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### ▶ To cite this version:

Ulrik Brandes, Sabine Cornelsen, Barbara Pampel, Arnaud Sallaberry. Path-Based Supports for Hypergraphs. Journal of Discrete Algorithms, 2012, 14, pp.248-261. 10.1016/j.jda.2011.12.009 . hal-00735894

## HAL Id: hal-00735894 https://hal.science/hal-00735894

Submitted on 27 Mar 2013

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### Path-Based Supports for Hypergraphs

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#### 7 Abstract

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A path-based support of a hypergraph H is a graph with the same vertex set as H in which each hyperedge induces a Hamiltonian subgraph. While it is  $\mathcal{NP}$ -hard to decide whether a path-based support has a monotone drawing, to determine a path-based support with the minimum number of edges, or to decide whether there is a planar path-based support, we show that a path-based tree support can be computed in polynomial time if it exists.

<sup>8</sup> Keywords: graph algorithm, graph drawing, hypergraph, metro map layout

#### 9 1. Introduction

A hypergraph is a pair H = (V, A) where V is a finite set and A is a (multi-)set of non-empty subsets of V. The elements of V are called *vertices* and the elements of A are called *hyperedges*. A support (or host graph) of a hypergraph H = (V, A) is a graph G = (V, E) such that each hyperedge of H induces a connected subgraph of G, i.e., such that the graph  $G[h] := (h, \{e \in E, e \subseteq h\})$  is connected for every  $h \in A$ . See Fig. 1(b) for an example.

Applications for supports of hypergraphs are, e.g., in hypergraph coloring [2, 3], databases [4], or hypergraph drawing [5, 6, 7, 8]. E.g., see Fig. 1 for an application of a support for designing Euler diagrams. An *Euler diagram* of a hypergraph H = (V, A)is a drawing of H in the plane in which the vertices are drawn as points and each hyperedge  $h \in A$  is drawn as a simple closed region containing the points representing the vertices in h and not the points representing the vertices in  $V \setminus h$ . There are various well-formedness conditions for Euler diagrams, see e.g. [9, 8].

Recently, many papers have been devoted to the problem of deciding which classes of hypergraphs admit what kind of supports. It can be tested in linear time whether a hypergraph has a support that is a tree [10], a path or a cycle [7]. It can be decided in polynomial time whether a hypergraph has a tree support with bounded degrees [7] or a cactus support [11]. A minimum weighted tree support can be computed in polynomial time [12]. It is  $\mathcal{NP}$ -complete to decide whether a hypergraph has a planar support [5], a compact support [5, 6] or a 2-outerplanar support [7]. A support with the minimum

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A preliminary version of this paper was presented at the 21st International Workshop on Combinatorial Algorithms (IWOCA 2010) and appears in the corresponding proceedings [1].

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Figure 1: Three representations of the hypergraph H = (V, A) with hyperedges  $h_1 = \{v_1, v_2\}, h_2 = \{v_2, v_3\}, h_3 = \{v_3, v_4\}, h_4 = \{v_4, v_5\}, h_5 = \{v_5, v_6\}, h = \{v_2, v_3, v_4, v_5\}, h' = \{v_2, v_3, v_4, v_5, v_7\}, and V = \{v_1, \dots, v_7\}.$ 

number of edges can be computed in polynomial time if the hypergraph is closed under
 intersections [7]. If the set of hyperedges is closed under intersections and differences, it
 can be decided in polynomial time whether the hypergraph has a planar or outerplanar
 support [11].

In this paper we consider a restriction on the subgraphs of a support that are induced 34 by the hyperedges. A support G of a hypergraph H = (V, A) is called *path-based* if the 35 subgraph G[h] contains a Hamiltonian path for each hyperedge  $h \in A$ , i.e., G[h] contains a 36 path that contains each vertex of h. This definition was motivated by by the aesthetics of 37 metro map layouts. I.e., the hyperedges could be visualized as lines along the Hamiltonian 38 path in the induced subgraph of the support like the metro lines in a metro map. See 39 Fig. 2 for examples of metro maps, Fig. 3 for an example of natural sciences drawn in the 40 metro map anthology, and Fig. 1(c) and 6(f) for a representation of some hyperedges in 41 such a metro map like drawing. For metro map layout algorithms see, e.g., [13, 14]. 42

We briefly consider monotone, planar, and minimum path-based supports. Our main result is a characterization of those hypergraphs that have a path-based tree support and a polynomial time algorithm for constructing path-based tree supports if they exist. E.g., Fig. 1 shows an example of a hypergraph H = (V, A) that has a tree support but no path-based tree support. However, the tree support in Fig. 1(b) is a path-based tree support for  $(V, A \setminus \{V\})$ .

The contribution of this paper is as follows. In Section 2 we give the necessary defi-49 nitions. We then briefly discuss monotone path-based supports in Section 3 and mention 50 that finding a minimum path-based support or deciding whether there is a planar path-51 based support, respectively, is  $\mathcal{NP}$ -hard. We consider path-based tree supports in Sect. 4. 52 In Section 4.1 we review a method for computing tree supports using the Hasse diagram. 53 In Section 4.2 we show how to apply this method to test whether a hypergraph has a 54 path-based tree support and if so how to compute one in polynomial time. Finally, in 55 Section 4.3 we discuss the run time of our method. 56



Figure 2: Local train map of Zurich (www.zvv.ch) and the metro map of Amsterdam (www.amsterdam.info). In (b) the union of all lines forms a tree.

#### 57 2. Preliminaries

In this section, we give the necessary definitions that were not already given in the 58 introduction. Throughout this paper let H = (V, A) be a hypergraph. We denote by 59 n = |V| the number of vertices, m = |A| the number of hyperedges, and  $N = \sum_{h \in A} |h|$ 60 the sum of the sizes of all hyperedges of a hypergraph H. The size of the hypergraph H is 61 then N + n + m. A hypergraph is a graph if all hyperedges contain exactly two vertices. 62 A hypergraph H = (V, A) is closed under intersections if  $h_1 \cap h_2 \in A \cup \{\emptyset\}$  for  $h_1, h_2 \in A$ . 63 We say that two hyperedges  $h_1, h_2$  overlap if  $h_1 \cap h_2 \neq \emptyset$ ,  $h_1 \not\subseteq h_2$ , and  $h_2 \not\subseteq h_1$ . A 64 hypergraph H = (V, A) is *connected* if for any pair of vertices  $v, w \in V$  there is a sequence 65 of hyperedges  $h_1, \ldots, h_\ell \in A$  such that  $v \in h_1, w \in h_\ell$ , and  $h_i \cap h_{i+1} \neq \emptyset, i = 1, \ldots, \ell - 1$ . 66 The Hasse diagram of a hypergraph H = (V, A) is the directed acyclic graph with 67 vertex set  $A \cup \{\{v\}; v \in V\}$  and there is an edge  $(h_1, h_2)$  if and only if  $h_2 \subsetneq h_1$  and there 68 is no set  $h \in A$  with  $h_2 \subsetneq h \subsetneq h_1$ . Fig. 1(a) shows an example of a Hasse diagram. Let 69 (v, w) be an edge of a directed acyclic graph. Then we say that w is a *child* of v and v 70 a parent of w. For a descendant d of v there is a directed path from v to d while for an 71 ancestor a of v there is a directed path from a to v. A source does not have any parents, 72 a sink no children and an inner vertex has at least one parent and one child. 73

#### 74 3. Path-Based Supports

In a metro map like drawing of a hypergraph vertices are drawn as disjoint simple closed regions in the plane and each hyperedge h is drawn as a curve  $C_h$  with the end points within the regions of different vertices of h, visiting the region of every vertex of h exactly once, not visiting the vertices not in h, and such that the pieces of  $C_h$  within the region of a vertex or between two such regions are simple. A path-based support of a hypergraph H = (V, A) is a graph G such that G[h] contains a spanning path for every hyperedge  $h \in A$ .

On one hand, a metro map like drawing of a hypergraph H = (V, A) induces a pathbased support G = (V, E) of H: For a hyperedge  $h \in A$  let  $p_h : v_1, \ldots, v_{|h|}$  be the



Figure 3: A map of modern science (www.crispian.net).

sequence of vertices of h in the order in which they are visited by the curve representing h. Starting with an empty set E add for every hyperedge  $h \in A$  with  $p_h : v_1, \ldots, v_{|h|}$  the edges  $\{v_{i-1}, v_i\}, i = 2, \ldots, |h|$  to E. On the other hand, if we have a path-based support G of H and we fix for every hyperedge  $h \in A$  a spanning path  $p_h$  of G[h] then this induces a metro map like drawing of H.

In order to have a readable metro map like drawing of a hypergraph it is typically desirable to draw any curve representing a hyperedge without self intersection or even monotone.

#### 92 3.1. Monotone Path-Based Supports

A drawing of a graph is a mapping of each vertex to a distinct point in the plane and 93 of each edge to a simple curve between the image of its adjacent vertices not containing 94 the image of any other vertex. In a straight-line drawing of a graph each edge is drawn 95 as a line segment. Given a drawing of G, a path p of G is monotone with respect to a 96 straight line  $\ell$  - called the *axis of monotonicity* – if every line perpendicular to  $\ell$  intersects 97 the drawing of p in at most one point. Note that a path p in a straight-line drawing is 98 monotone with respect to the axis  $\ell$  if and only if the orthogonal projections of the vertices 99 of p on  $\ell$  appear along  $\ell$  in the order induced by p. 100

Let G = (V, E) be a path-based support of a hypergraph H = (V, A). A drawing of G is monotone with respect to H if for each hyperedge  $h \in A$  there is a spanning path  $p_h$  of G[h] and a straight line  $\ell_h$  such that  $p_h$  is monotone with respect to the axis  $\ell_h$ . Gis a monotone path-based support of H if G has a monotone drawing with respect to H.

**Remark 1.** If G has a monotone drawing with respect to a hypergraph H then G has a straight-line drawing that is monotone with respect to H with the same axes of monotonicity.

PROOF. Let a drawing  $\mathcal{D}$  of G that is monotone with respect to H = (V, A) be given and let  $p_h, h \in A$  be a spanning path of G[h] that is monotone with respect to the axis  $\ell_h$ . If for each edge  $\{v, w\}$  of G the line segment between v and w does not contain any vertex of G other than v or w then the straight-line drawing of G in which the vertices are mapped to the same points as in  $\mathcal{D}$  is monotone with respect to H. Consider now for two vertices v, w in a hyperedge h the distances  $\operatorname{dist}_h(v, w)$  between the orthogonal projections of v and w to  $\ell_h$ . Let  $\Delta$  be the minimum of all distances  $\operatorname{dist}_h(v, w)$  over all  $h \in A$  and  $v, w \in h$  with  $v \neq w$ . Let  $0 < \varepsilon \leq \Delta/3$ .

Consider now the vertices of V in an arbitrary order  $v_1, \ldots, v_n, n = |V|$ . For  $k = 1, \ldots, n$ , we can now place  $v_k$  on the circle with radius  $\varepsilon$  around the position of  $v_k$  in  $\mathcal{D}$ but not on the intersection with the line through the already fixed drawings of  $v_i$  and  $v_j$ ,  $1 \leq i < j < k$ . The corresponding straight-line drawing is monotone with respect to Hwith the axes of monotonicity  $\ell_h, h \in A$ .

121 **Remark 2.** Not every path based support of a hypergraph is monotone.

122  $\ell \leq 5, i < k, \{i, j\} \cap \{k, \ell\} = \emptyset$  be an index set representing unordered pairs of disjoint 123 edges of the complete graph  $K_5$ . Let  $V_I = \{v_i; i = 1, \ldots, 5\} \cup \{v_{i,j,k,\ell,x}; (i,j,k,\ell) \in V_i\}$ 124  $I, x = 1, \dots, 3\}, \text{ let } h_{ijk\ell} = \{v_i, v_{i,j,k,\ell,1}, v_j, v_{i,j,k,\ell,2}, v_k, v_{i,j,k,\ell,3}, v_\ell\}, (i, j, k, \ell) \in I, \text{ let } i_{ijk\ell} \inI, \text{$ 125  $A_I = \{h_{ijk\ell}; (i, j, k, \ell) \in I\}, \text{ and let } H_I = (V_I, A_I). \text{ Let } E \text{ contain the edges } \{v_i, v_{i,j,k,\ell,1}\},$ 126  $\{v_{i,j,k,\ell,1}, v_j\}, \{v_j, v_{i,j,k,\ell,2}\}, \{v_{i,j,k,\ell,2}, v_k\}, \{v_k, v_{i,j,k,\ell,3}\}, \{v_{i,j,k,\ell,3}, v_\ell\}$  for  $(i, j, k, \ell) \in I$ . 127 The resulting path-based support G = (V, E) of  $H_I$  is shown in Fig. 4(a). Note that 128  $G[h_{ijk\ell}]$  is a path for any hyperedge  $h_{ijk\ell} \in A$  visiting the vertices  $v_i, v_j, v_k, v_\ell$  in this 129 order. Consider now any drawing of G. Since a  $K_5$  is not planar, there are two straight 130 line segments  $\overline{v_i v_j}, \overline{v_k, v_\ell}, (i, j, k, \ell) \in I$  that intersect. Hence, the path  $G[h_{ijk\ell}]$  cannot be 131 drawn monotonously. 132

<sup>133</sup> **Remark 3.** Every hypergraph has a monotone path-based support.

**PROOF.** Order the vertices of H = (V, A) with respect to an arbitrary ordering <. The 134 support  $G_{\leq} = (V, E_{\leq})$  of H with respect to the ordering  $\leq$  is constructed as follows. 135 For each hyperedge  $\{v_1, \ldots, v_k\} \in A$  with  $v_1 < \cdots < v_k$  the edge set  $E_{\leq}$  contains the 136 edges  $\{v_{i-1}, v_i\}, i = 1, \dots, k$ . Assume now that in a drawing of  $G_{\leq}$  the x-value of a 137 vertex v is smaller than the x-value of the vertex w if v < w and that the edges are 138 drawn monotonously in x-direction. Then for each hyperedge  $h = \{v_1, \ldots, v_k\} \in A$  with 139  $v_1 < \cdots < v_k$  the path  $p_h: v_1, \ldots, v_k$  is drawn monotonously with respect to the x-axis. 140 See Fig 4(c) for an example. 141

Note that the problem of deciding whether a given support is a support with respect
to an ordering and if so, finding such an ordering, is closely related to the betweenness
problem [15].

- <sup>145</sup> **Theorem 1.** Given a support G of a hypergraph H
- 1. it is  $\mathcal{NP}$ -hard to decide whether G is a monotone path-based support of H, and
- <sup>147</sup> 2. it is  $\mathcal{NP}$ -complete to decide whether there exists an ordering < of the vertex set <sup>148</sup> such that G is the support of H with respect to <,

even if G has the minimum number of edges among all supports of H.



(c) minimum path-based support w.r.t. an ordering

Figure 4: Three different supports for the hypergraph  $H_I$  introduced in Section 3.1. The small black vertices are the vertices  $v_{\sigma,x}, \sigma \in I, x = 1, 2, 3$ . The thick red path indicates the hyperedge  $h_{1324}$ .

#### 150 PROOF.

1. Consider an instance of the strictly monotone trajectory drawing problem consisting 151 of a set of paths P on a set of vertices  $V_t$ . It is  $\mathcal{NP}$ -hard to decide whether the 152 vertices can be mapped to points in the plane such that each path is monotone with 153 respect to some axis (one for each path) [16]. 154 Consider the hypergraph H = (V, A) with V containing  $V_t$  and for each path  $p \in P$ 155 and each edge  $e \in p$  a vertex  $v_{ep}$ . The set A contains for each path  $p \in P$  a hyperedge 156  $h_p = \bigcup_{\{v,w\} \in p} \{v, v_{\{v,w\}p}, w\}$  as well as the hyperedges  $\{v, v_{\{v,w\}p}\}$  and  $\{v_{\{v,w\}p}, w\}$ 157 for each edge  $\{v, w\} \in p$ . The graph G = (V, E) with  $E = \bigcup_{p \in P} \bigcup_{e \in p} \{\{v, v_{ep}\}; v \in V\}$ 158 e is a path-based support of H and has the minimum number of edges among all 159 supports of H. G is monotone if and only if P is drawable with each path monotone 160 with respect to some axis. 161 Consider an instance of the betweenness problem consisting of a set of vertices  $V_b$ 162 and a set of constraints C. Each constraint  $c \in C$  consists of a sequence of three 163 vertices. It is  $\mathcal{NP}$ -complete to decide whether the vertices can be totally ordered 164 such that for each constraint c = (u, v, w) the vertex v is between the vertices u and 165 w [15]. 166 Consider the hypergraph H = (V, A) with V containing  $V_b$  and for each constraint 167  $c \in C$  vertices  $v_{c2}$  and  $v_{c4}$ . The set A contains for each  $c = (v_{c1}, v_{c3}, v_{c5}) \in C$  a 168 hyperedge  $h_c = \{v_{c1}, \dots, v_{c5}\}$  and hyperedges  $h_{ci} = \{v_{ci}, v_{c(i+1)}\}$  for  $1 \le i \le 4$ . 169 The graph G = (V, E) with  $E = \bigcup_{c \in C} \{h_{ci}; 1 \le i \le 4\}$  is a path-based support of 170 H and has the minimum number of edges among all supports of H. 171 There is an ordering < of V such that G is the support of H with respect to <172 if and only if for each constraint  $c = (v_{c1}, v_{c3}, v_{c5}) \in C$  the five vertices in  $h_v$  are 173 either ordered  $v_{c1} < v_{c2} < v_{c3} < v_{c4} < v_{c5}$  or  $v_{c5} < v_{c4} < v_{c3} < v_{c2} < v_{c1}$ . Since 174

the vertices  $v_{c2}$  and  $v_{c4}$  do not appear in a hyperedge  $h_{c'}$  for any constraint  $c \neq c'$ it follows that there is an ordering < of V such that G is the support of H with respect to < if and only if  $V_b$  can be totally ordered while satisfying all betweenness constraints in C.

#### 179 3.2. Minimum Path-Based Supports

Assuming that each hyperedge contains at least one vertex, each hypergraph H = (V, A) has a monotone path-based support G = (V, E) with at most N - m edges. Just take the support  $G_{\leq}$  with respect to an arbitrary ordering  $\leq$  of the vertex set V. It is, however,  $\mathcal{NP}$ -hard to find an ordering that minimizes the number of edges among all path-based supports of H with respect to an ordering of the vertex set [17].

Further, note that a path-based support that minimizes the number of edges among 185 all path-based support of a hypergraph H with respect to some ordering of the vertex 186 set might not be a path-based support of H with the minimum number of edges over all 187 path-based supports of H. E.g., consider the hypergraph  $H_I$  from the previous section 188 (Fig. 4) or the hypergraph H with hyperedges  $\{1, 2, 4\}$ ,  $\{1, 3, 4\}$ , and  $\{2, 3, 4\}$  for an 189 easier example: the unique minimum path-based support of H is a star centered at 4 190 which cannot be created from any ordering of the vertex set. The problem of finding a 191 minimum path-based support remains, however,  $\mathcal{NP}$ -hard. 192

Theorem 2. It is NP-hard to minimize the number of edges among all path-based supports (or among all monotone path-based supports) of a hypergraph – even if the hypergraph is closed under intersections.

PROOF. Reduction from Hamiltonian path. Let G = (V, E) be a graph. Let  $H = (V, E \cup \{V\} \cup \{\{v\}; v \in V\})$  and K = |E|. Note that any support of H contains G as a subgraph. Hence, H has a path-based support with at most K edges if and only if G is a path-based support of H which is true if and only if G contains a Hamiltonian path.  $\Box$ 

#### 200 3.3. Planar Path-Based Supports

A graph is *planar* if it has a drawing in which no pair of edges intersect but in common end points. For the application of Euler diagram like drawings, planar supports are of special interest. However, like for general planar supports, the problem of testing whether there is a path-based planar support is hard.

**Theorem 3.** It is  $\mathcal{NP}$ -complete to decide whether a hypergraph – even if it is closed under intersections – has a path-based planar support.

<sup>207</sup> PROOF. The support that Johnson and Pollak [5] constructed to prove that it is  $\mathcal{NP}$ -<sup>208</sup> complete to decide whether there is a planar support, was already path-based.  $\Box$ 

#### 209 4. Path-Based Tree Supports

In this section we show how to decide in polynomial time whether a given hypergraph has a path-based tree support. If such a support exists, it is at the same time a path-based support of minimum size, a monotone path-based support [18], and a planar path-based support. Moreover the intersection of any subset of hyperedges induces again a path in a path-based tree support. So far it is known how to decide in linear time whether there is a path-based tree support if  $V \in A$  [7].



Figure 5: Illustration of the augmented Hasse Diagram for the hypergraph H = (V, A) indicated in 5(a).  $\overline{A} = A \cup \{h_3^1, \{v_2\}, \dots, \{v_5\}\}$ . The two hyperedges  $h_1^2$  and  $h_2^2$  are both implied but no summery edges. They are not present in the augmented Hasse diagram. The summary hyperedge  $h_s$  is added to A'.

#### 216 4.1. Constructing a Tree Support from the Hasse Diagram

A support with the minimum number of edges and, hence, a tree support if one exists can easily be constructed from the Hasse diagram if the hypergraph is closed under intersections [7]. Note, however, that the number of intersections of any subset of hyperedges could be exponential in the size of the hypergraph.

To construct a tree support of an arbitrary hypergraph H = (V, A), it suffices to consider the *augmented Hasse diagram* – a representation of "necessary" intersections of hyperedges. The definition is as follows. First consider the smallest set  $\overline{A}$  of subsets of Vthat contains A and that is closed under intersections. Consider the Hasse diagram  $\overline{D}$  of  $\overline{H} = (V, \overline{A})$ . Note that any tree support of H is also a tree support of  $\overline{H}$ : The intersection of two subtrees is again a subtree.

Let  $h_1, \ldots, h_k$  be the children of a hyperedge h in  $\overline{D}$ . The hyperedge  $h \in \overline{A}$  is *implied* 227 if the hypergraph  $(h_1 \cup \cdots \cup h_k, \{h_1, \ldots, h_k\})$  is connected and *non-implied* otherwise. Let 228  $\{h_1,\ldots,h_k\}$  be a maximal subset of the children of a non-implied hyperedge in  $\overline{A}$  such 229 that  $(h_1 \cup \cdots \cup h_k, \{h_1, \ldots, h_k\})$  is connected. Then  $h_1 \cup \cdots \cup h_k$  is a summary hyperedge. 230 Note that a summary hyperedge might not be in  $\overline{A}$ . Let A' be the set of subsets of V 231 containing the summary hyperedges, the hyperedges in  $\overline{A}$  that are not implied, and the 232 sources of  $\overline{D}$ . For an example consider Fig. 5(c). In this example, the hyperedge  $h_s$  is a 233 summary hyperedge,  $h_1^3$  and  $h_1^1, \ldots, h_5^1$  are non-implied, and V is a source. 234

The augmented Hasse diagram of H is the Hasse diagram D' of H' = (V, A'). If *H* has a tree support, then the augmented Hasse diagram has  $\mathcal{O}(n+m)$  vertices and can be constructed in  $\mathcal{O}(n^3m)$  time [7] (without explicitly constructing the closure under intersection  $\overline{A}$ ). Further note that if H has a tree support and  $h \in A'$  is non-implied, then all children of h in D' are disjoint: Otherwise there would be a summary hyperedge between h and intersecting children.

If a tree support G = (V, E) of H exists, it can be constructed as follows [7]. Starting with an empty graph G, we proceed from the sinks to the sources of D'. If  $h \in A'$ is not implied, choose an arbitrary ordering  $h_1, \ldots, h_k$  of the children of h in D'. We assume that at this stage,  $G[h_i], i = 1, \ldots, k$  are already connected subgraphs of G. For  $j = 2, \ldots, k$ , choose vertices  $v_j \in \bigcup_{i=1}^{j-1} h_i, w_j \in h_j$  and add edges  $\{v_j, w_j\}$  to E.

If we want to construct a path-based tree support, then  $G[h_j], j = 1, ..., k$  are paths and as vertices  $v_{j+1}$  and  $w_j$  for the edges connecting  $G[h_j]$  to the other paths, we choose the end vertices of  $G[h_j]$ . The only choices that remain are the ordering of the children of h and the choice of which end vertex of  $G[h_j]$  is  $w_j$  and which one is  $v_{j+1}$ . The implied hyperedges give restrictions on how these choices might be done.

#### 251 4.2. Choosing the Connections: A Characterization

<sup>252</sup> When we want to apply the general method introduced in Sect. 4.1 to construct a <sup>253</sup> path-based tree support G, we need to make sure that we do not create vertices of degree <sup>254</sup> greater than 2 in G[h] when processing non-implied hyperedges contained in an implied <sup>255</sup> hyperedge h.

**Definiton 1 (Conflicting Hyperedges).** Two overlapping hyperedges  $h', h'' \in A'$  have a conflict if there is some hyperedge in A' that contains both h' and h''. Two overlapping hyperedges  $h', h'' \in A'$  have a conflict with respect to  $h \in A'$  if h' has a conflict with h'',  $h' \cap h'' \subseteq h$  and h is a child of h' or h''. In that case we say that h' and h'' are conflicting hyperedges of h. Let  $A'_h$  be the set of conflicting hyperedges of h. Let  $A'_h$  be the set of children  $h_i$  of h such that  $h \in A'_{h_i}$ .

E.g., consider the hypergraph in Fig. 5(c). Then  $h_1^3$  and  $h_1^1$  are both contained in the 262 hyperedge V and they both contain  $\{v_2\}$ . Hence, they have a conflict. Further, is the 263 intersection  $h_1^3 \cap h_1^1 = \{v_2\}$  contained in the child  $h_s$  of  $h_1^3$ . Hence,  $h_1^3$  has a conflict with 264  $h_1^1$  with respect to  $h_s$ . Similarly does  $h_1^3$  have a conflict with  $h_5^1$  with respect to  $h_s$  and 265 we have on one hand  $A'_{h_s} = \{h_1^1, h_2^1, h_1^3\}$ . On the other hand does  $h_s$  have a conflict with  $h_1^1$  with respect to  $h_2^1$  and with  $h_5^1$  with respect to  $h_4^1$  and we have  $A_{h_s}^c = \{h_2^1, h_4^1\}$ . Note 266 267 that there might be hyperedges that have a conflict but not with respect to any of their 268 children. As an example see the hyperedges  $h_1^4$  and  $h_2^4$  in Fig. 6(a). In the lemmas in this 269 section, we will prove that it suffices if the algorithm considers only conflicts with respect 270 to some child. 271

Assume now that H has a path-based tree support G and let  $h', h'' \in A'$  be such that h' and h'' have a conflict with respect to a child h of h''. Since  $h' \cup h''$  is contained in a hyperedge it follows that  $G[h' \cup h'']$  is the subgraph of a path. Since in addition h' and h''intersect and G[h'] and G[h''] are paths, it follows that  $G[h' \cup h'']$  is also a path. Hence, we have the following situation.

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Note especially that among the two end vertices of G[h''] exactly one is contained in h'and that this end vertex is also an end vertex of G[h]. This yields the following three types of restrictions on the connections of the paths.

- 1.  $G[h' \setminus h]$  and  $G[h'' \setminus h]$  must be paths that are attached to different end vertices of G[h].
- 283 2. Assume further that h''' does also have a conflict with h'' with respect to h. Then 284 both,  $G[h' \setminus h]$  and  $G[h''' \setminus h]$ , must be appended to the common end vertex of G[h]285 and G[h''].
- 3. Assume further that  $h_2, h_1 \in A_h^c, h_2 \neq h_1$ . Let  $h^i \in A_h'$  have a conflict with hwith respect to  $h_i, i = 1, 2$ , respectively. Then  $G[h^i \setminus h]$  has to be appended to the common end vertex of G[h] and  $G[h_i]$ . Hence,  $G[h^1 \setminus h]$  and  $G[h^2 \setminus h]$  must be appended to different and vertices of G[h].

E.g., consider the hypergraph H = (V, A) in Fig. 5(c). Then on one hand,  $h_1^1$  has a conflict with  $h_1^1$  and  $h_5^1$  with respect to  $h_s$ . Hence, by the first type of restrictions  $G[h_1^1 \setminus h_s]$  and  $G[h_5^1 \setminus h_s]$  must be appended to the same end vertex of  $G[h_s]$ , i.e. the end vertex of  $G[h_s]$ to which  $G[h_1^1 \setminus h_s]$  is not appended. On the other hand,  $h_1^1$  and  $h_s$  have a conflict with respect to  $h_2^1$ , while  $h_5^1$  and  $h_s$  have a conflict with respect to  $h_4^1$ . Hence, by the third type of restrictions it follows that  $G[h_1^1 \setminus h_s]$  and  $G[h_5^1 \setminus h_s]$  must be appended to different end vertices of G[h]. Hence, there is no path-based tree support for H.

<sup>298</sup> This motivates the following definition of conflict graphs.

<sup>299</sup> **Definiton 2 (Conflict Graph).** The conflict graph  $C_h, h \in A'$  is a graph on the vertex <sup>300</sup> set  $A'_h \cup A^c_h$ . The conflict graph  $C_h$  contains the following three types of edges.

301 1.  $\{h',h''\}, h',h'' \in A'_h$  if h' and h'' have a conflict with respect to h.

20.  $\{h', h_1\}, h' \in A'_h, h_1 \in A^c_h$  if  $h' \in A'_{h_1}$  and h' and h have a conflict with respect to  $h_1$ .

304 3.  $\{h_1, h_2\}, h_1, h_2 \in A_h^c, h_1 \neq h_2.$ 

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E.g., consider the hypergraph H = (V, A) in Fig. 5(c). Then the conflict graph  $C_{h_s}$  contains the edges  $\{h_1^3, h_5^1\}$ and  $\{h_1^3, h_1^1\}$  of type one, the edges  $\{h_2^1, h_1^1\}$  and  $\{h_4^1, h_5^1\}$ of type 2 and the edge  $\{h_2^1, h_4^1\}$  of type 3. (See the figure

on the right.) Hence,  $C_{h_s}$  contains a cycle of odd length, reflecting that there is no suitable assignment of the end vertices of  $G[h_s]$  to  $h_1^1, h_5^1$  and  $h_1^3$ .



- **Theorem 4.** A hypergraph H = (V, A) has a path-based tree support if and only if
- 307 1. H has a tree support,
- 2. no hyperedge contains three pairwise overlapping hyperedges  $h_1, h_2, h_3 \in A'$  with  $h_1 \cap h_2 = h_2 \cap h_3 = h_1 \cap h_3$ , and
- 310 3. all conflict graphs  $C_h, h \in A', |h| > 1$  are bipartite.

From the observations before the definition of the conflict graph it is clear that the conditions of Theorem 4 are necessary for a path-based tree support. In the remainder of this section, we prove that the conditions are also sufficient.

In the following assume that the conditions of Theorem 4 are fulfilled. We show 314 in Algorithm 1 how to construct a path-based tree support G of H. We consider the 315 vertices of the augmented Hasse diagram D' from the sinks to the sources in a reversed 316 topological order, i.e., we consider a hyperedge only if all its children in D' have already 317 been considered. During the algorithm, a conflicting hyperedge h' of a hyperedge h is 318 labeled with the end vertex v of G[h] if the path  $G[h' \setminus h]$  will be appended to v. We 319 will call this label side h(h'). Concerning the choice of the ordering of the children in 320 Line 8 of Algorithm 1: the sets  $A_h^c, h \in A'$  contain at most two hyperedges – otherwise 321 the subgraph of  $C_h$  induced by  $A_h^c$  contains a triangle and, hence, is not bipartite. 322

Algorithm 1 constructs a tree support G of H [7]. Before we show that G is a pathbased tree support, we illustrate the algorithm with an example. Consider the hypergraph H in Fig. 6. We show how the algorithm proceeds  $h_1^5$  and all its descendants in D'. For

#### Algorithm 1: Path-based tree support

**Input** : augmented Hasse diagram D' of a hypergraph H = (V, A)fulfilling the conditions of Theorem 4; conflict graphs  $C_h$  on vertex sets  $A'_h \cup A^c_h$ , h non-source vertex of D'**Output**: path-based tree support G = (V, E) of H Data : labels side<sub>h</sub>(h') indicating the end vertex of G[h] to which  $h' \setminus h$  should be appended begin  $E \leftarrow \emptyset;$ foreach vertex h of D' in a reversed topological order of D' do if  $h = \{v\}$  for some  $v \in V$  then foreach vertex h' of  $C_h$  do  $| \operatorname{side}_h(h') \leftarrow v;$ else 8 Let  $h_1, \ldots, h_k$  be the children of h such that  $h_2, \ldots, h_{k-1} \notin A_h^c$ ; if h is non-implied then Let  $w_i, v_{i+1}, i = 1, ..., k$  be the end vertices of  $G[h_i]$  such that •  $\operatorname{side}_{h_1}(h) = v_2$  if  $h \in A'_{h_1}$  and •  $\operatorname{side}_{h_k}(h) = w_k$  if  $h \in A'_{h_k}$ ; \_ Add the edges  $\{v_i, w_i\}, i = 2, \dots, k$  to E;else Let  $w_1 \neq v_{k+1}$  be the end vertices of G[h] such that •  $v_{k+1} \notin h_1$  and •  $w_1 \notin h_k;$ if  $h_1 \in A_h^c$  then  $\operatorname{side}_h(h_1) \leftarrow v_{k+1}$ ; if  $h_k \in A_h^c$  then  $\operatorname{side}_h(h_k) \leftarrow w_1$ ; Label the remaining vertices of  $C_h$  with  $v_{k+1}$  or  $w_1$ such that no two adjacent vertices have the same label; end



Figure 6: Illustration of Algorithm 1.

the hyperedges  $h_3^1, h_4^1, h_6^1$ , and  $h_8^1$  the conflict graphs are empty. For the other leaves we have

When operating  $h_2^2$  and  $h_3^2$ , respectively, we add edges  $\{v_4, v_5\}$  and  $\{v_5, v_6\}$ , respec-328 tively, to G. While the conflict graph of  $h_2^2$  does only contain  $h_5^1$  with  $\operatorname{side}_{h_2^2}(h_5^1) = v_4$ , the 329 assignment of side in  $C_{h_2^2}$  is illustrated in Fig. 6(b).  $h_4^2$  has a conflict with respect to the 330 children  $h_7^1$  and  $h_9^1$ . Hence, we add edges  $\{v_7, v_8\}$  and  $\{v_8, v_9\}$  to G. The conflict graph 331 of  $h_4^2$  is shown in Fig. 6(c). When operating  $h_1^3$  we can choose  $h_1 = h_3^2$  and  $h_2 = h_7^1$ , since 332  $\operatorname{side}_{h_3^2}(h_1^3) = v_6$  and  $\operatorname{side}_{h_7^1}(h_1^3) = v_7$ . We add the edge  $\{v_6, v_7\}$  to G. The conflict graph 333  $C_{h_1^3}$  is shown in Fig. 6(d). The hyperedge  $h_1^4$  is implied and we set side  $h_1^4(h_4^2) = v_4$ . We 334 can finally connect  $v_3$  to  $v_4$  or  $v_9$  when operating  $h_1^5$ . 335

To prove the correctness of Algorithm 1, it remains to show that all hyperedges of Hinduce a path in G. Since we included all inclusion maximal hyperedges of H in A', it suffices to show this property for all hyperedges in A'. We start with a technical lemma.

Lemma 5. Let h' and h'' be two overlapping hyperedges and let h' be not implied. Then there is a hyperedge  $h \in A'$  with  $h' \cap h'' \subseteq h \subsetneq h'$ .

PROOF. Let  $h_c \in \overline{A}$  be maximal with  $h' \cap h'' \subseteq h_c \subsetneq h'$ . The hyperedge  $h_c$  is a child of the non-implied hyperedge h' in  $\overline{D}$ . Consider the summary hyperedge h with  $h_c \subseteq h \subsetneq h'$ . By definition of A' it follows that  $h \in A'$ .

For an edge  $\{v, w\}$  of G let  $h_{vw}$  be the intersection of all hyperedges of A' that contain v and w. Note that  $h_{vw}$  is not implied since v and w are contained in different children of  $h_{vw}$  in  $\overline{D}$  and  $\{v, w\}$  is an edge of the tree support G of  $\overline{H}$ . Hence,  $h_{vw} \in A'$ .

Lemma 6. Let Conditions 1-3 of Theorem 4 be fulfilled and let G = (V, E) be the graph computed in Algorithm 1. Let  $h', h'' \in A'$  have a conflict with respect to a child h of h' and let G[h'] and G[h''] be paths. Then

- 1. side<sub>q</sub>(h'') = side<sub>h</sub>(h'') for all  $g \in A'$  with  $h' \cap h'' \subseteq g \subseteq h$ ,
- 351 2. side<sub>h</sub>(h'')  $\in h''$ ,
- 352 3. side<sub>h</sub>(h'') is an end vertex of G[h'],
- 353 4.  $G[h' \setminus h'']$  is a path, and
- 5. side<sub>h</sub>(h'') is adjacent in G to a vertex of  $h'' \setminus h'$ .

PROOF. We prove the lemma by induction on the sum of the steps in which h' and h''were considered in Algorithm 1. If h' and h'' had been considered in the first two steps, then at least one of them is a leaf of D' and, hence, h' and h'' have no conflict. So there is nothing to show. Let now h' and h'' be considered in later steps. Let  $h'' \in A'$  have a conflict with h' with respect to a child h of h' and let G[h'] and G[h''] be paths.

**1.** + **2.** if  $h' \cap h'' \in A'$ : There is nothing to show if  $h = h' \cap h''$ . So let  $h_1$  be the child of h with  $h_1 \supseteq h' \cap h''$ . Then h, h'' have a conflict with respect to  $h_1$ . Hence,  $C_h$  contains the path  $h', h'', h_1$ . By the inductive hypothesis on Property 3, it



Figure 7: Illustration of the proof of Lemma 6.3.

follows that  $\operatorname{side}_{h_1}(h'')$  is an end vertex of G[h], and especially, that  $h_1$  and h share an end vertex. By construction it follows that  $\operatorname{side}_h(h_1)$  is the end vertex of hthat is not in  $h_1$ . Hence,  $\operatorname{side}_h(h'') \in h_1$  and  $\operatorname{side}_{h_1}(h'') = \operatorname{side}_h(h'')$ . By the inductive hypothesis it follows that  $\operatorname{side}_g(h'') = \operatorname{side}_h(h'')$  for  $h \cap h'' \subseteq g \subseteq h_1$ . Since the labels in  $\operatorname{side}_{h' \cap h''}(.)$  are the end vertices of  $G[h' \cap h'']$ , it follows that  $\operatorname{side}_h(h'') \in h' \cap h'' \subset h''$ .

- 1. + 2. + 5. if  $h' \cap h'' \notin A'$ : Let  $h''_1 \subseteq h''$  be minimal with  $h' \cap h'' \subset h''_1$ . Since h' and  $h''_1$  overlap, there is an edge  $\{v, w\} \in E$  such that  $v \in h' \cap h''$  and  $w \in h''_1 \setminus h'$ . We show that  $side_h(h'') = v$ .
- By Lemma 5 there is a child  $h_c$  of  $h_{vw}$  that contains  $h \cap h_{vw}$ . Since  $v \in h \cap h_{vw}$ , it follows that  $w \notin h_c$  and, hence, v is an end vertex of  $h_c$ .

Note that by the minimality of  $h_1''$  it follows that  $h' \cap h'' \not\subseteq h_{vw}$ . Since G[h''], G[h']374 are paths, it follows that  $h_c \subsetneq h$  and, hence,  $h_c = h \cap h_{vw}$ . Let  $h_p$  be minimal with 375  $h_c \subsetneq h_p \subseteq h$ . Then  $h_p, h_{vw}$  have a conflict with respect to  $h_c$  and it follows from 376 the inductive hypothesis on Property 5 that  $\operatorname{side}_{h_c}(h_{vw}) = v$ . Let  $h'_c$  be maximal 377 with  $h_c \subseteq h'_c \subsetneq h$ . By the inductive hypothesis on Property 1 it follows that 378  $\operatorname{side}_{h'_c}(h_{vw}) = v$ . Since  $h, h_{vw}$  have a conflict with respect to  $h'_c$ , it follows by the 379 inductive hypothesis on Property 3 that v is an end vertex of h. In  $C_h$  there is the 380 path  $h'_c, h_{vw}, h', h''$ . By construction, side<sub>h</sub> $(h'_c)$  is the end vertex of h that is not in 381  $h'_c$ . Hence, side<sub>h</sub> $(h_{vw}) = side_h(h'') = v$ . 382

**3.:** Let  $v = \operatorname{side}_h(h'')$ . By the construction in Algorithm 1, v is an end vertex of G[h'] if h' is non-implied. So assume that h' is implied and that v is not an end vertex of G[h']. Let  $w \in h' \setminus h$  be a neighbor of v in G. By Property 2, it follows that  $v \in h''$ . Let  $h_c$  be the child of  $h_{vw}$  that contains  $h_{vw} \cap h''$ . By the inductive hypothesis on Property 4, it follows that  $G[h_{vw} \setminus h'']$  is a path that contains w but not v. Hence,  $h_c = h_{vw} \cap h'' = h_{vw} \cap h$ . Let  $h'_1, h''_1 \in A'$  be minimal with  $h' \supseteq h'_1 \supsetneq h' \cap h''$  and  $h'' \supseteq h'_1 \supsetneq h' \cap h''$ , respectively. Assume first that  $h' \cap h'' \in A'$ . Then  $C_{h' \cap h''}$  contains the triangle  $h_{vw}, h'_1, h''_1, h_{vw}$  and, hence, is not bipartite.

Assume now that  $h' \cap h'' \notin A'$ . By the already proven part of Property 5 it follows 392 that there is an edge  $\{v, x\}$  of G with  $x \in h''_1 \setminus h$ . We have  $h_c = h_{vw} \cap h'' \supseteq h_{vw} \cap h_{vx}$ . 393 Further, the child of  $h_{vx}$  that contains  $h_{vx} \cap h$  equals  $h_{vx} \cap h$ . Since  $h''_1$  is implied 394 and  $h_{vx}$  not, it follows that  $h''_1 \neq h_{vx}$  and, hence,  $h_{vx} \not\supseteq h' \cap h''$ . Hence, either 395  $h_{vx} \cap h \subseteq h_{vw} \cap h$  or  $h_{vw} \cap h \subsetneq h_{vx} \cap h \subsetneq h' \cap h''$ . In the first case let  $h_1 \in A'$ 396 be minimal with  $h_{vw} \cap h \subsetneq h_1 \subseteq h$ . Then there is the triangle  $h_{vw}, h_{vx}, h_1, h_{vw}$  in 397  $C_{h\cap h_{vw}}$ . In the latter case let  $h_1 \in A'$  be minimal with  $h_{vx} \cap h \subsetneq h_1 \subseteq h$ . Then 398 there is the triangle  $h_{vw}, h_{vx}, h_1, h_{vw}$  in  $C_{h \cap h_{vx}}$ . 399

400 **4.:** By the inductive hypothesis  $G[h \setminus h'']$  is a path. Further, h and h' share side<sub>h</sub> $(h'') \in h''$ 401 as a common end vertex. By the precondition of the lemma, G[h'] is a path. Hence, 402  $G[h' \setminus h'']$  is a path.

**5.** if  $h' \cap h'' \in A'$ : If  $h \neq h' \cap h''$ , let  $h_1$  be the child of h with  $h' \cap h'' \subseteq h_1$ . By the inductive hypothesis  $\operatorname{side}_{h_1}(h'')$  is adjacent in G to a vertex of  $h'' \setminus h = h'' \setminus h'$  and by Property 1,  $\operatorname{side}_{h_1}(h'') = \operatorname{side}_{h}(h'')$ .

If  $h = h' \cap h''$ , let  $h''_1 \in A'$  be minimal with  $h \subsetneq h''_1 \subseteq h''$ . Applying Property 3 with  $h''_1$  as "h''" and h' as "h''" reveals that side<sub>h</sub>(h') is an end vertex of  $G[h''_1]$ . Since  $G[h''_1]$  is a path, it follows that some vertex of  $h''_1 \setminus h$  is adjacent to side<sub>h</sub>(h'').  $\Box$ 

Lemma 7. If Conditions 1-3 of Theorem 4 are fulfilled, then all hyperedges in A' induce a path in the graph G constructed in Algorithm 1.

<sup>411</sup> PROOF. Again, we prove the lemma by induction on the step in which h was considered <sup>412</sup> in Algorithm 1. There is nothing to show if h had been considered in the first step. So <sup>413</sup> assume that  $h \in A'$  and that G[h] contains a vertex v of degree greater than two.

Let  $u_1, u_2, u_3$  be the first three vertices connected to v in G. Let  $h_i = h_{vu_i}, i = 1, 2, 3$ . Then  $h_1, h_2, h_3$  are all three contained in h and its intersection contains v. Hence, any two of them have a conflict if and only if one of them is not contained in the other. A case distinction reveals that we wouldn't have appended all three,  $u_1, u_2$  and  $u_3$ , to v.

<sup>418</sup>  $h_2 = h_3$ : Since  $h_3$  contains no vertex of degree higher than two, it follows that  $u_1 \notin h_3$ , <sup>419</sup>  $h_3 \cap h_1 = \{v\}$ . Hence,  $h_1$  and  $h_3$  have a conflict with respect to the common child <sup>420</sup>  $\{v\}$ , contradicting that v is added in the middle of  $h_3$ .

421  $h_1 = h_2$  or  $h_1 = h_3$ : These cases are analogous to the first case.

 $h_1 \subsetneq h_3$ : Like in the first case it follows that  $u_2 \notin h_3$ . Let  $h'_i, i = 2, 3$  be the child of 422  $h_i$  that contains v. Then  $h_2$  and  $h_3$  have a conflict with respect to  $h'_i$ , i = 2, 3. 423 Since we add the edge  $\{v, u_i\}$  to G when we process  $h_i$ , it follows on one hand that 424  $\operatorname{side}_{h'}(h_i) = v$ . On the other hand, since  $h_1$  is contained in  $h_3$  and  $v \in h_1$ , it follows 425 that  $h_1 \subseteq h'_3$ . Hence,  $h'_3$  has more than one vertex. If  $h'_3 \neq h_3 \cap h_2$ , then v is the only 426 end vertex of  $G[h'_3]$  that is contained in  $h_2$ . By Lemma 6 Property 2 it follows that 427  $\operatorname{side}_{h'_2}(h_2) = v$  and hence,  $\operatorname{side}_{h'_2}(h_3) \neq v$ . If  $h'_3 = h_3 \cap h_2$ , let  $v' \neq v$  be the other 428 end vertex of  $h'_2$ . Since we know that  $\operatorname{side}_{h'_2}(h_2) = v$ , it follows that  $\operatorname{side}_{h'_2}(h_3) = v'$ . 429 Hence, by Lemma 6 Property 1 we can conclude that  $\operatorname{side}_{h'_2}(h_3) = v'$ . In both cases 430 we have a contradiction. 431



Figure 8: Illustration of Algorithm 2. Computation of the potential conflicts for  $h_1^4$ 

432  $h_1 \subseteq h_2$  or  $h_2 \subseteq h_3$ : These cases are analogous to the third case.

<sup>433</sup>  $h_1, h_2, h_3$  pairwise overlapping: Then  $h_1 \cap h_2 = h_2 \cap h_3 = h_1 \cap h_3 = \{v\}$ . Hence, <sup>434</sup> Condition 2 of Theorem 4 is not fulfilled.

This completes the proof of Theorem 4. We conclude this section with the following corollary.

<sup>437</sup> Corollary 8. Algorithm 1 computes a path-based tree support of a hypergraph H if H <sup>438</sup> has a path-based tree support, i.e., if and only if the conditions of Theorem 4 are fulfilled.

#### 439 4.3. Conflict Computation and Run Time

In this section we show how to efficiently compute the conflicts and give an upper bound for the run time of testing whether a hypergraph has a path-based tree support and, if it exists, of constructing one.

Representing the hyperedges as sorted lists of their elements, all conflicts can be determined straight-forwardly in  $\mathcal{O}(n^3(n+m))$  time. In the following, we show how this time bound can be improved.

We first compute candidates for conflicting pairs of hyperedges, which in the case 446 of hypergraphs having a path-based tree support turn out to be a superset of the set 447 of all conflicts. The idea is, that all potential conflicts lie on a path from an ances-448 tor of h to one of h's descendants. The method can be found as pseudocode in Algo-449 rithm 2. First, all ancestors of h are marked in Procedure ANCESTOR(h). The procedures 450 DESCENDANT $(h_c, h_c), h_c$  child of h label those descendants of a child  $h_c$  of h with DESC $(h_c)$ 451 that are not descendants of any other child of h. Finally, after all calls of UP-SEARCH a 452 hyperedge h' is labeled CONFLICT $(h_c)$  if and only if h' is (1) a descendant of an ancestor 453 of h, but neither a descendant nor an ancestor of h, (2) an ancestor of a descendant of 454 the child  $h_c$  of h that is not a descendant of another child of h, and (3) if all descendants 455 h' are descendants of  $h_c$  or of at least two children of h. This implies especially that h 456 and h' have a conflict, if h' is labeled CONFLICT $(h_c)$ . 457

Before we show that h' is labeled CONFLICT $(h_c)$  if h and h' have a conflict with respect to  $h_c$ , we illustrate Algorithm 2 with an example. Figure 8 shows the computation of potential conflicts for the hyperedge  $h_1^4$  of the hypergraph H from Figure 6(a). The different methods are colored.  $h_5^2$  is the only hyperedge that can be in conflict with  $h_1^4$ with respect to a child of  $h_1^4$  and if so, with respect to  $h_4^2$ .

#### Algorithm 2: Conflict Computation.

**Input** : augmented Hasse diagram D' of a hypergraph; vertex h**Output:** for each child  $h_c$  of h the vertices h' with LABEL $(h') = \text{CONFLICT}(h_c)$ Data : there are the following vertex labels  $LABEL(h') = ANC \text{ iff } h \subsetneq h'$ LABEL(h') = NOT-ANC only if  $h \cup h'$  not contained in any source of D' $LABEL(h') = DESC(h_c)$  iff  $h' \subseteq h_c$  for exactly one child  $h_c$  of hLABEL(h') = MULTI-DESC iff h' is contained in more than one child of hLABEL(h') = NOT-CONFLICT only if  $h \cap h'$  not contained in any child of hand  $h \cup h'$  contained in some source of D'LABEL(h') = CONFLICT $(h_c)$  only if  $h_c \cap h' \neq \emptyset$  for a child  $h_c$  of hand  $h \cup h'$  contained in some source of D'ANCESTOR(vertex h') begin foreach parent h'' of h' do  $LABEL(h'') \leftarrow ANC;$ ANCESTOR(h'');  $\mathbf{end}$ DESCENDANT(vertex h', vertex  $h_c$ ) begin if  $LABEL(h') = DESC(h'_c), h_c \neq h'_c$  then | LABEL $(h') \leftarrow$  MULTI-DESC; else | LABEL $(h') \leftarrow \text{DESC}(h_c);$ foreach child h'' of h' do if  $LABEL(h'') \neq MULTI-DESC$  then | DESCENDANT $(h'', h_c)$ ; end UP-SEARCH(vertex h', vertex  $h_c$ ) begin for each parent h'' of h' do if  $LABEL(h'') \in \{\emptyset, CONFLICT(h'_c), h'_c \neq h_c\}$  then | UP-SEARCH $(h'', h_c)$ ; if  $LABEL(h') = CONFLICT(h'_c), h_c \neq h'_c$  then | LABEL $(h') \leftarrow$  NOT-CONFLICT; else if  $LABEL(h') \neq DESC(h_c)$  then if  $LABEL(h'') \in \{CONFLICT(h_c), ANC, NOT-CONFLICT\}$  then | LABEL $(h') \leftarrow \text{CONFLICT}(h_c);$ if  $LABEL(h') \neq CONFLICT(h_c)$  then | LABEL $(h') \leftarrow$  NOT-ANC; end begin clear all labels;  $LABEL(h) \leftarrow NOT-CONFLICT;$ ANCESTOR(h); foreach child  $h_c$  of h do DESCENDANT $(h_c, h_c)$ ; foreach vertex  $h_d$  of D' with LABEL $(h_d) \in \{\text{DESC}(h_c); h_c \text{ child of } h\}$  do

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| UP-SEARCH $(h_d, h_c)$ ;

end



Figure 9: h' and h do not have a conflict with respect to any child of h but the label of h' is  $CONFLICT(h_1)$  in the end of Algorithm 2 applied to h. All descendants of  $h_2$  (black vertices) are labeled MULTI-DESC.

Lemma 9. Let D' be the augmented Hasse diagram of a hypergraph that has a path-based tree support and let h' and h have a conflict with respect to a child  $h_c$  of h. Then the label of h' in the end of Algorithm 2 applied to D' and h is CONFLICT( $h_c$ ).

<sup>466</sup> PROOF. Let G be a path-based tree support of a hypergraph and let h' and h have a <sup>467</sup> conflict with respect to a child  $h_c$  of h.

1. Let v be the end vertex of G[h] that is contained in h'. Then v and all its ancestors on the path from  $\{v\}$  to  $h_c$  are labeled  $\text{DESC}(h_c)$  (and not MULTI-DESC).

4702. If there was a descendant of h' labeled DESC $(h'_c)$  for a child  $h'_c \neq h_c$  of h, then h'471contains a vertex of h that is not a contained in  $h_c$ . Hence,  $h_c$  does not contain  $h \cap h'$ ,472contradicting that h and h' have a conflict with respect to  $h_c$ . Hence, Algorithm 2473does not label h' with NOT-CONFLICT.

For a path P in D' from h' to v let  $h_P$  be the first vertex on P that is labeled DESC $(h_c)$ . Assume that among all such vertices  $h_P$  is the one to which the procedure UP-SEARCH of Algorithm 2 is applied first. Then UP-SEARCH $(h_P, h_c)$  labels h' with CONFLICT $(h_c)$ .  $\Box$ 

Note, however, that the converse of Lemma 9 is not true. More precisely, if h' is labeled CONFLICT $(h_c)$  in the end of Algorithm 2 applied to h then indeed do h and h' have a conflict, but  $h_c$  does not have to contain  $h \cap h'$ . The reason for this is that all descendants of h' that are no descendants of  $h_c$  are descendants of several children of h and, hence, labeled MULTI-DESC. Fig. 9 shows an example.

Theorem 10. It can be tested in  $\mathcal{O}(n^3m)$  time whether a hypergraph has a path-based tree support and if so, such a support can be constructed within the same time bounds.

<sup>484</sup> PROOF. Let H be a hypergraph. First test in linear time whether there is a tree support <sup>485</sup> for H [10]. Let D' be the augmented Hasse diagram of H. The method works in four <sup>486</sup> steps.

- <sup>487</sup> 1. Start with an empty array conflict indexed with pairs of inner vertices of D'. Set <sup>488</sup> conflict<sub> $h,h'</sub> <math>\leftarrow h_c$  if and only if h' is labeled CONFLICT( $h_c$ ) in the end of Algorithm 2 <sup>489</sup> applied to D' and h.</sub>
- 2. For each pair h, h' of inner vertices of D', test whether conflict<sub>h,h'</sub> contains  $h \cap h'$ . Otherwise set conflict<sub> $h,h'</sub> <math>\leftarrow \emptyset$ . Now, if H has a path-based tree support, then h, h'has a conflict with respect to the child  $h_c$  of h if and only if  $h_c = \text{conflict}_{h,h'}$ .</sub>
- 493 3. Apply Algorithm 1 to compute a support G. If the algorithm stops without com-494 puting a support, then H does not have a path-based tree support.

495 4. Test whether every hyperedge induces a path in G. If not, H does not have a 496 path-based tree support.

<sup>497</sup> D' has  $\mathcal{O}(n+m)$  vertices,  $\mathcal{O}(n^2 + nm)$  edges, and can be computed in  $\mathcal{O}(n^3m)$  time if <sup>498</sup> H has a tree support [7]. Algorithm 2 visits every edge of D' at most twice and, hence, <sup>499</sup> runs in  $\mathcal{O}(n^2 + nm)$  time for each of the  $\mathcal{O}(n)$  inner vertices of D'.

We may assume that the hyperedges are given as sorted lists of their elements. If not given in advance, these lists could straight forwardly be computed from D' in  $\mathcal{O}(n^3 + mn^2)$ time by doing a graph search from each leaf. Now, for each of the  $\mathcal{O}(n^2)$  pairs h, h' of inner vertices, it can be tested in  $\mathcal{O}(n)$  time whether conflict<sub>h,h'</sub> contains  $h \cap h'$ .

The sum of the sizes of all conflict graphs is in  $\mathcal{O}(n^2)$ . Hence, Algorithm 1 runs in  $\mathcal{O}(n^2 + mn)$  time. For each of the  $\mathcal{O}(m)$  hyperedges h, it can be tested in  $\mathcal{O}(n)$  time whether G[h] is a path. Hence, the overall run time is dominated by the computation of the augmented Hasse diagram and is in  $\mathcal{O}(n^3m)$ .

#### 508 5. Conclusion

We have introduced path-based supports for hypergraphs. Hence, as a new model, 509 we considered a restriction on the appearance of those subgraphs of a support that are 510 induced by the hyperedges. We have discussed that monotone path-based supports are 511 desirable. We have shown that it is  $\mathcal{NP}$ -hard to decide whether a given path-based 512 support is monotone or to find a path-based support with the minimum number of edges. 513 Further, it is  $\mathcal{NP}$ -complete to decide whether there is a planar path-based support. As a 514 main result, we characterized those hypergraphs that have a path-based tree support and 515 we gave an algorithm that computes a path-based tree support in  $\mathcal{O}(n^3m)$  run time if it 516 exists. Our algorithm completed the paths for the hyperedges in the order in which they 517 appear in a reversed topological ordering of the augmented Hasse diagram. To connect 518 these subpaths in the right order, we introduced a conflict graph for each hyperedge h519 and colored its vertices with the end vertices of the path induced by h. 520

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