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Cudennec *et al*. (2004) propose an original theoretical GIUH model (Rodriguez-Iturbe and Valdès, 1979) following from a coupling of general symmetry assumptions and self-similarity of river networks. This model has the originality to involve two additional disciplines: statistical physics and fractal geometry. Their paper aims to derive theoretical expressions of probability density functions (p.d.f.s) of the river network's hydraulic lengths.

Cudennec *et al*. (2004) clarify the p.d.f.s of hydraulic lengths in two major steps. First, by using the Strahler (1957) scheme, the authors can make an isotropy assumption on the reduced hydraulic lengths of a river network. This assumption leads to the p.d.f. of the hydraulic lengths l_i [Equation (29) in Cudennec *et al*. (2004)]:

$$\text{p.d.f. } (l_i) = \frac{1}{\sqrt{\pi}} \sqrt{\frac{\lambda}{r^{i-1}}} \frac{1}{\sqrt{l_i}} e^{-\frac{\lambda l_i}{r^{i-1}}} \tag{1}$$

Equation (1) [or Equation (45) in Cudennec *et al*. (2004)] is a Gamma law with parameters $\alpha = 1/2$ and $\beta_i = r_l^{i-1}/\lambda$. One can easily calculate that $\lambda = r^{i-1}/2\overline{l_i}$, where $\overline{l_i}$ is the mathematical expectation of l_i.

Our comment does not concern the first step, which is original and relevant, but the second one.

The second step aims at calculating the p.d.f. of L knowing that $L = \sum_{i=1}^{n} l_i$ [Equation (6) in Cudennec *et al*. (2004)]. A classical way to calculate p.d.f. (L) would have been to use the following property. If one has n independent random variables $l_i, i = 1, 2, \ldots, n$ with respective p.d.f. (l_i), then the random variable L defined such as $L = \sum_{i=1}^{n} l_i$ has the following p.d.f.:

$$\text{p.d.f. } (L) = \text{p.d.f. } (l_1) * \text{p.d.f. } (l_2) * \cdots * \text{p.d.f. } (l_n) \tag{2}$$

where * is the convolution integral.

Cudennec *et al*. (2004) did not choose this way, but directed their reasoning towards physical considerations, aimed at expressing the variable L according to l_i. The authors' result [Equation (45) in Cudennec *et al*.

(2004)] is that p.d.f. (L) is also a Gamma law with parameters $\alpha = n/2$ and $\beta = 2\overline{L}/n$:

$$\text{p.d.f. } (L) = \left(\frac{n}{2\overline{L}}\right)^{\frac{n}{2}} \frac{L^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)} e^{-\frac{nL}{2\overline{L}}} \tag{3}$$

According to the authors' assumptions, Equations (2) and (3) have to be equal. So, Cudennec *et al.*'s (2004) result concludes that the convolution integral of n Gamma laws with different parameters (β_i) also gives a Gamma law. This is definitely wrong. In fact, the use of an estimation [Equation (35) in Cudennec *et al*. (2004)] leads to a result [Equation (45) in Cudennec *et al*. (2004)] which is right only in the specific case and only for one value of the Horton's ratio $r_1 = 1$, which is indeed not a realistic value since $1{\cdot}5 \leq r_1 \leq 3{\cdot}5$ in natural river networks (Rodriguez-Iturbe and Rinaldo, 1997).

We propose therefore to give the complete mathematical solution of Cudennec *et al.*'s (2004) problematic. Let's use the classical way stated above: if $l_1, l_2, \ldots, l_n$ are n independent random variables distributed according to a Gamma law given by Equation (1), then what is the p.d.f. of the variable $L = \sum_{i=1}^{n} l_i$? The problematic can be solved using the calculation of the generalized characteristic functions (Sim, 1992) and the result is (details are not developed because of lack of space):

$$\begin{aligned} \text{p.d.f. } (L) &= \text{p.d.f. } (l_1) * \text{p.d.f. } (l_2) * \cdots * \text{p.d.f. } (l_n) \\ &= \left(\frac{\sum_{i=1}^{n} r_l^{i-\frac{n+1}{2}}}{2\overline{L}}\right)^{\frac{n}{2}} \frac{L^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)} e^{-\frac{\sum_{i=1}^{n} r_l^{i-n} L}{2\overline{L}}} \sum_{k=0}^{\infty} \frac{b_n(k)\left(\frac{n-1}{2}\right)_k}{k!\left(\frac{n}{2}\right)_k} \left(\frac{1-r_l^n}{2\overline{L} r_l^{n-1}} L\right)^k \end{aligned} \tag{4}$$

and

$$b_i(k) = \begin{cases} 1, & \text{if } i = 2 \\ \sum_{j=0}^{k} \dfrac{b_{i-1}(j)\left(\frac{i-2}{2}\right)_j (-k)_j}{j!\left(\frac{i-1}{2}\right)_j} (-r)^j, & \text{for } i = 3, 4, \ldots, n \end{cases}$$

and

$$(p)_k = p(p+1)(p+2)\ldots(p+k-1)$$

In order to compare easily Equations (45) in Cudennec *et al*. (2004) and (4), we choose, by making a change of variables, to draw the p.d.f.s of the variable $L/\overline{L}$. The p.d.f.s are drawn (Figure 1) for various values of the Horton's ratio r_1, and for various values of the Strahler's order n. One can see that the more n or r_1 increases, the more Equations (45) in Cudennec *et al*. (2004) and (4) are different. Equations (45) in Cudennec *et al*. (2004) and (4) are equal only for $r_1 = 1$, for any value of n.

Many mathematical ways are then possible to prove that Equation (45) in Cudennec *et al*. (2004) is a particular case of Equation (4) for Horton's ratio $r_1 = 1$. The simplest is by replacing $r_1 = 1$ in Equation (4), which easily leads to Equation (45) in Cudennec *et al*. (2004). A more rigorous method shows that $r_1 = 1$ is the unique solution to have Equations (45) in Cudennec *et al*. (2004) and (4) equal. This method, following from theorems of the inverse Fourier transform and probability theory, is not developed here because of lack of space, but can be obtained from the authors.

Consequently, one can say that Equation (45) in Cudennec *et al*. (2004) is only a partial result since the authors' model only takes into account networks with Horton's ratio equal to 1, which is an unrealistic value in natural cases. *De facto*, the authors' measures of Horton's ratio are valued, not surprisingly, between 1·5 and 3. So, what is the validity of the authors' conclusions concerning their model validation comparison

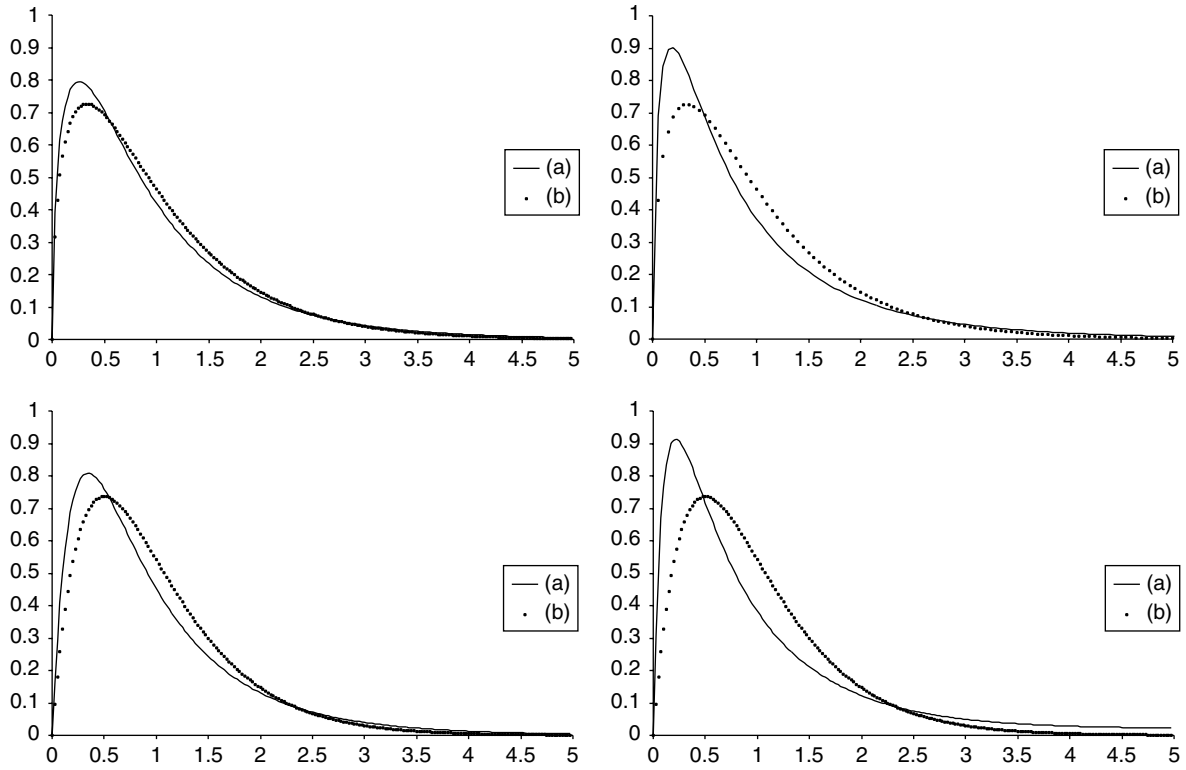

Figure 1. Comparison of the p.d.f.s: Equation (4) (a) and Equation (45) in Cudennec *et al*. (2004) (b). For top and bottom Figures, n is 3 and 4, successively and r_1 is 2 and 3, respectively for left and right figures

of experimental and theoretical p.d.f.s in Figure 5 of Cudennec *et al*. (2004)? The skewness of the model presented here [Equation (4)] is more negative than Equation (45) in Cudennec *et al*. (2004), but represents the complete mathematical solution of the authors' problematic. So, is the presented model [Equation (4)] still relevant to describe real river network structures?

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