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RECOVERING THE TWISTING FUNCTION IN A TWISTED WAVEGUIDE FROM THE DN MAP

MOURAD CHOULLI† AND ERIC SOCCORSI‡

ABSTRACT. We consider the inverse problem of determining the twisting function in a infinite cylindrical twisted waveguide from the corresponding DN map. This problem, which is naturally linked to some inverse anisotropic conductivity problem in a straight waveguide, remains generally open, unless the twisting function is assumed to be affine. Namely we prove Lipschitz stability in the determination of affine twisting functions from the DN map. This result still holds true upon substituting a suitable approximation of the DN map, provided the first derivative of the twisting is sufficiently close to some *a priori* fixed constant.

Key words: Dirichlet Laplacian, twisted infinite cylindrical waveguide, twisting function, DN map, stability estimate.

AMS subject classifications: 35R30.

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1. INTRODUCTION

Let ω be a bounded domain of \mathbb{R}^2 . To $\Omega = \omega \times \mathbb{R}$ and $\theta \in C^1(\mathbb{R})$ we associate the infinite twisted cylindrical domain

$$\Omega_\theta = \{(R_{\theta(x_3)}x', x_3); x' = (x_1, x_2) \in \omega, x_3 \in \mathbb{R}\},$$

where R_ξ denotes the rotation in \mathbb{R}^2 of angle $\xi \in \mathbb{R}$. Twisted waveguides modeled by Ω_θ exhibit interesting propagation properties such as the occurrence of propagating waveguide modes with phase velocities slower than those of similar modes in a straight waveguide. This explains why twisted waveguides are at the center of the attention of many theoretical and applied physicists (see e.g. [Ka, DR, KF, NZG, Sh, Wi, YM]). Moreover, it turns out that they are the source of challenging spectral and PDE problems, some of them having been extensively studied in the mathematical literature (see e.g. [BK, EKK, KK, KS, KZ1, KZ2]).

Nevertheless, the inverse problem of identifying the twisting function from the Dirichlet-to-Neumann (DN in short) map, has, to our knowledge, not been examined in this framework yet. The study of this open problem is actually the main purpose of the present article. Namely, we consider in this paper the following boundary value problem for the Laplacian in the twisted waveguide Ω_θ ,

$$(1.1) \quad \begin{cases} \Delta v(y) = 0, & y \in \Omega_\theta, \\ v(y) = g(y), & y \in \partial\Omega_\theta, \end{cases}$$

and we address the problem of recovering the twisting function θ (actually its first derivative) from the DN map

$$(1.2) \quad \tilde{\Lambda}_\theta : g \mapsto \nabla v(y) \cdot \nu(y),$$

where ν denotes the outer unit normal to $\partial\Omega_\theta$. More specifically, our aim is the stability issue for the problem of determining θ from $\tilde{\Lambda}_\theta$.

The DN map $\tilde{\Lambda}_\theta$ acts on functional spaces depending on θ . Therefore it is not well suited to the analysis of this inverse problem. This difficulty may be overcome by turning (1.1)-(1.2) into an equivalent system associated to some θ -independent DN map. To this purpose we introduce

$$T_\xi = \begin{pmatrix} R_\xi & 0 \\ 0 & 1 \end{pmatrix}$$

and put $u(x) = v(T_{\theta(x_3)}(x', x_3))$, $x = (x', x_3) \in \Omega$. By performing the change of variable $y = T_{\theta(x_3)}(x', x_3)$ in (1.1), we find by direct calculation that u is the solution to the following boundary value problem with an elliptic operator in the divergence form

$$(1.3) \quad \begin{cases} \operatorname{div}(A(x', \theta'(x_3))\nabla u) = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases}$$

where, $f(x) = g(T_{\theta(x_3)}(x', x_3))$, $x \in \partial\Omega$, and the matrix A is given by

$$A(x', t) = \begin{pmatrix} 1 + x_2^2 t^2 & -x_2 x_1 t^2 & -x_2 t \\ -x_2 x_1 t^2 & 1 + x_1^2 t^2 & x_1 t \\ -x_2 t & x_1 t & 1 \end{pmatrix}, \quad x' \in \omega, t \in \mathbb{R}.$$

Moreover, it is proved in section 5 that we have

$$(1.4) \quad \nabla v(y) \cdot \nu(y) = A(x', \theta'(x_3))\nabla u(x) \cdot \nu(x), \quad y = T_{\theta(x_3)}(x', x_3), \quad x = (x', x_3) \in \Omega,$$

in a formal sense. The above identity indicates that recovering θ from $\tilde{\Lambda}_\theta$ given by (1.2) is the same as determining θ from the following ¹DN map

$$\Lambda_\theta : f \mapsto A\nabla u \cdot \nu.$$

The major part of our work will therefore be devoted to studying Λ_θ in view of establishing stability in the identification of θ from Λ_θ . In light of (1.3)-(1.4) we notice that this is the same kind of inverse anisotropic conductivity problem, but stated here in an unbounded domain, as the one studied in a bounded domain by Alessandrini [A] and Alessandrini and Gaburro [AG1], [AG2] (see also Gaburro and Lionheart [GL]). However, it turns out that the usual monotonicity assumption on the conductivity, which is essential to the identification of A from the DN map in this approach, is not fulfilled by the matrix A under consideration. This is the main reason why the inverse problem associated to (1.3) remains open for general twisting functions $\theta \in C^1(\mathbb{R})$. However, in this paper we are able to prove Lipschitz stability in the determination of θ from the DN map when θ is an affine function. Moreover, if θ is sufficiently close to some arbitrarily fixed constant, we prove that the original DN map can be appropriately approximated by a DN map whose conductivity satisfies the monotonicity assumption.

The paper is organized as follows. Section 2 contains the definition and the main properties of Λ_θ , needed in the proofs of the coming sections. The first part of section 3 explains why the approach developed by Alessandrini and Gaburro in [AG1] and [AG2] does not apply to the problem under consideration. Nevertheless, we prove in the second part of section 3 that twisting functions close enough to some arbitrarily fixed constant may actually be identified by solving an inverse conductivity problem which satisfies the above mentioned monotonicity condition. Further, the particular case of affine twisting functions is examined in section 4. This is by means of the Fourier transform with respect to the variable x_3 , in order to bring the original problem into some anisotropic conductivity problem stated in ω . The corresponding conductivity matrix satisfies a weak monotonicity condition, allowing to claim stability in the determination of the twisting function from some suitable DN map. Finally, the original DN map $\tilde{\Lambda}_\theta$ is rigorously defined and linked to Λ_θ in section 5.

¹It is worth mentioning that the Neumann data measurements required by the definition of Λ_θ are taken on the boundary of the straight waveguide Ω where the twisting function θ is not directly accessible.

2. THE DN MAP

Solution to the boundary value problem (1.3). As we are dealing with an infinitely extended domain Ω , we start by defining the Sobolev spaces on $\partial\Omega$ that are needed in our analysis. To this end, for $s = 1/2$ or $s = 3/2$, we see that the trace operator

$$\begin{aligned} \tau : C_0^\infty(\mathbb{R}, H^{s+1/2}(\omega)) &\longrightarrow L^2(\mathbb{R}, H^s(\partial\omega)) \\ G &\mapsto [t \in \mathbb{R} \mapsto G(t, \cdot)|_{\partial\omega}] \end{aligned}$$

extends to a bounded operator, still denoted by τ , from $L^2(\mathbb{R}, H^{s+1/2}(\omega))$ into $L^2(\mathbb{R}, H^s(\partial\omega))$. Let

$$\tilde{H}^s(\partial\Omega) = \{g \in L^2(\mathbb{R}, H^s(\partial\omega)); \text{ there exists } G \in H^{s+1/2}(\Omega) \text{ such that } \tau G = g\}.$$

$\tilde{H}^s(\partial\Omega)$ is a Banach space for the quotient norm:

$$\|g\|_{\tilde{H}^s(\partial\Omega)} = \inf\{\|G\|_{H^{s+1/2}(\Omega)}; G \in H^{s+1/2}(\Omega) \text{ is such that } \tau G = g\}.$$

For the sake of clarity we write $G = g$ on $\partial\Omega$ instead of $\tau G = g$ in the sequel.

Let us now establish the following useful extension lemma from the above definitions.

Lemma 2.1. *Let $g \in \tilde{H}^s(\partial\Omega)$ for $s = 3/2$ or $s = 1/2$. Then there exists $G \in H^{s+1/2}(\Omega)$ such that $G = g$ on $\partial\Omega$ and*

$$(2.1) \quad \|G\|_{H^{s+1/2}(\Omega)} \leq 2\|g\|_{\tilde{H}^s(\partial\Omega)}.$$

Let us now make the following remark on the uniform ellipticity of A , where A denotes either $A(x', t)$ or $A(x', \theta(x_3))$, as defined in the previous section. For all $\zeta \in \mathbb{R}^3$, $x' \in \omega$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} A(x', t)\zeta \cdot \zeta &= \zeta_1^2 + \zeta_2^2 + \zeta_3^2 - 2tx_2\zeta_1\zeta_3 + 2tx_1\zeta_2\zeta_3 + t^2(x_2\zeta_1 - x_1\zeta_2)^2 \\ &= \zeta_1^2 + \zeta_2^2 + (\zeta_3 + t(x_2\zeta_1 - x_1\zeta_2))^2, \quad x' = (x_1, x_2) \in \omega, t \in \mathbb{R}, \end{aligned}$$

by a straightforward computation. For every $x' \in \omega$ and $t \in \mathbb{R}$, this entails that $A(x', t)\zeta \cdot \zeta = 0$ if and only if $\zeta = 0$. Therefore, since $\bar{\omega} \times [\underline{t}, \bar{t}]$ is compact for all real numbers $\underline{t} < \bar{t}$, there exists $\lambda \geq 1$, depending on ω , \underline{t} and \bar{t} , such that we have

$$(2.2) \quad \lambda^{-1}|\zeta|^2 \leq A(x', t)\zeta \cdot \zeta \leq \lambda|\zeta|^2 \text{ for all } x' \in \omega, t \in [\underline{t}, \bar{t}], \zeta \in \mathbb{R}^3.$$

In order to define the DN map associated to the boundary value problem (1.3), we first need to solve this later. To this end, pick $f \in \tilde{H}^{1/2}(\partial\Omega)$ and $F \in H^1(\Omega)$ such that $F = f$ on $\partial\Omega$. In light of (2.2) and the Lax-Milgram lemma, there is a unique $v \in H_0^1(\Omega)$ solving the variational problem

$$(2.3) \quad \int_{\Omega} A \nabla v \cdot \nabla w dx = - \int_{\Omega} A \nabla F \cdot \nabla w dx, \text{ for all } w \in H_0^1(\Omega).$$

Hence $u = v + F$ is the unique weak solution to the boundary value problem (1.3). That is, u satisfies the first equation in (1.3) in the distributional sense and the second equation in the trace sense. By taking $w = v$ in (2.3), we get $\|v\|_{H^1(\Omega)} \leq C\|F\|_{H^1(\Omega)}$ from (2.2) and Poincaré's inequality², whence

$$\|u\|_{H^1(\Omega)} \leq C\|F\|_{H^1(\Omega)},$$

where C denotes some generic positive constant depending on ω . Finally, by choosing $F \in H^1(\Omega)$ in accordance with Lemma 2.1 so that $\|F\|_{H^1(\Omega)} \leq 2\|f\|_{\tilde{H}^{1/2}(\partial\Omega)}$, we find out that

$$(2.4) \quad \|u\|_{H^1(\Omega)} \leq C\|f\|_{\tilde{H}^{1/2}(\partial\Omega)}.$$

Definition of the DN map. Prior to defining the DN map we introduce the following $H(\text{div})$ -type space,

$$H(\text{div}_A, \Omega) = \{P \in L^2(\Omega)^3; \text{div}(AP) \in L^2(\Omega)\},$$

and prove that the operator $P \in C_0^\infty(\bar{\Omega}) \mapsto AP \cdot \nu \in C^\infty(\partial\Omega)$ can be extended to a bounded operator from $H(\text{div}_A, \Omega)$ into the space $\tilde{H}^{-1/2}(\partial\Omega)$ dual of $\tilde{H}^{1/2}(\partial\Omega)$.

²Which holds true for Ω since ω is bounded.

Proposition 2.1. *Let $P \in H(\operatorname{div}_A, \Omega)$. Then $AP \cdot \nu \in \tilde{H}^{-1/2}(\partial\Omega)$ and we have*

$$(2.5) \quad \|AP \cdot \nu\|_{\tilde{H}^{-1/2}(\partial\Omega)} \leq C(\|P\|_{L^2(\Omega)} + \|\operatorname{div}(AP)\|_{L^2(\Omega)}),$$

for some constant $C = C(\omega, \theta) > 0$. In addition, the following identity

$$(2.6) \quad \langle AP \cdot \nu, g \rangle = \int_{\Omega} G \operatorname{div}(AP) dx + \int_{\Omega} A \nabla G \cdot P dx,$$

holds true for any $g \in \tilde{H}^{1/2}(\partial\Omega)$ and $G \in H^1(\Omega)$ such that $G = g$ on $\partial\Omega$. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\tilde{H}^{1/2}(\partial\Omega)$ and its dual $\tilde{H}^{-1/2}(\partial\Omega)$.

Proof. We first consider the case of $P \in C_0^\infty(\bar{\Omega})^3$. Fix $g \in \tilde{H}^{1/2}(\partial\Omega)$ and choose $G \in H^1(\Omega)$ such that $G = g$ on $\partial\Omega$. Since P has a compact support, we have

$$(2.7) \quad \int_{\Omega} G \operatorname{div}(AP) dx = - \int_{\Omega} A \nabla G \cdot P dx + \int_{\partial\Omega} g AP \cdot \nu d\sigma,$$

from Green's formula, whence

$$\left| \int_{\partial\Omega} g AP \cdot \nu d\sigma \right| \leq C \|G\|_{H^1(\Omega)} (\|P\|_{L^2(\Omega)} + \|\operatorname{div}(AP)\|_{L^2(\Omega)}).$$

By taking the infimum over $\{G \in H^1(\Omega), G = g \text{ on } \partial\Omega\}$ in the right hand side of above estimate, we find that (2.5) holds true for every $P \in C_0^\infty(\bar{\Omega})^3$.

Further, pick $P \in H(\operatorname{div}_A, \Omega)$. The set $C_0^\infty(\bar{\Omega})^3$ being dense in $H(\operatorname{div}_A, \Omega)$, as can be seen by mimicking the proof of Theorem 2.4 in [GR], we may find a sequence $(P_k)_k$ in $C_0^\infty(\bar{\Omega})^3$ converging to P in $H(\operatorname{div}_A, \Omega)$. Moreover, due to (2.5), $(AP_k \cdot \nu)_k$ is a Cauchy sequence in $\tilde{H}^{-1/2}(\partial\Omega)$. Therefore $(AP_k \cdot \nu)_k$ has a limit in $\tilde{H}^{-1/2}(\partial\Omega)$, which is denoted by $AP \cdot \nu$, and (2.6) follows readily from (2.7). \square

Let u denote the $H^1(\Omega)$ -solution to (1.3). By applying Proposition 2.1 to $P = \nabla u$, we deduce from (2.4) that

$$\Lambda_\theta : f \mapsto A \nabla u \cdot \nu$$

is well defined as a bounded operator from $\tilde{H}^{1/2}(\partial\Omega)$ into $\tilde{H}^{-1/2}(\partial\Omega)$. Moreover the following identity

$$(2.8) \quad \langle \Lambda_\theta f, g \rangle = \int_{\Omega} A \nabla u \cdot \nabla G dx,$$

holds true for all $g \in \tilde{H}^{1/2}(\partial\Omega)$ and $G \in H^1(\Omega)$ such that $G = g$ on $\partial\Omega$.

Further, by taking $G = v$ in (2.8), where v is the solution to (1.3) with f replaced by g , we find out that

$$\langle \Lambda_\theta f, g \rangle = \int_{\Omega} A \nabla u \cdot \nabla v dx = \int_{\Omega} \nabla u \cdot A \nabla v dx.$$

Therefore we have obtained that

$$(2.9) \quad \langle \Lambda_\theta f, g \rangle = \langle f, \Lambda_\theta g \rangle, \text{ for all } f, g \in \tilde{H}^{1/2}(\partial\Omega).$$

This proves that $\Lambda_\theta^*|_{\tilde{H}^{1/2}(\partial\Omega)} = \Lambda_\theta$, where $\tilde{H}^{1/2}(\partial\Omega)$ is identified with a subspace of its bidual space.

Finally, for $i = 1, 2$, put $A_i = A(x', \theta_i(x_3))$ and $\Lambda_i = \Lambda_{\theta_i}$, and let $u_i \in H^1(\Omega)$, $i = 1, 2$, be a weak solution to the equation

$$\operatorname{div}(A_i \nabla u_i) = 0 \text{ in } \Omega.$$

By applying (2.8) to $f = u_i|_{\partial\Omega}$ and $g = u_{3-i}|_{\partial\Omega}$, $i = 1, 2$, we get that

$$\langle \Lambda_1 u_1, u_2 \rangle = \int_{\Omega} A_1 \nabla u_1 \cdot \nabla u_2 dx \text{ and } \langle \Lambda_2 u_2, u_1 \rangle = \int_{\Omega} A_2 \nabla u_2 \cdot \nabla u_1 dx.$$

In light of (2.9), this yields:

$$(2.10) \quad \langle (\Lambda_1 - \Lambda_2) u_1, u_2 \rangle = \int_{\Omega} (A_1 - A_2) \nabla u_1 \cdot \nabla u_2 dx.$$

Restriction of the DN map. We turn now to establishing some useful smoothness property for the restriction of Λ_θ to $\tilde{H}^{3/2}(\partial\Omega)$. We assume for this purpose that $\Omega_1 = \omega \times (-1, 1)$ has H^2 -regularity property, i.e. that for every $F \in L^2(\Omega)$ and any matrix-valued function $C = (C_{ij}(x))_{1 \leq i, j \leq 3}$ with coefficients in $W^{1, \infty}(\Omega_1)$ satisfying the ellipticity condition

$$\exists \alpha > 0, C(x)\xi \cdot \xi \geq \alpha|\xi|^2, \text{ for all } \xi \in \mathbb{C}^3, x \in \Omega_1,$$

the following boundary value problem

$$\begin{cases} \operatorname{div}(C\nabla w) = F & \text{in } \Omega_1, \\ w = 0 & \text{on } \partial\Omega_1, \end{cases}$$

has a unique solution $w \in H^2(\Omega_1)$ obeying

$$\|w\|_{H^2(\Omega_1)} \leq C(\alpha, M)\|F\|_{L^2(\Omega_1)}.$$

Here $C(\alpha, M) > 0$ is a constant depending only on α , $M = \max_{1 \leq i, j \leq 3} \|C_{ij}\|_{W^{1, \infty}(\Omega_1)}$ and ω .

Notice that Ω_1 has H^2 -regularity property if and only if this is the case for $\Omega_a = \omega \times (-a, a)$ and some $a > 0$. Moreover we recall from [Gr] that Ω_1 has H^2 -regularity property provided ω is convex.

Having said that we may now prove the following claim, which is a cornerstone in the derivation of smoothness properties for the restriction of Λ_θ to $\tilde{H}^{3/2}(\partial\Omega)$.

Theorem 2.1. *Assume that $\theta \in C^{1,1}(\mathbb{R})$ and that Ω_1 has H^2 -regularity property. Then for any $f \in \tilde{H}^{3/2}(\partial\Omega)$, the boundary value problem (1.3) admits a unique solution $u \in H^2(\Omega)$. Moreover there is a constant $C > 0$, depending only on $\|\theta\|_{C^{1,1}(\mathbb{R})}$ and ω , such that we have:*

$$(2.11) \quad \|u\|_{H^2(\Omega)} \leq C\|f\|_{\tilde{H}^{3/2}(\partial\Omega)}.$$

Proof. Since $f \in \tilde{H}^{3/2}(\partial\Omega)$ we may choose $F \in H^2(\Omega)$ in accordance with Lemma 2.1 so that $F = f$ on $\partial\Omega$ and

$$(2.12) \quad \|F\|_{H^2(\Omega)} \leq 2\|f\|_{\tilde{H}^{3/2}(\partial\Omega)}.$$

Put $\Psi = \operatorname{div}(A\nabla F)$. Then, due to (2.2) there is a unique $u_0 \in H_0^1(\Omega)$ satisfying simultaneously

$$(2.13) \quad \int_{\Omega} A\nabla u_0 \cdot \nabla v dx = \int_{\Omega} \Psi v dx, \text{ for all } v \in H_0^1(\Omega),$$

and

$$(2.14) \quad \|u_0\|_{H^1(\Omega)} \leq C_0\|\Psi\|_{L^2(\Omega)},$$

for some constant $C_0 > 0$ depending on ω and $M = \|\theta\|_{C^{1,1}(\mathbb{R})}$.

Further, let $\xi_n \in C_0^\infty(-n+1, n+1)$, $n \geq 1$, be such that $\xi_n = 1$ in $[-n, n]$, $\|\xi_n'\|_\infty \leq 1/2$ and $\|\xi_n''\|_\infty \leq 1/2$. Then for every $v \in H_0^1(\Omega)$, we get by direct calculation that

$$\int_{\Omega} A\nabla(\xi_n u_0) \cdot \nabla v dx = \int_{\Omega} A\nabla u_0 \cdot \nabla(\xi_n v) dx - \int_{\Omega} (A\nabla u_0 \cdot \nabla \xi_n) v dx + \int_{\Omega} (A\nabla \xi_n \cdot \nabla v) u_0 dx.$$

An integration by parts in the last term of the right hand side of the above identity providing

$$\int_{\Omega} (A\nabla \xi_n \cdot \nabla v) u_0 dx = - \int_{\Omega} (A\nabla \xi_n \cdot \nabla u_0) v dx - \int_{\Omega} \operatorname{div}(A\nabla \xi_n) u_0 v dx,$$

we find out that

$$\int_{\Omega} A\nabla(\xi_n u_0) \cdot \nabla v dx = \int_{\Omega} A\nabla u_0 \cdot \nabla(\xi_n v) dx - \int_{\Omega} (A\nabla u_0 \cdot \nabla \xi_n) v dx - \int_{\Omega} (A\nabla \xi_n \cdot \nabla u_0) v dx - \int_{\Omega} \operatorname{div}(A\nabla \xi_n) u_0 v dx.$$

Since A is symmetric, it follows from this and (2.13) that

$$\int_{\Omega} A\nabla(\xi_n u_0) \cdot \nabla v dx = \int_{\Omega} \Psi \xi_n v dx - 2 \int_{\Omega} (A\nabla \xi_n \cdot \nabla u_0) v dx - \int_{\Omega} \operatorname{div}(A\nabla \xi_n) u_0 v dx, \text{ for all } v \in H_0^1(\Omega).$$

Therefore, by recalling that $\Omega_a = \omega \times (-a, a)$ for any $a > 0$, the function $\xi_n u_0 \in H_0^1(\Omega_{n+1})$ is solution to the variational problem

$$(2.15) \quad \int_{\Omega_{n+1}} A \nabla(\xi_n u_0) \cdot \nabla v dx = \int_{\Omega_{n+1}} \tilde{\Psi} v dx, \text{ for all } v \in H_0^1(\Omega_{n+1}),$$

with

$$\tilde{\Psi} = \Psi \xi_n - 2A \nabla \xi_n \cdot \nabla u_0 - \operatorname{div}(A \nabla \xi_n) u_0.$$

The next step of the proof is to make the change of variables $(x', x_3) \in \Omega_{n+1} \mapsto (x', y_3) = (x', 1/(n+1)x_3) \in \Omega_1$ in (2.15). To this end, we set

$$J_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/n \end{pmatrix}, \quad n \geq 1,$$

and for all $(x', y_3) \in \Omega_1$, we introduce the following notations:

$$\begin{aligned} \underline{A}(x', y_3) &= 1/(n+1) J_{n+1} A(x', (n+1)y_3) J_{n+1}, \\ \underline{\xi}(y_3) &= \xi_n((n+1)y_3), \\ \underline{u}(x', y_3) &= u_0(x', (n+1)y_3), \\ w_n(x', y_3) &= \xi_n((n+1)y_3) u_0(x', (n+1)y_3), \\ \underline{\operatorname{div}}(P(x', y_3)) &= \partial_{x_1} P_1(x', y_3) + \partial_{x_2} P_2(x', y_3) + 1/(n+1) \partial_{y_3} P_3(x', y_3), \\ \underline{\Psi}(x', y_3) &= 1/(n+1) \left[\Psi(x', (n+1)y_3) - 2J_{n+1} A(x', (n+1)y_3) J_{n+1} \nabla \underline{\xi}(y_3) \cdot \nabla \underline{u}(x', y_3) \right. \\ &\quad \left. - \operatorname{div}(A(x', (n+1)y_3) J_{n+1} \nabla \underline{\xi}(y_3)) \underline{u}(x', y_3) \right]. \end{aligned}$$

By performing the above mentioned change of variables, we find out that $w_n \in H_0^1(\Omega_1)$ is the solution of the variational problem

$$\int_{\Omega_1} \underline{A} \nabla w_n \cdot \nabla v dx = \int_{\Omega_1} \underline{\Psi} v dx, \text{ for all } v \in H_0^1(\Omega_1).$$

As Ω_1 has H^2 -regularity property by assumption, we get by straightforward computations that $\|w_n\|_{H^2(\Omega_1)} \leq C(M, \omega) \|\underline{\Psi}\|_{L^2(\Omega_1)}$. On the other hand, since $\|\underline{\Psi}\|_{L^2(\Omega_1)} \leq (n+1)^{-3/2} C(M, \omega) \|\Psi\|_{L^2(\Omega)}$ from (2.14), it holds true that $\|w_n\|_{H^2(\Omega_1)} \leq (n+1)^{-3/2} C(M, \omega) \|\Psi\|_{L^2(\Omega)}$, whence

$$(2.16) \quad \|\xi_n u_0\|_{H^2(\Omega)} \leq C(M, \omega) \|\Psi\|_{L^2(\Omega)},$$

by (2.14). Therefore, upon eventually extracting a subsequence of $(\xi_n u_0)_n$, we may assume that it converges weakly to \tilde{u} in $H^2(\Omega)$. On the other hand since $(\xi_n u_0)_n$ converges to u_0 in $L^2(\Omega)$ then $u_0 \in H^2(\Omega)$ and $(\xi_n u_0)_n$ thus converges weakly to u_0 in $H^2(\Omega)$. Further, the norm $\|\cdot\|_{H^2(\Omega)}$ being lower semi-continuous, we have

$$\|u_0\|_{H^2(\Omega)} \leq \liminf_n \|\xi_n u_0\|_{H^2(\Omega)} \leq C(M, \omega) \|\Psi\|_{L^2(\Omega)},$$

from (2.16). Bearing in mind that $\|\Psi\|_{L^2(\Omega)} \leq C(M, \omega) \|F\|_{H^2(\Omega)}$ and $\|F\|_{H^2(\Omega)} \leq 2\|f\|_{\tilde{H}^{3/2}(\partial\Omega)}$, it follows from the above inequality and from (2.12) that

$$\|u_0\|_{H^2(\Omega)} \leq C(M, \omega) \|f\|_{\tilde{H}^{3/2}(\partial\Omega)}.$$

As a consequence $u = u_0 + F \in H^2(\Omega)$ is the unique solution to (1.3). Moreover it satisfies

$$\|u\|_{H^2(\Omega)} \leq C(M, \omega) \|f\|_{\tilde{H}^{3/2}(\partial\Omega)},$$

which completes the proof. \square

For all Ω and θ obeying the assumptions of Theorem 2.1, the mapping

$$\Lambda_\theta : f \in \tilde{H}^{3/2}(\partial\Omega) \mapsto \partial_\nu u \in L^2(\mathbb{R}; H^{1/2}(\partial\omega)),$$

where u denotes the unique $H^2(\Omega)$ -solution to (1.3), is thus well defined by Theorem 2.1.

Further, by using that $C_0^\infty(\mathbb{R}; H^2(\omega))$ is dense in $H^2(\Omega)$, we find out that the trace operator

$$\tilde{\tau} : w \in H^2(\Omega) \mapsto \partial_\nu w \in L^2(\mathbb{R}; H^{1/2}(\partial\omega)),$$

is bounded. From this and (2.11) then follows that $\|\Lambda_\theta\| \leq C(M, \omega)$ as a linear bounded operator from $\tilde{H}^{3/2}(\partial\Omega)$ into $L^2(\mathbb{R}; H^{1/2}(\partial\omega))$.

3. CAN WE DETERMINE THE TWISTING FUNCTION FROM THE DN MAP?

Analysis of the problem for general twisting functions. In view of examining the problem of the identification of θ from Λ_θ we consider a nonempty open subset γ of $\partial\omega$ and we put $\Gamma = \gamma \times (-2L, 2L)$ for some fixed $L > 0$. Next we introduce the two following functional spaces

$$(3.1) \quad \tilde{H}_\Gamma^{1/2}(\partial\Omega) = \{f \in \tilde{H}^{1/2}(\partial\Omega); \text{supp } f \subset \Gamma\},$$

and³

$$(3.2) \quad \mathcal{W}(\mathbb{R}) = \{\theta \in W_{loc}^{2,\infty}(\mathbb{R}); \theta' \in W^{1,\infty}(\mathbb{R})\}.$$

Further, for $i = 1, 2$, we pick $\theta_i \in \mathcal{W}(\mathbb{R})$ and set $A_i = A(x', \theta'_i(x_3))$, $\Lambda_i = \Lambda_{\theta_i}$. In light of (2.10), we have

$$(3.3) \quad \langle (\Lambda_1 - \Lambda_2)u_1, u_2 \rangle = \int_\Omega (A_1 - A_2) \nabla u_1 \cdot \nabla u_2 dx,$$

for any function $u_i \in H^1(\Omega)$, $i = 1, 2$, which is a weak solution to the equation $\text{div}(A_i \nabla u_i) = 0$ in Ω . Assuming that

$$(3.4) \quad \theta_1(x_3) = \theta_2(x_3), \quad |x_3| > L,$$

we may rewrite (3.3) as

$$(3.5) \quad \langle (\Lambda_1 - \Lambda_2)u_1, u_2 \rangle = \int_{\Omega^L} (A_1 - A_2) \nabla u_1 \cdot \nabla u_2 dx,$$

with $\Omega^L = \omega \times (-L, L)$.

Our analysis is essentially based on the method built in [AG2] for bounded domains. We shall now adapt it to the case of the infinite waveguide Ω . To this purpose we put

$$\Gamma_\rho = \{x \in \Gamma; \text{dist}(x, \partial\Gamma) > \rho\} \text{ and } U_\rho = \{x \in \mathbb{R}^3; \text{dist}(x, \Gamma_\rho) < \rho/4\},$$

for all $\rho \in (0, \rho_0]$, where ρ_0 is some characteristic constant defined in [AG2] which depends only on ω and L . Upon eventually shortening ρ_0 , we may assume without loss of generality that $\Gamma_0 = \gamma_0 \times [-L, L] \subset \Gamma_\rho$ for some $\gamma_0 \Subset \gamma$. Then, in view of [AG2], we can find a Lipschitz domain Ω_ρ satisfying simultaneously:

$$\Omega \subset \Omega_\rho, \quad \Gamma_0 \subset \partial\Omega \cap \Omega_\rho \Subset \Gamma \text{ and } \text{dist}(x, \partial\Omega_\rho) \geq \rho/2 \text{ for all } x \in U_\rho.$$

Moreover we know from [AG1][Section 3] that there exists a unitary C^∞ vector field $\tilde{\nu}$, defined in some suitable neighborhood of $\partial\omega \times (-2L, 2L)$, which is non tangential to $\partial\Omega$ and points to the exterior of Ω . For $x^0 \in \overline{\Gamma_\rho}$, the point $z_\tau = x^0 + \tau\tilde{\nu}$ obeys $C\tau \leq \text{dist}(z_\tau, \partial\Omega) \leq \tau$ for all $0 < \tau \leq \tau_0$, according to [AG1][Lemma 3.3], where C and τ_0 are two positive constants depending only on Ω , λ , \underline{t} and \bar{t} .

In light of [HK], the operator $\text{div}(A_i \nabla \cdot)$, $i = 1, 2$, has a Dirichlet Green function $G_i = G_i(x, y)$ on Ω_ρ . More specifically, $G_i(x) = G_i(x, z_\tau)$, $i = 1, 2$, is the solution to the boundary value problem:

$$(3.6) \quad \begin{cases} \text{div}(A_i \nabla G_i) = -\delta(x - z_\tau) & \text{in } \mathcal{D}'(\Omega_\rho), \\ G_i = 0 & \text{on } \partial\Omega_\rho. \end{cases}$$

Let us prove now that the claim of [AG2][Corollary 3.4] remains essentially unchanged for the unbounded domain Ω_ρ arising in this framework.

³We may choose $\mathcal{W}(\mathbb{R}) = \{\theta \in \mathcal{D}'(\mathbb{R}); \theta' \in W^{1,\infty}(\mathbb{R})\}$ as well.

Lemma 3.1. *There exist two constants $\tau_0 = \tau_0(\Omega, \rho) > 0$ and $C = C(\Omega, \lambda, \underline{t}, \bar{t})$ such that the restriction $G_i(\cdot, z_\tau)$ to Ω , $i = 1, 2$, belongs to $H^1(\Omega)$, and satisfies*

$$(3.7) \quad \|G_i(\cdot, z_\tau)\|_{H^1(\Omega)} \leq C\tau^{-1/2}, \quad 0 < \tau \leq \tau_0.$$

Proof. Choose $\tau_0 = \tau_0(\Omega, \rho)$ so small relative to ρ that $\text{dist}(z_\tau, \partial\Omega_\rho) > \tau$ for all $\tau \in (0, \tau_0]$. Since the ball centered at z_τ with radius $\tau/3$, noted $B_{\tau/3}(z_\tau)$, is embedded in $\Omega_\rho \setminus \bar{\Omega}$, we have

$$(3.8) \quad \|G_i(\cdot, z_\tau)\|_{Y^{2,1}(\Omega)} \leq \|G_i(\cdot, z_\tau)\|_{Y^{2,1}(\Omega_\rho \setminus B_{\tau/3}(z_\tau))} \leq C\tau^{-1/2}, \quad 0 < \tau \leq \tau_0,$$

directly from [HK][Formula (4.45)]. Here C is some positive constant depending only on Ω , λ , \underline{t} and \bar{t} , and the space

$$Y^{2,1}(\Omega) = \{u \in L^6(\Omega); \nabla u \in L^2(\Omega)^3\},$$

is endowed with the norm $\|v\|_{Y^{2,1}(\Omega)} = \|v\|_{L^6(\Omega)} + \|\nabla v\|_{L^2(\Omega)^3}$.

Finally, Ω having the Poincaré inequality property since ω is bounded, we may actually deduce (3.7) from (3.8). \square

Now, as a Green function is a Levi function, it behaves locally like a parametrix. Hence, by applying [M][Formula (8.4)], G_i , $i = 1, 2$, can be brought into the form

$$(3.9) \quad G_i(x) = C(\det(A_i(z_\tau)))^{-1/2} (A(z_\tau)^{-1}(x - z_\tau) \cdot (x - z_\tau))^{-1/2} + R_i(x),$$

where $C > 0$ is a constant and the reminder R_i obeys the condition:

$$\exists(r_0, \alpha) \in \mathbb{R}_+^* \times (0, 1), \quad |R_i(x)| + |x - z_\tau| |\nabla R_i| \leq C|x - z_\tau|^{-1+\alpha}, \quad x \in \Omega_\rho, |x - z_\tau| \leq r_0.$$

Since

$$\text{div}(A_i \nabla G_i) = 0 \quad \text{in } \Omega,$$

in the weak sense, we may apply (3.5), getting:

$$\langle (\Lambda_1 - \Lambda_2)G_1, G_2 \rangle = \int_{\Omega^L} (A_1 - A_2) \nabla G_1 \cdot \nabla G_2 dx.$$

Moreover, taking into account that $G_i|_{\partial\Omega} \in \tilde{H}^{1/2}(\partial\Omega)$, we deduce from the above identity that

$$\int_{\Omega^L} (A_1 - A_2) \nabla G_1 \cdot \nabla G_2 dx \leq \|\Lambda_1^\Gamma - \Lambda_2^\Gamma\| \|G_1\|_{\tilde{H}^{1/2}(\partial\Omega)} \|G_2\|_{\tilde{H}^{1/2}(\partial\Omega)},$$

where Λ_i^Γ , $i = 1, 2$, is the restriction of Λ_i to the closed subspace $\tilde{H}_\Gamma^{1/2}(\partial\Omega)$. As a consequence we have

$$(3.10) \quad \int_{\Omega^L} (A_1 - A_2) \nabla G_1 \cdot \nabla G_2 dx \leq C \|\Lambda_1^\Gamma - \Lambda_2^\Gamma\| \|G_1\|_{H^1(\Omega)} \|G_2\|_{H^1(\Omega)}.$$

Now, pick $x^0 \in \Gamma_0$ such that $|\theta'_1(x_3^0) - \theta'_2(x_3^0)| = \|\theta'_1 - \theta'_2\|_{L^\infty(-L, L)}$. Actually, we may assume without loss of generality that we have $|\theta'_1(x_3^0) - \theta'_2(x_3^0)| = \theta'_1(x_3^0) - \theta'_2(x_3^0)$. In view of (3.9) and [AG2][Formula (4.2)], the main term in the left hand side of (3.10) has the following expression

$$\int_{B(z_\tau, \rho) \cap \Omega} \frac{[A^{-1}((x^0)', t_0) - A^{-1}((x^0)', s_0)](x - z_\tau) \cdot (x - z_\tau)}{[P_0(x - z_\tau) \cdot (x - z_\tau)]^{3/2} [Q_0(x - z_\tau) \cdot (x - z_\tau)]^{3/2}} dx,$$

with $t_0 = \theta'_1(x_3^0)$, $s_0 = \theta'_2(x_3^0)$, $P_0 = A^{-1}(z'_\tau, t_0)$ and $Q_0 = A^{-1}(z'_\tau, s_0)$, z_τ being the same as in above, i.e. $z_\tau = x^0 + \tau \tilde{\nu}$ for some $\tau \in \mathbb{R}_+^*$. The main ingredient in the analysis developed in [AG2] is the ellipticity condition [AG2][Formula (2.5)] imposed on $\partial_t A(x', t)$. Indeed, this assumption entails

$$(3.11) \quad \int_{B(z_\tau, \rho) \cap \Omega} \frac{[A^{-1}((x^0)', t_0) - A^{-1}((x^0)', s_0)](x - z_\tau) \cdot (x - z_\tau)}{[P_0(x - z_\tau) \cdot (x - z_\tau)]^{3/2} [Q_0(x - z_\tau) \cdot (x - z_\tau)]^{3/2}} dx \geq C\tau(t_0 - s_0),$$

for some constant $C > 0$, which leads ultimately to the desired result.

Unfortunately, the ellipticity condition [AG2][Formula (2.5)] is not fulfilled by $\partial_t A(x', t)$ in this framework. This can be seen from the following explicit expression

$$\begin{aligned}\lambda_1 &= 0, \\ \lambda_2 &= |x'|^2 t - \sqrt{|x'|^4 t^2 + |x'|^2}, \\ \lambda_3 &= |x'|^2 t + \sqrt{|x'|^4 t^2 + |x'|^2},\end{aligned}$$

of the eigenvalues of $\partial_t A(x', t)$, showing that the spectrum of $\partial_t A(x', t)$ has a negative component for $x' \in \partial\omega$. Moreover, it can be noticed that, due to the occurrence of this negative eigenvalue, the weak monotonicity assumption [AG1][Formula (5.7)] is not satisfied by the conductivity matrix under consideration either. Therefore, the approach developed in [AG2] does not apply to the problem under study. This explains why the determination of the twisting function from the corresponding DN map remains an open problem in the general case.

Nevertheless, we shall see in the following section that this is not the case for affine twisting functions anymore. But, prior to examining this peculiar framework, we shall now deduce from the above reasoning, upon preliminarily substituting a suitable matrix A^* for A , that twisting functions which are close to some *a priori* fixed constant value may well be identified from the DN map.

The case of twisting functions close to a constant value. Put

$$A^*(x', t) = t \begin{pmatrix} 1 + x_2^2 & -x_2 x_1 & -x_2 \\ -x_2 x_1 & 1 + x_1^2 & x_1 \\ -x_2 & x_1 & 1 \end{pmatrix}, \quad x' \in \omega, \quad t \in \mathbb{R},$$

and denote by Λ_θ^* the DN map Λ_θ where $A^*(x', \theta'(x_3))$ is substituted for $A(x', \theta'(x_3))$. Then, by arguing as in the derivation of (2.10), we obtain that

$$(3.12) \quad \langle (\Lambda_\theta - \Lambda_\theta^*)u, u^* \rangle = \int_\Omega (A(x', \theta'(x_3)) - A^*(x', \theta'(x_3))) \nabla u \cdot \nabla u^* dx,$$

for all $u, u^* \in H^1(\Omega)$ satisfying $\operatorname{div}(A(x', \theta'(x_3)) \nabla u) = \operatorname{div}(A^*(x', \theta'(x_3)) \nabla u^*) = 0$ in the weak sense in Ω . For f (resp. g) in $\tilde{H}^{1/2}(\partial\Omega)$ let u (resp. u^*) be the solution to (1.3) (resp. (1.3)) where (A^*, g) is substituted to (A, f) . Applying (3.12) we thus get that $|\langle (\Lambda_\theta - \Lambda_\theta^*)f, g \rangle| \leq \|\theta' - 1\|_\infty \|u\|_{H^1(\Omega)} \|u^*\|_{H^1(\Omega)}$, which entails $|\langle (\Lambda_\theta - \Lambda_\theta^*)f, g \rangle| \leq C \|\theta' - 1\|_\infty \|f\|_{\tilde{H}^{1/2}(\partial\Omega)} \|g\|_{\tilde{H}^{1/2}(\partial\Omega)}$, by (2.4), where $C = C(\omega)$ is some positive constant. As a consequence we have

$$(3.13) \quad \|\Lambda_\theta - \Lambda_\theta^*\|_{\mathcal{L}(\tilde{H}^{-1/2}(\partial\Omega), \tilde{H}^{1/2}(\partial\Omega))} \leq C \|\theta' - 1\|_\infty,$$

showing that Λ_θ^* is a suitable approximation of Λ_θ provided θ' is sufficiently close ⁴ to 1.

Actually, the main benefit of dealing with Λ_θ^* instead of Λ_θ in the inverse problem of determining θ' from Λ_θ , boils down to the fact that

$$\partial_t A^*(x', t) = A^*(x', 1) = A(x', 1), \quad \text{for all } x' \in \omega \text{ and } t \in \mathbb{R},$$

so the ellipticity condition [AG2][Formula (2.5)] required by the method developed in [AG2], is fulfilled by A^* . In light of the reasoning developed in the first part of this section, we thus derive the following result by repeating the arguments used in the proof of [AG2][Theorem 2.2]:

Theorem 3.1. *Let $L > 0$ and $M > 0$. Assume that $\theta_i \in \mathcal{W}(\mathbb{R})$ obey (3.4) for $i = 1, 2$, the space $\mathcal{W}(\mathbb{R})$ being the same as in (3.2), and that*

$$\|\theta'_i\|_{W^{1,\infty}(\mathbb{R})} \leq M.$$

Then there exists a constant $C = C(M, \omega, L) > 0$ such that we have

$$\|\theta'_1 - \theta'_2\|_{L^\infty(\mathbb{R})} \leq C \|(\Lambda_{\theta_1}^*)^\Gamma - (\Lambda_{\theta_2}^*)^\Gamma\|_{\mathcal{L}(\tilde{H}^{-1/2}(\partial\Omega), \tilde{H}^{1/2}(\partial\Omega))},$$

where $(\Lambda_{\theta_i}^)^\Gamma = (\Lambda_{\theta_i}^*)|_{\tilde{H}_\Gamma^{1/2}(\partial\Omega)}$, $i = 1, 2$, and $\tilde{H}_\Gamma^{1/2}(\partial\Omega)$ is defined in (3.1).*

⁴Note that 1 can be replaced by any constant μ by only substituting μA^* for the matrix A^* in the above reasoning.

4. THE CASE OF AFFINE TWISTING FUNCTIONS

In this section we address the case of affine twisting functions by means of the partial Fourier transform \mathcal{F}_{x_3} with respect to the variable x_3 . This is suggested by the translational invariance of the system under consideration in the infinite direction x_3 , arising from the fact that the matrix $A(x', \theta'(x_3))$ appearing in (1.3) is independent of x_3 in this peculiar case.

In the sequel, we note ξ the Fourier variable associated to x_3 and we write \widehat{w} instead of $\mathcal{F}_{x_3}w$ for every function $w = w(x', x_3)$:

$$\widehat{w}(x', \xi) = (\mathcal{F}_{x_3}w)(x', \xi), \quad x' \in \omega, \quad \xi \in \mathbb{R}.$$

The first step of the method is to re-express the system (1.3) in the Fourier plane $\{(x', \xi), x' \in \omega, \xi \in \mathbb{R}\}$.

Rewriting the boundary value problem in the Fourier domain. We start with the two following useful technical lemmas.

Lemma 4.1. *For every $w \in H^1(\Omega)$ it holds true that $\widehat{\partial_{x_j} w} = \partial_{x_j} \widehat{w}$, $j = 1, 2$.*

Proof. Fix $j = 1, 2$. For every $\varphi \in C_0^\infty(\omega)$ and $\psi \in \mathcal{S}(\mathbb{R})$, we have

$$\int_{\omega} \varphi(x') \left(\int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' = \int_{\mathbb{R}} \widehat{\psi}(x_3) \left(\int_{\omega} \partial_{x_j} w(x', x_3) \varphi(x') dx' \right) dx_3,$$

from Fubini's theorem. By integrating by parts in the last integral, we obtain

$$\int_{\omega} \partial_{x_j} w(x', x_3) \varphi(x') dx' = - \int_{\omega} w(x', x_3) \partial_{x_j} \varphi(x') dx', \quad \text{a.e. } x_3 \in \mathbb{R},$$

so we get

$$\begin{aligned} \int_{\omega} \varphi(x') \left(\int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' &= - \int_{\mathbb{R}} \widehat{\psi}(x_3) \left(\int_{\omega} w(x', x_3) \partial_{x_j} \varphi(x') dx' \right) dx_3 \\ &= - \int_{\omega} \partial_{x_j} \varphi(x') \left(\int_{\mathbb{R}} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx'. \end{aligned}$$

Further, the operator \mathcal{F}_{x_3} being selfadjoint in $L^2(\mathbb{R})$, it holds true that

$$\int_{\mathbb{R}} w(x', x_3) \widehat{\psi}(x_3) dx_3 = \int_{\mathbb{R}} \widehat{w}(x', \xi) \psi(\xi) d\xi, \quad \text{a.e. } x' \in \omega,$$

whence

$$\begin{aligned} \int_{\omega} \varphi(x') \left(\int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' &= - \int_{\omega} \partial_{x_j} \varphi(x') \left(\int_{\mathbb{R}} \widehat{w}(x', \xi) \psi(\xi) d\xi \right) dx' \\ &= \int_{\omega} \varphi(x') \left(\int_{\mathbb{R}} \partial_{x_j} \widehat{w}(x', \xi) \psi(\xi) d\xi \right) dx', \end{aligned}$$

by integrating by parts. From the density of $C_0^\infty(\omega)$ in $L^2(\omega)$, the above identity entails that

$$\int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 = \int_{\mathbb{R}} \partial_{x_j} \widehat{w}(x', \xi) \psi(\xi) d\xi, \quad \text{a.e. } x' \in \omega,$$

for every $\psi \in \mathcal{S}(\mathbb{R})$. From this, the selfadjointness of \mathcal{F}_{x_3} and the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$, then follows that $\widehat{\partial_{x_j} w} = \partial_{x_j} \widehat{w}$. \square

Lemma 4.2. *Let $C = (C_{kl})_{1 \leq k, l \leq 3} \in W^{1, \infty}(\omega)^{3 \times 3}$ be such that $(C_{kl}(x'))_{1 \leq k, l \leq 3}$ is symmetric for any $x' \in \omega$. Then every $w \in H^1(\Omega)$ obeying*

$$(4.1) \quad \int_{\Omega} C \nabla w \cdot \nabla v dx = 0 \quad \text{for all } v \in H_0^1(\Omega),$$

satisfies the equation

$$(4.2) \quad -\operatorname{div}_{x'}(\widetilde{C}(x') \nabla_{x'} \widehat{w}) + P(x', \xi) \cdot \nabla_{x'} \widehat{w} + q(x', \xi) \widehat{w} = 0 \quad \text{in } \mathcal{D}'(\Omega),$$

where

$$\begin{aligned}\tilde{C}(x') &= (C_{ij}(x'))_{1 \leq i, j \leq 2} \\ P(x', \xi) &= -i2\xi \begin{pmatrix} C_{31}(x') \\ C_{32}(x') \end{pmatrix} \\ q(x', \xi) &= -i\xi \operatorname{div}_{x'} \begin{pmatrix} C_{31}(x') \\ C_{32}(x') \end{pmatrix} + \xi^2 C_{33}(x').\end{aligned}$$

Moreover, if $w \in H^2(\Omega)$ is solution to (4.1) then the identity (4.2) holds true for a.e. $(x', \xi) \in \Omega$.

Proof. Choose $v = \varphi \otimes \widehat{\psi}$ in (4.1), where $\varphi \in C_0^\infty(\omega)$ and $\psi \in \mathcal{S}(\mathbb{R})$, so we have:

$$(4.3) \quad \sum_{k,l=1,2,3} \int_{\Omega} C_{kl}(x') \partial_{x_k} w(x', x_3) \partial_{x_l} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = 0.$$

For $1 \leq k, l \leq 2$, we notice that

$$(4.4) \quad \begin{aligned} \int_{\Omega} C_{kl}(x') \partial_{x_l} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 &= \int_{\omega} C_{kl}(x') \partial_{x_k} \varphi(x') \left(\int_{\mathbb{R}} \partial_{x_l} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' \\ &= \int_{\Omega} C_{kl}(x') \partial_{x_l} \widehat{w}(x', \xi) \partial_{x_k} (\varphi \otimes \psi)(x', \xi) dx' d\xi, \end{aligned}$$

directly from Lemma 4.1. Further, for all $l = 1, 2$, it holds true that

$$\begin{aligned} \int_{\Omega} C_{3l}(x') \partial_{x_l} w(x', x_3) \partial_{x_3} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 &= \int_{\omega} C_{3j}(x') \varphi(x') \left(\int_{\mathbb{R}} \partial_{x_l} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' \\ &= \int_{\omega} C_{3l}(x') \varphi(x') \left(\int_{\mathbb{R}} \partial_{x_l} w(x', x_3) \widehat{(-i\xi)\psi}(x_3) dx_3 \right) dx'. \end{aligned}$$

Therefore we have

$$(4.5) \quad \int_{\Omega} C_{3l}(x') \partial_{x_l} w(x', x_3) \partial_{x_3} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\Omega} C_{3l}(x') (-i\xi) \partial_{x_l} \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi.$$

Next, for $k = 1, 2$, we may write that

$$\begin{aligned} \int_{\Omega} C_{k3}(x') \partial_{x_3} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 &= \int_{\omega} C_{k3}(x') \partial_{x_k} \varphi(x') \left(\int_{\mathbb{R}} \partial_{x_3} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' \\ &= - \int_{\omega} C_{k3}(x') \partial_{x_k} \varphi(x') \left(\int_{\mathbb{R}} w(x', x_3) \widehat{\psi}'(x_3) dx_3 \right) dx', \end{aligned}$$

with

$$\int_{\mathbb{R}} w(x', x_3) \widehat{\psi}'(x_3) dx_3 = - \int_{\mathbb{R}} w(x', x_3) \widehat{(i\xi)\psi}(x_3) dx_3 = - \int_{\mathbb{R}} (i\xi) \widehat{w}(x', \xi) \psi(\xi) d\xi,$$

hence

$$\int_{\Omega} C_{k3}(x') \partial_{x_3} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\Omega} C_{k3}(x') (i\xi) \widehat{w}(x', \xi) \partial_{x_k} (\varphi \otimes \psi)(x', \xi) dx' d\xi.$$

By integrating by parts in the right hand side of the above identity we obtain that

$$(4.6) \quad \begin{aligned} \int_{\Omega} C_{k3}(x') \partial_{x_3} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 &= \int_{\Omega} C_{k3}(x') (-i\xi) \partial_{x_k} \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi \\ &+ \int_{\Omega} \partial_{x_k} C_{k3}(x') (-i\xi) \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi. \end{aligned}$$

Finally, bearing in mind that $\partial_{x_3} (\varphi \otimes \widehat{\psi}) = \varphi \otimes \widehat{(-i\xi)\psi}$ and noticing that

$$\int_{\mathbb{R}} \partial_{x_3} w(x', x_3) \widehat{(-i\xi)\psi}(x_3) dx_3 = - \int_{\mathbb{R}} w(x', x_3) \widehat{(-i\xi)^2 \psi}(x_3) dx_3 = - \int_{\mathbb{R}} \widehat{w}(x', \xi) (-i\xi)^2 \psi(\xi) d\xi,$$

we find out that

$$(4.7) \quad \int_{\Omega} C_{33}(x') \partial_{x_3} w(x', x_3) \partial_{x_3} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\Omega} C_{33}(x') (-i\xi)^2 \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi.$$

Now, putting (4.3)-(4.7) together, we end up getting that

$$\langle -\operatorname{div}_{x'}(\widetilde{C}(x') \nabla_{x'} \widehat{w}) + P(x', \xi) \cdot \nabla_{x'} \widehat{w} + q(x', \xi) \widehat{w}, \Phi \rangle = 0, \quad \Phi \in C_0^\infty(\omega) \otimes C_0^\infty(\mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $C_0^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$. From this and the density of $C_0^\infty(\omega) \otimes C_0^\infty(\mathbb{R})$ in $C_0^\infty(\Omega)$ then follows that

$$-\operatorname{div}_{x'}(\widetilde{C}(x') \nabla_{x'} \widehat{w}) + P(x', \xi) \cdot \nabla_{x'} \widehat{w} + q(x', \xi) \widehat{w} = 0 \quad \text{in } \mathcal{D}'(\Omega),$$

which completes the proof. \square

In the remaining of this section we assume that $\theta(x_3) = ax_3 + b$, where a and b are two fixed real numbers. With the help of Lemma 4.2 we may re-express (1.3) in the Fourier plane. For the sake of simplicity, we shall write $A_a(x')$ instead of $A(x', \theta(x_3))$, as we have $\theta'(x_3) = a$ for all $x_3 \in \mathbb{R}$.

For $g \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} g(x_3) dx_3 = 1$ and for $h \in H^{1/2}(\partial\omega)$, we consider the $H^1(\Omega)$ -solution u to (1.3), with:

$$f(x', x_3) = g(x_3)h(x'), \quad x' \in \partial\omega, \quad x_3 \in \mathbb{R}.$$

Since u is solution to (4.1) with $C = A_a \in W^{1,\infty}(\omega)^{3 \times 3}$, we deduce from Lemma 4.2 that $\widehat{u} \in L^2(\mathbb{R}; H^1(\omega))$ is solution to the system

$$(4.8) \quad \begin{cases} -\operatorname{div}_{x'}(\widetilde{A}_a(x') \nabla_{x'} \widehat{u}(x', \xi)) - 2ia\xi x'^{\perp} \cdot \nabla_{x'} \widehat{u} + \xi^2 \widehat{u} = 0 & \text{in } \mathcal{D}'(\Omega), \\ \widehat{u}(\cdot, \xi) = \widehat{g}(\xi)f & \text{on } \partial\omega, \text{ for all } \xi \in \mathbb{R}, \end{cases}$$

where

$$x'^{\perp} = (-x_2, x_1) \quad \text{and} \quad \widetilde{A}_a(x') = \begin{pmatrix} 1 + x_2^2 a^2 & -x_2 x_1 a^2 \\ -x_2 x_1 a^2 & 1 + x_1^2 a^2 \end{pmatrix}.$$

We turn now to examining the variational problem associated to (4.8).

Analysis of the variational problem associated to (4.8). Let us consider the bilinear form

$$\begin{aligned} \mathcal{A}_\xi[(v, w), (\varphi, \psi)] &= \int_{\omega} \widetilde{A}_a \nabla v \cdot \nabla \varphi dx' - 2a\xi \int_{\omega} x'^{\perp} \cdot \nabla w \varphi dx' + \xi^2 \int_{\omega} v \varphi dx' \\ &\quad + \int_{\omega} \widetilde{A}_a \nabla w \cdot \nabla \psi dx' + 2a\xi \int_{\omega} x'^{\perp} \cdot \nabla v \psi dx' + \xi^2 \int_{\omega} w \psi dx', \quad (v, w), (\varphi, \psi) \in \mathcal{H}, \end{aligned}$$

defined on the Hilbert space $\mathcal{H} = H_0^1(\omega) \times H_0^1(\omega)$ endowed with the norm

$$\|(v, w)\|_{\mathcal{H}} = (\|\nabla v\|_{L^2(\omega)}^2 + \|\nabla w\|_{L^2(\omega)}^2)^{1/2}.$$

Taking into account that

$$\widetilde{A}_a(x') \zeta \cdot \zeta \geq |\zeta|^2, \quad \text{for all } \zeta \in \mathbb{R}^2 \text{ and } x' \in \omega,$$

and that

$$2a|\xi| \int_{\omega} |x'^{\perp} \cdot \nabla v w| dx' \leq a^2 \delta^2 \int_{\omega} |\nabla v|^2 dx' + \xi^2 \int_{\omega} w^2 dx', \quad (v, w) \in \mathcal{H},$$

where $\delta = \max_{x' \in \omega} |x'|$, it is easy to see that

$$(4.9) \quad \mathcal{A}_\xi[(v, w), (v, w)] \geq (1 - a^2 \delta^2) \|(v, w)\|_{\mathcal{H}}^2.$$

Let us fix $a_0 > 0$ so small that $\alpha = 1 - a_0^2 \delta^2 > 0$. Then, due to the above estimate, the bilinear form \mathcal{A}_ξ is α -elliptic for every $\xi \in \mathbb{R}$, provided we have $|a| \leq a_0$. For each $\Phi \in C(\mathbb{R}; \mathcal{H}')$ and every $\xi \in \mathbb{R}$, there is thus a unique $(v(\xi), w(\xi)) \in \mathcal{H}$ satisfying

$$(4.10) \quad \mathcal{A}_\xi[(v(\xi), w(\xi)), (\varphi, \psi)] = \langle \Phi(\xi), (\varphi, \psi) \rangle \quad \text{for all } (\varphi, \psi) \in \mathcal{H},$$

by Lax-Milgram's lemma. From this then follows that

$$(4.11) \quad \begin{aligned} & \mathcal{A}_{\xi+\eta}[(v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi)), (\varphi, \psi)] \\ &= \mathcal{A}_{\xi}[(v(\xi), w(\xi)), (\varphi, \psi)] - \mathcal{A}_{\xi+\eta}[(v(\xi), w(\xi)), (\varphi, \psi)] + \langle \Phi(\xi+\eta) - \Phi(\xi), (\varphi, \psi) \rangle, \end{aligned}$$

for each $\xi, \eta \in \mathbb{R}$ and $(\varphi, \psi) \in \mathcal{H}$. Further, by noticing through elementary computations that

$$\mathcal{A}_{\xi+\eta}[(v, w), (\varphi, \psi)] = \mathcal{A}_{\xi+\eta}[(v, w), (\varphi, \psi)] - 2a\eta \int_{\omega} x'^{\perp} \cdot (\varphi \nabla_{x'} w - \psi \nabla_{x'} v) dx' + \eta(2\xi + \eta) \int_{\omega} (v\varphi + w\psi) dx',$$

for every $(v, w), (\varphi, \psi) \in \mathcal{H}$, we deduce from (4.11) and Poincaré's inequality that there is a constant $C = C(\xi, \omega, a_0) > 0$ satisfying

$$(4.12) \quad \begin{aligned} & \mathcal{A}_{\xi}[(v(\xi), w(\xi)), (v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi))] \\ & - \mathcal{A}_{\xi+\eta}[(v(\xi), w(\xi)), (v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi))] \\ & \leq C|\eta| \|(v(\xi), w(\xi))\|_{\mathcal{H}} \|(v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi))\|_{\mathcal{H}}, \end{aligned}$$

for all $\xi \in \mathbb{R}$ and $\eta \in [-1, 1]$. In light of (4.9), (4.11) written with $(\varphi, \psi) = (v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi))$ and (4.12), we thus find out that

$$\alpha \|(v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi))\|_{\mathcal{H}} \leq C|\eta| \|(v(\xi), w(\xi))\|_{\mathcal{H}} + \|\Phi(\xi+\eta) - \Phi(\xi)\|_{\mathcal{H}'}$$

This proves that $(v, w) \in C(\mathbb{R}; \mathcal{H})$. Moreover, we obtain

$$(4.13) \quad \|(v(\xi), w(\xi))\|_{\mathcal{H}} \leq (1/\alpha) \|\Phi(\xi)\|_{\mathcal{H}'}, \quad \xi \in \mathbb{R},$$

directly from (4.9)-(4.10). Further, it is easy to check for $\Phi \in C^1(\mathbb{R}; \mathcal{H}')$ that $(v'(\xi), w'(\xi)) \in C(\mathbb{R}, \mathcal{H})$ is the solution to the variational problem

$$\mathcal{A}_{\xi}[(v'(\xi), w'(\xi)), (\varphi, \psi)] = \langle \Phi_0(\xi), (\varphi, \psi) \rangle + \langle \Phi'(\xi), (\varphi, \psi) \rangle \quad \text{for all } (\varphi, \psi) \in \mathcal{H},$$

where we have set

$$\langle \Phi_0(\xi), (\varphi, \psi) \rangle = 2a \int_{\omega} x'^{\perp} \cdot (\varphi \nabla_{x'} w(\xi) - \psi \nabla_{x'} v(\xi)) dx' + 2\xi \int_{\omega} (v(\xi)\varphi + w(\xi)\psi) dx'.$$

Using (4.13) and putting $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, we deduce from the above estimate that

$$\|(v'(\xi), w'(\xi))\|_{\mathcal{H}} \leq C(\langle \xi \rangle \|\Phi(\xi)\|_{\mathcal{H}'} + \|\Phi'(\xi)\|_{\mathcal{H}'}), \quad \xi \in \mathbb{R},$$

for some constant $C = C(a_0, \omega) > 0$. Similarly, if $\Phi \in C^2(\mathbb{R}; \mathcal{H}')$ then the same reasoning shows that $(v, w) \in C^2(\mathbb{R}, \mathcal{H})$ and

$$\|(v''(\xi), w''(\xi))\|_{\mathcal{H}} \leq C(\langle \xi \rangle^2 \|\Phi(\xi)\|_{\mathcal{H}'} + \langle \xi \rangle \|\Phi'(\xi)\|_{\mathcal{H}'} + \|\Phi''(\xi)\|_{\mathcal{H}'}), \quad \xi \in \mathbb{R}.$$

Hence we have obtained the:

Proposition 4.1. *For every $\Phi \in H^2(\mathbb{R}, \mathcal{H}')$ such that $\langle \xi \rangle^{2-j} \Phi^{(j)} \in L^2(\mathbb{R}, \mathcal{H}')$, $j = 0, 1$, the variational problem (4.10) admits a unique solution $(v, w) \in H^2(\mathbb{R}; \mathcal{H})$ satisfying*

$$(4.14) \quad \|(v, w)\|_{H^2(\mathbb{R}; \mathcal{H})} \leq C \left(\sum_{j=0}^2 \|\langle \xi \rangle^{2-j} \Phi^{(j)}\|_{L^2(\mathbb{R}; \mathcal{H}')} \right),$$

for some constant $C = C(\omega, a_0) > 0$. The above assumptions on Φ are actually satisfied whenever $\Phi = \widehat{\Psi}$ for some $\Psi \in H^2(\mathbb{R}; \mathcal{H}')$ such that $x_3 \Psi \in H^1(\mathbb{R}; \mathcal{H}')$ and $x_3^2 \Psi \in L^2(\mathbb{R}; \mathcal{H}')$. Moreover, the estimate (4.14) reads

$$(4.15) \quad \|(v, w)\|_{H^2(\mathbb{R}; \mathcal{H})} \leq C \left(\sum_{j=0}^2 \|x_3^j \Psi\|_{H^{2-j}(\mathbb{R}; \mathcal{H}')} \right),$$

in this case.

Armed with Proposition 4.1 we may now tackle the analysis of the solution to (4.8).

Some useful properties of the solution to (4.8) and consequences. Pick $F \in H^1(\omega)$ such that $F = f$ on $\partial\omega$ and $\|F\|_{H^1(\omega)} \leq C(\omega)\|f\|_{H^{1/2}(\partial\omega)}$. Let \tilde{u}^r (resp. \tilde{u}^i) denote the real (resp. imaginary) part of $\tilde{u} = \hat{u} - \hat{g}(\xi)F = \tilde{u}^r + i\tilde{u}^i$. Since the Fourier transform \hat{u} of the $H^1(\Omega)$ -solution u to (1.3) is actually solution to (4.8), we get by direct calculation that $(\tilde{u}^r, \tilde{u}^i)$ is solution to the variational problem (4.10), where

$$(4.16) \quad \langle \Phi(\xi), (\varphi, \psi) \rangle = -\mathcal{A}_\xi[(\hat{g}^r F, \hat{g}^i F), (\varphi, \psi)],$$

and \hat{g}^r (resp. \hat{g}^i) stands for the real (resp. imaginary) part of \hat{g} . Further, in light of (4.16) we check out using elementary computations that

$$(4.17) \quad \|\Phi(\xi)\|_{\mathcal{H}'} \leq C\langle \xi \rangle^2 |\hat{g}(\xi)| \|f\|_{H^{1/2}(\partial\omega)}$$

$$(4.18) \quad \|\Phi'(\xi)\|_{\mathcal{H}'} \leq C(\langle \xi \rangle^2 |\hat{g}'(\xi)| + \langle \xi \rangle |\hat{g}(\xi)|) \|f\|_{H^{1/2}(\partial\omega)}$$

$$(4.19) \quad \|\Phi''(\xi)\|_{\mathcal{H}'} \leq C(\langle \xi \rangle^2 |\hat{g}''(\xi)| + \langle \xi \rangle |\hat{g}'(\xi)| + |\hat{g}(\xi)|) \|f\|_{H^{1/2}(\partial\omega)},$$

for some constant $C = C(a_0, \omega) > 0$. Therefore we have $\langle \xi \rangle^j \Phi^{(2-j)} \in L^2(\mathbb{R})$ for $j = 0, 1, 2$, provided $\langle \xi \rangle^{4-j} \hat{g}^j \in L^2(\mathbb{R})$, this later condition being ensured by the assumption $x_3^j g \in H^{4-j}(\mathbb{R})$. From this and Proposition 4.1 then follows the:

Corollary 4.1. *Assume that $g \in H^4(\mathbb{R})$ is such that $x_3 g \in H^3(\mathbb{R})$, $x_3^2 g \in H^2(\mathbb{R})$ and $\int_{\mathbb{R}} g(x_3) dx_3 = 1$. Then it holds true that $\hat{u} \in H^2(\mathbb{R}; H^1(\omega))$. Moreover we have $u \in L^1(\mathbb{R}; H^1(\omega))$ and $U = \hat{u}(\cdot, 0) = \int_{\mathbb{R}} u(\cdot, x_3) dx_3 \in H^1(\omega)$ is the variational solution to the following boundary value problem*

$$\begin{cases} \operatorname{div}_{x'}(\tilde{A}_a \nabla_{x'} U) = 0 & \text{in } \omega \\ U = f & \text{on } \partial\omega. \end{cases}$$

In view of (4.8) and (4.10), we deduce from (4.17)-(4.19) that

$$\begin{aligned} \|\operatorname{div}_{x'}(\tilde{A}_a \nabla_{x'} \hat{u}(\cdot, \xi))\|_{L^2(\omega)} &\leq C\langle \xi \rangle^4 |\hat{g}(\xi)| \|f\|_{H^{1/2}(\partial\omega)} \\ \|\partial_\xi \operatorname{div}_{x'}(\tilde{A}_a \nabla_{x'} \hat{u}(\cdot, \xi))\|_{L^2(\omega)} &\leq C(\langle \xi \rangle^5 |\hat{g}'(\xi)| + \langle \xi \rangle^4 |\hat{g}(\xi)|) \|f\|_{H^{1/2}(\partial\omega)} \\ \|\partial_\xi^2 \operatorname{div}_{x'}(\tilde{A}_a \nabla_{x'} \hat{u}(\cdot, \xi))\|_{L^2(\omega)} &\leq C(\langle \xi \rangle^6 |\hat{g}(\xi)| + \langle \xi \rangle^5 |\hat{g}'(\xi)| + \langle \xi \rangle^4 |\hat{g}''(\xi)|) \|f\|_{H^{1/2}(\partial\omega)}, \end{aligned}$$

for some positive constant C depending only on a_0 and ω . This combined with Corollary 4.1 yields:

Proposition 4.2. *Let $g \in H^6(\mathbb{R})$ be such that $x_3 g \in H^5(\mathbb{R})$, $x_3^2 g \in H^4(\mathbb{R})$ and $\int_{\mathbb{R}} g(x_3) dx_3 = 1$. Then we have $\tilde{A}_a \nabla_{x'} \hat{u} \cdot \nu(x') \in H^2(\mathbb{R}; H^{-1/2}(\partial\omega))$ and thus $A_a \nabla u \cdot \nu(x) \in L^1(\mathbb{R}; H^{-1/2}(\partial\omega))$, with*

$$\tilde{A}_a \nabla_{x'} U \cdot \nu(x') = \tilde{A}_a \nabla_{x'} \hat{u}(\cdot, 0) \cdot \nu(x') = \int_{\mathbb{R}} A_a \nabla u(\cdot, x_3) \cdot \nu(x') dx_3 \in H^{-1/2}(\partial\omega).$$

In light of Proposition 4.2 we now introduce the two following DN maps:

$$\begin{aligned} \Lambda_a &: f \in H^{1/2}(\partial\omega) \mapsto A_a \nabla u(\cdot, x_3) \cdot \nu(x) \in L^1(\mathbb{R}; H^{-1/2}(\partial\omega)) \\ \tilde{\Lambda}_a &: f \in H^{1/2}(\partial\omega) \mapsto \tilde{A}_a \nabla_{x'} U \cdot \nu(x') \in H^{-1/2}(\partial\omega). \end{aligned}$$

These two operators are bounded, and they satisfy the estimate

$$(4.20) \quad \|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\|_{\mathcal{L}(H^{1/2}(\partial\omega), H^{-1/2}(\partial\omega))} \leq \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(H^{1/2}(\partial\omega), L^1(\mathbb{R}; H^{-1/2}(\partial\omega)))},$$

where, for the sake of simplicity, we write Λ_j (resp. $\tilde{\Lambda}_j$) for Λ_{a_j} (resp. $\tilde{\Lambda}_{a_j}$), $j = 1, 2$.

Finally, the last step in the analysis of the case of affine twisting functions involves noticing by direct calculation that the matrix $\partial_a \tilde{A}(x', a)$ has two eigenvalues $\lambda_0 = 0$ and $\lambda_1 = |x'|^2$. Hence, by mimicking the proof of [AG1][CLAIM, page 169], we obtain the:

Theorem 4.1. *Let $a_0 > 0$ and let g be the same as in Proposition 4.2. Assume moreover that $1 - a_0^2 \delta^2 > 0$. Then there exists a constant $C > 0$, depending only on a_0 and ω , such that the following stability estimate*

$$|a_1 - a_2| \leq C \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(H^{1/2}(\partial\omega), L^1(\mathbb{R}; H^{-1/2}(\partial\omega)))},$$

holds true whenever

$$|a_1|, |a_2| \leq a_0.$$

5. THE DN MAP FOR THE ORIGINAL PROBLEM

We start by defining the trace space for functions in $H^1(\Omega_\theta)$. To this purpose we set for all $L > 0$,

$$\Omega_\theta^L = \{(R_{\theta(x_3)}x', x_3); x' = (x_1, x_2) \in \omega, x_3 \in (-L, L)\}$$

and

$$\Gamma_\theta^L = \{(R_{\theta(x_3)}x', x_3); x' = (x_1, x_2) \in \partial\omega, x_3 \in [-L, L]\}.$$

Given $L > 0$ it is obvious that $u \in H^1(\Omega_\theta^L)$ for every $u \in H^1(\Omega_\theta)$ hence $u|_{\partial\Omega_\theta^L} \in H^{1/2}(\partial\Omega_\theta^L)$. Putting

$$H^{1/2}(\Gamma_\theta^L) = \{h = g|_{\Gamma_\theta^L} \text{ in } L^2(\Gamma_\theta^L); g \in H^{1/2}(\partial\Omega_\theta^L)\},$$

we thus have $u|_{\Gamma_\theta^L} \in H^{1/2}(\Gamma_\theta^L)$. Here $H^{1/2}(\Gamma_\theta^L)$ is equipped with its natural quotient norm

$$\|h\|_{H^{1/2}(\Gamma_\theta^L)} = \inf\{\|g\|_{H^{1/2}(\partial\Omega_\theta^L)}; g|_{\Gamma_\theta^L} = h\}.$$

Further we introduce

$$H_{\text{loc}}^{1/2}(\partial\Omega_\theta) = \{h \in L_{\text{loc}}^2(\partial\Omega_\theta); h|_{\Gamma_\theta^L} \in H^{1/2}(\Gamma_\theta^L) \text{ for all } L > 0\},$$

and define the following subspace of $H_{\text{loc}}^{1/2}(\partial\Omega_\theta)$:

$$\tilde{H}^{1/2}(\partial\Omega_\theta) = \{h \in H_{\text{loc}}^{1/2}(\partial\Omega_\theta); \text{ there exists } v \in H^1(\Omega_\theta) \text{ such that } v|_{\partial\Omega_\theta} = h\}.$$

Here and henceforth $v|_{\partial\Omega_\theta} = h$ means that the identity $v|_{\Gamma_\theta^L} = h|_{\Gamma_\theta^L}$ holds in the trace sense for every $L > 0$.

It is not hard to see that $\tilde{H}^{1/2}(\partial\Omega_\theta)$ is a Banach space for the quotient norm:

$$\|h\|_{\tilde{H}^{1/2}(\partial\Omega_\theta)} = \inf\{\|v\|_{H^1(\Omega_\theta)}; v|_{\partial\Omega_\theta} = h\}.$$

Let us now consider the mapping

$$\begin{aligned} I_\theta : C_0^1(\partial\Omega_\theta) &\longrightarrow C_0^1(\partial\Omega) \\ g &\mapsto f = g \circ \varphi_\theta, \end{aligned}$$

where, for the sake of shortness, we note $\varphi_\theta(x) = T_{\theta(x_3)}(x', x_3)$ for $x \in \bar{\Omega}$. Pick g in $C_0^1(\partial\Omega_\theta)$ and choose $v \in C_0^1(\mathbb{R}^3)$ such that $v|_{\partial\Omega_\theta} = g$. Since $v|_{\Omega_\theta} \in H^1(\Omega_\theta)$ we get that $g \in \tilde{H}^{1/2}(\partial\Omega_\theta)$ from the above identity. Moreover for all $v \in H^1(\Omega_\theta)$ obeying $v|_{\partial\Omega_\theta} = g$ the function $u = v|_{\Omega_\theta} \circ \varphi_\theta \in H^1(\Omega)$ by [Bre][Proposition 9.6], and

$$\|I_\theta g\|_{\tilde{H}^{1/2}(\partial\Omega)} \leq \|u\|_{H^1(\Omega)} \leq C(\omega, \theta) \|v\|_{H^1(\Omega_\theta)}.$$

As a consequence we have

$$(5.1) \quad \|I_\theta g\|_{\tilde{H}^{1/2}(\partial\Omega)} \leq C(\omega, \theta) \|g\|_{\tilde{H}^{1/2}(\partial\Omega_\theta)} \text{ for any } g \in C_0^1(\partial\Omega_\theta).$$

Let us now consider $g \in \tilde{H}^{1/2}(\partial\Omega_\theta)$ and $v \in H^1(\Omega_\theta)$ such that $v|_{\partial\Omega_\theta} = g$. For any sequence $(v_n)_n \in C_0^1(\mathbb{R}^3)$ such that $v_n|_{\Omega_\theta} \mapsto v$ in $H^1(\Omega_\theta)$ as $n \rightarrow +\infty$, it is clear that

$$\|g - g_n\|_{\tilde{H}^{1/2}(\partial\Omega_\theta)} \leq \|v - v_n\|_{H^1(\Omega_\theta)},$$

provided $g_n = v_n|_{\partial\Omega_\theta}$. Hence $(g_n)_n$ converges to g in $\tilde{H}^{1/2}(\partial\Omega_\theta)$.

For all $n \geq 1$, put $f_n = I_\theta g_n = g_n \circ \varphi_\theta$ and $u_n = v_n \circ \varphi_\theta$. Since $f_n = u_n|_{\partial\Omega}$, we see that

$$\|f_n - f_m\|_{\tilde{H}^{1/2}(\partial\Omega)} \leq \|u_n - u_m\|_{H^1(\Omega)} \leq C(\omega, \theta) \|v_n - v_m\|_{H^1(\Omega_\theta)}.$$

Therefore $(f_n)_n$ is a Cauchy sequence in $\tilde{H}^{1/2}(\partial\Omega)$ hence $f = \lim_n f_n \in \tilde{H}^{1/2}(\partial\Omega)$. Set $f = I_\theta g$. Then, in view of (5.1), I_θ extends to a bounded operator, still denoted by I_θ , from $\tilde{H}^{1/2}(\partial\Omega_\theta)$ into $\tilde{H}^{1/2}(\partial\Omega)$.

Arguing as above, we find out that the mapping

$$\begin{aligned} J_\theta : C_0^1(\partial\Omega) &\longrightarrow C_0^1(\partial\Omega_\theta) \\ f &\mapsto g = f \circ \psi_\theta, \end{aligned}$$

where $\psi_\theta = \varphi_\theta^{-1}$, extends to a bounded operator, which is still called J_θ , from $\tilde{H}^{1/2}(\partial\Omega)$ into $\tilde{H}^{1/2}(\partial\Omega_\theta)$.

Evidently, we have $I_\theta J_\theta f = f$ for all $f \in C_0^1(\partial\Omega)$ and $J_\theta I_\theta g = g$ for all $g \in C_0^1(\partial\Omega_\theta)$. Therefore $J_\theta = I_\theta^{-1}$, by density of $C_0^1(\partial\Omega)$ (resp. $C_0^1(\partial\Omega_\theta)$) in $\tilde{H}^{1/2}(\partial\Omega)$ (resp. $\tilde{H}^{1/2}(\partial\Omega_\theta)$).

Next, by reasoning in the same way as in the derivation of (1.3), we prove with the help of the Lax-Milgram lemma that the boundary value problem (1.1) has a unique solution $v \in H^1(\Omega_\theta)$ for every $g \in \tilde{H}^{1/2}(\partial\Omega_\theta)$. Moreover the operator $\tilde{\Lambda}_\theta$ is well defined as a bounded operator from $\tilde{H}^{1/2}(\partial\Omega_\theta)$ into its dual space $\tilde{H}^{-1/2}(\partial\Omega_\theta)$. Similarly to Λ_θ , it can be checked that $\tilde{\Lambda}_\theta$ is characterized by the following identity

$$\langle \tilde{\Lambda}_\theta g, h \rangle = \int_{\Omega_\theta} \nabla v \cdot \nabla H dy,$$

which holds true for all $h \in \tilde{H}^{1/2}(\partial\Omega_\theta)$ and all $H \in H^1(\Omega_\theta)$ such that $H|_{\partial\Omega_\theta} = h$. By performing the change of variable $y = \varphi_\theta(x)$ in the last integral, we thus get that

$$\langle \tilde{\Lambda}_\theta g, h \rangle = \int_{\Omega} A \nabla u \cdot \nabla (H \circ \varphi_\theta) dx,$$

where u is the solution to the boundary value problem (1.3) with $f = I_\theta g$. Therefore we have

$$\langle \tilde{\Lambda}_\theta g, h \rangle = \langle \Lambda_\theta I_\theta g, I_\theta h \rangle,$$

which means that $\tilde{\Lambda}_\theta = I_\theta^* \Lambda_\theta I_\theta$, or equivalently that $\Lambda_\theta = J_\theta^* \tilde{\Lambda}_\theta J_\theta$.

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