

Analytical model for a Geomorphologic Instantaneous Unit Hydrograph (AGIUH)

C. Fleurant ^{a,*} B. Kertiwa ^b B. Roland ^a

^a*Département Paysage, Institut National d'Horticulture, Angers, France.*

^b*Center for Soil and Agroclimate Research, Bogor, Indonesia.*

Abstract

The rainfall-runoff modelling of a river basin can be divided into two processes: the production function and the transfer function. The production function determines the proportion of gross rainfall actually involved in the runoff. The transfer function spreads the net rainfall over time and space in the river basin. Such transfer function can be modelled through an approach of the geomorphologic instantaneous unit hydrograph type (GIUH). The effectiveness of geomorphological models is actually revealed in rainfall-runoff modeling, where hydrologic data are desperately lacking, just as in ungauged basins. These models make it possible to forecast the hydrograph shape and runoff variation versus time at the basin outlet. This article is an introduction to a new GIUH model which proves to be simple and analytical. Its geomorphological parameters are easily available on a map or from a DEM. This model is based on general hypotheses on symmetry which provide it with multi-scale versatile characteristics. After having validated the model in river basins of very different nature and size, we present an application of this model for rainfall-runoff modelling. Since parameters are determined relying on real geomorphological data, no calibration is necessary, and it is then possible to carry out rainfall-runoff simulations in ungauged river basins.

1 Introduction

The purpose of rainfall-runoff modelling is the conversion of rainfall into runoff at the river basin outlet. This complex process is the result of the combination of multiple factors involved at various spatial scales. The process pattern made by hydrologists usually includes two functions: the production and the transfer functions. The production function makes it possible to determine the fallen rain (gross rainfall) which is actually involved in the runoff at the river outlet. The transfer function works out the runoff at the outlet according to the net rainfall.

In the present article, we will exclusively focus on the transfer function and its geomorphological characteristics.

The transfer function can be understood through an overall approach (Chow et al., 1988): the instantaneous unit hydrograph theory (Sherman, 1932; Nash, 1957; Dooge, 1959; Lienhard, 1964; Rodriguez-Iturbe and Valdès, 1979; Duchesne et al., 1997). The instantaneous unit hydrograph theory (Sherman, 1932) is a determinist model which relies on three major hypotheses:

- A univocal relationship between rainfalls and runoff, in the case of rainfalls having the same intensity and duration features;

* Corresponding author: Cyril Fleurant, Département Paysage, Institut National d'Horticulture, 2 rue Le Nôtre, 49045 Angers cedex 01, France. Email: cyril.fleurant@inh.fr, tel.: 00 33 2 41 22 54 98, fax: 00 33 2 41 22 55 28.

- Linearity between unit rainfalls of different intensity and the corresponding runoffs;
- Uniformity of rainfalls over the whole surface of the river basin.

The three hypotheses are quite restrictive. However, the unit hydrograph method has a major advantage: its simplicity; which is why it has been used for many years on quite a large number of river basins throughout the world (Chow et al., 1988). The unit hydrograph method is based on three basic principles:

- The unit rainfall is a rainfall impulse by definition, which means that it is short-lasting compared to the basin lag, homogeneous in space and with a sufficient intensity to generate a runoff surge;
- For a unit rainfall involving a runoff depth h (m), the relation between the rate of flow by unit area $q(t)$ (m/s), the unit hydrograph $T(t)$ (s⁻¹) and h is:

$$q(t) = h(t)T(t) \quad (1)$$

- The additionality property applying to the unit hydrograph is a linear element. Therefore, if a complex rainfall is divided into a succession of unit rainfalls, it is possible to calculate the corresponding hydrograph as follows:

$$q(t) = \int_0^t i(\tau)T(t - \tau)d\tau \quad (2)$$

Where $q(t)$ (m/s) is the rate of flow by unit area, $i(t)$ (m/s) the net rainfall, rainfall in excess of infiltration and interception, uniformly falling on the drainage basin and $T(t)$ (s⁻¹) is a transfer function, and more precisely the probability density function (*pdf*) of time transfer on the drainage basin

(Gupta et al., 1980). Consequently, the difficulties to express the rate of flow $q(t)$ are twofold: what is the value of the net rainfall $i(t)$ and which function $T(t)$ should be chosen ?

Choosing the characteristic IUH of a river basin is a delicate issue. Its experimental measurement with real rainfall events is always worked out in gauged river basins but remains difficult to carry out. Consequently, theoretical approaches for identifying the suitable IUH have been used. For example, Nash (1957, 1959, 1960) conceptualized the river basin and compared it to a series of n identical linear reservoirs. Its analytical model for unit hydrograph is a two-parameter Gamma law: n the number of cascade reservoirs and τ the usual release rate from one reservoir into another. The n and τ parameters have no physical reality and must be determined relying on previous accounts of rainfall-runoff events. This is one of the major drawbacks of Nash's model and other similar models (Dooge, 1959; Lienhard, 1964), which cannot therefore be used in non-gauged river basins.

If the hydrograph depends on rainfall, it also depends strongly on the geomorphological features of the river basin (shape, slope, organisation of the river network). Therefore, determining a geomorphological transfer function such as the Geomorphologic Instantaneous Unit Hydrograph (GIUH, Rodriguez-Iturbe and Valdès (1979)), or the model described in the present article, makes it possible to give a physical meaning to the unit hydrograph which has been constructed on the basis of parameters set according to the river basin's geometry. Such geomorphological models are based on the analyses of the morphological characteristics of river basins and more particularly river networks.

The purpose of this article is to present a new analytical model for geomorphological transfer function, that is to say, a theoretical law giving the distribution of hydraulic pathways according to the river basin geomorphology. This model was already introduced in Fleurant and Boulestreau (2005) and is a theoretical coupling of quantitative geomorphology and hydrology. Here we present some applications of the model. We will compare this theoretical law to the distribution of real hydraulic pathways in two river basins. By way of an applied example, we will also show how to use this geomorphological transfer function as an Analytical Geomorphologic Instantaneous Unit Hydrograph (AGIUH) and thus make rainfall-runoff simulations.

2 An analytical geomorphological transfer function

The geomorphological theory of unit hydrograph has been formulated by Rodriguez-Iturbe and Valdès (1979). This model is called GIUH (Geomorphologic Instantaneous Unit Hydrograph) and provides the times of transfer according to the geomorphology of the drainage basin. It can be expressed as follows (Rodriguez-Iturbe and Rinaldo, 1997):

$$T(t) = \sum_{\gamma \in \Gamma} p(\gamma) f_{\gamma}(t) \quad (3)$$

$p(\gamma)$ is the probability to have a γ path, γ being any path covered by a raindrop in the stream network, and Γ is the set of all the possible paths γ . $f_{\gamma}(t)$ (s^{-1}) is the probability density function of the travel time of the path γ . The GIUH model $T(t)$ (equation 3) is the frequency distribution of travel times from points within the drainage basin to the outlet.

Rosso (1984) shows that the equation (3) can be fitted by a Gamma law whose parameters depends on Horton's parameters.

The GIUH model has generated a wealth of literature over the last twenty five years. For most modifications, authors put forward the hypothesis that the response to rainfall input coming out of each drain is either an exponential function (Rodriguez-Iturbe and Valdès, 1979), or a single-valued function (Gupta et al., 1980; Gupta and Waymire, 1983) or else a Gamma function (van der Tak and Bras, 1990; Jin, 1992; Kirchner et al., 2001). In all cases, these hypotheses are empirical and therefore influenced by experimental observations.

The model described in the present article is different from the GIUH by Rodriguez-Iturbe and Valdès (1979), and there are two main reasons for this:

- It is based on general hypotheses on symmetry combined with the fractal properties of branched structures; thus it is able to be used to branched structures in general (Duchesne et al., 2002; Fleurant et al., 2004), and not to stream networks exclusively;
- The reasoning is carried out on the pathways' lengths rather than on the times of transfer to the basin.

The model introduced here uses findings of research carried out on stream networks (Duchesne et al., 1997; Cudennec, 2000; Cudennec et al., 2004).

The focus is to describe networks structure thanks to Strahler stream ordering (1964), and to apply Horton's laws (1945) to a geomorphological statistics rationale, in order to elaborate a descriptive theoretical model of network lengths.

Well before Mandelbrot (1988) developed his fractal theory; branching networks and more particularly stream networks structure had been studied very accurately. In fact, geological scientists and hydrologists such as Horton (1945), Schumm (1956), Strahler (1957) or Shreve (1969) took an interest in analysing the complex ordering of these networks. They made topological and hydraulic analyses (Kirshen and Bras, 1983) which can be applied to all branching networks that are three-dimensional and organised into a hierarchy. Many classification systems have been put forward (Horton, 1945) but we have decided upon Strahler's system (1952) which is the most widely used. The classification system (Figure 1) is as follows:

- headwaters are first order stream segments;
- when two stream segments within the same order i merge, the stream segment resulting from this confluence is within order $i + 1$;
- when two stream segments within different orders, i and j merge, the stream segment resulting from this confluence is within order $\max(i, j)$.

Strahler's classification thus makes it possible to organise the different segments of a stream network into a hierarchy. Consequently, the stream outlet will have the highest index value, corresponding to the river network order.

Hydrologists relied on this classification to put forward general geometric laws concerning the ordering of stream networks. Among them, Horton's laws (1945), describe the way stream networks are organised. One of these laws expresses the so-called length ratio $R_L = \frac{\bar{l}_i}{\bar{l}_{i-1}}$, also known as Horton's ratio. A great number of experimental studies on stream networks (LaBarbera and Rosso, 1987; Feder, 1988; LaBarbera and Rosso, 1989;

Tarboton et al., 1990; LaBarbera and Rosso, 1990; Rosso et al., 1991) reveal that this ratio is rather stable and fluctuate between 1.5 and 3.5. A brief outline of the model is given hereafter, while the detailed description can be found in the appendix.

2.1 *Definition of hydraulic lengths*

Before giving the theoretical expression of probability density function of hydraulic lengths, it is necessary to define this variable accurately. As shown in Figure 1, if we take an indefinite point on the river bassin which represents a rain drop, the path to be covered between this point and the outlet successively goes over channels of increasing orders. The hydraulic length is defined as follows:

$$L = l_0 + \sum_{k=1}^n l_k \quad (4)$$

l_k (m) is the length of the channel in k order and n is the order of the river network. l_0 (m) is the length on the hillslope. Hence, the hydraulic length is the added lengths n of the channels.

The hydraulic length L (m), in the river network, may be calculated using a vector with n components (l_1, l_2, \dots, l_n) . It is importante to notice that some lenght can be nill: a first-order channel might join a third order one directly, so l_2 would be zero in that particular instance.

2.2 The transfer function of the drains

The rationale of Duchesne et al. (1997), Cudennec (2000) and Cudennec et al. (2004), founded on the hypotheses on symmetry combined with fractal geometry, leads to the probability density function of the drains' hydraulic lengths of order k that we have reformulated as follows (see appendix for details):

$$pdf(l_k) = \frac{1}{\sqrt{2\pi\overline{l_k}}} \frac{1}{\sqrt{\overline{l_k}}} e^{-\frac{l_k}{2\overline{l_k}}} \quad k=1, 2, \dots, n \quad (5)$$

A Gamma distribution with $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2\overline{l_k}}$ can be recognised here and named $\Gamma\left(l_k, \frac{1}{2}, \frac{1}{2\overline{l_k}}\right)$, where $\overline{l_k}$ is the mean of l_k .

2.3 The geomorphological transfer function in the stream network

We put forward the hypothesis that this above distribution law (equation 5) is an appropriate model for representing the hydraulic pathways of individual drains, which is consistent with research work by van der Tak and Bras (1990), Jin (1992) and Kirchner et al. (2001).

Now, the step consists of determining the probability density distribution for L , knowing that $L = \sum_{k=1}^n l_k$. The most usual way to calculate the probability density function of L , is to use the following property (Feller, 1971):

If we have n independent random variables l_k , $k = 1, 2, \dots, n$ of probability density function $pdf(l_k)$ then the random variable L defined as $L = \sum_{k=1}^n l_k$

has a probability density function $pdf(L)$ such as:

$$pdf(L) = pdf(l_1) \star pdf(l_2) \star \dots \star pdf(l_n) \quad (6)$$

Where \star is the convolution integral. Full explanation of the convolution operator \star is given in appendix.

Equation (6) can be calculated very easily with numerical calculation software, though an analytical solution exists that is explained in this paper.

The result of the convolution integral for n Gamma distributions is obtained by working out the generalised characteristic function (Mathai, 1982; Moschopoulos, 1985) and we can thus assert the following theorem (Sim, 1992):

Let l_1, l_2, \dots, l_n , be n independent random distributed according to a Gamma distribution of the following type $\Gamma\left(l_k, \frac{1}{2}, \frac{1}{2\bar{l}_k}\right)$, then the probability density for $L = \sum_{k=1}^n l_k$, is:

$$\begin{aligned} pdf(L) &= pdf(l_1) \star pdf(l_2) \star \dots \star pdf(l_n) \\ &= \Gamma\left(l_1, \frac{1}{2}, \frac{1}{2\bar{l}_1}\right) \star \Gamma\left(l_2, \frac{1}{2}, \frac{1}{2\bar{l}_2}\right) \star \dots \star \Gamma\left(l_n, \frac{1}{2}, \frac{1}{2\bar{l}_n}\right) \\ &= \frac{1}{\sqrt{2^n \prod_{i=1}^n \bar{l}_i}} \frac{L^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)} e^{\left(-\frac{L}{2\bar{l}_n}\right)} \sum_{k=0}^{\infty} \frac{b_n(k) \left(\frac{n-1}{2}\right)_k}{k! \left(\frac{n}{2}\right)_k} \left(\frac{L}{2(\bar{l}_n - \bar{l}_{n-1})}\right)^k \quad (7) \end{aligned}$$

with

$$b_i(k) = \begin{cases} 1, & \text{if } i = 2 \\ \sum_{j=0}^k \frac{b_{i-1}(j) \left(\frac{i-2}{2}\right)_j (-k)_j}{j! \left(\frac{i-1}{2}\right)_j} \left(\frac{\bar{l}_{i-1} - \bar{l}_{i-2} \bar{l}_{i-1}}{\bar{l}_{i-1} - \bar{l}_i \bar{l}_{i-2}}\right)^j & \text{for } i = 3, 4, \dots, n \end{cases}$$

and

$$(x)_k = x(x+1)(x+2)\dots(x+k-1) \quad (8)$$

The equation (7) is a geomorphological transfer analytical function since it represents the distribution of hydraulic pathways through which water travels inside the river network. Moreover, the parameters of this analytical function are geomorphological:

- The Strahler order n , represents the rate of the hierarchical system of the river network;
- The lengths \overline{l}_k (m), are the average hydraulic lengths for each Strahler order.

In fact, this transfer function is equivalent to a Width function Rodriguez-Iturbe and Rinaldo (1997) since it gives the probability of having a given distance between a definite point of the stream network and the outlet of the drainage basin.

2.4 *The transfer function applied to the hillslopes*

First, the impact of time transfer applied to the slopes was not considered by the GIUH model. Then, van der Tak and Bras (1990) took this impact into account by convoluting a transfer function on the basin slopes thanks to the GIUH model. The transfer function applied to the basin slopes is a Gamma $pdf(l_0) = \Gamma(l_0, \alpha_h, \beta_h)$ law.

The α_h shape parameter can be determined experimentally and can reach values ranging from 1.2 to 3.1 (van der Tak and Bras, 1990). The value of

the scale parameter is $\beta_h = \frac{\alpha}{l_0}$ so that the $\Gamma(l_0, \alpha_h, \beta_h)$ integral is still equal to 1 (Figure 2).

3 Methodology and experimental data

In order to compare this geomorphological transfer analytical function (equation 7) with data from real river basins, it is first necessary to assess the parameters' value of the model (Strahler order n and average lengths l_k within the different orders k). Then we will just have to compare the graph of the function (7) with that of the experimental Width function of the studied river basins.

3.1 Assessment of the model's parameters

The parameters of the model are of geomorphological nature and easily available, these are the Strahler order (n) of the river basin and the drains' average lengths of order k ($\overline{l_k}$). These parameters can be assessed either with numerical data from the river basin (DEM, digitalisation), combined with a retrieval software for geomorphological data (ArcView, MapWindow), or manually with a map of the stream network. In the second case, the procedure is the following: first, the river basin is divided into a number of points representing raindrops falling uniformly on the river basin surface. The path of the raindrops down the hill slope to the stream network is determined using the contour lines; it is represented by length l_0 on Figure 1. Finally, the distance from this point to the river basin outlet is assessed by combining the covered lengths with the different Strahler order; they

correspond to lengths l_1, l_2, \dots, l_n .

For each point (or raindrop) we have the lengths l_1, l_2, \dots, l_n - some of them are obviously equal to zero - corresponding to the distance covered by raindrop through the river network, within the different Strahler orders.

Thus, we can calculate the average distance $\overline{l_1}, \overline{l_2}, \dots, \overline{l_n}$ covered for all the raindrops.

A sensitivity study of this methodology was carried out by Kertiwa (2004), revealing that it is not necessary to have a large number of points, that is to say too detailed a grid. In fact, Kertiwa's study on several river basins highlights the fact that shifting from a 3000-point grid to a 30- point grid involves very few measurement errors exceeding 10% of the estimated $\overline{l_k}$ averages.

As for the Strahler order n of the river basin, it is obtained even more quickly by reading the map and applying the Strahler's rules specified before. This sensitivity study highlights the fact that the parameters of the model are manually available quite rapidly.

3.2 Experimental data

In order to compare the analytical equation of the geomorphological transfer function (equation 7) with reality, we have chosen two very different river basins with regard to scale and morphology (Figures 3 and 4).

The Bunder river basin is located in Indonesia, its surface area is $29 \times 10^4 \text{m}^2$ and its Strahler order is 3. It is situated in the Wonosari region, 80 km north of Yogyakarta, the province capital, $07^\circ 53\text{S}$ and $110^\circ 32\text{E}$.

The river basin of Saint-Michel Mont-Mercure is located in France (Vendée),

its surface area is $673 \times 10^4 \text{m}^2$, and its Strahler order is 5. Its precise location is Vendée, $46^\circ 48' 3'' \text{N}$ and $0^\circ 54' 8'' \text{W}$.

The model parameters resulting from the experimental river basins are manually determined with the above described method, and appear in table 1.

4 Results and applications

4.1 *The geomorphological transfer function*

The comparison between the model (equation 7) and the experimental data is shown in Figure 5. The model parameters are given in table 1 for each river basin and equation (7) can thus be compared with the experimental Width function of the river basins.

It can be noted that, even though it is far from being perfect, the analytical function of geomorphological transfer provides the experimental Width function with a relatively good trend. It should be noticed that no calibration is necessary since the geomorphological parameters of the transfer function are directly derived from the experimental basins. Here is the major asset of such a function, which is based on general laws on symmetry, and repeats the hydraulic paths distribution (Width function) of the river basin, the geomorphological features of the latter being known (Strahler order and average lengths of hydraulic paths within the different orders).

Moreover, it can be noted that the combination of the symmetry hypotheses with fractal geometry (see model development in appendix), makes it possible to skip scale problems. In fact, although the Saint-Michel river basin

has a surface area which is more than twenty times as big as that of Bunder, this scale aspect does not necessarily limit the use of the model.

4.2 The AGIUH model

Equation (2) shows the relation between the GIUH, rainfall and hydrograph. The function $T(t)$, corresponds to the distribution of rainfall transfer times over the whole river basin. And yet, the analytical geomorphological transfer function (equation 7) specified in the present article is a theoretical Width function; consequently, it determines the lengths distribution covered by rain throughout the whole river basin. It is thus possible to shift from function $pdf(L)$ to function $T(t)$, supposing (Beven and Wood, 1993) an average runoff speed \bar{v} in the stream network and \bar{v}_h in the hillslopes. Then the geomorphological transfer function (equation 7) becomes an analytical GIUH (AGIUH) that reads:

$$T(t) = pdf\left(\frac{L}{\bar{v}}\right) \star pdf\left(\frac{l_0}{\bar{v}_h}\right) \quad (9)$$

$$= \frac{1}{\sqrt{(2\bar{v})^n \prod_{i=1}^n \bar{l}_i}} \frac{t^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)} e^{\left(-\frac{\bar{v}t}{2\bar{l}_n}\right)} \sum_{k=0}^{\infty} \frac{b_n(k) \left(\frac{n-1}{2}\right)_k}{k! \left(\frac{n}{2}\right)_k} \left(\frac{\bar{v}t}{2(\bar{l}_n - \bar{l}_{n-1})}\right)^k \star pdf\left(\frac{l_0}{\bar{v}_h}\right)$$

with $b_n(k)$ and $(x)_k$ given in equation (7).

The aim of this paper is not to discuss the rainfall-runoff modelling.

However, it seems important to describe an application of such AGIUH.

4.3 *Data watershed*

To keep as close as possible to the theory of unit hydrograph, the smaller river basin (Bunder) is chosen in order to fulfil the hypothesis of rainfall homogeneity. The used rainfall-runoff data (Kartiwa, 2004) correspond to two short events. The gross rainfall data are assessed with a tipping bucket rain gauge, allowing measurements with an accuracy to within 0.2 mm. Measurements are recorded every 6 minutes. Water depths at the river outlet are measured with an ultrasound sensor, which gives the distance between the sensor and the depth of runoff. The transformation of water depth into runoff is carried out with a rating curve which has been previously established.

The region is situated in an inter-tropical zone and has an equatorial type of climate under the influence of monsoon. It is characterised by an alternation of a rainy season lasting 6 months in average (from November to April) and a dry season starting from May until October. Average annual rainfall is between 1500 and 3000 mm.

4.4 *Production function*

In order to determine the net rainfall which transforms gross rainfall into rainfall runoff, it is necessary to assess the runoff deficit. This runoff deficit may be analysed more precisely by studying the infiltration of water into the soil. Several methods are available to assess the infiltration rate of water: Green and Ampt (1911), Horton (1933), Philip (1957) or Holtan (1961). We have chosen Horton's law because in our present state of knowledge, we

believe it is the most appropriate method to explain the hydraulic functioning of our specific case: the Indonesian drainage basin Kertiwa (2004).

This law makes it possible to express the instant capacity for infiltration according to time, in the case of saturating rainfall, in the form of (Horton, 1933):

$$f(t) = f_c + (f_0 - f_c) e^{-kt} \quad (10)$$

with

$f(t)$: infiltration capacity at time t (mm/mn)

f_0 : initial infiltration capacity (mm/mn)

f_c : final infiltration capacity (mm/mn)

k : constant (mn^{-1})

t : time (mn)

The value of the mass infiltration (F) is then worked out in the case of saturating rainfall:

$$F(t) = f_c t + \left(\frac{f_0 - f_c}{k} \right) (1 - e^{-kt}) \quad (11)$$

To make the most of Horton's law for modelling, it is necessary to start from the equation linking f and F to be able to express (accurately enough) at any time the instant capacity for infiltration according to the amount of water already infiltrated into the ground, whether rainfall is saturating or not:

$$f(t) = f_0 - k (F(t) - f_c t) \quad (12)$$

Net rainfall (RE , Rainfall Excess) is then worked out using the following formula:

$$RE(t) = R(t) - (f_0 - k(F(t) - f_c t)) \quad (13)$$

The prevailing types of soils are Mollisols of the haplustolls type on the one hand and haplustolls and Inceptisols of the ustropets and ustropets type on the other hand. Land-use is mainly gardens (37.8%) and agroforestry (30%), rice fields/groundnut cultivation (24%), 8.2% of the basin's surface are residential areas. Infiltration rate of water has been measured using the submersion method (double-ring infiltrometer) (Hills, 1970). The measuring points are chosen for each type of land-use. After setting the measured infiltration with the equation of Horton's law, the three parameters f_0 , f_c and k of Horton's law, are determined (table 2).

4.5 Model parameters

The averages hydraulic pathways within the various orders are respectively $\bar{l}_1 = 39$ m, $\bar{l}_2 = 68$ m and $\bar{l}_3 = 336$ m, which represents an average of $\bar{L} = 443$ m for all the whole river network.

The theoretical probability density function of the hillslopes hydraulic pathways $pdf(l_0) = \Gamma(l_0, \alpha_h, \beta_h)$ is adjusted to experimental data (Figure 2). The shape parameter α_h is 1.6 and matches van der Tak and Bras (1990) observations.

The average stream flow velocity in the river network has been determined (Kartiwa, 2004) in the field with a pygmy current meter. Value of the hillslope velocity is calculated by using Manning's semi empirical equation

(Chow, 1959):

$$\overline{v_h} = \frac{D^{\frac{2}{3}} S^{\frac{1}{2}}}{n_m} \quad (14)$$

Where D is the average depth of runoff on the hillslope and S is the average slope. The experimental values of the parameters related to slope appear in table 2. Concerning the average depth of water, the value of parameter D , is very hard to work out. Consequently, we have fixed a value for D making the value of $\overline{v_h}$ consistent with the observed runoff rates on the studied basin slopes. The Manning parameter n_m is $n_m = 0.075$ ("heavy brush"). These values lead to $\overline{v_h} = 0.17$ m/s.

Field method provide results in the order of 1 m/s and 0.9 m/s respectively. As for the simulations (Figure 6), we have set an average velocity $\overline{v} = 0.95$ m/s for the stream network.

For each of the two rainfall-runoff events in the Bunder river basin, the net rainfall determined by the production function is convoluted with the AGIUH model following equation (2), in order to simulate a hydrograph. This simulated hydrograph is then compared with the real hydrograph (Figure 6). The simulation quality is assessed thanks to the Nash and Sutcliffe (1970) coefficient:

$$F = 1 - \frac{\sum (Q_{exp} - Q_{sim})^2}{\sum (Q_{exp} - \overline{Q_{exp}})^2} \quad (15)$$

Where, Q_{exp} (m^3/s) is the actual flow rate, Q_{sim} (m^3/s) is the simulated flow rate and $\overline{Q_{exp}}$ (m^3/s) is the average value of the actual flow rate.

A higher than 0.8 Nash-Sutcliffe coefficient means a good simulation which is close to the stream measurements; this is the case of our simulations resulting in coefficient values of 0.97 and 0.98.

5 Conclusion and prospects

We have described here an analytical geomorphological transfer function representing the theoretical distribution of hydraulic lengths in a river basin. This model is based on the assumption that the reduced lengths distributions ($\frac{L_k}{R_L}$) in order k (Strahler order) are similar and represented by a Gamma law with $(\frac{1}{2}, \frac{1}{2L_k})$ parameters. The analytical geomorphological transfer function results from the convolution of these n Gamma laws (equations 6 and 7); n being the total Strahler order in the river basin. This theoretical result is an analytical function depending on $n + 1$ geomorphological parameters which are easily available by simply reading a map or analysing digital data (DEM for example). These parameters are: n , the Strahler order in the stream network and the average lengths $\overline{l}_1, \overline{l}_2, \dots, \overline{l}_n$ from the drains to the different network orders.

After explaining the approach to obtain this analytical function (equation 7), we have tested our theoretical model in two very different river basins, as far as their morphological nature and scale were concerned. The results reveal that this theoretical function, equivalent to a Width function by definition, highlights the real structure of both experimental river basins, without any calibration being implemented. Such function makes it possible to give a fairly accurate description of the morphological structure of the stream network in the river basin.

Then, we have put forward an applied example of this analytical geomorphological transfer function for the rainfall-runoff modelling. In fact, by assuming that flow velocity is homogenous through the stream network, it

is possible to convert this analytical geomorphological transfer function into a AGIUH (Analytical Geomorphologic Instantaneous Unit Hydrograph, equation 9), and thus to model the conversion of rainfall into runoff in a river basin.

Here are the major conclusions that can be drawn from this article:

- We have described the development of a new analytical GIUH model, whose geomorphological parameters can easily and rapidly be determined;
- The transfer function, based on the fractal features of stream networks, can be used at multiple scales and thus makes it possible to validate the model on river basins having quite different scales;
- The hypotheses on symmetry Duchesne et al. (1997), which constitute the foundations of the model, make it universal and versatile. In this particular case, the model provides a description of the geomorphometric structure of stream networks; but it may be used for branched structures in general Duchesne et al. (2002); Fleurant et al. (2004);
- Since the analytical model of transfer function is parameterized with geomorphological variables derived from the real river basin, the corresponding AGIUH makes it possible to carry out rainfall-runoff simulations, through a simple convolution with rainfall and with no calibration. Consequently, the model proves to be a useful tool for anticipating floods and studying the resulting hydrological response even in ungauged river basins.

These findings pave the way for additional validation work of the model, at various scales and with regards to even more varied hydrological applications. Actually, the relation which exists between the river basin's

geomorphology and the rainfall-runoff response is a significant issue for the understanding of hydro-geomorphological processes (Poole et al., 2002; Yair and Raz-Yassif, 2004).

Acknowledgements

Review of Anne Bouillon, Keith Beven and two referees have helped to improve and to clarify this manuscript.

Appendix

Detailed calculations concerning the present model

The determination of both $pdf(l_k)$ and $pdf(L)$ needs some mathematical developments that are presented in this section.

The distribution model of the l_k drains' hydraulic lengths (equation 5) is based on two hypotheses on symmetry, which are combined with the fractal nature of the stream networks (Duchesne et al., 1997; Cudennec, 2000; Cudennec et al., 2004):

- The independence hypotheses of the distribution function of components reduced by the morphometric length ($\frac{l_k}{R_L}$). According to this hypothesis, the distribution function of a component l_k only depends on l_k , not on any of the other components l_j , k being different from j ;
- The isotropy hypothesis of distributions. One considers that the distribution law of l_k is isotropic. Because of the self-similarity nature of

tree structure and Horton's law, the k order component is larger, on average, than the $k - 1$ order component, because $\overline{l_k} = R_L \overline{l_{k-1}}$. The isotropy hypothesis should then be applied, with the reduced components of branch lengths $\frac{l_k}{R_L^{k-1}}$ as coordinates of the symbolic space.

Such hypotheses lead to the distribution the l_k drains' hydraulic lengths (Duchesne et al., 1997; Cudennec, 2000; Cudennec et al., 2004):

$$pdf(l_k) = \frac{1}{\sqrt{\pi}} \sqrt{\frac{\lambda}{R_L^{k-1}}} \frac{1}{\sqrt{l_k}} e^{-\frac{\lambda l_k}{R_L^{k-1}}} \quad (16)$$

The constant λ may be explained by expressing the l_k components average, named $\overline{l_k}$, it gives successively:

$$\begin{aligned} \overline{l_k} &= \int_0^{+\infty} l_k pdf(l_k) dl_k \\ &= \frac{1}{\sqrt{\pi}} \sqrt{\frac{\lambda}{R_L^{k-1}}} \int_0^{+\infty} \frac{l_k}{\sqrt{l_k}} e^{-\frac{\lambda l_k}{R_L^{k-1}}} dl_k \\ &= \frac{1}{\sqrt{\pi}} \sqrt{\frac{\lambda}{R_L^{k-1}}} \int_0^{+\infty} l_k^{\frac{1}{2}} e^{-\frac{\lambda l_k}{R_L^{k-1}}} dl_k \\ &= \frac{1}{\sqrt{\pi}} \sqrt{\frac{\lambda}{R_L^{k-1}}} \Gamma\left(\frac{1}{2} + 1\right) \left(\frac{R_L^{k-1}}{\lambda}\right)^{\frac{3}{2}} \\ &= \frac{R_L^{k-1}}{2\lambda} \end{aligned}$$

After replacing this result in equation (16) we have:

$$pdf(l_k) = \frac{1}{\sqrt{2\pi \overline{l_k}}} \frac{1}{\sqrt{l_k}} e^{-\frac{l_k}{2\overline{l_k}}} \quad (17)$$

A Gamma distribution with $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2\overline{l_k}}$ can be recognised here and is named $\Gamma\left(l_k, \frac{1}{2}, \frac{1}{2\overline{l_k}}\right)$.

The second step consists of determining the probability density distribution

for L , knowing that $L = \sum_{k=1}^n l_k$. The most usual way to calculate the distribution $pdf(L)$, is to use the following property (Feller, 1971):

If we have n independent random variables l_k , $k = 1, 2, \dots, n$ of probability density function $pdf(l_k)$ then the random variable L defined as $L = \sum_{k=1}^n l_k$ has a probability density function $pdf(L)$ such as:

$$pdf(L) = pdf(l_1) \star pdf(l_2) \star \dots \star pdf(l_n) \quad (18)$$

Where \star is the convolution integral. To clarify this complex mathematical process, here is a simplest example of what a convolution integral with only two functions is:

If $f(x)$ and $g(x)$ are two functions:

$$\begin{aligned} h(x) &= f(x) \star g(x) \\ &= \int_{-\infty}^{+\infty} f(y)g(x-y)dy \end{aligned}$$

Convolution integral of two Gamma distributions

Let us work out the convolution integral of two Gamma distributions with β parameters which are different in pairs, we then try to calculate $\Gamma(\alpha, \beta_1) \star \Gamma(\alpha, \beta_2)$. To do this calculation, we will use the characteristic Gamma distribution functions. In fact, to convolute both Gamma distributions means that we are looking for the probability density function $pdf(L)$ for L such as $L = l_1 + l_2$ where l_1 and l_2 have a probability density function represented by the functions $\Gamma(\alpha, \beta_1)$ and $\Gamma(\alpha, \beta_2)$. Moreover, the characteristic functions, named $\varphi_{l_k}^*$, have the following property:

$$\varphi_{l_1+l_2}^* = \varphi_{l_1}^* \varphi_{l_2}^* \quad (19)$$

The characteristic function of a random variable, named l_k variables here, is defined as a mathematical expectation of the exponential function of this random variable; in the present case we have therefore:

$$\begin{aligned}
\varphi_{l_k}^*(s) &= E \left[e^{-sl_k} \right] \\
&= \int_0^{+\infty} e^{-sl_k} pdf(l_k) dl_k \\
&= \frac{1}{\Gamma(\alpha)\beta_k^\alpha} \int_0^{+\infty} l_k^{\alpha-1} e^{-l_k \left(s + \frac{1}{\beta_k} \right)} dl_k \\
&= \frac{1}{(1 + s\beta_k)^\alpha}
\end{aligned}$$

Here on has:

$$\varphi_{l_1+l_2}^*(s) = \frac{1}{[(1 + s\beta_1)(1 + s\beta_2)]^\alpha}$$

Concerning our case, $\alpha = \frac{1}{2}$, then function $pdf(L)$ such as $L = l_1 + l_2$ can be worked out thanks to Laplace transform tables:

$$pdf(L) = \frac{1}{\sqrt{\beta_1\beta_2}} e^{-\frac{\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)L}{2}} I_0 \left(\frac{\left(\frac{1}{\beta_2} - \frac{1}{\beta_1}\right)L}{2} \right) \quad (20)$$

Where $I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} + \frac{x^6}{2^2 4^2 6^2} + \dots$ is the modified Bessel function of the first kind and zero order. The result of the probability density function of morphometric lengths L is therefore:

$$pdf(L) = \frac{1}{2\sqrt{l_1 l_2}} e^{-\frac{\left(\frac{1}{l_1} + \frac{1}{l_2}\right)L}{4}} I_0 \left(\frac{\left(\frac{1}{l_2} - \frac{1}{l_1}\right)L}{4} \right) \quad (21)$$

Convolution integral of n Gamma distributions

The result of the convolution integral for n Gamma distributions is obtained by working out the generalised characteristic functions (Mathai, 1982;

Moschopoulos, 1985; Sim, 1992):

Let l_1, l_2, \dots, l_n , be n independent random distributed according to a

Gamma distribution of the following type:

$$\Gamma(l, \alpha_i, \beta_i) = \frac{\beta_i^{\alpha_i} l^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-\beta_i l} \quad (22)$$

Then the probability density function for $L = \sum_{i=1}^n l_i$, is:

$$pdf(L) = \left(\prod_{i=1}^n \beta_i^{\alpha_i} \right) \frac{L^{\sum_{i=1}^{n-1} \alpha_i - 1}}{\Gamma\left(\sum_{i=1}^n \alpha_i\right)} e^{-\beta_n L} \sum_{k=0}^{+\infty} \frac{b_n(k) \left(\sum_{i=1}^{n-1} \alpha_i\right)_k}{k! \left(\sum_{i=1}^n \alpha_i\right)_k} [(\beta_n - \beta_{n-1}) L]^k \quad (23)$$

where

$$b_i(k) = \begin{cases} 1, & \text{if } i = 2 \\ \sum_{j=0}^k \frac{b_{i-1}(j) \left(\sum_{p=1}^{i-2} \alpha_p\right)_j (-k)_j}{j! \left(\sum_{p=1}^{i-1} \alpha_p\right)_j} C_i^j & \text{for } i = 3, 4, \dots, n \end{cases}$$

with

$$C_i = \frac{\beta_{i-2} - \beta_{i-1}}{\beta_i - \beta_{i-1}} \quad (24)$$

and

$$(x)_k = x(x+1)(x+2) \dots (x+k-1) \quad (25)$$

By returning to our initial notations, the probability density function of

morphometric lengths is defined by the following equation:

$$pdf(L) = \frac{1}{\sqrt{2^n \prod_{i=1}^n \bar{l}_i}} \frac{L^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)} e^{\left(-\frac{L}{2\bar{l}_n}\right)} \sum_{k=0}^{\infty} \frac{b_n(k) \left(\frac{n-1}{2}\right)_k}{k! \left(\frac{n}{2}\right)_k} \left(\frac{L}{2(\bar{l}_n - \bar{l}_{n-1})}\right)^k \quad (26)$$

with

$$b_i(k) = \begin{cases} 1, & \text{if } i = 2 \\ \sum_{j=0}^k \frac{b_{i-1}(j) \left(\frac{i-2}{2}\right)_j (-k)_j}{j! \left(\frac{i-1}{2}\right)_j} \left(\frac{\bar{l}_{i-1} - \bar{l}_{i-2}}{\bar{l}_{i-1} - \bar{l}_i} \frac{\bar{l}_{i-1}}{\bar{l}_{i-2}}\right)^j & \text{for } i = 3, 4, \dots, n \end{cases}$$

References

- Beven, K, Wood, E., 1993. Channel network hydrology, chapter Flow routing and the hydrological response of channel networks. K. Beven and M. Kirkby (eds), John Willey and Sons.
- Chow, V., 1959. Open-Channel Hydraulics. McGraw-Hill, New York.
- Chow, V., Maidment, D., Mays, L., 1988. Applied Hydrology. New York, McGraw-Hill.
- Cudennec, C., 2000. Description mathématique de l'organisation du réseau hydrographique et modélisation hydrologique. Ph.D. thesis, Ecole Nationale Supérieure d'Agronomie de Rennes.
- Cudennec, C., Fouad, Y., Sumarjo Gatot, I., Duchesne, J., 2004. A Geomorphological explanation of the unit hydrograph concept. Hydrological Processes, **18**, 603–621.
- Dooge, J., 1959. A general theory of the unit hydrograph. J. Geophys. Res., **64**(2), 241–256.
- Duchesne, J., Cudennec, C., Corbierre, V., 1997. Relevance of the H2U model to predict the discharge of a catchment. Water Science and Technology, **36**, 169–175.
- Duchesne, J., Raimbault, P., Fleurant, C., 2002. Towards a universal law of tree morphometry by combining fractal geometry and statistical physics. Proceeding: Emergent Nature, Patterns, growth and scaling in the sciences, pages 93–102.
- Feder, J., 1988. Fractals. Plenum Press.
- Feller, W., 1971. An introduction to probability theory and its applications. Wiley.
- Fleurant, C., Duchesne, J., Raimbault, P., 2004. An allometric model for

- trees. *J. Theor. Biology*, **227**, 137–147.
- Fleurent, C., Boulestreau, P., 2005. Comment on 'Cudennec, Fouad, Sumarjo gatot, Duchesne, A Geomorphological explanation of the unit hydrograph concept, *Hydrological Processes*, 18 (2004), 603-621. *Hydrological Processes*, **19**, 541–543.
- Green, W.H., Ampt, G.A., 1911. Studies on soils physics: 1, flow of air and water through soils. *J. Agric. Sci.*, **4**, 1–24.
- Gupta, V. Waymire, E., 1983. On the formulation of an analytical approach to hydrologic response and similarity at the basin scale. *J. Hydrology*, **65**, 95–123.
- Gupta, V., Waymire, E., Wang, C., 1980. A representation of an instantaneous unit hydrograph from geomorphology. *Water Resources Res.*, **16**(5), 855–862.
- Hills, R.C., 1970. The determination of the infiltration capacity on field soils using the cylinder infiltrometer. *B.G.R.G., Technical Bull.*, **3**, 24.
- Holtan, H.N., 1961. A concept for infiltration estimates in watershed engineering. *U.S. Dep. Agric. Agric. Res. Serv. Bull.*, **41-51**, 25.
- Horton, R.E., 1933. The role of infiltration in the hydrologic cycle. *EOS Trans A.G.U.*, **14**, 446–460.
- Horton, R., 1945. Erosional development of streams and their drainage basins; hydrological approach to quantitative morphology. *Bulletin of the Geological Society of America*, **56**, 275–370.
- Jin, C., 1992. A deterministic gamma-type geomorphologic instantaneous unit hydrograph based on path types. *Water Resources Res.*, **28**(2), 479–486.
- Kartiwa, B., 2004. Modélisation du fonctionnement hydrologique des bassins versants. Application sur des bassins versants de Java et Sumatra

- (Indonésie). Ph.D. thesis, Institut National d'Horticulture.
- Kirchner, J., Feng, X., Neal, C., 2001. Catchment-scale advection and dispersion as a mechanism for fractal scaling in stream tracer concentrations. *J. Hydrology*, **254**, 82–101.
- Kirshen, D., Bras, R., 1983. The linear channel and its effects on the geomorphological evolution. *J. Hydrology*, **65**, 175–208.
- LaBarbera, P., Rosso, R., 1987. Fractal geometry of river networks. *EOS Transaction AGU*, **68**, 12–76.
- LaBarbera, P., Rosso, R., 1989. On the fractal dimension of stream networks. *Water Resources Res.*, **25**(4), 735–741.
- LaBarbera, P., Rosso, R., 1990. On the fractal dimension of stream networks. *Water Resources Res.*, **26**, 2245–2248.
- Lienhard, J., 1964. A statistical mechanical prediction of the dimensionless UH. *J. Geophys. Res.*, **69**, 5231–5238.
- Mandelbrot, B., 1988. *Fractal geometry of nature*. Hardcover.
- Mathai, A., 1982. Storage capacity of a dam with gamma type inputs. *Ann. Inst. Statist. Math.*, **34**, 591–597.
- Moschopoulos, P., 1985. The distribution of the sum of independent gamma random variables. *Ann. Inst. Statist. Math.*, **37**, 541–544.
- Nash, J., 1957. The form of the instantaneous unit hydrograph. *International Association of Scientific Hydrology*, **42**, 112–114.
- Nash, J., 1959. Synthetic determination of unit hydrograph parameters. *J. Geophys. Res.*, **64**(1), 112–114.
- Nash, J., 1960. A unit hydrograph study, with particular reference to british catchments. *Proc. Inst. Civ. Eng.*, **17**, 249–282.
- Nash, J., Sutcliffe, J., 1970. River flow forecasting through conceptual models: 1. discussion and principles. *J. Hydrology*, **10**, 282–290.

- Philip, J., 1957. The theory of infiltration : 1. The infiltration equation and its solution. *Soil Science*, **83**, 345–357.
- Poole, G., Stanford, J. A., Frissell, C. A., Running, S. W., 2002. Three-dimensional mapping of geomorphic controls on flood-plain hydrology and connectivity from aerial photos. *Geomorphology*, **48**(4), 329–347.
- Rodriguez-Iturbe, I., Rinaldo, A., 1997. *Fractal river basins*. Cambridge University Press.
- Rodriguez-Iturbe, I., Valdès, J., 1979. The geomorphologic structure of hydrologic response. *Water Resources Res.*, **15**(6), 1409–1420.
- Rosso, R., 1984. Nash model relation to horton order ratios. *Water Resources Res.*, **20**, 914–920.
- Rosso, R., Bacchi, B., LaBarbera, P., 1991. Fractal relation of mainstream length to catchment area in river networks. *Water Resources Res.*, **27**(3), 381–387.
- Schumm, S., 1956. Evolution of drainage system and slopes in badlands at pearth amboy. *Bulletin of the Geological Society of America*, **87**, 597–646.
- Sherman, L., 1932. Streamflow from rainfall by unit-graph method. *Engineering News-Record*, pages 501–505.
- Shreve, R., 1969. Stream lengths and basin areas in topologically random channel networks. *Journal of Geology*, **77**, 397–414.
- Sim, C., 1992. Point processes with correlated gamma interarrival times. *Statistics and Probability Letters*, **15**, 135–141.
- Strahler, A., 1957. quantitative analysis of watershed geomorphology. *Transaction AGU*, **38**, 913–920.
- Tarboton, D., Bras, R., Rodriguez-Iturbe, I., 1990. Comment on "on the fractal dimension of stream networks" by P. LaBarbera and R. Rosso.

Water Resources Res., **26**, 2243–2244.

van der Tak, L., Bras, R., 1990. Incorporating hillslope effects into the geomorphologic instantaneous unit hydrograph. Water Resources Res., **26**(10), 2393–2400.

Yair, A., Raz-Yassif, N., 2004. Hydrological processes in a small arid catchment: scale effects of rainfall and slope length. Geomorphology. in press.

Figures and Tables

Figure 1: The concept of Stralher's hierarchical classification system. The hydraulic length is the added lengths of channel up to an indefinite rain drop of the river basin. Here, the path between a rain drop of the river basin and the outlet passes through first, second and third orders.

Figure 2: Distributions of the hydraulic lengths on the hillslopes of Bunder and the corresponding probability density function $pdf(l_0)$.

Figure 3: Bunder watershed's map, Wonosari, Yogyakarta, Indonesia. The basin of Bunder is a 3 Strahler's order river basin with only $29 \times 10^4 \text{m}^2$ of area.

Figure 4: Saint-Michel river basin's map, France. The basin of Saint-Michel is a 5 Strahler's order river basin with $673 \times 10^4 \text{m}^2$ of area.

Figure 5: Comparison between the experimental pdfs of the hydraulic pathways (experimental Width functions) of the drainage basins of Bunder (up - dash line) and Saint-Michel (down - dash line) and the corresponding theoretical probability density functions (equation 7). Simulated parameters are given in table 1.

Figure 6: Rainfall-runoff simulation on the river basin of Bunder. Observed rainfall-runoff data for event November 14, 2001 at Bunder. Infiltration is calculated by an original method from Kertiwa, 2004. Simulations are carried out using the AGIUH (equation 9) convoluted within net rainfall according to equation (2). Model parameters are given in the table 1 and the average water velocity on the whole river basin is $\bar{v} = 0.95 \text{ m/s}$.

River basin	Area ($\times 10^4 \text{m}^2$)	Strahler order	\overline{l}_1 (m)	\overline{l}_2 (m)	\overline{l}_3 (m)	\overline{l}_4 (m)	\overline{l}_5 (m)
Saint-Michel	673	5	149	240	1275	426	329
Bunder	29	3	39	62	329		

Table 1

Geomorphological characteristics of the studied river basins.

Lad use	area	f_0	f_c	k
	(%)	(mm.mn ⁻¹)	(mm.mn ⁻¹)	(mn ⁻¹)
garden	37.8	1.149	0.106	0.244
Agroforestry	30.0	1.335	0.107	0.253
Rice/Arachid	24.0	1.278	0.144	0.136
settlement	8.2	0.490	0.143	0.068
mean value		1.181	0.118	0.206

Table 2

Values of the Horton's parameters on the watershed of Bunder.

Model parameters	value
n	3
$\overline{l_0}$ (m)	57
\overline{v} (ms ⁻¹)	0.95
$\overline{v_h}$ (ms ⁻¹)	0.17
$\overline{l_1}$ (m)	39
$\overline{l_2}$ (m)	68
$\overline{l_3}$ (m)	336

Table 3

Values of the simulation parameters corresponding to the model presented in this article. the model has four common parameters: the Strahler order of the drainage basin (n), the average rate of flow in the stream network (\overline{v}), the average runoff rate on the slopes ($\overline{v_h}$) and the average hydraulic lengths on the slopes ($\overline{l_0}$).

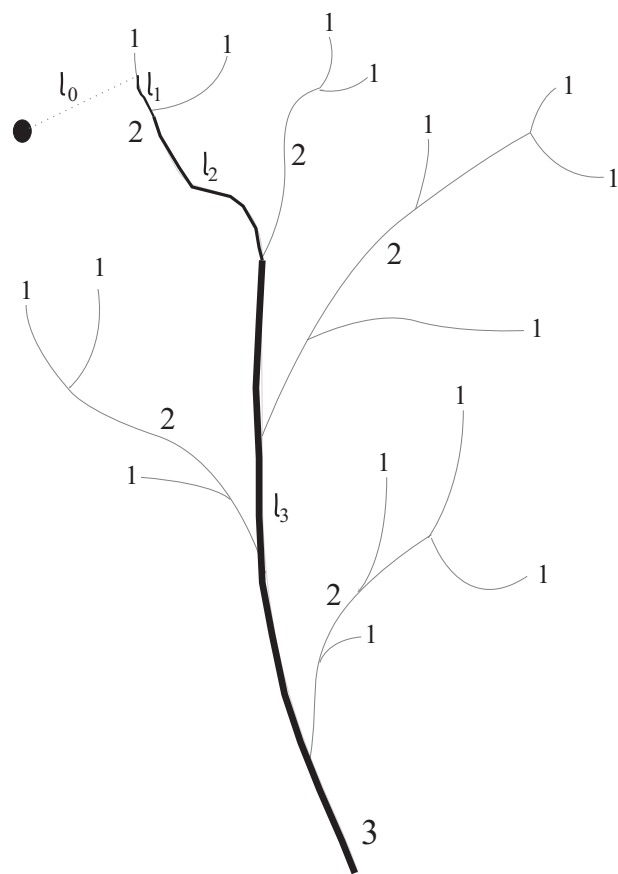


Fig. 1. C. Fleurant

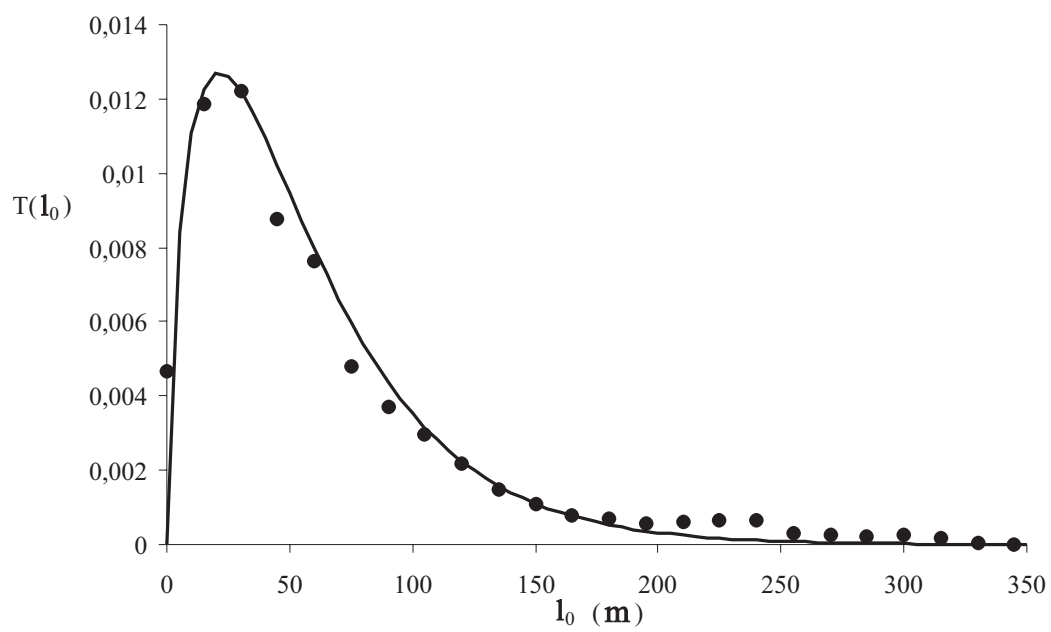


Fig. 2. C. Fleurant

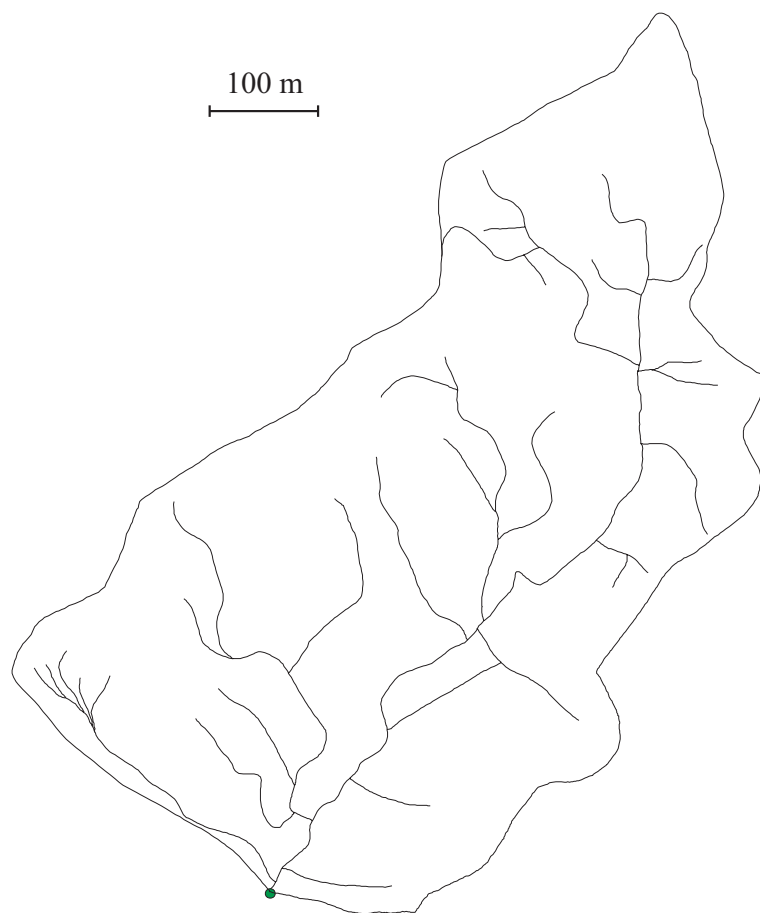


Fig. 3. C. Fleurant

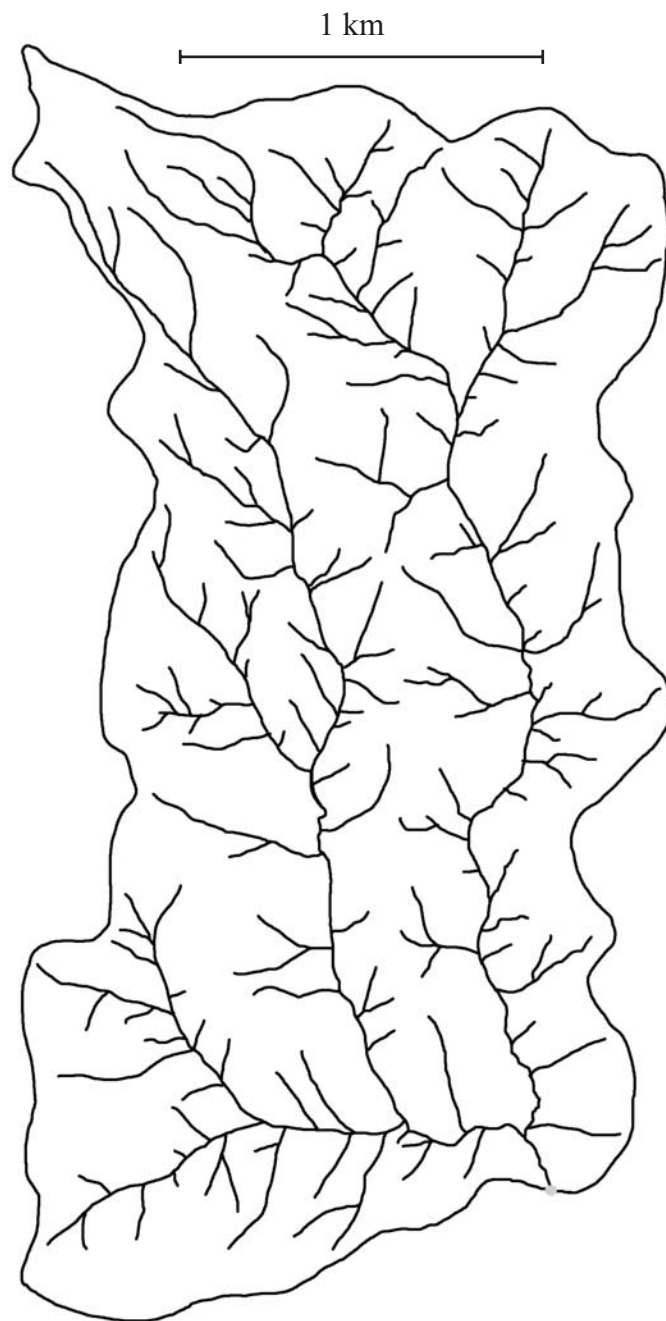


Fig. 4. C. Fleurant

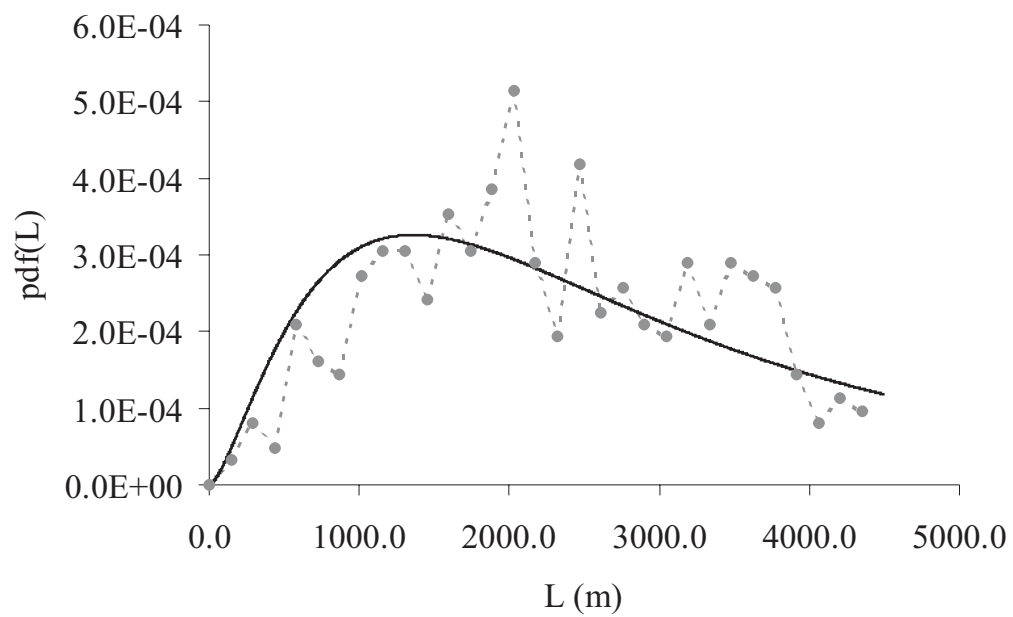
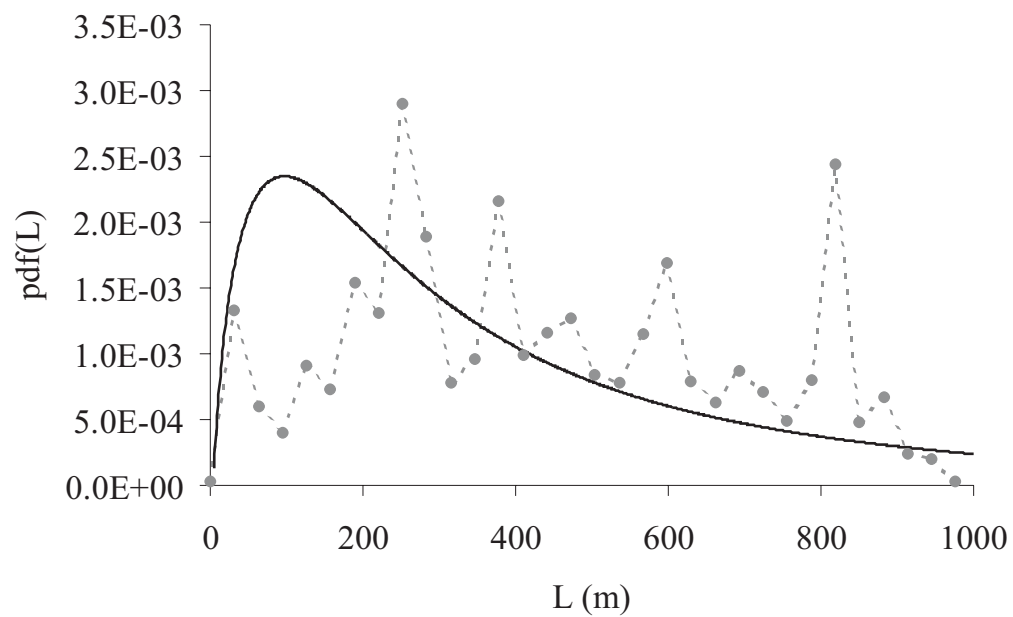


Fig. 5. C. Fleurant

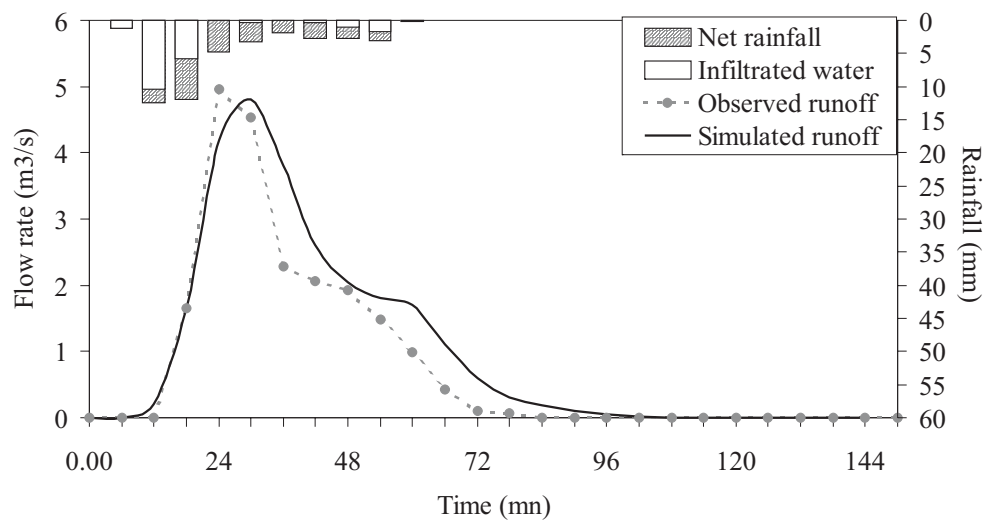
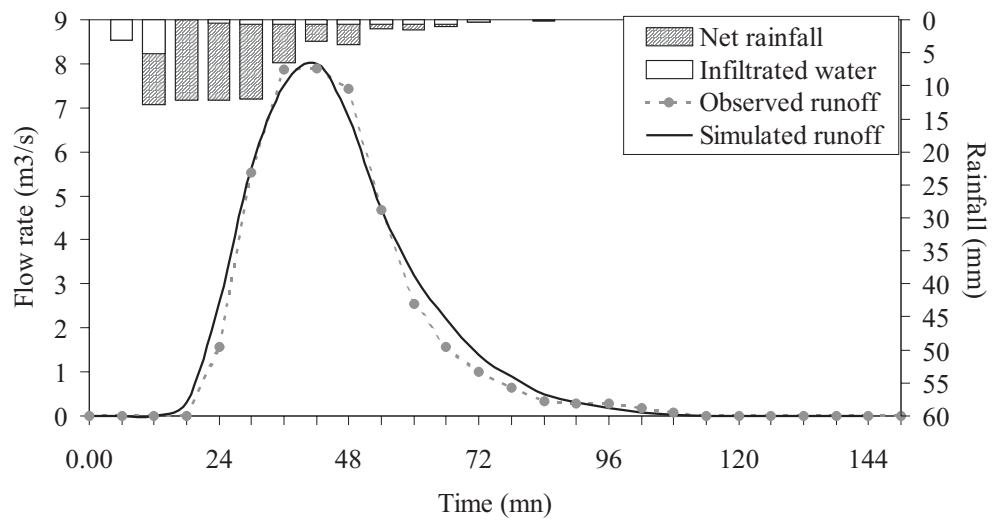


Fig. 6. C. Fleurant