transform of some  $\text{div}(u_i)$ ,  $i = 2, 3$ , we have  $d_i(x') = d_i(x)$  for  $u'_i(x') = 0$  by **II.6**(iii). This leads to:

$$\epsilon(x') \leq \min\left\{\frac{\text{ord}_{x'}(u'_2{}^{a(2,j)-jd_2} u'_3{}^{a(3,j)-jd_3} G_j(1, u'_2, u'_3))}{j}, 2 \leq j \leq m\right\} \leq \epsilon(x) \quad (3)$$

with the convention:  $\text{ord}_{x'}(u_i^a) = 0$  when  $a \in \mathbb{Q}_+$  and  $u'_i(x') \neq 0$ ,  $\text{ord}_{x'}(u_i^a) = a$  when  $u'_i(x') = 0$ . This proves  $(-\tau(x'), \epsilon(x')) \leq (-\tau(x), \epsilon(x))$ .

Assume that  $(-\tau(x'), \epsilon(x')) = (-\tau(x), \epsilon(x))$ . Then

$$\text{ord}_{x'} \left( \prod_{\text{div}(u_i) \subset E} u_i^{-m!(a(i,j)-jd_i)} F_j(1, u'_2, u'_3)^{\frac{m!}{j}} \right) = \text{deg} \left( \prod_{\text{div}(u_i) \subset E} U_i^{-m!(a(i,j)-jd_i)} F_j^{\frac{m!}{j}} \right)$$

for each  $j$  with  $F_j \neq 0$ . By [H3, Theorems 2 and 3], this means that  $x'$  lies on

$$\text{Proj}k(x)[Z, U_1, U_2, U_3]/(Z, \mathcal{V}) \subset \text{Proj}k(x)[Z, U_1, U_2, U_3] = \mathbb{P}_{k(x)}^3. \quad (4)$$

This proves the first assertion of (ii) in this case. All other assertions are easy consequences of (3) and of its explicitation (4).

Assume now that  $u'_2(x')u'_3(x') \neq 0$ . If  $x'$  is rational over  $x$ , i.e.  $u'_2(x') = \lambda \in k(x)$ ,  $u'_3(x') = \mu \in k(x)$ , we have

$$I_{x'} \equiv (Z', G_j(1, V_2 - \lambda, V_3 - \mu)^{\frac{m!}{j}}, 2 \leq j \leq m) \text{ mod } (U'_1), \quad (5)$$

where  $I_{x'}$  is the refined tangent ideal of  $x'$  (cf. **IV.1(ii)**). This proves the last assertion of (ii) in this case. Finally, if  $x'$  is not rational over  $x$ , then  $\dim(\mathcal{V}) = 1$ . We get

$$\mathcal{V} = \langle aU_1 + bU_2 + cU_3 \rangle, \quad a, b, c \in k(x), \quad (b, c) \neq (0, 0).$$

If  $b \neq 0$ , we take  $v_2 := a + bu'_2 + cu'_3 \text{ mod } (u_1)$  and we get by **IV.1.1**

$$I_{x'} = (Z', V_2^{m!\epsilon(x)}) \text{ mod } (U'_1), \quad (6)$$

which proves the last assertion of (ii) in this case. All other assertions are easy as in the previous case.

We now consider blowing up along a curve  $Y = V(z, u_1, u_2)$ .

By [H3, thm.3, p.331], if  $m(x') = m(x)$ , then  $x'$  lies on the strict transform of  $z = 0$ . The variables  $u_1, u_2$  play symmetric roles; so after reordering, it can be assumed that  $x'$  belongs to the affine chart  $\text{Spec}R[z/u_1, u_2/u_1, u_3] \subset Z'$ . Let

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3).$$

We can choose a r.s.p. at  $x'$  in the following way: if  $x'$  is the origin, take  $(z', u'_1, u'_2, u'_3)$ ; otherwise take  $(z', u'_1, v_2, u'_3)$  where  $v_2 = \sum_a \lambda_a u'_3{}^a$ ,  $\lambda_a \in R$  a unit or zero (the sum is finite) whose residue  $\sum_a \bar{\lambda}_a U_3{}^a \in k(x)[U_3]$  is an irreducible polynomial.

With notations (1), since  $V(z, u_1, u_2)$  is  $\epsilon$ -permissible, we have

$$a(3, j) = jd_3, \quad G_j \in k(x)[U_1, U_2].$$

Let  $h' := h/u_1^m$  define the strict transform of  $h$ . Equation (2) gets replaced by

$$\text{in}_{v'}(h') = \Gamma_0 Z'^m + \sum_{2 \leq j \leq m} Z'^{m-j} U_1^{j(\delta(x)-1-d_3)} u_2^{a_2} u_3^{jd_3} \Gamma_j \in \text{gr}_{v'}(R') = R'/(z', u'_1)[Z', U'_1], \quad (2^*)$$

with  $\Gamma_j \in R'/(z', u'_1)$  whose residue in  $R'/(z', u'_1, u'_3)$  is  $G_j(1, u'_2)$  for  $2 \leq j \leq m$ ,  $\Gamma_0$  a unit.

By (2\*),  $\delta(x) - 1 - d_3$  is the minimum value of the first coordinate of points in  $\Delta(h', u'_1, v_2, u'_3, z')$ . As in (1) there is no  $Z'^{m-1}$ , each vertex of  $\Delta(h', u'_1, v_2, u'_3, z')$  with first coordinate equal to  $\delta(x) - 1 - d_3$  is not solvable. Then

$$d_1(x') = \delta(x) - 1 - d_3,$$

and for at least one vertex  $(x'_1 = \delta(x) - 1 - d_3, x'_2, x'_3)$ , we have

$$x'_2 + x'_3 \leq \min\left\{\frac{\text{ord}_{x'}(u_2^{a_2} u_3^{a_3} G_j(1, u'_2))}{j}, 2 \leq j \leq m\right\}.$$

By theorem **II.6**(iii), we have  $d_3(x') = d_3$ . If  $u'_2(x') \neq 0$ , this gives

$$\epsilon(x') \leq \min\left\{\frac{\text{ord}_{x'}(u_3^{a_3 - d_3} G_j(1, u'_2))}{j}, 2 \leq j \leq m\right\} \leq \epsilon(x).$$

If  $u'_2(x') = 0$ , we also have  $d_2(x') = d_2$  by **II.6**(iii). This leads to:

$$\epsilon(x') \leq \text{inh}\left\{\frac{\text{ord}_{x'}(u_2^{a_2 - d_2} u_3^{a_3 - d_3} G_j(1, u'_2))}{j}, 2 \leq j \leq m\right\} \leq \epsilon(x)$$

with the convention:  $\text{ord}_{x'}(u_i^a) = 0$  when  $a \in \mathbb{Q}^+$  and  $u_i(x') \neq 0$ ,  $\text{ord}_{x'}(u_i^a) = a$  when  $u_i(x') = 0$ .

If  $\epsilon(x') = \epsilon(x)$ , then

$$\text{ord}_{x'} \left( \prod_{\text{div}(u_i) \subset E} u_i^{-m!(a(i,j) - jd_i)} G_j(1, u'_2)^{\frac{m!}{j}} \right) = \text{deg} \left( \prod_{\text{div}(u_i) \subset E} U_i^{-m!(a(i,j) - jd_i)} G_j^{\frac{m!}{j}} \right)$$

for each  $j$  with  $G_j \neq 0$ . By [H3, Theorems 2 and 3], this means that  $x'$  lies on

$$\text{Proj}k(x)[Z, U_1, U_2]/(Z, \mathcal{V}) \subset \text{Proj}k(x)[Z, U_1, U_2], \quad (4*)$$

where the latter is identified with  $\pi^{-1}(x) \subset Z'$ . The proof now runs parallel to the case  $Y = \{x\}$ .

**PROPOSITION V.4** *With assumption as in V.3, assume  $e(x) = 3$  and  $Y = \{x\}$ . Then  $x'$  is not very near  $x$ , i.e.  $\iota(x') < \iota(x)$  (definition **IV.2**).*

*Proof.* If  $\rho(x) = 3$ , this follows from **V.3**(ii), since  $\text{Proj}k(x)[Z, U_1, U_2, U_3]/(Z, \mathcal{V}) = \emptyset$ .

When  $\rho(x) = 2$ , we have  $t(x) < 2$  necessarily: otherwise we should have

$$E \subseteq \text{div}(u_1), \mathcal{V} = \langle U_2, U_3 \rangle \text{ mod } (U_1).$$

By a linear change on the free variables  $(u_2, u_3)$ , we would get

$$\mathcal{V} = \langle U_2, U_3 \rangle,$$

i.e.  $e(x) = 2$ , a contradiction.

When  $\rho(x) = 2$ ,  $t(x) = 1$ , we can choose parameters such that

$$E = \text{div}(u_1 u_2), \mathcal{V} = \langle U_3, \alpha U_1 + U_2 \rangle, \alpha \in k(x)^\times.$$

By proposition **V.3**(ii), we have

$$\mathcal{V}' \equiv \langle U'_3, V'_2 \rangle \text{ mod } (U'_1),$$

with  $E' = \text{div}(u'_1)$ ,  $V'_2 = \text{in}_{x'}(a + u_2/u_1)$ , where  $a \in R$  is a preimage of  $\alpha$ . Then  $\rho(x') \geq t(x') \geq 2$ .

When  $\rho(x) = 2$ ,  $t(x) = 0$ , then, up to a permutation on  $u_1, u_2, u_3$ , we have  $E = \text{div}(u_1 u_2 u_3)$  and

$$\mathcal{V} = \langle U_1 + \alpha U_2, \beta U_2 + U_3, \rangle, \quad \alpha \in k(x)^\times, \quad \beta \in k(x).$$

By proposition **V.3**(ii), we can take  $\pi^{-1}(x) = \text{div}(u_2)$  locally at  $x'$ , and r.s.p.

$$(z', v'_1, u'_2, v'_3) := (z/u_2, u_1/u_2 + a, u_2, u_3/u_2 + b),$$

where  $a, b \in R$  are preimages of  $\alpha, \beta$ . In particular we get  $E \subseteq \text{div}(u'_2 u'_3)$ . On the other hand, we have

$$\mathcal{V}' \equiv \langle V'_1, V'_3 \rangle \pmod{(U'_2)},$$

and this proves that  $t(x') \geq 1$ .

When  $\rho(x) = 1$ , then  $e(x) = 3$  implies  $E = \text{div}(u_1 u_2 u_3)$  (so  $t(x) = 0$ ) and

$$\mathcal{V} = \langle \alpha U_1 + \beta U_2 + U_3, \rangle, \quad \alpha, \beta \in k(x)^\times$$

up to renumbering parameters. By proposition **V.3**(ii), we can choose  $E' \subseteq \text{div}(u'_1 u'_2)$ , say  $\pi^{-1}(x) = \text{div}(u_1)$  locally at  $x'$  and r.s.p.

$$(z', u'_1, v'_2, v'_3) := (z/u_1, u_1, v'_2, a + b u_2/u_1 + u_3/u_1),$$

at  $x'$ , where  $a, b \in R$  are preimages of  $\alpha, \beta$ . Since  $V'_3 \in \mathcal{V}' \oplus \langle U_1 \rangle$ , we get  $t(x') \geq 1$ .

**PROPOSITION V.5** *Let  $x$  satisfy the conclusion of proposition **III.1** (ii) and assume  $e(x) = 2$ . There exists a finite sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n), \quad (1)$$

where  $x_0 = x$ ,  $x_i \in X_i$  ( $X_i$  denoting the strict transform of  $X$ ),  $0 \leq i \leq n$ , is the center of  $v$ , each blowing up center is  $Y_i = \{x_i\}$  such that  $\iota(x') < \iota(x)$ .

*Proof.* First assume that  $\rho(x) = 1$ . Then  $t(x) = 0$ ,  $\text{div}(u_2 u_3) \subseteq E$  and we have  $\mathcal{V} = \langle \alpha U_2 + U_3, \rangle$ ,  $\alpha \in k(x)^\times$  after possibly renumbering parameters. If  $x'$  does not belong to the strict transform of  $\text{div}(u_2)$ , we can take  $\pi^{-1}(x) = \text{div}(u_2)$  locally at  $x'$ , and r.s.p.

$$(z', v'_1, u'_2, v'_3) := (z/u_2, v'_1, u_2, a + u_3/u_2),$$

where  $a \in R$  is a preimage of  $\alpha$ . In particular we get  $E' \subseteq \text{div}(u'_1 u'_2)$ , with  $u'_1 = u_1/u_2$ . On the other hand, we have

$$\mathcal{V}' \equiv \langle V'_3 \rangle \pmod{(U'_2)},$$

whence  $t(x') \geq 1$ , so  $\iota(x') < \iota(x)$ . Assume now that  $x'$  belongs to the strict transform of  $\text{div}(u_2)$ . We can take  $\pi^{-1}(x) = \text{div}(u_1)$  locally at  $x'$ , and r.s.p.

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1).$$

We get  $E = \text{div}(u'_1 u'_2 u'_3)$  and

$$\mathcal{V}' \equiv \langle \alpha U'_2 + U'_3 \rangle \pmod{(U'_1)}.$$

If  $\iota(x') = \iota(x)$ , then  $\mathcal{V}' = \langle \alpha U'_2 + U'_3 \rangle$  and iterate. Since the valuation  $v$  has rank one, say  $v(u_2) < n v(u_1)$  for some  $n > 0$ , the process stops after iterating  $n$  times.

Assume that  $\rho(x) = 2$ . Then  $\mathcal{V} = \langle U_2, U_3 \rangle$  after possibly renumbering parameters. We can take  $\pi^{-1}(x) = \text{div}(u_1)$  locally at  $x'$  and r.s.p.

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1).$$

We get  $\text{div}(u'_1) \subseteq E'$  and

$$\mathcal{V}' \equiv \langle U'_2, U'_3 \rangle \pmod{(U'_1)},$$

hence  $\langle U'_2 + \alpha_2 U'_1, U'_3 + \alpha_3 U'_1 \rangle \subseteq \mathcal{V}'$  for some  $\alpha_2, \alpha_3 \in k(x)$ . If  $\iota(x') = \iota(x)$ , then equality holds; moreover  $\alpha_i = 0$  whenever  $\text{div}(u_i) \subseteq E$ ,  $i = 2$  or  $i = 3$ .

Iterating, there exists a *regular* formal curve  $\widehat{Y} \subset X$  passing through all points  $x, x_1 := x', \dots, x_n$ , taking  $Y_i = \{x_i\}$  for each  $i \geq 0$ . By standard arguments,  $\widehat{Y} \subseteq \text{Sing}_{m(x)}(X)$ . Our assumptions (beginning of section IV) force  $\widehat{Y} \subset E$ , say  $\widehat{Y} \subset \text{div}(u_2)$ . One concludes as in the case  $\rho(x) = 1$ .

## VI PROOF OF THE MAIN THEOREM.

By theorem **II.9**, a reduction in  $m = m(x)$  can be achieved when  $\epsilon(x, E) = 0$  for some normal crossings divisor  $E \subseteq \text{div}(u_1 u_2 u_3)$ . The previous section (propositions **V.4** and **V.5**) reduces theorem **0.3** to the only case  $\epsilon(x) > 0$ ,  $e(x) = 1$ . There remains to prove the following:

**THEOREM VI.1** *Let  $x$  satisfy the conclusion of proposition **III.1(ii)** (w.r.t.  $E \subseteq \text{div}(u_1 u_2 u_3)$ ) and assume  $\epsilon(x) > 0$ ,  $e(x) = 1$ . There exists a finite sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n), \quad (1)$$

where  $x_0 = x$ ,  $x_i \in X_i$  ( $X_i$  denoting the strict transform of  $X$ ),  $0 \leq i \leq n$ , is the center of  $v$ , each blowing up center  $Y_i \subset X_i$  is permissible in Hironaka's sense, such that

(i)  $\iota(x_n) < \iota(x)$ , and

(ii)  $x_n$  satisfies the conclusion of proposition **III.1(ii)** (w.r.t. the strict transform  $E_n$  of  $E$  in  $Z_n$ ) if  $((m(x_n), \tau(x_n)) = (m(x), \tau(x))$  and  $\epsilon(x_n) > 0$ ).

The proof is long, needing new invariants and the control of their behavior under permissible blowing ups. There are two different cases:

- (i)  $t(x) = 0$ ,  $e(x) = 1$ ,
- (ii)  $t(x) = e(x) = 1$ .

In both cases, we choose the indices so that  $\mathcal{V} = \langle U_3 \rangle$ . We assume that

- (P1)  $(z, u)$  is fully prepared, and
- (P2)  $E \subseteq \text{div}(u_1 u_2 u_3)$ .

**VI.2** A new invariant  $B$ , preparation of the free variable (case (ii)).

Let us remind the convention  $d_i(x) = 0$  for  $\text{div}(u_i) \not\subseteq E$ ,  $1 \leq i \leq 3$ . In particular,  $d_3(x) = 0$  in case (ii). For  $B \in \mathbb{Q}_+$ , define the monomial valuation  $v_B$  by

$$v_B(z) = 1, \quad v_B(u_3) := \frac{1}{\epsilon(x) + d_3 + \frac{d_1 + d_2}{B}} =: Bv_B(u_1) = Bv_B(u_2).$$

We choose  $B \in \mathbb{N} \cup \{+\infty\}$  maximal such that, up to the multiplication by an element of  $k(x)^\times$ ,  $\text{in}_{v_B}(h)$  takes the following form:

$$\text{in}_{v_B}(h) = Z^m + \sum_{1 \leq j \leq m} Z^{m-j} \Phi_j(U_1, U_2, U_3),$$

with

$$\deg_{U_3}(\Phi_j) \leq j(d_3 + \epsilon(x)), \quad 1 \leq j \leq m,$$

$$\Phi_j(U_1, U_2, U_3) = U_1^{jd_1} U_2^{jd_2} U_3^{jd_3} (\lambda_j U_3^{j\epsilon(x)} + P_j(U_1, U_2, U_3)) \quad (1)$$

with  $\lambda_j \in k(x)$  and  $(jd_1, jd_2, j\epsilon(x)) \in \mathbb{N}^3$  whenever equality holds; furthermore, equality holds for some  $j$ ,  $2 \leq j \leq m$ .

Note that we necessarily have  $\Phi_j(U_1, U_2, U_3) \neq \lambda_j U_3^{j\epsilon(x)}$  for some  $j$  if  $B < +\infty$ , since  $B$  is taken to be maximal. Moreover, since  $(z, u)$  is fully prepared and  $\mathcal{V} = \langle U_3 \rangle$ , we necessarily have  $B \geq 1$  and  $\deg_{U_3}(\Phi_1) < d_3 + \epsilon(x)$ .

This construction builds up a face of  $\Delta(h, u, z)$  with equation

$$\frac{x_1 + x_2}{B(\epsilon(x) + d_3) + d_1 + d_2} + \frac{x_3}{(\epsilon(x) + d_3 + \frac{d_1 + d_2}{B})} = 1,$$

for some  $B$  which contains the point  $\mathbf{x} := (d_1, d_2, \epsilon(x) + d_3)$  and at least another point.

Let  $p$  be the projection

$$p : \mathbb{R}^3 - \{\mathbf{x}\} \longrightarrow \{x_3 = 0\}.$$

For analytic computations, note that if  $M = z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3}$  is a monomial appearing with nonzero coefficient in some expansion **II.1**(1) of  $h$  and  $j \geq 1$ , then  $M$  defines the point  $\mathbf{x}_M$ :

$$M = z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3} \leftrightarrow \mathbf{x}_M = \left( \frac{a_1}{j}, \frac{a_2}{j}, \frac{a_3}{j} \right) \in \Delta(h, u, z),$$

and

$$p(\mathbf{x}_M) = \left( d_1 + \frac{\frac{a_1}{j} - d_1}{d_3 + \epsilon(x) - \frac{a_3}{j}}, d_2 + \frac{\frac{a_2}{j} - d_2}{d_3 + \epsilon(x) - \frac{a_3}{j}} \right). \quad (2)$$

Then  $B + d_1 + d_2$  is the minimum value  $x_1 + x_2$  for points in  $p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\})$ . We define  $\Delta_2(h; u_1, u_2; u_3) \subseteq (\mathbb{R}_+)^2$  by the formula

$$(d_1, d_2) + \Delta_2(h; u_1, u_2; u_3) := p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\}).$$

The main idea is that  $\Delta_2(h; u_1, u_2; u_3)$  acts as the characteristic polyhedron of a surface singularity and in the following, we mimic [CJS],[2], all these following Hironaka.

In case (ii) ( $\text{div}(u_3) \not\subseteq E$ ), we will require two extra conditions (to be achieved in **VI.3** below by possibly changing  $u_3$ ):

(P3) *there is no homogeneous  $P \in k(x)[U_1, U_2]$ ,  $P \neq 0$ , such that*

$$\text{in}_{v_B}(h) = Z^m + \sum_{2 \leq j \leq m} Z^{m-j} \lambda_j U_1^{jd_1} U_2^{jd_2} (U_3 + P(U_1, U_2))^{j\epsilon(x)}, \quad (3)$$

*with the convention  $\lambda_j = 0$  when  $(jd_1, jd_2, j\epsilon(x)) \notin \mathbb{N}^3$ ;*

(P4) *if  $B < +\infty$ , let  $\mathbf{x}_2 = (d_1 + A(1), \beta + d_2)$  be the vertex of  $\Delta_2(h; u_1, u_2; u_3)$  with minimal first coordinate. Then  $\mathbf{x}_2$  does not vanish by changing  $u_3$  to  $u_3 + \gamma u_1^\alpha u_2^\beta$ ,  $\gamma \in \mathbb{R}$ ,  $\gamma$  invertible.*

**PROPOSITION VI.3** *With assumptions as above, there exist  $(z, u_1, u_2, u_3)$ ,  $z, u_3 \in \widehat{R}$  such that (P1)(P2) and (P3)(P4) (in case (ii) with  $B < \infty$ ) are satisfied.*

*Proof.* The conditions (P1)(P2) can be achieved easily. If (P3) or (P4) is not achieved, we make a translation on  $u_3$ : we replace  $u_3$  by  $u_3 + \sum_{a_1, a_2} \gamma_{a_1, a_2} u_1^{a_1} u_2^{a_2}$ ,  $\gamma_{a_1, a_2} \in R$ ,  $(a_1, a_2) \in p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\})$ . To achieve (P3), we take

$$\sum_{a_1, a_2} \overline{\gamma_{a_1, a_2}} U_1^{a_1} U_2^{a_2} := P(U_1, U_2),$$

$P(U_1, U_2)$  as in (3), which makes  $B$  increase if (P3) is not achieved. To achieve (P4) we change  $u_3$  to  $u_3 + \gamma u_1^\alpha u_2^\beta$  as in (P4)), which makes  $(A(1), \beta)$  strictly increase for the lexicographical ordering .

In both cases, this translation makes  $\Delta_2(h; u_1, u_2; u_3)$  smaller. These translations may spoil (P1), so each must be followed by a translation on  $z$  to get again (P1). This translation makes  $\Delta_2(h; u_1, u_2; u_3)$  not bigger. The process may be infinite, but since  $\Delta_2(h; u_1, u_2; u_3)$  gets smaller at each step, this converges to some  $z, u_3 \in \widehat{R}$ .

**DEFINITION VI.3.1** *With assumptions as above, a r.s.p.  $(z, u_1, u_2, u_3)$ ,  $z, u_3 \in \widehat{R}$  such that (P1)(P2) and (P3)(P4) (in case (ii) with  $B < \infty$ ) are satisfied is said to be well prepared. For such  $(z, u_1, u_2, u_3)$ , the number  $B$  defined above is denoted by  $B(z, u_1, u_2, u_3)$  or  $B(x)$  for short, even if it may depend on the choice of  $(z, u_1, u_2, u_3)$ .*

**VI.4.** We begin the proof of Theorem VI.1 by the special case  $B(x) = \infty$ .

When  $B(z, u_1, u_2, u_3) = \infty$ ,  $\Delta(h, u, z)$  has only one vertex with coordinates  $(d_1, d_2, \epsilon(x) + d_3)$ . Since  $\epsilon(x) > 0$ , we have  $\text{div}(u_3) \not\subseteq E$ , hence  $t(x) = 1$  (case (ii)) and  $E \subseteq \text{div}(u_1 u_2)$ ,  $d_3 = 0$ . The proof is a variation of that of theorem **II.9**, checking carefully the algebraicity of the blowing up centers.

It has been assumed from section IV on that  $\text{Sing}_m(X) \subseteq E$ , so  $\epsilon(x, E) < 1$  necessarily since  $V(z, u_3) \subseteq \text{Sing}_m(X)$  otherwise.

By blowing up the surfaces  $V(z, u_i)$ ,  $\text{div}(u_i) \subseteq E$ , it can be assumed w.l.o.g. that  $d_i < 1$ . Similarly, it can be assumed that  $d_1 + d_2 < 1$  by blowing up  $V(z, u_1, u_2)$ .

Assume that  $V(z, u_i, u_3) \subseteq \text{Sing}_m(X) \subseteq E$ , i.e.  $d_i + \epsilon(x) \geq 1$ ,  $i = 1$  or  $i = 2$ . Then  $C_i := V(z, u_i, u_3)$  is a formal irreducible component of  $\text{Sing}_m(X)$ . By excellence, its Zariski closure  $\overline{C}_i$  is a curve on  $X$ . On the other hand,  $\overline{C}_i$  is contained in  $V(z, u_i)$ , so  $C_i$  itself is a curve on  $X$ . By blowing up  $C_i$ , we may assume that  $d_i + \epsilon(x) < 1$ ,  $i = 1, 2$ .

At this point, we have reached the situation of theorem **II.9**(1) and the proof therein extends without changes: we eventually get reduction in  $(m(x), \tau(x))$  by blowing up closed points. We observe that theorem **VI.1** can also be phrased as follows in this case:  $E_n$  can be enlarged to a new normal crossings divisor  $F_n$  such that  $\epsilon(x_n, F_n) = 0$ .

From now on, we assume that

$$e(x) = 1, \quad B(x) < \infty. \quad (\text{Hyp})$$

**DEFINITION VI.4.1** (New invariants) *We define  $A_1, \beta$  by:  $(d_1 + A_1, \beta + d_2)$  is the vertex of minimal first coordinate of*

$$p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\}).$$

*We define  $A_2$  by:  $d_2 + A_2$  is the minimal second coordinate of the points of*

$$p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\}).$$

*We define  $C(u, z)$  (or  $C(x)$  for short) by;*

$$C(u, z) = B(u, z) - A_1 - A_2.$$

Finally, we define  $\gamma(u, z)$  (or  $\gamma(x)$  for short) as follows:

- (i)  $\gamma(u, z) := \lceil \beta(u, z) \rceil \geq 0$  if  $(E \subseteq \text{div}(u_1)$  and  $t(x) = 1)$ ;
- (ii)  $\gamma(u, z) := \lceil \beta(u, z) \rceil \geq 0$  if  $(E \subseteq \text{div}(u_1 u_3)$  and  $t(x) = 0)$ ;
- (iii)  $\gamma(u, z) := 1 + \lfloor C(u, z) \rfloor \geq 1$  otherwise, i.e. if  $(E = \text{div}(u_1 u_2)$  and  $t(x) = 1)$  or if  $(E = \text{div}(u_1 u_2 u_3)$  and  $t(x) = 0)$ .

**PROPOSITION VI.4.2** (Behaviour of the new invariants under blowing up along an  $\epsilon$ -permissible curve). Assume that (Hyp) is true,  $(z, u)$  is well prepared and let

$$\mathcal{C}_i := V(z, u_i, u_3), \quad i = 1 \text{ or } i = 2.$$

Assume that  $\mathcal{C}_i$  is  $\epsilon$ -permissible in  $\widehat{X} = \text{Spec}(\widehat{R}/(h))$ , for some  $i, i = 1, 2$ , then:

- (i)  $\epsilon(x) + d_3 + d_i \geq 1$ ,
- (ii)  $\mathcal{C}_i$  is algebraic, i.e., if in achieving (P3)(P4), we get  $z, u_3 \in \widehat{R}$ , then there exists a curve in  $\text{Spec } R$  whose formal completion is  $V(z, u_i, u_3)$ .
- (iii) let  $\pi_i : (Z', x') \rightarrow (Z, x)$  be the blowing up along  $\mathcal{C}_i$ ,  $X' \subset Z'$  the strict transform of  $X$  and  $x' \in X'$  the center of  $v$ , with  $\iota(x') = \iota(x)$ . Then:
- (iv) if  $i = 1$  and  $x'$  is the point of  $Z'$  with parameters

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2, u_3/u_1),$$

these are well-prepared parameters and

$$\beta(x') = \beta(x), \quad A_1(x') = A_1(x) - 1, \quad A_2(x') = A_2(x),$$

$$d_1(x') = d_1(x) + \epsilon(x) + d_3(x) - 1, \quad d_2(x') = d_2(x), \quad d_3(x') = d_3(x);$$

- (v) if  $i = 2$  and  $x'$  is the point of  $Z'$  with parameters

$$(z', u'_1, u'_2, u'_3) := (z/u_2, u_1, u_2, u_3/u_2),$$

these are well-prepared parameters and

$$\beta(x') = \beta(x) - 1, \quad A_1(x') = A_1(x), \quad A_2(x') = A_2(x) - 1,$$

$$d_2(x') = d_1(x) + \epsilon(x) + d_3(x) - 1, \quad d_1(x') = d_1(x).$$

*Proof of (i).* Condition (i) is equivalent to  $h \in (z, u_3, u_i)^m$ .

*Proof of (ii)(iii).* Let us note that (ii) is clear when  $\text{div}(u_3) \subset E$ , because in that case, we do not make (P3)(P4),  $z, u_1, u_3 \in R$ . When  $\text{div}(u_3) \not\subset E$ , we will prove that

- (ii)'  $\mathcal{C}_i$  is the only analytic branch in  $\text{div}(u_i) \cap \text{Sing}_m(X) \cap \{y \in X : \epsilon(y) > 0\}$  not contained in  $\text{div}(u_j)$ ,  $j = 1, 2, j \neq i$ .

By **II.10 II.11**, this will prove (ii). We compute  $\pi_i : X' \subset Z' \rightarrow X' \subset Z'$ . By symmetry, we suppose  $i = 1$ . Let us expand:

$$h = \sum_{A, m-j, 0 \leq j \leq m} C_{A, m-j} z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3},$$

$C_{A, m-j} \in R$ ,  $C_{A, m-j}$  invertible or zero,  $C_{A, m-j} \neq 0 \Rightarrow a_1 + a_2 + a_3 \geq j\delta(x)$ ,  $a_i \geq jd_i$ ,  $i = 1, 2, 3$ ,  $C_{0,0}$  invertible.



Since  $h \equiv \delta z^m \pmod{(u_1, u_2, u_3)}$ ,  $\delta \in R$  a unit,  $X' \cap \text{Spec}R[u_1/z, u_3/z] \subset Z'$  does not contain the point  $(z, u_1/z, u_2, u_3/z)$ . Assume now that  $x'$  belongs to the affine chart  $\text{Spec}R[z/u_3, u_1/u_3] \subset Z'$ . Let

$$(z', u'_1, u'_2, u'_3) := (z/u_3, u_1/u_3, u_2, u_3).$$

We have

$$h' := u_3^{-m} h = \sum C_{A, m-j} z'^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_1+a_3-j}, \quad (1)$$

$$h' = C_{0,m} z'^m + \sum_{2 \leq j \leq m} \gamma_j u_1^{j d_1} u_2^{j d_2} u_3^{j(d_1+d_3+\epsilon(x)-1)} \pmod{I^+(z', u'_1, u_2, u_3)}, \quad (2)$$

where,  $\gamma_j \in R$ ,  $\gamma_j$  invertible or zero,  $\gamma_j = 0$  when one exponent is not integer,

$$\gamma_j = C_{j d_1, j d_2, j(d_3+\epsilon(x)), m-j} \pmod{\mathfrak{M}}$$

when  $\gamma_j$  is invertible and  $I^+(z', u'_1, u_2, u_3)$  is generated by

$$z'^{m+1}, z'^{m-j} u_1^a u_2^b u_3^c,$$

with  $1 \leq j \leq m$ ,  $a \geq j d_1$ ,  $b \geq j d_2$ ,  $c \geq j(d_1+d_3+\epsilon(x)-1)$ ,  $a+b+c > j(d_1+d_2+d_1+d_3+\epsilon(x)-1)$ .

Note that (2) implies  $d_3(x') \geq d_1+d_3+\epsilon(x)-1$ , in fact there is equality. Otherwise, by [H2], there would exist  $t = z' + \gamma u_3^e$ ,  $e \geq d_1+d_3+\epsilon(x)-1$ ,  $\gamma \in R$ ,  $\text{ord}_{u_i} \gamma \geq d_i$ ,  $i = 1, 2$ , with  $h' = C_{0,0} t^m \pmod{I^+(t, u'_1, u_2, u_3)}$ . As  $I^+(z', u'_1, u_2, u_3) = I^+(t, u'_1, u_2, u_3)$  and, in (2), there is no term in  $z'^{m-1}$ , this is impossible. As  $d_1(x') = d_1(x)$  and  $d_2(x') = d_2(x)$ , by (2), we get  $\epsilon(x') = 0$ : there is no  $x'$  very near  $x$  in this chart. This gives the first statement in (iv). This gives also (ii)', because if there was a curve in  $\text{div}(u_2) \cap \text{Sing}_m(X) \cap \{y \in X : \epsilon(y) > 0\}$ , the strict transform of this curve would have a non empty intersection with our affine chart and there would exist in this chart some  $x'$  with  $\epsilon(x') \geq 1$ .

*Proof of (iv).* Now  $x' \in \text{Spec}R[z/u_1, u_3/u_1] \subset Z'$  is the point with parameters  $(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2, u_3/u_1)$ . Then, using the notations of (1),

$$h' := u_1^{-m} h = \sum C_{A, m-j} z'^{m-j} u_1^{a_1+a_3-j} u_2^{a_2} u_3^{a_3},$$

$\Delta(h', u', z')$  is obtained as follows: take the convex hull of the set  $\{(a+c-1, b, c) | (a, b, c) \in \Delta(h, u, z)\}$  and add  $\mathbb{R}^{+3}$ , then

$$\partial(\Delta(h', u', z')) \subset \{(a+c-1, b, c) | (a, b, c) \in \partial(\Delta(h, u, z))\},$$

$$\text{in}(h', \Delta')_{u', z'} = \overline{C_{0,m} Z'^m} + \sum_{2 \leq j \leq m, A/j \in \partial(\Delta(h', u', z'))} \lambda_{j,A} Z'^{m-j} U_1^{a_1+a_3-j} U_2^{a_2} U_3^{a_3},$$

where the  $\lambda_{j,A} \in k(x)$  are defined by:

$$\text{in}(h, \Delta)_{u, z} = \overline{C_{0,m} Z^m} + \sum_{2 \leq j \leq m, \frac{A}{j} \in \partial(\Delta(h, u, z))} \lambda_{j,A} Z^{m-j} U_1^{a_1+a_3-j} U_2^{a_2} U_3^{a_3}.$$

Let  $\mathcal{M}'$  be the set of monomials  $M' = z'^{m-j} u_1^{a_1+a_3-j} u_2^{a_2} u_3^{a_3}$  which appear with a non zero coefficient in the expansion of  $h'$ , let  $\mathcal{M}$  be the set of monomials  $M = z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3}$  which appear with a non zero coefficient in the expansion of  $h$ :

$$d_1(x') = \text{inh}_{M' \in \mathcal{M}'} \left( \frac{a_1 + a_3 - j}{j} \right) = d_1 + d_3 + \epsilon(x) - 1, d_i(x') = \text{inh}_{M' \in \mathcal{M}'} \left( \frac{a_i}{j} \right) = d_i, \quad i = 2, 3.$$

As  $x'$  is very near  $x$ ,  $\epsilon(x) = \epsilon(x')$ ,  $\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x)$ . The only point on the first side of  $\Delta(h', u', z')$  is

$$(d_1(x'), d_2(x), d_3(x) + \epsilon(x))$$

let  $p'$  be the projection on  $x_3 = 0$  from this vertex. A monomial  $M'$  defines a point  $(\frac{a_1+a_3}{j}-1, \frac{a_2}{j}, \frac{a_3}{j})$  that we call also  $M'$ , when  $a_3 < d_3(x) + \epsilon(x)$ ,

$$p'(M') = (d_1(x') + \frac{\frac{a_1+a_3}{j}-1-d_1(x')}{d_3(x)+\epsilon(x)-\frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j}-d_2(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}}),$$

as  $\frac{\frac{a_1+a_3}{j}-1-d_1(x')}{d_3(x)+\epsilon(x)-\frac{a_3}{j}} = \frac{\frac{a_1}{j}-d_1(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}} - 1$  and by **VI.3(2)**

$$p(M) = (d_1(x) + \frac{\frac{a_1}{j}-d_1(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j}-d_2(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}}).$$

We get

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x')),$$

i.e. the polyhedron  $p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\})$  translated by the vector  $-(d_1(x'), d_2(x'))$ , from

$$p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$$

by making an horizontal translation of  $-1$ . This gives the other assertions of (iv). Mutatis mutandis, we get (v).

**PROPOSITION VI.4.3** (Behaviour of the new invariants under blowing up a closed point). *Assume that (Hyp) is true and  $(z, u)$  is well prepared. Let  $\pi_i : (Z', x') \rightarrow (Z, x)$  be the blowing up along  $x$ ,  $X' \subset Z'$  the strict transform of  $X$  and  $x' \in X'$  the center of  $v$ , with  $\iota(x') = \iota(x)$ . Then*

- (i)  $x'$  belongs to the strict transform of  $V(z, u_3)$ ,
- (ii) if  $x' \in \text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$  is the point

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1),$$

these parameters are well-prepared and

$$\beta(x') \leq \beta(x), \quad A_1(x') = B(x) - 1, \quad A_2(x') = A_2(x), \quad C(u', z') \leq C(u, z),$$

$$d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1, \quad d_2(x') = d_2(x), \quad d_3(x') = d_3(x);$$

(iii) if  $x' \in \text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$  and  $x' \neq (z', u'_1, u'_2, u'_3)$ , then

$$\beta(x') \leq 1 + \left\lfloor \frac{C(u, z)}{2} \right\rfloor, \quad A_1(x') = d(x) - 1,$$

$$d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1, \quad d_2(x') = 0, \quad d_3(x') = d_3(x),$$

where  $\lfloor \cdot \rfloor$  denotes lower integral part. If moreover  $(E \subseteq \text{div}(u_1 u_3))$  and  $0 < \beta(x)$ , then  $\beta(x') \leq \beta(x)$ .

We have

$$\gamma(x') \leq \gamma(x).$$

More precisely: if  $(x'$  is not rational over  $x$  and  $\gamma(x) \geq 3$ ), then  $\gamma(x') < \gamma(x)$ ; if  $(\gamma(x') = \gamma(x) = 2$  and  $x'$  is not rational over  $x$ ), then  $\beta(x) = 2$ ,  $\text{div}(u_2) \not\subseteq E$  and  $\beta(x') < \beta(x) = 2$ ;  
(ii) if  $x' \in \text{Spec}R[z/u_2, u_1/u_2, u_3/u_2] \subset Z'$  is the point with parameters

$$(z', u'_1, u'_2, u'_3) := (z/u_2, u_1/u_2, u_2, u_3/u_2),$$

these are well prepared parameters and

$$\beta(x') = \beta(x) + A_1(x) - 1, \quad A_1(x') = A_1(x), \quad A_2(x') = B(x) - 1,$$

$$d_2(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1, \quad d_1(x') = d_1(x),$$

$$\gamma(x') \leq \gamma(x), \quad C(u', z') \leq \frac{\beta(x)}{2}.$$

*Proof.* (i) is a consequence of **V.3(ii)** and **V.3(3)**.

*Proof of (ii).* Write

$$h = \sum C_{A, m-b} z^{m-b} u_1^{a_1} u_2^{a_2} u_3^{a_3}, \quad C_{A, m-b} \in R^\times \text{ or } C_{A, m-b} = 0,$$

where the sum runs along  $b \leq m$ ,  $A = 0$  when  $b = 0$ , and  $A = (a_1, a_2, a_3) \in b\Delta(h, u, z)$ . Then

$$h' := u_1^{-m} h = \sum C_{A, m-j} z'^{m-j} u_1^{a_1+a_2+a_3-j} u_2^{a_2} u_3^{a_3},$$

and  $\Delta(h', u', z')$  is obtained as follows: take the convex hull of the set

$$\{(a + b + c - 1, b, c) + \mathbb{R}_+^3 \mid (a, b, c) \in \Delta(h, u, z)\}.$$

Let  $\mathcal{M}'$  be the set of monomials  $M' = z'^{m-j} u_1^{a_1+a_2+a_3-j} u_2^{a_2} u_3^{a_3}$  which appear with a non zero coefficient in the expansion of  $h'$ , let  $\mathcal{M}$  be the set of monomials  $M = z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3}$  which appear with a non zero coefficient in the expansion of  $h$ :

$$d_1(x') = \inf_{M' \in \mathcal{M}'} \left( \frac{a_1 + a_2 + a_3 - j}{j} \right) = d_1 + d_2 + d_3 + \epsilon(x) - 1,$$

$$d_i(x') = \inf_{M' \in \mathcal{M}'} \left( \frac{a_i}{j} \right) = d_i(x), \quad i = 2, 3.$$

As  $x'$  is very near to  $x$ ,  $\epsilon(x) = \epsilon(x')$ ,  $\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x)$ . The only point on the first side of  $\Delta(h', u', z')$  is

$$(d_1(x'), d_2(x), d_3(x) + \epsilon(x)).$$

Let  $p'$  be the projection on  $x_3 = 0$  from this vertex. A monomial  $M'$  corresponds to a point  $\mathbf{x}_{M'}$   $(\frac{a_1+a_2+a_3}{j} - 1, \frac{a_2}{j}, \frac{a_3}{j})$ . When  $a_3 < d_3(x) + \epsilon(x)$ ,

$$p'(M') = (d_1(x') + \frac{\frac{a_1+a_2+a_3}{j} - 1 - d_1(x')}{d_3(x) + \epsilon(x) - \frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j} - d_2(x)}{d_3(x) + \epsilon(x) - \frac{a_3}{j}}),$$

as  $\frac{\frac{\alpha_1 + \alpha_3}{j} - 1 - d_1(x')}{d_3(x) + \epsilon(x) - \frac{\alpha_3}{j}} = \frac{\frac{\alpha_1}{j} - d_1(x)}{d_3(x) + \epsilon(x) - \frac{\alpha_3}{j}} - 1$  and by **VI.3**(2)

$$p(M) = (d_1(x) + \frac{\frac{\alpha_1}{j} - d_1(x)}{d_3(x) + \epsilon(x) - \frac{\alpha_3}{j}}, d_2(x) + \frac{\frac{\alpha_2}{j} - d_2(x)}{d_3(x) + \epsilon(x) - \frac{\alpha_3}{j}}).$$

So we get

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x'))$$

from

$$p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$$

as follows: take the convex hull of the set

$$\{(a + b - 1, b) + \mathbb{R}_+^2 | (a, b) \in p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))\}.$$

These are the usual transformation laws of the characteristic polyhedra of surfaces see the appendix of H. Hironaka in [3]. To get the other assertions of (ii), the proof runs along the same lines as **VI.4.2**(1).

*Proof of (iv).* Mutatis mutandis, we get all assertions of (iv), except the last line that we prove now. In fact, we get

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x'))$$

from

$$p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$$

as follows: take the convex hull of the set  $\{(a, a + b - 1) + \mathbb{R}_+^2 | (a, b) \in p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))\}$ . We get  $A_1(x') = A_1(x)$ ,  $\beta(x') = \beta(x) + A_1(x) - 1$  and  $A_2(x) = d(x) - 1$ .

Let us denote by  $(\alpha_2, \beta_2)$  and  $(\alpha_3, \beta_3)$  with  $\alpha_2 \leq \alpha_3$ , the coordinates of the (maybe equal) vertices of the first side of  $p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$ .

Then  $(\alpha_2, \alpha_2 + \beta_2 - 1) = (\alpha_2, B(x) - 1)$  is the vertex of smaller second coordinate of

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x')).$$

Note that  $(A_1(x), A_1(x) + \beta(x) - 1)$  is the vertex of smaller first coordinate of

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x')).$$

All this leads to:

$$A_1(x') = A_1(x), \quad A_2(x') = B(x) - 1,$$

$$C(u', z') \leq \beta(x') - A_2(x') = A_1(x) + \beta(x) - 1 - (B(x) - 1) = \beta(x) - (B(x) - A_1(x)),$$

$$C(u', z') \leq \alpha_2 - A_1(x') = \alpha_2 - A_1(x) \leq \alpha_2 + \beta_2 - A_1(x) = B(x) - A_1(x).$$

Then either  $B(x) - A_1(x) \leq \frac{\beta(x)}{2}$ , then  $C(u', z') \leq \frac{\beta(x)}{2}$  by the last inequality; or  $B(x) - A_1(x) > \frac{\beta(x)}{2}$ , then  $C(u', z') < \frac{\beta(x)}{2}$  by the first of the two inequalities just above. The inequality  $\gamma(x') \leq \gamma(x)$  is left to the reader.

*Proof of (iii).* Recall the notations and assumptions of **VI.2**(1). We write

$$\Phi_j(U_1, U_2, U_3) = U_1^{jd_1} U_2^{jd_2} U_3^{jd_3} (\lambda_j U_3^{j\epsilon(x)} + \sum_{i \in \mathbb{Q}^+} U_1^{a(i,j)} U_2^{b(i,j)} U_3^{j\epsilon(x)-i} Q_{i,j}(U_1, U_2)) \quad (1)$$

with  $\lambda_j \in k(x)$ ,  $\lambda_j = 0$  if  $(jd_1, jd_2, j\epsilon(x)) \notin \mathbb{N}^3$ . In this expansion, we take:

$$Q_{i,j} \in k(x)[U_1, U_2], \quad Q_{i,j} = 0 \text{ or } (U_1 \nmid Q_{i,j} \text{ and } U_2 \nmid Q_{i,j}),$$

$$Q_{i,j} = 0 \text{ when } (jd_1 + a(i,j), jd_2 + b(i,j), jd_3 + j\epsilon(x) - i) \notin \mathbb{N}^3.$$

Note that at least one  $Q_{i,j}$ ,  $2 \leq j \leq m$  is nonzero and at least one  $\lambda_{j'}$ ,  $2 \leq j' \leq m$  is nonzero.

By definition of  $C(u, z)$ , when  $Q_{i,j} \neq 0$ ,  $\deg(Q_{i,j}) \leq iC(u, z)$ , where  $\deg$  is the usual homogeneous degree. When  $Q_{i,j} \neq 0$ , let us denote  $d(i, j) = \deg(Q_{i,j})$ . Then we have, with natural notations, the relation:

$$v_B(u_3^{j\epsilon(x)}) = v_B(U_1^{a(i,j)} U_2^{b(i,j)} U_3^{j\epsilon(x)-i} Q_{i,j}(U_1, U_2))$$

$$j\epsilon(x)v_B(u_3) = (j\epsilon(x) - i)v_B(u_3) + (a(i, j) + b(i, j) + d(i, j))v_B(u_1)$$

$$j\epsilon(x)v_B(u_3) = (j\epsilon(x) - i)v_B(u_3) + (a(i, j) + b(i, j) + d(i, j))\frac{v_B(u_3)}{B},$$

which leads to:

$$a(i, j) + b(i, j) + d(i, j) - j(d_1 + d_2) = iB. \quad (2)$$

Then, in the expansion of  $U_3^{j\epsilon(x)+jd_3-i} U_1^{a(i,j)} U_2^{b(i,j)} Q_{i,j}(U_1, U_2)$ , the monomial with non zero coefficient and minimal exponent in  $U_1$  is

$$U_1^{a(i,j)} U_2^{iB-a(i,j)} U_3^{j\epsilon(x)+jd_3-i}$$

which gives the point (cf. **VI.2(2)**)

$$\left( d_1 + \frac{\frac{a(i,j)}{j} - d_1}{d_3 + \epsilon(x) - \frac{jd_3 + j\epsilon(x) - i}{j}}, d_2 + \frac{\frac{iB - a(i,j)}{j} - d_1}{d_3 + \epsilon(x) - \frac{jd_3 + j\epsilon(x) - i}{j}} \right)$$

in  $p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\})$ . As

$$d_1 + \frac{\frac{a(i,j)}{j} - d_1}{d_3 + \epsilon(x) - \frac{jd_3 + j\epsilon(x) - i}{j}} = d_1 + \frac{a(i, j) - jd_1}{i},$$

we deduce that

$$A_1(u, z) = \inf \left\{ \frac{a(i, j) - jd_1}{i} \mid 2 \leq j \leq m(x), 0 < i \leq j\epsilon(x), j\epsilon(x) + jd_3 - i \in \mathbb{N}, i \in \mathbb{Q}, Q_{i,j} \neq 0 \right\}. \quad (3)$$

Similarly,

$$A_2(u, z) = \inf \left\{ \frac{b(i, j) - jd_1}{i} \mid 2 \leq j \leq m(x), 0 < i \leq j\epsilon(x), j\epsilon(x) + jd_3 - i \in \mathbb{N}, i \in \mathbb{Q}, Q_{i,j} \neq 0 \right\} \quad (4)$$

and, finally, by (2), when  $Q_{i,j} \neq 0$ ,

$$d(i, j) = i(B - \frac{a(i, j) - jd_1}{i} - \frac{b(i, j) - jd_1}{i}) \leq iC(u, z). \quad (5)$$

Since  $x' \in \text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ ,  $x'$  is not the origin

$$(z', u'_1, u'_2, u'_3) = (z/u_1, u_1, u_2/u_1, u_3/u_1),$$

and  $x'$  belongs to the strict transform of  $V(z, u_3)$  then  $z'(x') = u'_1(x') = u'_3(x') = 0$ . We complete  $(z', u'_1, u'_3)$  to a r.s.p.  $(z', u'_1, v', u'_3)$  at  $x'$  where

$$v' = u'_2{}^n + \sum_{0 \leq a \leq n-1} \mu_a u'_2{}^{n-a}, \quad \mu_a = 0 \text{ or } \mu_a \in R^\times,$$

for some irreducible polynomial

$$P := U_2^n + \sum_{0 \leq a \leq n-1} \bar{\mu}_a U_2^{n-a} \in k(x)[U_2].$$

The following lemma will end the proof of **VI.4.3**(iii).

**LEMMA VI.4.4** *With hypotheses and notations as in **VI.4.3**(iii), let  $d := [k(x') : k(x)]$ . We have:*

- (i)  $A_1(x') = B(u, z) - 1$ ;
- (ii) if  $\text{div}(u_3) \subset E$ , then

$$\beta(x') \leq \frac{C(u, z)}{d} \leq \frac{\beta(x)}{d}$$

- (iii) in general,

$$\beta(x') < 1 + \left\lfloor \frac{C(u, z)}{d} \right\rfloor, \quad (1)$$

- (iv) if  $(E \subseteq \text{div}(u_1), 0 < \beta(x)$  and  $x'$  is rational over  $x$ ), then

$$\beta(x') \leq \beta(x).$$

*Proof.* As  $x'$  is very near to  $x$ , we have  $\epsilon(x) = \epsilon(x')$ ,  $\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x)$ . As  $x'$  is on the strict transform of  $\text{div}(u_3)$  and not on the strict transform of  $\text{div}(u_2)$ , we get:

$$d_2(x') = 0, \quad d_3(x') = d_3(x).$$

With notations as in the proof of **VI.4.3**(ii):

$$h = \sum C_{A, m-b} z^{m-b} u_1^{a_1} u_2^{a_2} u_3^{a_3}, \quad C_{A, m-b} \in R^\times \text{ or } C_{A, m-b} = 0,$$

where the sum runs along  $b \leq m$ ,  $A = 0$  when  $b = 0$ ,  $A = (a_1, a_2, a_3) \in b\Delta(h, u, z)$ ,

$$h' := u_1^{-m} h = \sum C_{A, m-j} z'^{m-j} u_1^{a_1+a_2+a_3-j} u_2'^{a_2} u_3'^{a_3}.$$

Up to multiplying  $h$  by an unit, we may assume  $\overline{C_{0,m}} = 1 \in k(x)$ . Then, with the notations of **II.2**, we have

$$\begin{aligned} \delta(x) &= d_1 + d_2 + d_3 + \epsilon(x), \\ \text{in}_{\delta, u, z}(h) &= Z^m + \sum_{2 \leq j \leq m} \mu_j Z^{m-j} U_1^{jd_1} U_2^{jd_2} U_3^{jd_3 + j\epsilon(x)}, \quad \mu_j \in k(x), \end{aligned} \quad (1)$$

$\mu_j = 0$  whenever  $(jd_1, jd_2, jd_3 + j\epsilon(x)) \notin \mathbb{N}^3$ ,  $\mu_j = \overline{C_{jd_1, jd_2, jd_3 + j\epsilon(x), m-j}} \in k(x)$  otherwise. This leads to

$$h' = C_{0,m} z'^m + \sum_{2 \leq j \leq m} z'^{m-j} C_{jd_1, jd_2, jd_3 + j\epsilon(x), m-j} u_1^{j(d_1 + d_2 + d_3 + \epsilon(x) - 1)} u_2^{jd_2} u_3^{jd_3 + j\epsilon(x)} + h'_1 \quad (2)$$

where  $h'_1 \in \{z'^{m-j} u_1^{a(j)}, j \in \mathbb{N}, a(j) > j(d_1 + d_2 + d_3 + \epsilon(x) - 1) = j(\delta(x) - 1)\}$ . As a consequence,

$$(\delta(x) - 1, 0, d_3(x) + \epsilon(x))$$

is the vertex of smallest first coordinate of  $\Delta(h', u'_1, v', u'_3, t)$  and is not solvable. In the preparation, we may replace  $z'$  by  $t = z' + \lambda u_1'^a$  with  $a \geq \delta(h) - 1$ , but, this cannot erase the vertex  $(\delta(x) - 1, 0, d_3(x) + \epsilon(x))$ . We get

$$d_1(x') = \delta(x) - 1.$$

Let us study the projection of  $\Delta(h', u'_1, v', u'_3, t) \cap \{x'_3 < d_3 + \epsilon(x')\}$  on  $x'_3 = 0$ , in particular we are interested in the vertex of smallest first coordinate of this projection. Let  $w$  be the monomial valuation on  $R' := \mathcal{O}_{X', x'}$  defined by

$$\begin{aligned} w(z') &= 1, \quad w(u'_3) = \frac{1}{\epsilon(x) + d_3 + \frac{d_1 + d_2}{B(u, z) - 1}}, \\ w(u'_1) &= \frac{1}{(B(u, z) - 1)(\epsilon(x) + d_3) + d_1 + d_2} = \frac{1}{B(u, z) - 1} w(u'_3). \end{aligned}$$

There is an expansion

$$\text{in}_w(h') = Z'^m + \sum_{2 \leq j \leq m} Z'^{m-j} \Phi'_j(U'_1, U'_3) \in \text{gr}_w(R') = R' / (z', u'_1, u'_3)[Z', U'_1, U'_3],$$

where

$$\begin{aligned} \Phi'_j(U'_1, U'_3) &= \lambda_j U_1'^{j(\delta(x) - 1)} \overline{u_2'}^{jd_2} U_3'^{j\epsilon(x) + jd_3} + \\ &\sum_{0 < i \leq j\epsilon(x), j\epsilon(x) + jd_3 - i \in \mathbb{N}, i \in \mathbb{Q}} U_3'^{j\epsilon(x) + jd_3 - i} U_1'^{i} \overline{u_2'}^{b(i, j) + d(i, j) + j\epsilon(x) + jd_3 - i - j} \overline{u_2'}^{b(i, j)} Q_{i, j}(1, \overline{u_2'}), \end{aligned}$$

where  $d(i, j) = \deg(Q_{i, j}) \overline{u_2'}$  is the image of  $u_2'$  in  $\frac{R'}{(z', u_1, u_3)} = k(x)[\overline{u_2'}]_{(\overline{v'})}$ ,  $\overline{v'}$  being the image of  $v'$  in  $\frac{R'}{(z', u_1, u_3)}$ . Let us recall that

$$\text{in}_w(h') \in \text{gr}_w(R') := \bigoplus_{r \in \mathbb{Q}_{\geq 0}} \frac{I_r}{I_r^+},$$

with  $I_r = \{a \in R' \mid w(a) \geq r\}$ ,  $I_r^+ = \{a \in R' \mid w(a) > r\}$ . By **VI.4.3(2)**,  $\text{in}_w(h') =$

$$Z'^m + \sum_{2 \leq j \leq m} Z'^{m-j} U_1'^{j(\delta(x) - 1)} [\lambda_j U_3'^{j\epsilon(x) + jd_3} + U_3'^{j\epsilon(x) + jd_3 - i} U_1'^i \overline{u_2'}^{b(i, j) + d(i, j) + j\epsilon(x) + jd_3 - i - j} \overline{u_2'}^{b(i, j)} Q_{i, j}(1, \overline{u_2'})]. \quad (3)$$

This means that

$$\frac{1}{(B(u, z) - 1)(\epsilon(x) + d_3) + d_1 + d_2} x'_1 + \frac{1}{\epsilon(x) + d_3 + \frac{d_1 + d_2}{B(u, z) - 1}} x'_3 = 1$$

is the defining equation of a face of  $\Delta(h', u'_1, v', u'_3)$ .

**VI.4.4.1** When  $\text{div}(u_3) \subset E$ , then  $\text{div}(u'_3) \subset E'$ , we have just to make (P2) in the preparation, we may replace  $z'$  by  $t = z' + r$ ,  $r \in R'$  and, as  $Z'^{m-1}$  does not appear in (2),  $w(r) > 1$ ,  $w(t) = w(z') = 1$ . This means that

$$\frac{1}{(B(u, z) - 1)(\epsilon(x) + d_3) + d_1 + d_2} x'_1 + \frac{1}{\epsilon(x) + d_3 + \frac{d_1 + d_2}{B(u, z) - 1}} x'_3 = 1$$

is the defining equation of a face of  $\Delta(h', u'_1, v', u'_3, t)$ . By **VI.2(2)**,

$$A_1(u'_1, v', u'_3, t) = B(u, z) - 1 \text{ and } \beta(x') = \inf\{\text{ord}_{x'}(Q_{i,j}(1, \overline{u'_2}))/id\}.$$

By **VI.4.3(5)**,  $\beta(x') \leq C(u, z)/d$  and this gives **VI.4.4** in the case  $\text{div}(u_3) \subset E$ .

**VI.4.4.2** From now on,  $\text{div}(u_3) \not\subset E$ , in particular  $d_3(x) = 0$ . Then, to get (P1),..., (P4), we may replace  $z'$  by  $t = z' + r$ ,  $r \in R'$  and, as  $Z'^{m(x)-1}$  does not appear in (2),  $w(r) > 1$ ,  $w(t) = w(z') = 1$ . We possibly have to make the projection of  $\Delta(h', u'_1, v', u'_3, t) \cap \{x'_3 < d_3 + \epsilon(x')\}$  on  $x'_3 = 0$  smaller by changing  $u'_3$  to  $v_3 = u'_3 + \lambda u'_1{}^a$  with  $a \geq B(u, z) - 1$  and  $\lambda \in R'$ ,  $\lambda$  not divisible by  $u'_1$ .

Assume that  $a > B(u, z) - 1$  (this is always the case when  $B(u, z) \notin \mathbb{N}$ ). Then  $\text{in}_w(v_3) = \text{in}_w(u'_3)$ , we get  $A_1(u'_1, v', u'_3, t) = B(u, z) - 1$  and

$$\beta(x') = \inf\{\text{ord}_{x'}(Q_{i,j}(1, \overline{u'_2}))/id\}.$$

By **VI.4.3(5)**,

$$\beta(x') \leq C(u, z)/d$$

which gives **VI.4.4** in this case.

**VI.4.4.3** From now on,

$$B(u, z) \in \mathbb{N}, \quad a = B(u, z) - 1.$$

If there exists a couple  $(i, j_0)$  such that in (3) above

$$\lambda_{j_0} = 0 \text{ and } Q_{i, j_0} \neq 0,$$

then the translations  $t = z' + r$  and  $v_3 = u'_3 + \lambda u'_1{}^a$  will not modify the term

$$U_3'^{j_0 \epsilon(x) - i_0} U_1'^{i_0 (B(u, z) - 1)} \overline{u_2'}^{b(i_0, j_0)} Q_{i_0, j_0}(1, \overline{u_2'})$$

with  $i_0 := \min\{i : Q_{i, j_0} \neq 0\}$ . More precisely, in the expansion

$$\text{in}_w(h') = T^m + \sum_{2 \leq j \leq m} T^{m-j} U_1'^{j(\delta(x)-1)} [\lambda_j V_3^{j\epsilon(x)} + \mu_{i,j} U_3'^{j\epsilon(x)-i} U_1'^{i(B(u, z)-1)} \times V'^{e(i,j)}],$$

$\mu_{i,j} \in \frac{R'}{(t, u'_1, u'_3)}$ ,  $e(i, j) \in \mathbb{N}$ , we will have

$$\overline{u_2'}^{b(i_0, j_0)} Q_{i_0, j_0}(1, \overline{u_2'}) = \mu_{i_0, j_0} \times V'^{e(i_0, j_0)}.$$



Then

$$\beta(x') \leq \text{ord}_{x'}(Q_{i_0, j_0}(1, \overline{u'_2}))/id,$$

which, by **VI.4.3**(5), gives

$$A_1(x') = B(u, z) - 1, \quad \beta(x') \leq C(u, z)/d$$

and implies **VI.4.4** in this case.

**VI.4.4.4** From now on, we assume the implication:

$$Q_{i,j} \neq 0 \Rightarrow \lambda_j \neq 0.$$

In particular, we have  $j\epsilon(x) \in \mathbb{N}$ ,  $j\delta(x) \in \mathbb{N}$  and all the indices  $i$  in (2)(3) are integers. Let us define

$$F_j \in \text{gr}_{v_B}(R) = k(x)[U_1, U_2, U_3, Z]$$

by

$$F_j = \lambda_j U_3^{j\epsilon(x)} + \sum_{1 \leq i \leq j\epsilon(x)-1} U_3^{j\epsilon(x)-i} U_1^{a(i,j)} U_2^{b(i,j)} Q_{i,j}(U_1, U_2).$$

$$F'_j \in \text{gr}_w(R') = \frac{R'}{(u'_1, u'_3, z')} [U'_1, U'_3, Z'] = \frac{R'}{(u'_1, u'_3, z')} [U'_1, U'_3, T]$$

by

$$F'_j = \lambda_j U'_3^{j\epsilon(x)} + \sum_{1 \leq i \leq j\epsilon(x)-1} U'_3^{j\epsilon(x)-i} U'_1^{i(B(u,z)-1)} \overline{u'_2}^{b(i,j)} Q_{i,j}(1, \overline{u'_2}),$$

so (3) can be rewritten:

$$\text{in}_w(h') = T^{m(x)} + \sum_{2 \leq j \leq m(x)} T^{m(x)-j} U'_1^{j(\delta(x)-1)} F'_j. \quad (3')$$

The preceding remarks rewrite  $j\epsilon(x) \notin \mathbb{N} \Rightarrow F_j = 0, F'_j = 0$ . Let

$$G_j = F_j^{\frac{m! \epsilon(x)}{j\epsilon(x)}}, \quad G'_j = F'_j{}^{\frac{m! \epsilon(x)}{j\epsilon(x)}} \quad 2 \leq j \leq m, \quad j\epsilon(x) \in \mathbb{N},$$

$\deg_{U_3}(G_j) = m! \epsilon(x)$  or  $G_j = 0$ , and  $\deg_{U'_3}(G'_j) = m! \epsilon(x)$  or  $G'_j = 0$ .

Let  $\mu_1, \mu_2 \in k(x)$ ,  $j_1, j_2$ ,  $2 \leq j_1, j_2 \leq m$ , let

$$G = \mu_1 G_{j_1} + \mu_2 G_{j_2} = \mu_{m! \epsilon(x)} U_3^{m! \epsilon(x)} + \sum_{1 \leq i \leq m! \epsilon(x)-1} U_3^{m! \epsilon(x)-i} U_1^{a(i)} U_2^{b(i)} Q_i(U_1, U_2),$$

where  $Q_i = 0$  or  $Q_i$  neither divisible by  $U_1$  nor by  $U_2$ . Let us denote  $d(i) := \deg(Q_i)$ . Assume that for some  $i$ ,  $Q_i \neq 0$ , then, by **VI.4.3**(5),

$$d(i) = i(B - \frac{a(i)}{i} - \frac{b(i)}{i}) \leq iC(u, z). \quad (4)$$

Assume that not all  $G_j$ 's are collinear in the  $k(x)$ -vector space  $\text{gr}_{v_B}(R)$ . Then there is some  $G \neq 0$  as above with  $\lambda = 0$ . Let

$$G = \mu_1 G_{j_1} + \mu_2 G_{j_2} = \sum_{1 \leq i \leq m! \epsilon(x)-1} U_3^{m! \epsilon(x)-i} U_1^{a(i)} U_2^{b(i)} Q_i(U_1, U_2),$$

with some  $Q_i \neq 0$ . Let  $i_0 := \min\{i : Q_{i_0} \neq 0\}$ . Let

$$G' = \lambda G'_{j_1} + \mu G'_{j_2} = \sum_{1 \leq i \leq m! \epsilon(x) - 1} U_3^{m! \epsilon(x) - i} U_1^{i(B(u,z) - 1)} \overline{u_2}^{-b(i)} Q_i(1, u'_2).$$

Replacing  $U'_3$  by  $V_3$ , we get

$$G' = \mu_1 G'_{j_1} + \mu_2 G'_{j_2} = V_3^{m! \epsilon(x) - i_0} U_1^{i_0(B(u,z) - 1)} \overline{u_2}^{-b(i_0)} Q_{i_0}(1, u'_2) + H', \quad \deg_{V_3} H' < m! \epsilon(x) - i_0.$$

Then

$$A_1(x') = B(u, z) - 1, \quad \beta(x') \leq C(u, z)/d$$

which implies VI.4.4 in this case.

**VI.4.4.5** From now on, we assume that all  $G_j$ 's are collinear in the  $k(x)$ -vector space  $\text{gr}_{v_B}(R)$ .

By (P3) for  $(z, u)$ , any  $G_j \neq 0$  is not collinear to a  $(m! \epsilon(x))^{\text{th}}$ -power, any  $F_j \neq 0$  is not collinear to a  $(j \epsilon(x))^{\text{th}}$ -power. Take some  $F_j \neq 0$ , and let

$$j \epsilon(x) = p^e q, \quad (p, q) = 1. \quad (5)$$

Let  $v_3 = u'_3 + \lambda u_1'^a$ , with  $a \geq B(u, z) - 1$  and  $\lambda \in R'$ ,  $\lambda$  not divisible by  $u_1'$ . Let

$$\bar{\lambda} \in \frac{R'}{(z', u_1, u'_3)} = k(x)[\overline{u_2}]_{v'}, \quad b := \text{ord}_{v'}(\bar{\lambda}), \quad \beta_0 := \min_i \left( \frac{\text{ord}_{v'}(Q_{i,j}(1, u'_2))}{i} \right) \leq C(u, z)/d.$$

When  $b < \beta_0$ , we have

$$A_1(x') = B(u, z) - 1, \quad \beta(x') = b < C(u, z)/d.$$

When  $b > \beta_0$ , we have

$$A_1(x') = B(u, z) - 1, \quad \beta(x') = \beta_0 \leq C(u, z)/d.$$

When  $b = \beta_0$  and there exists  $i < p^e$  such that  $Q_{i,j} \neq 0$ , say  $i_0$  is the smallest such  $i$ , we get

$$F'_j = \lambda_j V_3^{j \epsilon(x)} + V_3^{j \epsilon(x) - i_0} U_1^{i_0(B(u,z) - 1)} \overline{u_2}^{-b(i_0, j)} Q_{i_0, j}(1, \overline{u_2}) + H'_j, \quad \deg_{V_3} H'_j < j \epsilon(x) - i_0,$$

$$A_1(x') = B(u, z) - 1, \quad \beta(x') \leq \frac{\text{ord}_{v'}(Q_{i_0, j}(1, u'_2))}{i_0} \leq C(u, z)/d.$$

When  $b = \beta_0$  and for  $i \leq p^e$   $Q_{i,j} = 0$ , then

$$F'_j = \lambda_j V_3^{j \epsilon(x)} + V_3^{j \epsilon(x) - p^e} U_1^{p^e(B(u,z) - 1)} \lambda^q + H'_j, \quad \deg_{V_3} H'_j < j \epsilon(x) - p^e,$$

$$A_1(x') = B(u, z) - 1, \quad \beta(x') \leq b = \beta_0 \leq C(u, z)/d.$$

When  $b = \beta_0$  and for  $i < p^e$   $Q_{i,j} = 0$  and  $\lambda_j^{-1} U_1^{a(p^e, j)} U_2^{b(p^e, j)} Q_{p^e, j}$  is a  $(p^e)^{\text{th}}$ -power, then

$$\lambda_j^{-1} U_1^{a(p^e, j)} U_2^{b(p^e, j)} Q_{p^e, j} = U_1^{p^e a} U_2^{p^e b} Q_0^{p^e},$$

with  $a \geq A_1(x)$ ,  $b \geq A_2(x)$  and  $B_0 := \deg(Q_0) \leq C(u, z)$ . Let  $W_3 := U_3 + U_1^a U_2^b Q_0$ , we get

$$F_j = \lambda_j W_3^{j\epsilon(x)} + \sum_{1+p^e \leq i \leq j\epsilon(x)-1} W_3^{j\epsilon(x)-i} U_1^{a_0(i,j)} U_2^{b_0(i,j)} Q_{0,i,j}(U_1, U_2),$$

$$a_0(i, j) \geq iA_1(x), \quad b_0(i, j) \geq iA_2(x), \quad d_0(i, j) := \deg(Q_{0,i,j}) \leq iC(u, z), \quad \text{or } Q_{0,i,j} = 0.$$

Let  $w_3 \in R$  such that  $\text{in}_{v_B}(w_3) = W_3$ , then, with  $w'_3 = w_3/u_1$ ,  $W'_3 = \text{in}_w(w'_3)$ :

$$F'_j = \lambda_j W'_3{}^{j\epsilon(x)} + \deg_{W'_3} < j\epsilon(x) - p^e.$$

Let  $v_3 = w'_3 + \lambda' u_1^{a'}$ , with  $a' \geq B(u, z) - 1$  and  $\lambda' \in R'$ ,  $\lambda'$  not divisible by  $u'_1$ . Then we conclude as above:

$$A_1(x') = B(u, z) - 1, \quad \beta(x') \leq b = \beta_0 \leq C(u, z)/d.$$

**VI.4.4.6** From now on, we assume **VI.4.4.2, VI.4.4.3, VI.4.4.4, VI.4.4.5** and for  $i < p^e$   $Q_{i,j} = 0$  and  $\lambda_j^{-1} U_1^{a(p^e, j)} U_2^{b(p^e, j)} Q_{p^e, j}$  is NOT a  $(p^e)^{\text{th}}$ -power (c.f.(5)). In particular  $e \geq 1$ . Let us recall the following elementary lemma [CP2, II.5.3.2].

**Lemma VI.4.4.7** Let  $F(U_1, U_2) \in k(x)[U_1, U_2]$  be a homogeneous polynomial of degree  $d_0 \geq 0$ , and  $a, b \in \mathbb{N}$  be such that  $U_1^a U_2^b F(U_1, U_2) \notin (k(x)[U_1, U_2])^p$ .

Let  $x' \in \text{Speck}(x)[\frac{U_2}{U_1}]$  be a closed point with ideal  $(v := P(1, \frac{U_2}{U_1}))$ ,  $P \in k(x)[U_1, U_2]$  a nonzero homogeneous irreducible polynomial of degree  $d := [k(x') : k(x)]$ , unitary in  $U_2$ .

Let  $A \in T' := k(x)[U_1, \frac{U_2}{U_1}]_{(U_1, v)}$  be such that  $U_1^{a+b+d_0}$  (resp.  $U_1^{a+b+d_0} v^b$ ) divides  $A^p$  in  $T'$  if  $P \neq U_2$  (resp.  $P = U_2$ ). There exists an integer  $c \geq 0$  such that

$$U_1^{a+b+d_0} \left(\frac{U_2}{U_1}\right)^b F\left(1, \frac{U_2}{U_1}\right) + A^p \equiv U_1^{a+b+d_0} \left(\frac{U_2}{U_1}\right)^b \gamma v^c \pmod{(U_1^{a+b+d_0+1} T')},$$

with  $\gamma$  invertible in  $T'$ . We have the following estimates for  $c$ :

- (i) if  $P \neq U_2$  (resp.  $P = U_2$ ), then  $c \leq 1 + \frac{d_0}{d}$  (resp.  $c \leq d_0$ );
- (ii) if  $P \neq U_2$ , then  $c < p(1 + \lfloor \frac{d_0}{pd} \rfloor)$  (equivalently: for every  $N \in \mathbb{N}$  such that  $\frac{d_0}{pd} < N$ , we have  $c < Np$ );
- (iii) if  $d_0 \geq 1$  and  $b = 0$ , then  $c \leq i$ .

Let  $f < e$  be the integer defined by:

$$\lambda_j^{-1} U_1^{a(p^e, j)} U_2^{b(p^e, j)} Q_{p^e, j} = Q_0^{p^h}, \quad Q_0 \text{ is not a } p^{\text{th}} \text{ - power.}$$

Let

$$Q_0 =: (U_1^a U_2^b F(U_1, U_2)),$$

with  $p^h a = a(p^e, j)$ ,  $p^h b = b(p^e, j)$ ,  $p^h d_0 = d(p^e, j)$ , where  $d_0 := \deg(Q_0)$ . In particular,

$$d_0 \leq p^{e-h} C(u, z).$$

Then,

$$F'_j = \lambda_j (V_3^{j\epsilon(x)} + V_3^{j\epsilon(x)-p^e} \lambda_j^{-1} (U_1^{p^e(B(u, z)-1)} \overline{u_2}^{-b(i_0, j)} Q_{i_0, j}(1, \overline{u_2}) + \overline{\lambda}^{p^e}) + \deg_{V_3} < j\epsilon(x) - p^e).$$

By lemma **VI.4.4.7**,  $(\overline{u_2}^{b(i_0,j)} Q_{i_0,j}(1, \overline{u_2}) + \overline{\lambda}^{p^e}) = (\gamma v'^c)^{(p^h)} \neq (0)$ , so

$$A_1(x') = B(u, z) - 1, \quad p^e \beta(x') \leq p^h c.$$

Furthermore, by (i) above,  $c \leq 1 + \frac{d_0}{d}$ , so:

$$p^e \beta(x') \leq p^h \left(1 + \frac{d_0}{d}\right) \leq p^h + p^h \frac{p^{e-h} C(u, z)}{d} = p^h + \frac{p^e C(u, z)}{d},$$

$$\beta(x') \leq \frac{1}{p} + \frac{C(u, z)}{d}.$$

By (ii),

$$p^e \beta(x') < p^h p \left(1 + \left\lfloor \frac{d_0}{pd} \right\rfloor\right) \leq p^{f+1} \left(1 + \left\lfloor \frac{p^{e-h} C(u, z)}{pd} \right\rfloor\right), \quad \beta(x') < 1 + \left\lfloor \frac{C(u, z)}{d} \right\rfloor,$$

which is **VI.4.4(iii)**. Now **VI.4.4(iv)** is a consequence of (iii) above.

**VI.4.5** Proof of Theorem VI.1: some cases with  $\gamma(u, z) = 1$ .

The strategy to make the proof is to make a list of different subcases covering this case, from the easiest to the most difficult and to prove them up to the former ones.

All cases ( $\beta(u, z) < 1$  and  $\text{div}(u_1 u_2) \not\subseteq E$ ) are covered by **VI.4.5.3** below. All cases with ( $\gamma(u, z) < 1$  and  $\text{div}(u_1 u_2) \subseteq E$ ) are dealt with in **VI.4.5.6**. This includes in particular all remaining cases with  $\beta(u, z) < 1$  since  $C(u, z) \leq \beta(u, z)$  for  $\text{div}(u_1 u_2) \subseteq E$  (see definition **VI.4.1**).

**LEMMA VI.4.5.1** *With assumptions as in VI.4.3, assume furthermore that*

$$A_1(u, z) < 1, \quad \beta(u, z) < 1. \tag{1}$$

*There exist well prepared parameters  $(z', u')$  at  $x'$  such that*

$$(A_1(u', z'), \beta(u', z')) <_{\text{lex}} (A_1(u, z), \beta(u, z)), \quad \text{and } \beta(u', z') < 1.$$

*Proof.* This is a direct consequence of lemma **VI.4.4**.

**LEMMA VI.4.5.2** *With assumptions as in VI.4.3, assume furthermore that*

$$\beta(u, z) < 1, \quad A_1(u, z) \geq 1, \quad (d_1(x) + d_3(x) + \epsilon(x) \geq 1 \text{ or } E \subseteq \text{div}(u_1 u_3)). \tag{2}$$

*Then  $\mathcal{C}_1 := V(z, u_1, u_3)$  is an  $\epsilon$ -permissible algebraic curve on  $X$ .*

*Let  $\pi : (Z', x') \rightarrow (Z, x)$  be the blowing up along  $\mathcal{C}_1$ ,  $X' \subset Z'$  the strict transform of  $X$  and  $x' \in X'$  the center of  $v$  and assume  $\iota(x') = \iota(x)$ . Then  $(z', u') = (z/u_1, u_1, u_2, u_3/u_1)$  are well prepared parameters at  $x'$  and we have*

$$(A_1(u', z'), \beta(u', z')) = (A_1(u, z) - 1, \beta(u, z)).$$

*Proof.* This follows from proposition **VI.4.2**.

**REMARK VI.4.5.3** Lemmas **VI.4.5.1** **VI.4.5.2** prove theorem **VI.1** when (Hyp) is true and

$$\beta(u, z) < 1, A_1(u, z) \geq 1, (d_3(x) + \epsilon(x) \geq 1 \text{ or } E \subset \text{div}(u_1 u_3)). \quad (3)$$

Indeed,  $d_3(x) + \epsilon(x) = d_3(x') + \epsilon(x')$  if  $\iota(x') = \iota(x)$  after blowing up. If  $E \subseteq \text{div}(u_1 u_3)$ , then  $E' \subseteq \text{div}(u'_1 u'_3)$ , so condition (3) remains stable after blowing up. A descending induction on  $A_1(u, z)$  ends the proof.

**VI.4.5.4** Proof of Theorem VI.1 in the case  $C(u, z) = 0$ .

In that special case, we have  $\beta(u, z) = A_2(u, z)$ . When  $A_2(u, z) < 1$  and  $A_1(u, z) < 1$ , **VI.4.5.1** gives the result. Let us see the other cases:

$$A_1(u, z) \geq 1 \text{ or } A_2(u, z) \geq 1. \quad (1)$$

*Case 1.*  $d_3(x) + \epsilon(x) \geq 1$ . We may assume  $A_2(u, z) \geq 1$  by symmetry on  $u_1, u_2$ . Then  $\mathcal{C} := (z, u_2, u_3)$  is an  $\epsilon$ -permissible algebraic curve by proposition **VI.4.2** and we get

$$A_1(x') = A_1(x), \quad A_2(x') = A_2(x) - 1, \quad C(u', z') = 0$$

after blowing up along  $\mathcal{C}$  if  $\iota(x') = \iota(x)$ . A descending induction on  $A_2(x)$  and **VI.4.5.2** give the result. From now on, we assume

$$d_3(x) + \epsilon(x) < 1. \quad (2)$$

*Case 2.*  $d_3(x) + d_i(x) + \epsilon(x) < 1$ ,  $i = 1$  and  $i = 2$ . We blow up  $\{x\}$  in this case. By proposition **VI.4.3**(ii) or (iv), we have

$$d_3(x') = d_3(x), \quad d_2(x') = d_2(x), \quad d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1 < d_1(x), \quad C(u', z') = 0,$$

$$\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x') < \delta(x)$$

if  $\iota(x') = \iota(x)$  and  $x'$  is the origin of a chart. Otherwise, lemma **VI.4.4**(ii)(iii) gives **VI.4.5.2**(2) at  $x'$  for some well prepared r.s.p.  $(z', u')$ .

*Case 3.*  $d_3(x) + \epsilon(x) < 1$ ,  $d_3(x) + d_i(x) + \epsilon(x) \geq 1$  for some  $i = 1$  or  $i = 2$ ,  $A_j(x) \geq 1$ ,  $j = 1, 2$ . We choose an  $\epsilon$ -permissible blowing up center  $Y$  as follows:

if  $V(z, u_i, u_3)$ , for  $i = 1, 2$  are  $\epsilon$ -permissible, then  $Y := (z, u_i, u_3)$  with

$$(A_i(x), d_i(x)) \geq (A_{i'}(x), d_{i'}(x)), \quad \{i, i'\} = \{1, 2\};$$

if  $V(z, u_i, u_3)$  is  $\epsilon$ -permissible for a unique  $i \in \{1, 2\}$ , then  $Y := (z, u_i, u_3)$ ;

if  $V(z, u_i, u_3)$  is not  $\epsilon$ -permissible for  $i \in \{1, 2\}$ , then  $Y := \{x\}$ .

Let  $n(x) := 2$  if  $(A_1(x), d_1(x)) = (A_2(x), d_2(x))$ ,  $n(x) := 1$  otherwise. If  $\iota(x') = \iota(x)$ , we claim that  $x'$  satisfies **VI.4.6.3**(2) or falls into cases 1,2 above, or there is a well prepared r.s.p.  $(z', u')$  at  $x'$  with  $C(u', z') = 0$  and

$$(\max_{i=1,2}\{A_i(x)\}, \max_{i=1,2}\{d_i(x)\}, n(x)) <_{\text{lex}} (\max_{i=1,2}\{A_i(x')\}, \max_{i=1,2}\{d_i(x')\}, n(x')). \quad (3)$$

Note that this ends the proof of the case  $C(u, z) = 0$ , since (3) can repeat but finitely many times. To prove the claim, first assume that  $Y = (z, u_1, u_3)$ . By proposition **VI.4.2**, we have

$$A_1(x') = A_1(x) - 1, \quad A_2(x') = A_2(x), \quad d_1(x') = d_1(x) + d_3(x) + \epsilon(x) - 1 < d_1(x), \quad d_2(x') = d_2(x)$$

and the result is clear. The case  $Y = (z, u_2, u_3)$  is similar.

Assume now that  $Y = \{x\}$ . By symmetry on  $u_1, u_2$ , we assume  $A_2(x) \geq 1$ . If  $x' \in \text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$  and  $x'$  is the point with parameters  $(z/u_1, u_1, u_2/u_1, u_3/u_1)$  (origin of the first chart), we get

$$A_1(x') = A_1(x) + A_2(x) - 1, \quad A_2(x') = A_2(x), \quad d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1, \quad d_2(x') = d_2(x).$$

Since  $A_2(x) \geq 1$  and  $V(z, u_2, u_3)$  is not  $\epsilon$ -permissible, we have

$$d_2(x) + d_3(x) + \epsilon(x) < 1, \quad d_1(x) + d_3(x) + \epsilon(x) \geq 1 \quad \text{and} \quad A_1(x) < 1.$$

We get  $d_1(x) > d_2(x)$  and  $d_1(x') = d_1(x) + d_2(x) + d_3(x)\epsilon(x) - 1 < d_1(x)$ ,  $A_1(x') < A_1(x) \leq A_2(x') = A_2(x)$  which proves the claim.

If  $x' \in \text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$  and  $x'$  is not the above point, we have **VI.4.5.2(2)** at  $x'$  for some well prepared r.s.p.  $(z', u')$  at  $x'$  by lemma **VI.4.4(ii)(iii)**.

If  $x' \in \text{Spec}R[z/u_2, u_1/u_2, u_3/u_2] \subset Z'$  and  $x'$  is the point with parameters  $(z/u_2, u_1/u_2, u_2, u_3/u_2)$  (origin of the second chart), we get

$$A_2(x') = A_1(x) + A_2(x) - 1, \quad A_1(x') = A_1(x), \quad d_2(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1, \quad d_1(x') = d_1(x).$$

We have  $A_1(x) < 1$ : otherwise, as  $V(z, u_1, u_3)$  is not  $\epsilon$ -permissible, this would imply  $d_1(x) + d_3(x) + \epsilon(x) < 1$ ,  $d_2(x) + d_3(x) + \epsilon(x) \geq 1$ , hence  $V(z, u_2, u_3)$   $\epsilon$ -permissible since  $A_2 \geq 1$ : a contradiction. We now get  $A_1(x) < 1 \leq A_2(x)$  and  $A_2(x') = A_1(x) + A_2(x) - 1 < A_2(x)$  which completes the proof of the claim.

**VI.4.5.5** Proof of Theorem **VI.1** in the case  $C(u, z) < 1$ ,  $\text{div}(u_1 u_2) \subseteq E$ .

We perform the sequence of local blowing ups

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n) \longleftarrow \cdots,$$

where  $x_0 = x$ ,  $x_i \in X_i$  ( $X_i$  denoting the strict transform of  $X$ ),  $0 \leq i \leq n$ , is the center of  $v$ , each blowing up center is  $Y_i = \{x_i\}$ .

If  $\iota(x_1) = \iota(x)$ , and is not the origin of a chart (*viz.* case 3 in **VI.4.5.4**), then  $x_1$  verifies the assumptions of **VI.4.5.1** by lemma **VI.4.4(iii)**.

Assume now that  $\iota(x_i) = \iota(x)$  and  $x_i$  is the origin of a chart for all  $i \geq 0$ . By **VI.4.3(ii)(iv)**,  $x_i$  verifies the assumptions of **VI.4.5.5** and  $C(x_{i+1}) \leq C(x_i)$  for all  $i \geq 0$ . It is then a very well known fact that  $C(x_i) = 0$  for  $i \gg 0$ , i.e. the assumptions of **VI.4.5.4** are satisfied.

**VI.4.5.6** End of the proof of Theorem **VI.1**. As our invariants  $C(u, z)$ ,  $\beta(u, z)$  are discrete, the next lemma shows that we will reach one of the cases (ii)  $\beta(u, z) < 1$  or (iii)  $C(u, z) < 1$ . This ends the proof of theorem **VI.1** (see comments right after **VI.4.5.**).

**LEMMA VI.4.5.7.** *With assumptions as in **VI.4.3**, consider the sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n) \longleftarrow \cdots,$$

where  $x_0 = x$ ,  $x_i \in X_i$  ( $X_i$  denoting the strict transform of  $X$ ),  $0 \leq i \leq n$ , is the center of  $v$ , each blowing up center is  $Y_i = \{x_i\}$ .

Assume that  $\iota(x_i) = \iota(x)$  for all  $i \geq 0$ . There exists some  $i \geq 0$  and a well prepared r.s.p.  $(z_i, u_{1,i}, u_{2,i}, u_{3,i})$  at  $x_i$  (w.r.t. the reduced inverse image of  $E$  in  $Z_i$ ) such that one of the following holds:

- (i)  $\gamma(u_i, z_i) < \gamma(u, z)$ ;
- (ii)  $\beta(u_i, z_i) < 1$ ;
- (iii)  $C(u_i, z_i) < 1$ ;

*Proof.* This breaks up in three cases:

*Case 1.* for all  $i \geq 0$ , the point  $x_i$  is the origin of one of the two charts of **VI.4.3**, i.e. we are always in one of the cases **VI.4.3(ii)(iv)**. Then  $C(u_i, z_i) = 0$  for  $i \gg 0$  (see **VI.4.5.5** above).

*Case 2.* for all  $i \geq 0$ , ( $x_i$  is rational over  $x$  and belongs to the first chart), i.e.  $x_i$  is a rational point not on the strict transform of  $\text{div}(u_1)$ . By **VI.4.3(ii)(iii)**,  $x_1$  has a r.s.p. of the form  $(z/u_1, u_1, u_2/u_1 + \mu_1, u_3/u_1)$  for some  $\mu_1 \in R$ . A well prepared r.s.p. is of the form

$$z/u_1 + \lambda_1 u_1, u_1, u_2/u_1 + \mu_1, u_3/u_1 + \mu_2 u_1, \lambda_1, \mu_1, \mu_2 \in R,$$

with  $\mu_2 = 0$  if  $\text{div}(u_3) \subseteq E$ . Then there exists a regular formal curve  $\mathcal{C}$  of the form  $\mathcal{C} = V(\widehat{z}, \widehat{u}_2, \widehat{u}_3)$  on  $\text{Spec}(\widehat{R}/(h))$ , transverse to  $E_i$  for all  $i \geq 0$ ,  $\widehat{u}_3 = u_3$  if  $\text{div}(u_3) \subseteq E$ , whose strict transform goes through all points  $x_i$ ,  $i \geq 0$ . Necessarily  $\mathcal{C} \subseteq \text{Sing}_m(X)$ , so we may assume that  $\mathcal{C} \subset \text{div}(u_j) \subseteq E$  for  $j = 2$  or  $j = 3$ . In particular, we may take  $\widehat{u}_j = u_j$  for  $j = 2$  or  $j = 3$ . This implies that  $v(u_j) > v(u_1^n) = nv(u_1)$  for all  $n \geq 1$ : a contradiction, since our given valuation  $v$  has rank one.

*Case 3.*  $E \subseteq \text{div}(u_1 u_3)$  and we are not in case 2, i.e. there exists  $i_0 \geq 0$  such that either  $x_{i_0+1}$  is not rational over  $x_{i_0}$  or  $E_{i_0+1}$  has one more component than  $E_{i_0}$ ,  $i_0$  minimal. Suppose  $\beta(x_{i_0}) \geq 1$ .

If  $x_{i_0+1}$  is not rational over  $x_{i_0}$ , we get

$$\beta(x) \geq \beta(x_{i_0}) > \beta(x_{i_0+1})$$

by **VI.4.3(ii)(iii)** and **VI.4.4(iii)**: note that  $C(u, z) \leq \beta(u, z)$  since  $\text{div}(u_2) \not\subseteq E$  and

$$1 + \left\lfloor \frac{x}{2} \right\rfloor \leq x \text{ for every } x \geq 1.$$

If  $E_{i_0+1}$  has one more component than  $E_{i_0}$ , we have

$$C(u_{i_0+1}, z_{i_0+1}) \leq \frac{\beta(x_{i_0})}{2} \leq \frac{\beta(x)}{2}$$

by **VI.4.3(ii)(iii)(iv)**. This gives **VI.4.5.7(iii)** if  $1 \leq \beta(x) < 2$ .

Now,  $\gamma(x_{i_0+1}) = 1 + \lfloor C(u_{i_0+1}, z_{i_0+1}) \rfloor$ ,  $\gamma(x) = \lceil \beta(x) \rceil$ , so we get **VI.4.5.7(i)** if  $\beta(x) > 2$ .

Assume that

$$\beta(x_{i_0}) = \beta(x) = 2. \tag{1}$$

Since  $\gamma(x) = 2$ , we get **VI.4.5.7(i)** unless  $\gamma(x_i) = 2$  for  $i \geq 0$  by proposition **VI.4.3**.

Let  $i_1 > i_0$  be the largest index such that  $E_i$  has as many components as  $E_{i_0}$  for  $i_0 \leq i \leq i_1$ . We may assume  $i_1 < +\infty$  by case 1 and we have

$$\gamma(x_{i_1}) = 2 = 1 + \lfloor C(u_{i_1}, z_{i_1}) \rfloor.$$

By **VI.4.4(iii)**, we get  $\beta(x_{i_1+1}) < 2$ . Now the point  $x_{i_1+1}$  falls into case 2 above or into case 3 with (1) not satisfied. This concludes the proof in case 3.

The end of the proof of **VI.4.5.7** is just a logical game: we reach the assumption  $E \subseteq \text{div}(u_1 u_3)$  for some point  $x_i$ ,  $i \geq 0$  provided we are not in case 1.

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