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RECOVERING THE TWISTING FUNCTION IN A TWISTED WAVEGUIDE FROM THE DN MAP

MOURAD CHOULLI† AND ERIC SOCCORSI‡

ABSTRACT. We consider the determination of the twisting function in a twisted waveguide from the corresponding DN map. We relate this problem to an inverse anisotropic conductivity problem in a waveguide. We show that this later is still an open problem. In the special case of an affine twisting function, we prove a stability estimate for a reduced DN map.

Key words : twisted waveguide, twisting function, DN map, stability estimate.

AMS subject classifications : 35R30.

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1. INTRODUCTION

Let ω be a bounded domain of \mathbb{R}^2 and $\Omega = \omega \times \mathbb{R}$. The rotation in \mathbb{R}^2 of angle $\xi \in \mathbb{R}$ is denoted by R_ξ . To $\theta \in C^1(\mathbb{R})$ we associate

$$\Omega_\theta = \{(R_{\theta(x_3)}x', x_3); x' = (x_1, x_2) \in \omega, x_3 \in \mathbb{R}\}.$$

Consider the following BVP for the laplacian in the twisted waveguide Ω_θ :

$$(1.1) \quad \begin{cases} \Delta v(y) = 0 & \text{in } \Omega_\theta, \\ v(y) = g(y) & \text{on } \partial\Omega_\theta. \end{cases}$$

We are concerned by the problem of recovering the twisting function θ from the Dirichlet-to-Neumann (DN in short) map

$$\tilde{\Lambda}_\theta : g \rightarrow B(y)\nabla v(y) \cdot \nu(y),$$

where

$$B(y) = \begin{pmatrix} 1 & 0 & -y_2\theta'(y_3) \\ 0 & 1 & y_1\theta'(y_3) \\ 0 & 0 & 1 \end{pmatrix}.$$

Our aim is the stability issue for the problem of determining θ from $\tilde{\Lambda}_\theta$. As one can see, it is not convenient to work directly with $\tilde{\Lambda}_\theta$ because it acts on function spaces depending on θ . To overcome this difficulty, we will transform our original problem into a problem having a DN map acting on function spaces not depending on θ .

Set

$$T_\xi = \begin{pmatrix} R_\xi & 0 \\ 0 & 1 \end{pmatrix}$$

and $u(x) = v(T_{\theta(x_3)}(x', x_3))$, $x = (x', x_3) \in \Omega$. By straightforward computations we prove that u is the solution a BVP for an elliptic operator in the divergence form. Namely, we have

$$(1.2) \quad \begin{cases} \operatorname{div}(A(x', \theta'(x_3))\nabla u) = 0 & \text{in } \Omega, \\ u(x) = f(x) & \text{on } \partial\Omega. \end{cases}$$

Here, $f(x) = g(T_{\theta(x_3)}(x', x_3))$, $x \in \partial\Omega$, and the matrix A is given as follows:

$$A(x', t) = \begin{pmatrix} 1 + x_2^2 t^2 & -x_2 x_1 t^2 & -x_2 t \\ -x_2 x_1 t^2 & 1 + x_1^2 t^2 & x_1 t \\ -x_2 t & x_1 t & 1 \end{pmatrix}, \quad x' \in \omega, \quad t \in \mathbb{R}.$$

Moreover,

$$B(y)\nabla v(y) \cdot \nu(y) = A(x', \theta'(x_3))\nabla u(x) \cdot \nu(x).$$

This identity explain the choice of the boundary operator B for v . It is the one who gives the "right" Neumann condition for u .

The last identity says that, at least formally, recovering θ from $\tilde{\Lambda}_\theta$ is the same as recovering θ from the following DN map

$$\Lambda_\theta : f \rightarrow A\nabla u \cdot \nu.$$

The major part of our work will be concentrated on studying the properties of the DN map Λ_θ . The rest is devoted to the stability of the determination of θ from Λ_θ . Surprisingly, our inverse problem corresponds to an anisotropic conductivity problem in an unbounded domain having exactly the same form to the one already studied by Alessandrini [A] and Alessandrini and Gaburo [AG1], [AG2] in the case of a bounded domain (see also Gaburo and Lionheart [GL]). However, the monotonicity assumption on A is no longer valid in our case. Therefore, the problem in its full generality is still open. In the present work, we establish a Lipschitz stability estimate in the case of affine twisting functions and for a reduced DN map.

This paper is organized as follows. In section 2, we give the precise definition of the DN map Λ_θ and its regularity properties. We explain in section 3 why the results by Alessandrini and Gaburo in [AG1] and [AG2] cannot be adapted to our problem. We also show in this section that if the twisting function is close to a constant, then our original inverse problem is close, in some sense, to an inverse conductivity problem for which the Alessandrini's monotonicity assumption holds. The particular case of affine twisting functions is studied in section 4. Taking the Fourier transform with respect to the variable x_3 , we show that the original problem can be converted to a classical anisotropic conductivity problem on ω . We prove a stability estimate for a reduced DN map using that A satisfies a weaker form of the monotonicity assumption. Section 5 is devoted to the definition of the original DN map $\tilde{\Lambda}_\theta$ and the relationship between $\tilde{\Lambda}_\theta$ and Λ_θ .

2. THE DN MAP

We start with following extension lemma.

Lemma 2.1. (*extension lemma*) *Let $g \in H^{s+1/2}(\mathbb{R}; H^s(\partial\omega))$, $s = 3/2$ or $s = 1/2$. Then there exists $G \in H^{s+1/2}(\mathbb{R}; H^{s+1/2}(\omega))$ such that $G(t) = g(t)$ on $\partial\omega$ and*

$$(2.1) \quad \|G\|_{H^{s+1/2}(\mathbb{R}; H^{s+1/2}(\omega))} \leq C(\omega)\|g\|_{H^{s+1/2}(\mathbb{R}; H^s(\partial\omega))},$$

where $C(\omega)$ is a constant depending only on ω .

Proof. We give the proof for $s = 3/2$. The proof for $s = 1/2$ is similar. Let us first assume that $g \in C_0^\infty(\mathbb{R}; H^{3/2}(\partial\omega))$. Following [LM], for each $h \in H^2(\partial\omega)$, the BVP

$$(2.2) \quad \begin{cases} \Delta H = 0 & \text{in } \omega, \\ H = h & \text{on } \partial\omega \end{cases}$$

has a unique solution $H \in H^2(\omega)$ and there exists a constant $C(\omega)$, depending only on ω , such that

$$(2.3) \quad \|H\|_{H^2(\omega)} \leq C(\omega)\|h\|_{H^{3/2}(\partial\omega)}.$$

Let $G(t)$, $t \in \mathbb{R}$, be the solution of (2.2) corresponding to $h = g(t)$. Using that $G(t) - G(s)$, $s, t \in \mathbb{R}$, is the solution of the BVP (2.2) with $h = g(t) - g(s)$, we obtain from estimate (2.3)

$$\|G(t) - G(s)\|_{H^2(\omega)} \leq C(\omega)\|g(t) - g(s)\|_{H^{3/2}(\partial\omega)}.$$

Therefore $G \in C(\mathbb{R}; H^2(\omega))$ and, again from estimate (2.3),

$$(2.4) \quad \|G\|_{L^2(\mathbb{R}; H^2(\omega))} \leq C(\omega) \|g\|_{L^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

Next, Let $K(t)$ be the solution of the BVP (2.2) corresponding to $h = g'(t)$. As previously $K \in C(\mathbb{R}; H^2(\omega))$. Then, since $L(t, s) = G(t + s) - G(t) - sK(t)$, $t, s \in \mathbb{R}$, is the solution of (2.3) with $h = g(t + s) - g(s) - sg'(t)$, we have from estimate (2.3)

$$\|G(t) - G(s) - sK(t)\|_{H^2(\omega)} \leq C(\omega) \|g(t + s) - g(s) - sg'(t)\|_{H^{3/2}(\partial\omega)}.$$

Hence, $G \in C^1(\mathbb{R}; H^2(\omega))$, $G' = K$ and

$$(2.5) \quad \|G'\|_{L^2(\mathbb{R}; H^2(\omega))} \leq C(\omega) \|g'\|_{L^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

Repeating the above argument to G' in place of G , we find $G \in C^2(\mathbb{R}; H^2(\omega))$ and

$$(2.6) \quad \|G''\|_{L^2(\mathbb{R}; H^2(\omega))} \leq C(\omega) \|g''\|_{L^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

It follows from (2.4), (2.5) and (2.6),

$$(2.7) \quad \|G\|_{H^2(\mathbb{R}; H^2(\omega))} \leq C(\omega) \|g\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

Now, let $g \in H^2(\mathbb{R}; H^2(\partial\omega))$ and (g_n) a sequence in $C_0^\infty(\mathbb{R}; H^2(\partial\omega))$ converging to g in $H^2(\mathbb{R}; H^2(\partial\omega))$. By uniqueness, the extension (constructed as before) corresponding to $g_n - g_m$ is equal $G_n - G_m$, where G_n (resp. G_m) is the extension of g_n (resp. g_m). Therefore, we can apply (2.7). We find

$$\|G_n - G_m\|_{H^2(\mathbb{R}; H^2(\omega))} \leq C(\omega) \|g_n - g_m\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

Consequently, (G_n) is a Cauchy sequence in $H^2(\mathbb{R}; H^2(\omega))$. Let $G \in H^2(\mathbb{R}; H^2(\omega))$ be the limit of the sequence (G_n) in $H^2(\mathbb{R}; H^2(\omega))$. Using the continuity of the trace operator

$$W \in H^2(\mathbb{R}; H^2(\omega)) \rightarrow W|_{\partial\Omega} \in H^2(\mathbb{R}; H^2(\partial\omega)),$$

we deduce that G is an extension of g . Since we have estimate (2.7) for g_n , for each n , we can pass to the limit as n goes to infinity. The resulting estimate is exactly (2.1). \square

Let $s = 3/2$ or $s = 1/2$. Since the trace operator $w \in H^{s+1/2}(\omega) \rightarrow w|_{\partial\omega} \in H^s(\partial\omega)$ is bounded, then so is the following one

$$G \in H^{s+1/2}(\mathbb{R}; H^{s+1/2}(\omega)) \rightarrow G|_{\partial\Omega} \in H^{s+1/2}(\mathbb{R}; H^s(\omega)).$$

Therefore,

$$\|g\|_{H^{s+1/2}(\mathbb{R}; H^s(\omega))} = \inf\{\|G\|_{H^{s+1/2}(\mathbb{R}; H^{s+1/2}(\omega))}; G = g \text{ on } \partial\Omega\}$$

is a norm on $H^{s+1/2}(\mathbb{R}; H^s(\omega))$, equivalent to the original one. In the sequel, we use indifferently one of these two equivalent norms, each of them is denoted by the same symbol.

We need to make some assumptions insuring the uniform ellipticity of A . Here and henceforth, A is $A(x', t)$ or $A(x', \theta(x_3))$, as it was defined in the previous section. For $\zeta \in \mathbb{R}^3$, we have

$$\begin{aligned} A(x', t)\zeta \cdot \zeta &= \zeta_1^2 + \zeta_2^2 + \zeta_3^2 - 2tx_2\zeta_1\zeta_3 + 2tx_1\zeta_2\zeta_3 + t^2(x_2\zeta_1 - x_1\zeta_2)^2 \\ &= \zeta_1^2 + \zeta_2^2 + (\zeta_3 + t(x_2\zeta_1 - x_1\zeta_2))^2. \end{aligned}$$

Let $\underline{t} < \bar{t}$ be given. Since $A(x', t)\zeta \cdot \zeta = 0$ if and only if $\zeta = 0$, for any $x' \in \omega$, $t \in \mathbb{R}$, a compactness argument shows that there exist $\lambda \geq 1$, depending on ω , \underline{t} and \bar{t} such that

$$(2.8) \quad \lambda^{-1}|\zeta|^2 \leq A(x', t)\zeta \cdot \zeta \leq \lambda|\zeta|^2 \text{ for all } x' \in \omega, t \in [\underline{t}, \bar{t}], \zeta \in \mathbb{R}^3.$$

In order to define the DN map associated to the boundary value problem (1.2), we need first to solve this later. To this end, pick $f \in H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ and $F \in H^1(\Omega)$ such that $F = f$ on $\partial\Omega$, where we identified $H^1(\mathbb{R}, H^1(\omega))$ by $H^1(\Omega)$. Note that a such F exists by the previous lemma. From the Lax-Milgram lemma, there exists a unique $v \in H_0^1(\Omega)$ solving the variational problem

$$(2.9) \quad \int_{\Omega} A \nabla v \cdot \nabla w = - \int_{\Omega} A \nabla F \cdot \nabla w, \text{ for all } w \in H_0^1(\Omega).$$

Hence $u = v + F$ is the unique weak solution of the boundary value problem (1.2). That is, it satisfies the first equation in (1.2) in the distributional sense and the second equation in the trace sense. Moreover, taking $w = v$ in (2.9), we obtain with the help of Poincaré's inequality (which is valid for Ω because ω is bounded)

$$\|v\|_{H^1(\Omega)} \leq C\|F\|_{H^1(\Omega)}$$

and then

$$\|u\|_{H^1(\Omega)} \leq C\|F\|_{H^1(\Omega)}.$$

But F can be arbitrarily chosen in $H^1(\Omega)$ such that $F = f$ in $\partial\Omega$. Consequently,

$$(2.10) \quad \|u\|_{H^1(\Omega)} \leq C\|f\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))}.$$

Let us introduce the following $H(\operatorname{div}_A)$ -type space. Set

$$H(\operatorname{div}_A, \Omega) = \{P \in L^2(\Omega)^3; \operatorname{div}(AP) \in L^2(\Omega)\}.$$

Next, as usual the dual space of $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ is denoted by $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$.

Proposition 2.1. *Let $P \in H(\operatorname{div}_A, \Omega)$. Then $AP \cdot \nu \in H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$ and*

$$(2.11) \quad \|AP \cdot \nu\|_{H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))} \leq C(\|P\|_{L^2(\Omega)} + \|\operatorname{div}(AP)\|_{L^2(\Omega)}).$$

In addition, the following identity holds

$$(2.12) \quad \langle AP \cdot \nu, g \rangle = \int_{\Omega} G \operatorname{div}(AP) dx + \int_{\Omega} A \nabla G \cdot P dx,$$

for any $g \in H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ and $G \in H^1(\Omega)$ such that $G = g$ on $\partial\Omega$. Here, $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ and its dual $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$.

Proof. Consider first the case $P \in C_c^\infty(\overline{\Omega})^3$. Fix $g \in H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ and let $G \in H^1(\Omega)$ be arbitrary chosen such that $G = g$ on $\partial\Omega$. Since P has a compact support, we can use Green's formula. We have

$$(2.13) \quad \int_{\Omega} G \operatorname{div}(AP) dx = - \int_{\Omega} A \nabla G \cdot P dx + \int_{\partial\Omega} g AP \cdot \nu.$$

Hence

$$\left| \int_{\partial\Omega} g AP \cdot \nu d\sigma \right| \leq C\|G\|_{H^1(\Omega)} (\|P\|_{L^2(\Omega)} + \|\operatorname{div}(AP)\|_{L^2(\Omega)}).$$

Taking the infimum with respect to those $G \in H^1(\Omega)$ satisfying $G = g$ on $\partial\Omega$, we deduce (2.11).

Now a slight modification of the proof of Theorem 2.4 in [GR] shows that $C_0^\infty(\overline{\Omega})^3$ is dense in $H(\operatorname{div}_A, \Omega)$. Let then $P \in H(\operatorname{div}_A, \Omega)$ and (P_k) a sequence in $C_0^\infty(\overline{\Omega})^3$ converging to P in $H(\operatorname{div}_A, \Omega)$. From (2.11), we obtain that $(AP_k \cdot \nu)$ is a Cauchy sequence in $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$. Therefore $(AP_k \cdot \nu)$ has a limit, denoted by $AP \cdot \nu$, in $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$.

We complete the proof by noting that (2.12) is an immediate consequence of (2.13). \square

For simplicity, u will always denote the solution of (1.2). Applying the previous proposition to $P = \nabla u$, we obtain, in view of (2.10), that

$$\Lambda_\theta : f \rightarrow A \nabla u \cdot \nu$$

is well defined as a bounded operator from $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ into $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$ and we have the following formula

$$(2.14) \quad \langle \Lambda_\theta f, g \rangle = \int_{\Omega} A \nabla u \cdot \nabla G,$$

for any $g \in H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ and $G \in H^1(\Omega)$ such that $G = g$ on $\partial\Omega$.

In (2.14), if we choose $G = v$, v being the solution of (1.2) when f is replaced by g , then we have

$$\langle \Lambda_\theta f, g \rangle = \int_{\Omega} A \nabla u \cdot \nabla v = \int_{\Omega} \nabla u \cdot A \nabla v.$$

Therefore,

$$(2.15) \quad \langle \Lambda_\theta f, g \rangle = \langle f, \Lambda_\theta g \rangle, \text{ for all } f, g \in H^1(\mathbb{R}; H^{1/2}(\partial\omega)).$$

¹This means that the operator $P \in C_0^\infty(\overline{\Omega}) \rightarrow AP \cdot \nu \in C^\infty(\partial\Omega)$ can be extended to a bounded operator from $H(\operatorname{div}_A, \Omega)$ into $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$.

This means that $\Lambda_\theta^*|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))} = \Lambda_\theta$ when $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ is identified to a subspace of its bidual space.

For $i = 1, 2$, let $A_i = A(x', \theta_i(x_3))$ and set $\Lambda_i = \Lambda_{\theta_i}$. Let $u_i \in H^1(\Omega)$, $i = 1, 2$ be a weak solution of

$$\operatorname{div}(A_i \nabla u_i) = 0 \text{ in } \Omega.$$

An application of (2.14) with $f = u_i|_{\partial\Omega}$ and $g = u_{3-i}|_{\partial\Omega}$ yields

$$\begin{aligned} \langle \Lambda_1 u_1, u_2 \rangle &= \int_{\Omega} A_1 \nabla u_1 \cdot \nabla u_2, \\ \langle \Lambda_2 u_2, u_1 \rangle &= \int_{\Omega} A_2 \nabla u_2 \cdot \nabla u_1. \end{aligned}$$

In view of (2.15), these two identities imply the following one

$$(2.16) \quad \langle (\Lambda_1 - \Lambda_2) u_1, u_2 \rangle = \int_{\Omega} (A_1 - A_2) \nabla u_1 \cdot \nabla u_2.$$

Next, let us discuss a regularity property of Λ_θ when it is restricted to $H^2(\mathbb{R}; H^{3/2}(\partial\omega))$. We will say that $\Omega_1 = \omega \times (-1, 1)$ has H^2 -regularity property if for any matrix-valued function $C = (C_{ij}(x))$ with coefficients in $W^{1,\infty}(\Omega_1)$ satisfying an ellipticity condition

$$C(x)\xi \cdot \xi \geq \alpha|\zeta|^2, \text{ for all } \zeta \in \mathbb{C}^3, x \in \Omega_1$$

and $f \in L^2(\Omega)$, the BVP

$$\begin{cases} \operatorname{div}(C\nabla w) = f & \text{in } \Omega_1, \\ w = 0 & \text{on } \partial\Omega_1. \end{cases}$$

has a unique solution $w \in H^2(\Omega_1)$ such that, if

$$\max_{i,j} \|C_{ij}\|_{W^{1,\infty}(\Omega_1)} \leq M$$

then

$$\|w\|_{H^2(\Omega_1)} \leq C(\alpha, M) \|f\|_{L^2(\Omega_1)}.$$

Here $C(\alpha, M)$ is a constant depending only on α, M and ω .

Note that if ω is convex then Ω_1 has H^2 -regularity property (e.g. [Gr]) and Ω_1 has H^2 -regularity property if and only if $\Omega_a = \omega \times (-a, a)$ has H^2 -regularity property for an arbitrary $a > 0$.

Henceforth, we frequently identify $H^2(\Omega)$ by $H^2(\mathbb{R}; H^2(\omega))$.

Theorem 2.1. *Assume that $\theta \in C^{1,1}(\mathbb{R})$ and Ω_1 has H^2 -regularity property. Then for any $f \in H^2(\mathbb{R}; H^{3/2}(\partial\omega))$, the BVP (1.2) has a unique solution $u \in H^2(\Omega)$. Moreover if $\|\theta\|_{C^{1,1}(\mathbb{R})} \leq M$, for some $M > 0$, then*

$$(2.17) \quad \|u\|_{H^2(\Omega)} \leq C \|f\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

Here, C is a constant depending only on M and ω .

Proof. Fix $M > 0$. Let $\theta \in C^{1,1}(\mathbb{R})$, $\|\theta\|_{C^{1,1}(\mathbb{R})} \leq M$ and $f \in H^2(\mathbb{R}; H^{3/2}(\partial\omega))$. From Lemma 2.1, we know that there exists $F \in H^2(\mathbb{R}; H^{3/2}(\omega))$ such that $F = f$ on $\partial\Omega$ and

$$(2.18) \quad \|F\|_{H^2(\Omega)} \leq C(\omega) \|f\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

Set

$$\Psi = \operatorname{div}(A\nabla F),$$

We already know that there exists $u_0 \in H_0^1(\Omega)$ such that

$$(2.19) \quad \int_{\Omega} A\nabla u_0 \cdot \nabla v = \int_{\Omega} \Psi v, \text{ for all } v \in H_0^1(\Omega).$$

and

$$(2.20) \quad \|u_0\|_{H^1(\Omega)} \leq C_0(\omega) \|\Psi\|_{L^2(\Omega)}.$$

For each integer $n \geq 1$, let $\xi_n \in C_c^\infty[-(n+1), n+1]$ such that $\xi_n = 1$ in a neighborhood of $[-n, n]$ and $|\xi'|, |\xi''| \leq d$, where d is an universal constant. If $v \in H_0^1(\Omega)$, we have

$$A\nabla(\xi_n u) \cdot \nabla v = A\nabla u_0 \cdot \nabla(\xi_n v) - A\nabla u_0 \cdot \nabla \xi_n v + A\nabla \xi_n \cdot \nabla v u_0.$$

Hence,

$$\int_{\Omega} A \nabla(\xi_n u) \cdot \nabla v = \int_{\Omega} A \nabla u_0 \cdot \nabla(\xi_n v) - \int_{\Omega} A \nabla u_0 \cdot \nabla \xi_n v + \int_{\Omega} A \nabla \xi_n \cdot \nabla v u_0.$$

We integrate by parts the last term in the right hand side of the above identity. We obtain

$$\int_{\Omega} A \nabla \xi_n \cdot \nabla v u_0 = - \int_{\Omega} A \nabla \xi_n \cdot \nabla u_0 v - \int_{\Omega} \operatorname{div}(A \nabla \xi_n) u_0 v.$$

Therefore,

$$\int_{\Omega} A \nabla(\xi_n u) \cdot \nabla v = \int_{\Omega} A \nabla u_0 \cdot \nabla(\xi_n v) - \int_{\Omega} A \nabla u_0 \cdot \nabla \xi_n v - \int_{\Omega} A \nabla \xi_n \cdot \nabla u_0 v - \int_{\Omega} \operatorname{div}(A \nabla \xi_n) u_0 v.$$

In combination with (2.19) and using that A is symmetric, this last identity implies

$$\int_{\Omega} A \nabla(\xi_n u) \cdot \nabla v = \int_{\Omega} \Psi \xi_n v - 2 \int_{\Omega} A \nabla \xi_n \cdot \nabla u_0 v - \int_{\Omega} \operatorname{div}(A \nabla \xi_n) u_0 v.$$

In particular, $\xi_n u_0 \in H_0^1(\Omega_{n+1})$ is the solution of the following variational problem

$$(2.21) \quad \int_{\Omega_{n+1}} A \nabla(\xi_n u_0) \cdot \nabla v = \int_{\Omega_{n+1}} \tilde{\Psi} v, \text{ for all } v \in H_0^1(\Omega_{n+1}),$$

where $\Omega_n = \omega \times (-n, n)$ and

$$\tilde{\Psi} = \Psi \xi_n - 2A \nabla \xi_n \cdot \nabla u_0 - \operatorname{div}(A \nabla \xi_n) u_0.$$

We are now going to make a change of variables in (2.21). To this end, we need to introduce some notations

$$\begin{aligned} J_n &= \operatorname{diag}(1, 1, 1/n), \\ \bar{A} &= 1/(n+1) J_{n+1} A(x', (n+1)y_3) J_{n+1}, \quad y_3 \in (-1, 1) \\ \bar{\xi}(y_3) &= \xi_n((n+1)y_3), \quad y_3 \in (-1, 1), \\ \bar{u}(x', y_3) &= u_0(x', (n+1)y_3), \quad y_3 \in (-1, 1), \\ w_n(x', y_3) &= \xi_n((n+1)y_3) u_0(x', (n+1)y_3), \quad y_3 \in (-1, 1), \\ \bar{\operatorname{div}}(P(x', y_3)) &= \partial_{x_1} P_1(x', y_3) + \partial_{x_2} P_2(x', y_3) + 1/(n+1) \partial_{y_3} P_3(x', y_3), \\ \bar{\Psi}(x', y_3) &= 1/(n+1) \left[\Psi(x', (n+1)y_3) - 2J_{n+1} A(x', (n+1)y_3) J_{n+1} \nabla \bar{\xi}(y_3) \cdot \nabla \bar{u}(x', y_3) - \right. \\ &\quad \left. \bar{\operatorname{div}}(J_{n+1} A(x', (n+1)y_3) \nabla \bar{\xi}(y_3)) \bar{u}(x', y_3) \right]. \end{aligned}$$

Making the change of variables $(x', x_3) \in \Omega_{n+1} \rightarrow (x', y_3) = (x', 1/(n+1)x_3) \in \Omega_1$ in (2.21), we obtain that $w_n \in H_0^1(\Omega_1)$ is the solution of the variational problem

$$\int_{\Omega_1} \bar{A} \nabla w_n \cdot \nabla v = \int_{\Omega_1} \bar{\Psi} v, \text{ for all } v \in H_0^1(\Omega_1).$$

As Ω_1 is assumed to have H^2 -regularity property, we obtain by straightforward computations

$$\|w_n\|_{H^2(\Omega_1)} \leq C(M, \omega) \|\bar{\Psi}\|_{L^2(\Omega_1)}.$$

On the other hand, we prove

$$\|\bar{\Psi}\|_{L^2(\Omega_1)} \leq (n+1)^{-3/2} C(M, \omega) \|\Psi\|_{L^2(\Omega)}.$$

Therefore,

$$(2.22) \quad \|w_n\|_{H^2(\Omega_1)} \leq (n+1)^{-3/2} C(M, \omega) \|\Psi\|_{L^2(\Omega)}.$$

Returning to $\xi_n u_0$, we easily obtain from (2.20) and (2.22) that

$$(2.23) \quad \|\xi_n u_0\|_{H^2(\Omega)} \leq C(M, \omega) \|\Psi\|_{L^2(\Omega)}.$$

Hence, substracting a subsequence if necessary, we assume that $\xi_n u$ converges weakly, in $H^2(\Omega)$, to $\tilde{u} \in H^2(\Omega)$. Or $\xi_n u_0$ converges to u_0 in $L^2(\Omega)$. Consequently, $u_0 \in H^2(\Omega)$ and $\xi_n u_0$ converges weakly, in $H^2(\Omega)$, to u_0 . As the norm $\|\cdot\|_{H^2(\Omega)}$ is lower semi-continuous, we have from (2.23)

$$\|u_0\|_{H^2(\Omega)} \leq \liminf_n \|\xi_n u_0\|_{H^2(\Omega)} \leq C(M, \omega) \|\Psi\|_{L^2(\Omega)}.$$

But, $\|\Psi\|_{L^2(\Omega)} \leq C(M, \omega) \|F\|_{H^2(\Omega)}$ and $\|F\|_{H^2(\Omega)} \leq C(\omega) \|f\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}$. Therefore

$$\|u_0\|_{H^2(\Omega)} \leq C(M, \omega) \|f\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

Consequently, $u = u_0 + F \in H^2(\Omega)$ is the unique solution of (1.2) and

$$\|u\|_{H^2(\Omega)} \leq C(M, \omega) \|f\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

The proof is then complete. \square

Let us recall that the trace operator $\tau : w \in H^2(\omega) \rightarrow \partial_\nu w \in H^{1/2}(\partial\omega)$ is bounded. Proceeding in a similar manner as in Lemma 2.1 and using that $C_c^\infty(\mathbb{R}; H^2(\omega))$ is dense in $H^2(\Omega)$, we prove that the trace operator

$$\tilde{\tau} : w \in H^2(\Omega) \rightarrow \partial_\nu w \in H^2(\mathbb{R}; H^{1/2}(\partial\omega))$$

is bounded and $\|\tilde{\tau}\| \leq \|\tau\|$.

To θ satisfying assumptions of Theorem 2.1, we associate the corresponding DN map Λ_θ defined as follows :

$$\Lambda_\theta : f \in H^2(\mathbb{R}; H^{3/2}(\partial\omega)) \rightarrow \partial_\nu u \in H^2(\mathbb{R}; H^{1/2}(\partial\omega)).$$

Then Λ_θ is bounded and estimate (2.17) imply $\|\Lambda_\theta\| \leq C(M, \omega)$.

3. THE DETERMINATION OF THE TWISTING FUNCTION FROM THE DN MAP: AN OPEN PROBLEM

Let γ be a nonempty open subset of $\partial\omega$. Fix $L > 0$ and introduce the following notations : $\Omega^L = \omega \times (-L, L)$, $\Gamma = \gamma \times (-L, L)$,

$$H_\Gamma^1(\mathbb{R}; H^{1/2}(\partial\omega)) = \{f \in H^1(\mathbb{R}; H^{1/2}(\partial\omega)); \text{supp} f \subset \Gamma\}.$$

and

$$\mathcal{W}(\mathbb{R}) = \{\theta \in W_{loc}^{2,\infty}(\mathbb{R}); \theta' \in W^{1,\infty}(\mathbb{R})\}^2$$

For $i = 1, 2$, let $A_i = A(x', \theta'_i(x_3))$, with $\theta_i \in \mathcal{W}(\mathbb{R})$, and set $\Lambda_i = \Lambda_{\theta_i}$. From (2.15), we know that

$$(3.1) \quad \langle (\Lambda_1 - \Lambda_2)u_1, u_2 \rangle = \int_\Omega (A_1 - A_2) \nabla u_1 \cdot \nabla u_2$$

for any $u_i \in H^1(\Omega)$, where u_i is a weak solution of

$$\text{div}(A_i \nabla u_i) = 0 \text{ in } \Omega.$$

Let us assume that $\theta_1 = \theta_2$ if $|x_3| > L$. Then (3.1) becomes

$$(3.2) \quad \langle (\Lambda_1 - \Lambda_2)u_1, u_2 \rangle = \int_{\Omega_L} (A_1 - A_2) \nabla u_1 \cdot \nabla u_2.$$

For $0 < \rho \leq \rho_0$, ρ_0 is some constant depending only on ω (see [AG2] for details), let

$$\begin{aligned} \Gamma_\rho &= \{x \in \Gamma; \text{dist}(x, \Gamma) > \rho\}, \\ U_\rho &= \{x \in \mathbb{R}^3; \text{dist}(x, \Gamma_\rho) < \rho/4\}. \end{aligned}$$

As it is said in [AG2], one can construct a Lipschitz domain Ω_ρ satisfying

$$\begin{aligned} \Omega &\subset \Omega_\rho^L, \quad \partial\Omega^L \cap \Omega_\rho^L \subset \subset \Gamma, \\ \text{dist}(x, \partial\Omega_\rho^L) &\geq \rho/2 \text{ for all } x \in U_\rho. \end{aligned}$$

We shall consider $\tilde{\nu}$, an unitary C^∞ vector field, defined near $\partial\Omega$, which is non tangential to $\partial\omega$ (a such vector field was constructed in [AG1]).

Let $x^0 \in \overline{\Gamma}_\rho^L$ and set $z_\tau = x^0 + \tau\tilde{\nu}$. From Theorem 3.3 in [AG2], for $i = 1, 2$, $G_i \in H^1(\Omega_\rho^L)$ the solution of the following boundary value problem

$$\begin{cases} \text{div}(A_i \nabla G_i) = -\delta(x - z_\tau) & \text{in } \Omega_\rho^L, \\ G_i = 0 & \text{on } \partial\Omega_\rho^L, \end{cases}$$

has the form

$$(3.3) \quad G_i(x) = C(\det(A_i(z_\tau)))^{-1/2} (A(z_\tau)^{-1}(x - z_\tau) \cdot (x - z_\tau))^{-1/2} + R_i(x),$$

where C is a constant and the reminder R_i satisfies

$$|R_i(x)| + |x - z_\tau| |\nabla R_i| \leq C|x - z_\tau|^{5+\alpha}$$

for every $x \in \Omega_\rho^L$, $|x - z_\tau| \leq r_0$, for some r_0 . Here C is a constant and $0 < \alpha < 1$.

²We can also take $\mathcal{W}(\mathbb{R}) = \{\theta \in \mathcal{D}'(\mathbb{R}); \theta' \in W^{1,\infty}(\mathbb{R})\}$.

Since $G_i|_{\omega \times \{\pm L\}} = 0$, the zero extension in Ω , denoted by \tilde{G}_i , belongs to $H^1(\Omega)$ and satisfies in the weak sense

$$\operatorname{div}(A_i \nabla \tilde{G}_i) = 0 \quad \text{in } \Omega.$$

It follows from (3.2)

$$\langle (\Lambda_1 - \Lambda_2) \tilde{G}_1, \tilde{G}_2 \rangle = \int_{\Omega_L} (A_1 - A_2) \nabla G_1 \cdot \nabla G_2 dx.$$

Since $\tilde{G}_i|_{\partial\Omega} \in H^1_\Gamma(\mathbb{R}; H^{1/2}(\partial\omega))$,

$$\int_{\Omega_L} (A_1 - A_2) \nabla G_1 \cdot \nabla G_2 dx \leq \|\Lambda_1^\Gamma - \Lambda_2^\Gamma\| \|\tilde{G}_1\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))} \|\tilde{G}_2\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))}.$$

Here, for $i = 1, 2$, Λ_i^Γ is the restriction of Λ_i to the closed subspace $H^1_\Gamma(\mathbb{R}; H^{1/2}(\partial\omega))$. Therefore

$$(3.4) \quad \int_{\Omega_L} (A_1 - A_2) \nabla G_1 \cdot \nabla G_2 dx \leq C \|\Lambda_1^\Gamma - \Lambda_2^\Gamma\| \|G_1\|_{H^1(\Omega^L)} \|G_2\|_{H^1(\Omega^L)}.$$

Fix $\tilde{x} \in \partial\omega$ and let $x_3^0 \in [-L, L]$ such that $|\theta'_1(x_3^0) - \theta'_2(x_3^0)| = \|\theta'_1 - \theta'_2\|_{L^\infty(-L, L)}$. Without loss of generality, we assume that $|\theta'_1(x_3^0) - \theta'_2(x_3^0)| = \theta'_1(x_3^0) - \theta'_2(x_3^0)$. In view of (3.3), it is shown in [AG2] that the main term of the left hand side of inequality (3.4) is the following one

$$\int_{B(x_\tau, \rho) \cap \Omega} \frac{[A^{-1}(\tilde{x}', t_0) - A^{-1}(\tilde{x}', s_0)] (x - x_\tau) \cdot (x - x_\tau)}{[P_0(x - x_\tau) \cdot (x - x_\tau)]^{3/2} [Q_0(x - x_\tau) \cdot (x - x_\tau)]^{3/2}} dx,$$

Here $t_0 = \theta'_1(x_3^0)$, $s_0 = \theta'_2(x_3^0)$, $P_0 = A^{-1}(\tilde{z}'_\tau, t_0)$ and $Q_0 = A^{-1}(\tilde{z}'_\tau, s_0)$.

The main ingredient of the approach in [AG2] is an ellipticity condition for $\partial_t A(x', t)$. This condition allows the authors in [AG2] to establish an estimate of the the form

$$(3.5) \quad \int_{B(x_\tau, \rho) \cap \Omega} \frac{[A^{-1}(\tilde{x}', t_0) - A^{-1}(\tilde{x}', s_0)] (x - x_\tau) \cdot (x - x_\tau)}{[P_0(x - x_\tau) \cdot (x - x_\tau)]^{3/2} [Q_0(x - x_\tau) \cdot (x - x_\tau)]^{3/2}} dx \geq C \tau^{n-2} (t_0 - s_0).$$

This last estimate leads immediately to the desired estimate. Unfortunately, the ellipticity condition of $\partial_t A(x', t)$ is no longer valid in our case. Worst, $\partial_t A(x', t)$ has a negative eigenvalue. Precisely, the eigenvalues of $\partial_t A(x', t)$ are the following

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= |x'|^2 t - \sqrt{|x'|^4 t^2 + |x'|^2}, \\ \lambda_3 &= |x'|^2 t + \sqrt{|x'|^4 t^2 + |x'|^2}. \end{aligned}$$

In other words, even the weak monotonicity assumption is not satisfied. In conclusion, the approach in [AG2] cannot be adapted to our problem. Therefore, the determination of the twisting function from the corresponding DN map is still an open problem in the general case. In the next section, we consider the particular case of an affine twisting function. But for the moment, let us show that when the twisting function is close to a constant, the original inverse problem is close, in some sense, to an inverse problem for which we prove a stability estimate. Let

$$A^*(x', t) = t \begin{pmatrix} 1 + x_2^2 & -x_2 x_1 & -x_2 \\ -x_2 x_1 & 1 + x_1^2 & x_1 \\ -x_2 & x_1 & 1 \end{pmatrix}, \quad x' \in \omega, t \in \mathbb{R}.$$

Denote by Λ_θ^* the DN map when $A(x', \theta(x_3))$ is changed to $A^*(x', \theta(x_3))$. Then, similarly to (2.16), we prove

$$(3.6) \quad \langle (\Lambda_\theta - \Lambda_\theta^*) u, u^* \rangle = \int_{\Omega} (A(x', \theta(x_3)) - A^*(x', \theta(x_3))) \nabla u \cdot \nabla u^*,$$

for any u and $u^* \in H^1(\Omega)$ satisfying respectively, in the weak sense,

$$\begin{aligned} \operatorname{div}(A(x', \theta(x_3)) \nabla u) &= 0, \quad \text{in } \Omega \\ \operatorname{div}(A^*(x', \theta(x_3)) \nabla u^*) &= 0, \quad \text{in } \Omega \end{aligned}$$

Assume that $\|\theta' - 1\|_\infty \leq \frac{1}{2}^3$. With the help of identity (3.6), a straightforward computation gives

$$\|\Lambda_\theta - \Lambda_\theta^*\|_{\mathcal{L}(H^{-1}(\mathbb{R}, H^{-1/2}(\partial\omega)), H^1(\mathbb{R}, H^{1/2}(\partial\omega)))} \leq C \|\theta' - 1\|_\infty,$$

³Note that 1 can be replaced by any constant.

where $C = C(\omega)$ is a constant.

Therefore, if θ' is sufficiently close to 1, we may replace our original inverse problem by the one consisting in the determination of θ' from $\Lambda_{\theta'}^*$.

Now since $\partial_t A^*(x', t) = A^*(x, 1) = A(x, 1)$, the ellipticity condition required in [AG2] is satisfied. Proceeding as before, we are able to prove the following theorem, where $(\Lambda_{\theta'}^*)^\Gamma$ is defined similarly to $\Lambda_{\theta'}^\Gamma$.

Theorem 3.1. *Let $\theta_1, \theta_2 \in \mathcal{W}(\mathbb{R})$ be such that $\theta'_1 = \theta'_2$, for $|x_3| > L$, and*

$$\|\theta'_1\|_{W^{1,\infty}(\mathbb{R})}, \|\theta'_2\|_{W^{1,\infty}(\mathbb{R})} \leq M,$$

where M is a given constant. Then, we find a constant $C = C(M, \omega, L)$ such that

$$\|\theta'_1 - \theta'_2\|_{L^\infty(\mathbb{R})} \leq C \|(\Lambda_{\theta_1}^*)^\Gamma - (\Lambda_{\theta_2}^*)^\Gamma\|_{\mathcal{L}(H^{-1}(\mathbb{R}, H^{-1/2}(\partial\omega)), H^1(\mathbb{R}, H^{1/2}(\partial\omega)))}.$$

4. THE CASE OF AFFINE TWISTING FUNCTIONS

In this section we discuss the particular case of affine twisting functions. In the sequel, if w is a function of variables (x', x_3) , \widehat{w} will denote its Fourier transform with respect to the variable x_3 . \widehat{w} will be a function of variables (x', ξ) .

Lemma 4.1. *Let $w \in H^1(\Omega)$. Then $\widehat{\partial_{x_j} w} = \partial_{x_j} \widehat{w}$, $j = 1, 2$.*

Proof. Let $\varphi \in C_0^\infty(\omega)$ and $\psi \in \mathcal{S}(\mathbb{R})$. From Fubini's theorem, we have

$$\int_\omega \varphi(x') dx' \int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 = \int_{\mathbb{R}} \widehat{\psi}(x_3) dx_3 \int_\omega \partial_{x_j} w(x', x_3) \varphi(x') dx'.$$

But,

$$\int_\omega \partial_{x_j} w(x', x_3) \varphi(x') dx' = - \int_\omega w(x', x_3) \partial_{x_j} \varphi(x') dx' \text{ a.e. } x_3 \in \mathbb{R}.$$

Hence,

$$\int_\omega \varphi(x') dx' \int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 = - \int_{\mathbb{R}} \widehat{\psi}(x_3) dx_3 \int_\omega w(x', x_3) \partial_{x_j} \varphi(x') dx'.$$

Therefore,

$$\int_\omega \varphi(x') dx' \int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 = - \int_\omega \partial_{x_j} \varphi(x') dx' \int_{\mathbb{R}} w(x', x_3) \widehat{\psi}(x_3) dx_3.$$

As the Fourier transform is a self-adjoint operator, we have

$$\int_{\mathbb{R}} w(x', x_3) \widehat{\psi}(x_3) dx_3 = \int_{\mathbb{R}} \widehat{w}(x', \xi) \psi(\xi) d\xi \text{ a.e. } x' \in \omega$$

and then

$$\begin{aligned} \int_\omega \varphi(x') dx' \int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 &= - \int_\omega \partial_{x_j} \varphi(x') dx' \int_{\mathbb{R}} \widehat{w}(x', \xi) \psi(\xi) d\xi \\ &= \int_\omega \varphi(x') dx' \int_{\mathbb{R}} \partial_{x_j} \widehat{w}(x', \xi) \psi(\xi) d\xi. \end{aligned}$$

From this, we deduce

$$\int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 = \int_{\mathbb{R}} \partial_{x_j} \widehat{w}(x', \xi) \psi(\xi) d\xi \text{ a.e. } x' \in \omega.$$

Then, we easily conclude that $\widehat{\partial_{x_j} w} = \partial_{x_j} \widehat{w}$. □

Lemma 4.2. *Let $C = (C_{kl}(x')) \in W^{1,\infty}(\omega)^{3 \times 3}$, with $(C_{ij}(x'))$ symmetric for any $x' \in \omega$, and $w \in H^1(\Omega)$ satisfying*

$$(4.1) \quad \int_\Omega C \nabla w \cdot \nabla v = 0 \text{ for all } v \in H_0^1(\Omega).$$

Then

$$\operatorname{div}(\widetilde{C}(x') \nabla_{x'} \widehat{w}) + P(x', \xi) \cdot \nabla_{x'} \widehat{w} + q(x', \xi) \widehat{w} = 0 \text{ in } \mathcal{D}'(\Omega),$$

where,

$$\begin{aligned}\tilde{C}(x') &= (C_{ij}(x'))_{1 \leq i, j \leq 2} \\ P(x', \xi) &= -i2\xi \begin{pmatrix} C_{31}(x') \\ C_{32}(x') \end{pmatrix} \\ q(x, \xi) &= -i\xi \operatorname{div}_{x'} \begin{pmatrix} C_{31}(x') \\ C_{32}(x') \end{pmatrix} - \xi^2 C_{33}(x').\end{aligned}$$

If in addition $w \in H^2(\Omega)$, then

$$(4.2) \quad \operatorname{div}(\tilde{C}(x') \nabla_{x'} \widehat{w}(x', \xi)) + P(x', \xi) \cdot \nabla_{x'} \widehat{w}(x', \xi) + q(x', \xi) \widehat{w}(x', \xi) = 0 \quad \text{a.e. } (x', \xi) \in \Omega.$$

Proof. Take $v = \varphi \otimes \widehat{\psi}$ in (4.1), where $\varphi \in C_0^\infty(\omega)$ and $\psi \in \mathcal{S}(\mathbb{R})$, we obtain

$$(4.3) \quad \sum_{k,l=1,2,3} \int_{\Omega} C_{kl}(x') \partial_{x_k} u(x', x_3) \partial_{x_l} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = 0.$$

If $k \neq 3$ and $l \neq 3$, we have

$$\int_{\Omega} C_{kl}(x') \partial_{x_l} u(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\omega} C_{kl}(x') \partial_{x_k} \varphi(x') dx' \int_{\mathbb{R}} \partial_{x_l} w(x', x_3) \widehat{\psi} dx_3.$$

Applying Lemma 4.1, we obtain

$$(4.4) \quad \int_{\Omega} C_{kl}(x') \partial_{x_l} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\Omega} C_{kl}(x') \partial_{x_l} \widehat{w}(x', \xi) \partial_{x_k} (\varphi \otimes \psi)(x', \xi) dx' d\xi.$$

For $l \neq 3$,

$$\begin{aligned} \int_{\Omega} C_{3l}(x') \partial_{x_l} w(x', x_3) \partial_{x_3} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 &= \int_{\omega} C_{3j}(x') \varphi(x') dx' \int_{\mathbb{R}} \partial_{x_l} w(x', x_3) \widehat{\psi}'(x_3) dx_3 \\ &= \int_{\omega} C_{3l}(x') \varphi(x') dx' \int_{\mathbb{R}} \partial_{x_l} w(x', x_3) (-i\xi) \widehat{\psi}(x_3) dx_3. \end{aligned}$$

As before, we easily deduce

$$(4.5) \quad \int_{\Omega} C_{3l}(x') \partial_{x_l} w(x', x_3) \partial_{x_3} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\Omega} C_{3l}(x') (-i\xi) \partial_{x_l} \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi.$$

For $k \neq 3$

$$\int_{\Omega} C_{k3}(x') \partial_{x_3} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\omega} C_{k3}(x') \partial_{x_k} \varphi(x') dx' \int_{\mathbb{R}} \partial_{x_3} w(x', x_3) \widehat{\psi}(x_3) dx_3.$$

Or,

$$\int_{\mathbb{R}} \partial_{x_3} w(x', x_3) \widehat{\psi}(x_3) dx_3 = - \int_{\mathbb{R}} w(x', x_3) \widehat{\psi}'(x_3) dx_3.$$

Hence,

$$\int_{\Omega} C_{k3}(x') \partial_{x_3} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = - \int_{\omega} C_{k3}(x') \partial_{x_k} \varphi(x') dx' \int_{\mathbb{R}} w(x', x_3) \widehat{\psi}'(x_3) dx_3.$$

Therefore, similarly to (4.5), we prove

$$\int_{\Omega} C_{k3}(x') \partial_{x_3} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\Omega} C_{k3}(x') (i\xi) \widehat{w}(x', \xi) \partial_{x_k} (\varphi \otimes \psi)(x', \xi) dx' d\xi.$$

An integration by parts leads

$$(4.6) \quad \begin{aligned} \int_{\Omega} C_{k3}(x') \partial_{x_3} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 &= \int_{\Omega} C_{k3}(x') (-i\xi) \partial_{x_k} \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi \\ &+ \int_{\Omega} \partial_{x_k} C_{k3}(x') (-i\xi) \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi. \end{aligned}$$

Finally, we prove in a straightforward manner

$$(4.7) \quad \int_{\Omega} C_{33}(x') \partial_{x_3} w(x', x_3) \partial_{x_3} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\Omega} C_{33}(x') (-i\xi)^2 \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi.$$

Now a combination of (4.3) -(4.7) yields, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $C_0^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$,

$$\langle \operatorname{div}(\tilde{C}(x') \nabla_{x'} \widehat{w}) + P(x', \xi) \cdot \nabla_{x'} \widehat{w} + q(x', \xi) \widehat{w}, \Phi \rangle, \quad \Phi \in C_0^\infty(\omega) \otimes C_0^\infty(\mathbb{R}).$$

Therefore, since $C_0^\infty(\omega) \otimes C_0^\infty(\mathbb{R})$ is dense in $C_0^\infty(\Omega)$,

$$\operatorname{div}(\tilde{C}(x')\nabla_{x'}\hat{w}) + P(x', \xi) \cdot \nabla_{x'}\hat{w} + q(x', \xi)\hat{w} = 0 \quad \text{in } \mathcal{D}'(\Omega)$$

and the proof is complete. \square

Let us now assume that the twisting function θ is an affine function: $\theta(x_3) = ax_3 + b$, where a and b are two constants. We shall use the following new notation $A_a(x') = A(x', \theta(x_3))$ (note that A doesn't depend on b). Fix $g \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ satisfying $\int_{\mathbb{R}} g(x_3)dx_3 = 1$. Then $\hat{u} \in L^2(\mathbb{R}; H^1(\omega))$ satisfies

$$(4.8) \quad \begin{cases} \operatorname{div}_{x'}(\tilde{A}_a(x')\nabla_{x'}\hat{u}(x', \xi)) - 2ia\xi x'^\perp \cdot \nabla_{x'}\hat{u} - \xi^2\hat{u} = 0 & \text{in } \mathcal{D}'(\Omega). \\ \hat{u}(\cdot, \xi) = \hat{g}(\xi)f & \text{on } \partial\omega, \text{ for all } \xi \in \mathbb{R}. \end{cases}$$

Here, $x'^\perp = (-x_2, x_1)$ and

$$\tilde{A}_a(x') = \begin{pmatrix} 1 + x_2^2 a^2 & -x_2 x_1 a^2 \\ -x_2 x_1 a^2 & 1 + x_1^2 a^2 \end{pmatrix}.$$

We define on the Hilbert space $\mathcal{H} = H_0^1(\omega) \times H_0^1(\omega)$ the following bilinear form

$$\begin{aligned} \mathcal{A}_\xi[(v, w), (\varphi, \psi)] &= \int_{\omega} \tilde{A}_a \nabla v \cdot \nabla \varphi dx' - 2a\xi \int_{\omega} x'^\perp \cdot \nabla w \varphi dx' + \xi^2 \int_{\omega} v \varphi dx' \\ &\quad + \int_{\omega} \tilde{A}_a \nabla w \cdot \nabla \psi dx' + 2a\xi \int_{\omega} x'^\perp \cdot \nabla v \psi dx' + \xi^2 \int_{\omega} w \psi dx', \end{aligned}$$

$(v, w), (\varphi, \psi) \in \mathcal{H}$.

Since,

$$\tilde{A}_a(x')\zeta \cdot \zeta \geq |\zeta|^2, \quad \text{for all } \zeta \in \mathbb{R}^2 \text{ and } x' \in \omega$$

and

$$\begin{aligned} 2a|\xi| \int_{\omega} |x'^\perp \cdot \nabla w v| dx' &\leq a^2 \delta^2 \int_{\omega} |\nabla w|^2 + \xi^2 \int_{\omega} v^2 dx' \\ 2a|\xi| \int_{\omega} |x'^\perp \cdot \nabla v w| dx' &\leq a^2 \delta^2 \int_{\omega} |\nabla v|^2 + \xi^2 \int_{\omega} w^2 dx', \end{aligned}$$

we have

$$\mathcal{A}_\xi[(v, w), (v, w)] \geq (1 - a^2 \delta^2) \|(v, w)\|_{\mathcal{H}}^2.$$

Here,

$$\|(v, w)\|_{\mathcal{H}} = (\|\nabla v\|_{L^2(\omega)}^2 + \|\nabla w\|_{L^2(\omega)}^2)^{1/2}.$$

Let $a_0 > 0$ be fixed. Under the assumption $1 - a_0^2 \delta^2 > 0$, the bilinear form \mathcal{A}_ξ is elliptic for any $|a| < a_0$ and $\xi \in \mathbb{R}$. Precisely, we have

$$(4.9) \quad \mathcal{A}_\xi[(v, w), (v, w)] \geq \alpha \|(v, w)\|_{\mathcal{H}}^2 \quad \text{for all } |a| < a_0, \xi \in \mathbb{R},$$

for some constant $\alpha = \alpha(\omega, a_0)$.

Let $\Phi \in C(\mathbb{R}; \mathcal{H}')$. From Lax-Milgram's lemma, there exists a unique $(v(\xi), w(\xi)) \in \mathcal{H}$ satisfying

$$(4.10) \quad \mathcal{A}_\xi[(v, w), (\varphi, \psi)] = \langle \Phi(\xi), (\varphi, \psi) \rangle \quad \text{for all } (\varphi, \psi) \in \mathcal{H}.$$

Let $\xi, \eta \in \mathbb{R}$ with $\eta \leq 1$. We have

$$(4.11) \quad \begin{aligned} \mathcal{A}_{\xi+\eta}\xi[(v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi)), (\varphi, \psi)] \\ = \{ \mathcal{A}_\xi[(v(\xi), w(\xi)), (\varphi, \psi)] - \mathcal{A}_{\xi+\eta}[(v(\xi), w(\xi)), (\varphi, \psi)] \} \\ \langle \Phi(\xi+\eta) - \Phi(\xi), (\varphi, \psi) \rangle \quad \text{for all } (\varphi, \psi) \in \mathcal{H}. \end{aligned}$$

Using Poincaré's inequality, we obtain easily

$$(4.12) \quad \begin{aligned} \mathcal{A}_\xi[(v(\xi), w(\xi)), (v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi))] \\ - \mathcal{A}_{\xi+\eta}[(v(\xi), w(\xi)), (v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi))] \\ \leq C|\eta| \|(v(\xi), w(\xi))\|_{\mathcal{H}} \|(v(\xi), w(\xi)), (v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi))\|_{\mathcal{H}}, \end{aligned}$$

for some $C = C(\xi, \omega, a_0)$.

A combination of (4.9), (4.11) with $(\varphi, \psi) = (v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi))$ and (4.12) leads

$$\alpha \|(v(\xi+\eta) - v(\xi), w(\xi+\eta) - w(\xi))\|_{\mathcal{H}} \leq C|\eta| \|(v(\xi), w(\xi))\|_{\mathcal{H}} + \|\Phi(\xi+\eta) - \Phi(\xi)\|_{\mathcal{H}'}$$

From this, we deduce immediately that $(v, w) \in C(\mathbb{R}; \mathcal{H})$. Moreover, the following estimate holds

$$(4.13) \quad \|(v(\xi), w(\xi))\|_{\mathcal{H}} \leq (1/\alpha)\|\Phi(\xi)\|_{\mathcal{H}'}, \quad \xi \in \mathbb{R}.$$

In a classical way, one can prove that if $\Phi \in C^1(\mathbb{R}; \mathcal{H}')$ then $(v'(\xi), w'(\xi))$ exists in $C(\mathbb{R}, \mathcal{H})$ and its is the solution of the following variational problem

$$\mathcal{A}[(v'(\xi), w'(\xi)), (\varphi, \psi)] = \langle \Phi_0(\xi), (\varphi, \psi) \rangle + \langle \Phi'(\xi), (\varphi, \psi) \rangle \quad \text{for all } (\varphi, \psi) \in \mathcal{H},$$

where,

$$\langle \Phi_0(\xi), (\varphi, \psi) \rangle = 2a \int_{\omega} x'^{\perp} \cdot \nabla w(\xi) \varphi dx' - 2\xi \int_{\omega} v(\xi) \varphi dx' - 2a \int_{\omega} x'^{\perp} \cdot \nabla v(\xi) \psi dx' - 2\xi \int_{\omega} w(\xi) \psi dx'.$$

In view of (4.13), we easily obtain

$$\|(v'(\xi), w'(\xi))\|_{\mathcal{H}} \leq C(\langle \xi \rangle \|\Phi(\xi)\|_{\mathcal{H}'} + \|\Phi'(\xi)\|_{\mathcal{H}'}), \quad \xi \in \mathbb{R}.$$

Here $C = C(a_0, \omega)$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

In a similar manner, $\Phi \in C^2(\mathbb{R}; \mathcal{H}')$ implies $(v, w) \in C^2(\mathbb{R}, \mathcal{H})$ and

$$\|(v''(\xi), w''(\xi))\|_{\mathcal{H}} \leq C(\langle \xi \rangle^2 \|\Phi(\xi)\|_{\mathcal{H}'} + \langle \xi \rangle \|\Phi'(\xi)\|_{\mathcal{H}'} + \|\Phi''(\xi)\|_{\mathcal{H}'}), \quad \xi \in \mathbb{R}.$$

This estimate, together with a density argument yield

Proposition 4.1. *Let $\Phi \in H^2(\mathbb{R}, \mathcal{H}')$ such that $\langle \xi \rangle^2 \Phi, \langle \xi \rangle \Phi' \in L^2(\mathbb{R}, \mathcal{H}')$. Then the variational problem (4.10) has a unique solution in $(v, w) \in H^2(\mathbb{R}; \mathcal{H})$ and there exist a constant $C = C(\omega, a_0)$ such that*

$$(4.14) \quad \|(v, w)\|_{H^2(\mathbb{R}; \mathcal{H})} \leq C \left(\sum_{j=0}^2 \|\langle \xi \rangle^{2-j} \Phi^{(j)}\|_{L^2(\mathbb{R}; \mathcal{H}')}\right).$$

The above assumptions on Φ are satisfied whenever $\Phi = \widehat{\Psi}$, where $\Psi \in H^2(\mathbb{R}; \mathcal{H}')$ satisfies $x_3 \Psi \in H^1(\mathbb{R}; \mathcal{H}')$ and $x_3^2 \Psi \in L^2(\mathbb{R}; \mathcal{H}')$. In this case (4.14) becomes

$$(4.15) \quad \|(v, w)\|_{H^2(\mathbb{R}; \mathcal{H})} \leq C \left(\sum_{j=0}^2 \|x_3^j \Psi\|_{H^{2-j}(\mathbb{R}; \mathcal{H}')}\right).$$

Let us come back to (4.8). Let $F \in H^1(\Omega)$ such that $F = f$ on $\partial\omega$ and $\|F\|_{H^1(\omega)} \leq C(\omega)\|f\|_{H^{1/2}(\omega)}$. Let $\tilde{u} = \tilde{u}^r + i\tilde{u}^i = \widehat{u} - \widehat{g}(\xi)F$. Then straightforward computations shows that $(\tilde{u}^r, \tilde{u}^i)$ is the solution of the variational problem (4.10) when

$$\langle \Phi(\xi), (\varphi, \psi) \rangle = -\mathcal{A}_{\xi}[(\widehat{g}^r F, \widehat{g}^i F), (\varphi, \psi)].$$

Here \widehat{g}^r and \widehat{g}^i are respectively the real part and the imaginary part of \widehat{g} .

Elementary calculations show, where $C = C(a_0, \omega)$ is a constant,

$$\begin{aligned} \|\Phi(\xi)\|_{\mathcal{H}'} &\leq C\langle \xi \rangle^2 |\widehat{g}(\xi)| \|f\|_{H^{1/2}(\partial\omega)} \\ \|\Phi'(\xi)\|_{\mathcal{H}'} &\leq C(\langle \xi \rangle^2 |\widehat{g}'(\xi)| + \langle \xi \rangle |\widehat{g}(\xi)|) \|f\|_{H^{1/2}(\partial\omega)} \\ \|\Phi''(\xi)\|_{\mathcal{H}'} &\leq C(\langle \xi \rangle^2 |\widehat{g}''(\xi)| + \langle \xi \rangle |\widehat{g}'(\xi)| + |\widehat{g}(\xi)|) \|f\|_{H^{1/2}(\partial\omega)}. \end{aligned}$$

As an immediate consequence of Proposition 4.1, we have

Corollary 4.1. *Assume that $g \in H^4(\mathbb{R})$ is such that $x_3 g \in H^3(\mathbb{R})$, $x_3^2 g \in H^2(\mathbb{R})$ and $\int_{\mathbb{R}} g(x_3) dx_3 = 1$. Then $\widehat{u} \in H^2(\mathbb{R}; H^1(\omega))$. In particular, $u \in L^1(\mathbb{R}; H^1(\omega))$ and $U = \widehat{u}(\cdot, 0) = \int_{\mathbb{R}} u(\cdot, x_3) dx_3 \in H^1(\omega)$ is the variational solution of the following boundary value problem*

$$\begin{cases} \operatorname{div}(\widetilde{A}_a \nabla U) = 0 & \text{in } \omega \\ U = f & \text{on } \partial\omega. \end{cases}$$

From the estimates above, one can derive the followings ones.

$$\begin{aligned} \|\operatorname{div}_{x'}(\widetilde{A} \cdot \nabla_{x'} \widehat{u}(\cdot, \xi))\|_{L^2(\omega)} &\leq C\langle \xi \rangle^4 |\widehat{g}(\xi)| \|f\|_{H^{1/2}(\partial\omega)} \\ \|\partial_{\xi} \operatorname{div}_{x'}(\widetilde{A} \cdot \nabla_{x'} \widehat{u}(\cdot, \xi))\|_{L^2(\omega)} &\leq C(\langle \xi \rangle^5 |\widehat{g}'(\xi)| + \langle \xi \rangle^4 |\widehat{g}(\xi)|) \|f\|_{H^{1/2}(\partial\omega)} \\ \|\partial_{\xi}^2 \operatorname{div}_{x'}(\widetilde{A} \cdot \nabla_{x'} \widehat{u}(\cdot, \xi))\|_{L^2(\omega)} &\leq C(\langle \xi \rangle^6 |\widehat{g}(\xi)| + \langle \xi \rangle^5 |\widehat{g}'(\xi)| + \langle \xi \rangle^4 |\widehat{g}''(\xi)|) \|f\|_{H^{1/2}(\partial\omega)}. \end{aligned}$$

In view of these estimates, we can state the following proposition.

Proposition 4.2. *Let $g \in H^6(\mathbb{R})$ be given such that $x_3 g \in H^5(\mathbb{R})$, $x_3^2 g \in H^4(\mathbb{R})$ and $\int_{\mathbb{R}} g(x_3) dx_3 = 1$. We have $\tilde{A}_a \cdot \nabla_{x'} \hat{u} \cdot \nu(x') \in H^2(\mathbb{R}; H^{-1/2}(\partial\omega))$ implying $A_a \nabla u \cdot \nu(x) \in L^1(\mathbb{R}; H^{-1/2}(\partial\omega))$, and then*

$$\tilde{A}U \cdot \nu(x') = \tilde{A} \nabla \hat{u}(\cdot, 0) \cdot \nu(x') = \int_{\mathbb{R}} A_a \nabla u(\cdot, x_3) \cdot \nu(x) \in H^{-1/2}(\partial\omega).$$

Consider the following DN maps:

$$\begin{aligned} \Lambda_a : f &\in H^{1/2}(\partial\omega) \rightarrow A_a \nabla u(\cdot, x_3) \cdot \nu(x) \in L^1(\mathbb{R}; H^{-1/2}(\partial\omega)) \\ \tilde{\Lambda}_a : f &\in H^{1/2}(\partial\omega) \rightarrow \tilde{A}U \cdot \nu(x') \in H^{-1/2}(\partial\omega). \end{aligned}$$

From Proposition 4.2, these operators are bounded and

$$(4.16) \quad \|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\|_{\mathcal{L}(H^{1/2}(\partial\omega), H^{-1/2}(\partial\omega))} \leq \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(H^{1/2}(\partial\omega), L^1(\mathbb{R}; H^{-1/2}(\partial\omega)))},$$

where, $\Lambda_j = \Lambda_{a_j}$ and $\tilde{\Lambda}_j = \tilde{\Lambda}_{a_j}$, $j = 1, 2$.

By a straightforward computation, we see that the eigenvalues of $\partial_a \tilde{A}(x', a)$ are $\lambda_0 = 0$ and $\lambda_1 = |x'|^2$. An adaptation of the proof of the claim in page 169 of [AG1] yields

Theorem 4.1. *Let $a_0 > 0$ be given and assume that g is as in Proposition 4.2. Under the assumption $1 - a_0^2 \delta^2 > 0$, there exists a constant $C > 0$ such that for any*

$$\begin{aligned} |a_1|, |a_2| &\leq a_0, \\ |a_1 - a_2| &\leq C \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(H^{1/2}(\partial\omega), L^1(\mathbb{R}; H^{-1/2}(\partial\omega)))}. \end{aligned}$$

5. THE DN MAP FOR THE ORIGINAL PROBLEM

First of all, we start by defining the trace space for functions from $H^1(\Omega_\theta)$. If $L > 0$, we set

$$\Omega_\theta^L = \{(R_{\theta(x_3)} x', x_3); x' = (x_1, x_2) \in \omega, x_3 \in (-L, L)\}$$

and

$$\Gamma_\theta^L = \{(R_{\theta(x_3)} x', x_3); x' = (x_1, x_2) \in \partial\omega, x_3 \in [-L, L]\}.$$

Let $u \in H^1(\Omega_\theta)$. Since $u \in H^1(\Omega_\theta^L)$, $u|_{\partial\Omega_\theta^L} \in H^{1/2}(\partial\Omega_\theta^L)$ and therefore $u|_{\Gamma_\theta^L} \in H^{1/2}(\Gamma_\theta^L)$. Here $H^{1/2}(\Gamma_\theta^L)$ is defined as follows

$$H^{1/2}(\Gamma_\theta^L) = \{h = g|_{\Gamma_\theta^L} \text{ (in the } L^2 \text{ sense); } g \in H^{1/2}(\partial\Omega_\theta^L)\}.$$

This space is equipped with its natural (quotient) norm

$$\|h\|_{H^{1/2}(\Gamma_\theta^L)} = \inf\{\|g\|_{H^{1/2}(\partial\Omega_\theta^L)}; g|_{\Gamma_\theta^L} = h\}.$$

Set

$$H_{loc}^{1/2}(\partial\Omega_\theta) = \{h \in L_{loc}^2(\partial\Omega_\theta); h|_{\Gamma_\theta^L} \in H^{1/2}(\Gamma_\theta^L) \text{ for any } L > 0\}$$

and consider the following subspace of $H_{loc}^{1/2}(\partial\Omega_\theta)$:

$$\tilde{H}^{1/2}(\partial\Omega_\theta) = \{h \in H_{loc}^{1/2}(\partial\Omega_\theta); \text{ there exists } v \in H^1(\Omega_\theta) \text{ such that } v|_{\partial\Omega_\theta} = h\}.$$

Here and henceforth $v|_{\partial\Omega_\theta} = h$ means $v|_{\Gamma_\theta^L} = h|_{\Gamma_\theta^L}$ in the trace sense for any $L > 0$.

One can check that $\tilde{H}^{1/2}(\partial\Omega_\theta)$ is a Banach space for the quotient norm

$$\|h\|_{\tilde{H}^{1/2}(\partial\Omega_\theta)} = \inf\{\|v\|_{H^1(\Omega_\theta)}; v|_{\partial\Omega_\theta} = h\}.$$

In what follows, for simplicity, we use the following notations

$$\varphi_\theta(x) = T_{\theta(x_3)}(x', x_3) \text{ and } \psi_\theta = \varphi_\theta^{-1}.$$

Introduce the mapping

$$\begin{aligned} I_\theta : C_0^1(\partial\Omega_\theta) &\longrightarrow C_0^1(\partial\Omega) \\ g &\longrightarrow f = g \circ \varphi_\theta. \end{aligned}$$

Pick $g \in C_0^1(\partial\Omega_\theta)$, let $v \in C_0^1(\mathbb{R}^3)$ such that $v|_{\partial\Omega_\theta} = g$ and $u = v|_{\Omega_\theta} \circ \varphi_\theta$. Then

$$\|I_\theta g\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))} \leq C(\omega) \|u\|_{H^1(\Omega)} \leq C(\omega, \theta) \|v\|_{H^1(\Omega_\theta)},$$

Therefore

$$(5.1) \quad \|I_\theta g\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))} \leq C(\omega, \theta) \|g\|_{\tilde{H}^{1/2}(\partial\Omega_\theta)} \quad \text{for any } g \in C_0^1(\partial\Omega_\theta).$$

Let $g \in \tilde{H}^{1/2}(\partial\Omega_\theta)$ and $v \in H^1(\Omega_\theta)$ such that $v|_{\partial\Omega_\theta} = g$. Pick a sequence $(v_n) \in C_0^1(\mathbb{R}^3)$ such that $v_n|_{\Omega_\theta} \rightarrow v$ in $H^1(\Omega_\theta)$. If $g_n = v_n|_{\partial\Omega_\theta}$, then

$$\|g - g_n\|_{\tilde{H}^{1/2}(\partial\Omega_\theta)} \leq \|v - v_n\|_{H^1(\Omega_\theta)}.$$

Hence, g_n converges to g in $\tilde{H}^{1/2}(\partial\Omega_\theta)$.

Let us set $f_n = I_\theta g_n = g_n \circ \varphi_\theta$ and $u_n = v_n \circ \varphi_\theta$. Since $f_n = u_n|_{\partial\Omega}$, we have

$$\|f_n - f_m\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))} \leq C(\omega) \|u_n - u_m\|_{H^1(\Omega)} \leq C(\Omega, \theta) \|v_n - v_m\|_{H^1(\Omega_\theta)}.$$

Consequently, (f_n) is a Cauchy sequence in $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ and then $f = \lim_n f_n$ exists in $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$. We set $f = I_\theta g$. In view of (5.1), we deduce that I_θ can be extended to a bounded operator, still denoted by I_θ , from $\tilde{H}^{1/2}(\partial\Omega_\theta)$ into $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$.

Similarly, the mapping

$$\begin{aligned} J_\theta : C_0^1(\partial\Omega) &\longrightarrow C_0^1(\partial\Omega_\theta) \\ f &\longrightarrow g = f \circ \psi_\theta. \end{aligned}$$

can be extended to a bounded operator, still denoted by J_θ from $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ into $\tilde{H}^{1/2}(\partial\Omega_\theta)$.

Clearly $I_\theta J_\theta f = f$ for any $f \in C_0^1(\partial\Omega)$ and $J_\theta I_\theta g = g$ for any $g \in C_0^1(\partial\Omega_\theta)$. Thus, $J_\theta = I_\theta^{-1}$ by density.

In the sequel $\tilde{H}^{-1/2}(\partial\Omega_\theta)$ will denote the dual of $\tilde{H}^{1/2}(\partial\Omega_\theta)$. As we have done for (1.2), we prove, with the help of the Lax-Milgram's lemma, that the boundary value problem (1.1) has a unique solution $v \in H^1(\Omega_\theta)$ if $g \in \tilde{H}^{1/2}(\partial\Omega_\theta)$ and the operator $\tilde{\Lambda}_\theta$ is well defined as bounded operator from $\tilde{H}^{1/2}(\partial\Omega_\theta)$ into $\tilde{H}^{-1/2}(\partial\Omega_\theta)$. As $\Lambda_\theta, \tilde{\Lambda}_\theta$ is characterized by the formula

$$\langle \tilde{\Lambda}_\theta g, h \rangle = \int_{\Omega_\theta} \nabla v \cdot \nabla H dy$$

for any $h \in \tilde{H}^{1/2}(\partial\Omega_\theta)$ and $H \in H^1(\Omega_\theta)$ such that $H|_{\partial\Omega_\theta} = h$. Making the change of variable $y = \varphi_\theta(x)$ in the last integral, we obtain

$$\langle \tilde{\Lambda}_\theta g, h \rangle = \int_{\Omega} A \nabla u \cdot \nabla (H \circ \varphi_\theta) dx$$

where u is the solution of the boundary value problem (1.2) with $f = I_\theta g$. That is, we have

$$\langle \tilde{\Lambda}_\theta g, h \rangle = \langle \Lambda_\theta I_\theta g, I_\theta h \rangle.$$

Then $\tilde{\Lambda}_\theta = I_\theta^* \Lambda_\theta I_\theta$ or equivalently $\Lambda_\theta = J_\theta^* \tilde{\Lambda}_\theta J_\theta$.

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