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Optimal portfolio liquidation with additional information

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Abstract

We consider the problem of how to optimally close a large asset position in a market with a linear temporary price impact. We take the perspective of an agent with a market opinion that translates into a (linear) drift in asset price dynamics. By appealing to classical stochastic control we derive explicit formulas for the closing strategy that minimizes a sum of execution costs and a quadratic risk functional. We then proceed by comparing agents observing a signal about the asset's future price with agents who do not see the signal. We compute explicitly the expected additional gain due to the signal, and perform a comparative statics analysis.

Introduction

For many companies it is part of day-to-day business to build up and close large asset positions on financial markets. For example, whenever a fund modifies its investment strategy, it will reduce the position of some of its assets, while

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enlarging the holdings of other ones. Energy companies have to unwind long positions of power and buy the commodities they need for the power generation.

Selling or buying a large amount of an asset in short time usually entails a price impact. This is why in practice financial institutions, from now on referred to as agents, frequently unwind large positions by splitting them into smaller parts and closing them successively. Spreading orders over time implies the price impact to be smaller. It entails, however, also price risk. Any liquidation, therefore, involves a trade-off between liquidation costs and risk.

Agents closing a large asset position are often guided by directional views. They may have a particular opinion about the current trend of the asset's price. Such an opinion can be the reason for deciding to close the position in the first place. Directional views are often based on assessments of market analysts. Trading houses have teams of analysts constantly observing markets. Analysts provide market assessments or even forecasts that are incorporated in the company's trading decisions. For example, power companies try to estimate, by performing computer-based optimizations, the marginal costs for generating the electricity that will be demanded in future years. By comparing the estimated marginal costs with actual forward market prices, they derive an opinion about the direction power prices will take. The directional opinion guides them in selling power on forward markets.

The main aim of the paper is to study the value of a market assessment *before* it is revealed. To put it differently, we look at the *expected additional value* of market expertise. To this end we introduce into our model an expert who obtains a signal about the asset's price at time T . If the expert passes the knowledge on to the agent having to close the position, then we say that the agent is *informed*; else she is *non-informed*. We use the technique of *filtration enlargements* for modeling the information flow of the informed agent.

We aim at comparing the optimal execution strategy of a non-informed agent with the one of an informed agent. We assume that the price signal is the asset price at T disturbed by an independent centered Gaussian noise. We suppose that any transaction has a linear absolute temporary (abbreviated by LAT in the pioneering paper [7]) impact on the asset's price. The fundamental (i.e. non-influenced) price process is assumed to be a Brownian motion, complemented by a *drift*. We suppose that the agent aims at maximizing the expected proceeds (resp. minimizing expected costs) from closing a position, while keeping a quadratic risk functional low.

We show that this problem is linked to to a problem of optimal closure with *linear drift*. We characterize optimal execution strategies by appealing to classical stochastic control theory. Profiting from the linear-quadratic model set-up we obtain explicit formulas for the value function and the optimal control.

With the optimal position process given in closed form, we prove an explicit formula for the additional expected gain from the signal. Besides, we perform a comparative statics analysis of the additional gain and show how market frictions limit the value of additional information.

We consider also the case where the signal reveals the asset's *exact* fundamental price at T , i.e. where the signal is not distorted by noise. We show that also in this case the additional gain is finite. The market would admit arbitrage if there were no market frictions. The price impact entailed by any trading implies that the gain from exactly knowing the fundamental price at a future date is only finite.

The value of a price signal has so far been studied mainly within utility maximization models. In [13] the authors calculate, also by employing filtration enlargements, the expected additional *logarithmic utility* of an investor possessing inside information. They do *not* consider market frictions and hence obtain that the additional utility is infinite if the exact asset's price at T is known to the investor. The model of [13] has been put forward in many succeeding paper, e.g. in [2], [4], [5].

Finally, if we interpret the linear *drift* as a directional view, we can directly study its dependence on the agent's directional view. We observe, for example, a competitive interplay between risk aversion and directional views: the influence of any opinion diminishes as risk aversion increases.

So far the liquidation literature has only briefly analyzed the impact market opinions on trading strategies. Almgren & Chriss [1] calculate optimal *deterministic* liquidation strategies, allowing for directional views. They assume that the agent's objective is to minimize a weighted sum of the mean and the variance of the proceeds. It is remarkable that the optimal strategy from [1] maximizes CARA utility - not only among all deterministic, but even among all *predictable* trajectories. This is shown in [16] for a time continuous version of the Almgren & Chriss model.

The paper [15] also studies the influence of price drifts on optimal liquidation strategies. The agent's objective functional, however, slightly differs from ours. Moreover, a more general semimartingale perspective is taken, leading to a more abstract representation of liquidation strategies in terms of conditional expectations. Our first result, Theorem 2.1, can be derived from the results in [15]. Nevertheless, we present a simple, direct and self-contained proof, based on classical verifications arguments (in contrast to variational arguments used in [15]). We describe the liquidation paths explicitly, which allows us, in a subsequent step, to perform a simple comparative statics analysis.

The paper is organized as follows : Section 1 is devoted to the presentation of the model, Section 2 studies the case of a linear price drift and obtains a closed form expression of the optimal position for the liquidation problem. Besides, we compute the value function for a non-informed or a informed agent. In Section 3, we aim at estimating the value of additional information from a risk-neutral agent's perspective before the information is revealed.

1 Informed and non-informed agents

Consider an agent who has to unwind a position of $X_0 \in \mathbb{R}$ shares of an asset until a time horizon $T > 0$. We assume that the non-influenced asset price S has dynamics as in the Bachelier model, i.e. it satisfies $S_t = S_0 + \sigma W_t$, where $\sigma > 0$ is a constant volatility and W a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that S is a *forward price* so that no discounting is needed. Note that S is a martingale with respect to the filtration (\mathcal{F}_t^W) generated by W [satisfying the usual conditions](#).

Suppose that there is an expert, e.g. a market analyst or an insider, who has obtained a signal about the asset price at time T . We model the signal as a random variable $G = S_T + N$, where N is independent of the price process and normally distributed with mean zero. Since G is Gaussian, it is equivalent to

the signal sent from a price $S_{T'}$, where $T' \geq T$. One can interpret the difference $T' - T$ as the variance of the signal's noise.

If the expert discloses the signal to the agent, then we say that the agent is *informed*. In this case the agent's information flow can be modeled as the following initial enlargement of the Brownian filtration:

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \sigma(S_{T'}), \quad 0 \leq t \leq T.$$

In case the expert does not pass on the signal, we say that the agent is non-informed. The information flow of the agent is then represented by the natural filtration (\mathcal{F}_t^W) .

Let (\mathcal{H}_t) be either the small filtration (\mathcal{F}_t^W) or the enlarged filtration (\mathcal{G}_t) . A *closing strategy* (or simply strategy) with respect to (\mathcal{H}_t) of a position $x \in \mathbb{R}$ at time $t \in [0, T)$ is a (\mathcal{H}_t) -predictable strategy $\xi = (\xi_u)$ satisfying $\int_t^T \xi_u du = x$. We interpret ξ_t as the *selling rate* at time $t \in [0, T]$. Given ξ , the total position at time $t \in [0, T]$ is given by

$$X_t = X_0 - \int_0^t \xi_s ds.$$

Notice that $X_T = 0$, i.e. the position is closed at T .

For technical reasons we impose the following integrability condition on the closing strategies: A strategy (ξ_u) , resp. its associated position process (X_t) , is called *admissible* if

(A1) the process ξ is L^2 -integrable, i.e. $E(\int_0^T \xi_u^2 du) < \infty$,

(A2) the family $\left(\left(\frac{X_t^*}{T-t} \right)^2 \right)_{0 \leq t \leq T}$ is uniformly integrable and $\lim_{t \rightarrow T} \frac{X_t^2}{T-t} = 0$,
a.s.

We denote by $\mathcal{A}^{\mathcal{H}}(t, x)$ the set of all admissible closing strategies of x at t .

We suppose that any transaction entails a price impact that is linear with respect to the selling rate. Moreover, the impact is assumed to be absolute and only instantaneous. Selling at a rate of ξ_t is thus possible only at the *realized price* of

$$\tilde{S}_t = S_t - \eta \xi_t,$$

where $\eta > 0$ is the price impact parameter.

The final revenues (possibly negative) of the liquidation operation when selling at a rate $(\xi_t)_{t \in [0, T]}$ are given by

$$R_T = \int_0^T \xi_u \tilde{S}_u du.$$

Notice that by the product formula we have

$$\begin{aligned} R_T &= \int_0^T \xi_u S_u du - \eta \int_0^T \xi_u^2 du \\ &= X_0 S_0 + \int_0^T X_u a(u, S_u) du + \int_0^T X_u \sigma dW_u - \eta \int_0^T \xi_u^2 du. \end{aligned} \quad (1)$$

Assumption (A2) guarantees that a position process X , associated to an admissible strategy ξ , is square integrable, and thus taking expectations in (1) we get

$$E(R_T) = X_0 S_0 + E \int_0^T (X_u a(u, S_u) - \eta \xi_u^2) du.$$

We assume that the agent aims at maximizing the sum of the expected value of the final revenues. For a closing strategy ξ , we define the target function by

$$J(t, x, s, \xi) = E \left[\int_t^T (X_u a(u, S_u) - \eta \xi_u^2) du \middle| X_t = x, S_t = s \right],$$

with $(t, x, s) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$. The value function, or simply the expected execution costs, of the informed agent is defined by

$$V^I(t, x, s) = \sup_{\xi \in \mathcal{A}^{\mathcal{G}}(t, x)} E \left[\int_t^T (X_u a(u, S_u) - \eta \xi_u^2) du \middle| X_t = x, S_t = s \right], \quad (2)$$

for $(t, x, s) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$. The value function of the *non-informed* agent is given by

$$V^N(t, x, s) = \sup_{\xi \in \mathcal{A}^{\mathcal{F}}(t, x)} E \left[\int_t^T (X_u a(u, S_u) - \eta \xi_u^2) du \middle| X_t = x, S_t = s \right]. \quad (3)$$

Remark 1.1. If the impact of the liquidation operation on the price dynamics is not only instantaneous, but lasts in the considered period, one can add to the model a so-called perpetual impact factor, depending on the total amount of the position closed up to time t . The form of the realized price dynamics is then :

$$\tilde{S}_t = S_t - \eta_t - c(X_0 - X_t),$$

where c is the permanent impact factor.

The final revenues in this case are given by

$$R_T = X_0 S_0 + \frac{1}{2} c X_0^2 + \int_0^T X_u a(u, S_u) du + \int_0^T X_u \sigma dW_u - \eta \int_0^T \xi_u^2 du. \quad (4)$$

The only difference to Equation (1) is the constant term $\frac{1}{2} c X_0^2$, and thus the optimization is not changed. The problem is identical.

2 Valuing the additional information

2.1 A priori and a posteriori signal value

We aim at estimating the value of additional information from the agent's perspective *before* the information is revealed. We first calculate the a posteriori value of the signal. The a priori value coincides with the expected a posteriori value and finally the problem can be reduced to a liquidation problem with a

linear drift.

The price dynamics under (\mathcal{G}_t) satisfies

$$dS_t = \sigma dW_t^{\mathcal{G}} + \sigma \frac{S_{T'} - S_t}{T' - t} dt, \quad (5)$$

where $W^{\mathcal{G}}$ is a Brownian motion with respect to (\mathcal{G}_t) (see e.g. [14]). In other words, conditionally to the signal $S_{T'}$, the agent perceives the price process as a Brownian motion with a linear drift

$$dS_t = \sigma dW_t^{\mathcal{G}} + (\alpha(t) + \beta(t)S_t)dt, \quad (6)$$

where $\alpha(t) = \sigma \frac{S_{T'}}{T' - t}$ and $\beta(t) = -\sigma \frac{1}{T' - t}$. The a posteriori value of the signal can be calculated by determining the optimal strategy for a price with dynamics (6).

Then, the optimization problem (2) is transformed into a problem of optimal closure for linear drifts. As in Shied [?], the optimal position can be determined in closed-form. The value function turns out to be a quadratic form of the position size and the price. Besides, we apply this result to compute the value function for the non-informed trader and the informed trader. This result can be derived from Shied's main result in [?], nevertheless we provide a simpler, direct and selfcontained proof in Annex.

The expected execution costs of a non-informed agent or an informed agent, and the coefficients of the value function of an informed trader, highlighted with the superscript I , are given as follows :

Theorem 2.1. *Let $T' > T$.*

The minimal expected execution costs for a non-informed agent are given by

$$V^N(0, x, s) = -\eta \frac{x^2}{T}.$$

The coefficients α and β are bounded and the minimal expected execution costs for an informed agent are given by

$$V^I(t, x, s) = b^I(t)x^2 + c^I(t)xs + d^I(t)s^2 + e^I(t)x + f^I(t)s + g^I(t), \quad (7)$$

The coefficients of the value function for the informed agent are given by

$$\begin{aligned} b^I(t) &= -\eta \frac{1}{T-t} \\ c^I(t) &= -\frac{1}{2} \frac{T-t}{T'-t} \\ d^I(t) &= \frac{1}{48\eta} \frac{(T-t)^3}{(T'-t)^2} \\ e^I(t) &= \frac{1}{2} \frac{T-t}{T'-t} S_{T'} \\ f^I(t) &= \frac{1}{\eta} \frac{S_{T'}}{T'-t} \left(\frac{1}{8}(T'-T)(T-t) - \frac{1}{24}(T'-t)^2 + \frac{1}{24} \frac{(T'-T)^3}{T'-t} \right) \\ g^I(t) &= \frac{S_{T'}^2}{\eta} \left(\frac{1}{12}(T'-T) - \frac{1}{16} \frac{(T'-T)^2}{T'-t} - \frac{1}{8}(T'-T) \frac{T-t}{T'-t} - \frac{1}{24}(T-t) - \frac{1}{48} \frac{(T'-T)^3}{(T'-t)^2} \right) \\ &\quad + \frac{\sigma^2}{48\eta} \left(\frac{(T'-T)^3}{T'-t} - \frac{3}{2}(T'-T)^2 + 3(T'-T)^2 \ln \left(\frac{T'-t}{T'-T} \right) \right) \\ &\quad + \frac{\sigma^2}{48\eta} \left(-3(T'-T)(T-t) + \frac{1}{2}(T'-t)^2 \right). \end{aligned}$$

Proof. The result follows from Theorem (A1) given in Annex. \square

2.2 The value of the additional information

The value of the additional information can be calculated via several methods. The most intuitive is to compute the expected additional gain of the informed agent. This is the purpose of the following theorem.

Theorem 2.2. *Let $T' > T$. The expected additional gain of the informed agent is given by*

$$E[V^I(0, s, x) - V^N(0, s, x)] = \frac{\sigma^2}{16\eta} \left((T' - T)^2 \ln \left(\frac{T'}{T' - T} \right) - TT' + \frac{3}{2}T^2 \right). \quad (8)$$

Remark 2.3. Notice that the expected additional gain does *not* depend on the price level s and the initial position x , but only on T and T' . In the following we denote the expected additional gain by $\Delta(T, T') = E[V^I(0, s, x) - V^N(0, s, x)]$. Sometimes we write $\Delta(T, T', \sigma, \eta)$ in order to stress its dependence on σ and η .

The following Lemma will be necessary to prove Theorem 2.2.

Lemma 2.4.

$$E \left[\left(\int_0^t \frac{W_T - W_u}{T - u} du \right)^2 \right] = 2t(1 + \ln(T)) - 2T \ln(T) + 2(T - t) \ln(T - t) \quad (9)$$

Proof. The product formula applied to the (\mathcal{G}_t) -semimartingales $X_t = \ln(T - t)$ and $Y_t = W_T - W_t$, $t \in [0, T]$, implies

$$- \int_0^t \frac{W_T - W_u}{T - u} du = \ln(T - t)(W_T - W_t) - \ln(T)W_T - \int_0^t \ln(T - u) dW_u, \quad (10)$$

and hence

$$\begin{aligned} E \left[\left(\int_0^t \frac{W_T - W_u}{T - u} du \right)^2 \right] &= \ln^2(T - t)(T - t) + \ln^2(T)T^2 + \int_0^t \ln^2(T - u) du \\ &\quad - 2 \ln(T - t) \ln(T)(T - t) - 2 \ln(T) \int_0^t \ln(T - u) du. \end{aligned}$$

A straightforward simplification of the integrals leads to Equation (9). \square

Proof of Theorem 2.2. By Theorem ?? the optimal strategy of the informed agent satisfies

$$\xi_t^* = \frac{X_t^*}{T - t} - \frac{1}{4\eta} \frac{T - t}{T' - t} (S_{T'} - S_t),$$

and the optimal position trajectory is given by

$$X_t^* = \frac{T - t}{T} \left(x + \frac{1}{4\eta} T \int_0^t \frac{S_{T'} - S_u}{T' - u} du \right).$$

The martingale property of the price process implies that the value function satisfies

$$\begin{aligned} & V^I(0, x, s) \\ &= -\eta \frac{x^2}{T} + E \int_0^T \left[\frac{3}{2} X_t^* \frac{S_{T'} - S_t}{T' - t} - \frac{1}{16\eta} \left(\int_0^t \frac{S_{T'} - S_u}{T' - u} du \right)^2 - \frac{1}{16\eta} \frac{(T-t)^2}{(T'-t)^2} (S_{T'} - S_t)^2 \right] dt \end{aligned} \quad (11)$$

Observe that

$$\begin{aligned} E(X_t^* (S_{T'} - S_t)) &= (T-t) \frac{1}{4\eta} E \left[(S_{T'} - S_t) \int_0^t \frac{S_{T'} - S_u}{T' - u} du \right] \\ &= (T-t) \frac{1}{4\eta} \int_0^t \frac{E[(S_{T'} - S_t) ((S_{T'} - S_t) + (S_t - S_u))]}{T' - u} du \\ &= \frac{\sigma^2}{4\eta} (T-t)(T'-t) \ln\left(\frac{T'}{T'-t}\right). \end{aligned} \quad (12)$$

Moreover, by Lemma 2.4,

$$E \left[\left(\int_0^t \frac{S_{T'} - S_u}{T' - u} du \right)^2 \right] = \sigma^2 [2t(1 + \ln(T')) - 2T' \ln(T') + 2(T' - t) \ln(T' - t)]. \quad (13)$$

Combining Equation (12) and (13) with (11) yields

$$\begin{aligned} E[V^I(0, x, s)] &= -\eta \frac{x^2}{T} + \frac{3\sigma^2}{8\eta} \int_0^T (T-t) \ln\left(\frac{T'}{T'-t}\right) dt \\ &\quad - \frac{\sigma^2}{8\eta} \int_0^T [t(1 + \ln(T')) - T' \ln(T') + (T' - t) \ln(T' - t)] dt \\ &\quad - \frac{\sigma^2}{16\eta} \int_0^T \frac{(T-t)^2}{(T'-t)} dt. \end{aligned} \quad (14)$$

Notice that

$$\int_0^T (T-t) \ln(T'-t) dt = \frac{1}{2} (T'-T)^2 \ln(T'-T) + TT' \ln(T') - \frac{1}{2} (T')^2 \ln(T') + \frac{1}{2} TT' - \frac{3}{4} T^2$$

and

$$\int_0^T (T'-t) \ln(T'-t) dt = -\frac{1}{2} (T'-T)^2 \ln(T'-T) + \frac{1}{4} (T'-T)^2 + \frac{1}{2} (T')^2 \ln(T') - \frac{1}{4} (T')^2$$

and

$$\int_0^T \frac{(T-t)^2}{(T'-t)} dt = (T'-T)^2 \ln\left(\frac{T'}{T'-T}\right) - 2(T'-T)T + TT' - \frac{1}{2} T^2.$$

A straightforward calculation shows that (14) simplifies to

$$E[V^I(0, x, s)] = -\eta \frac{x^2}{T} + \frac{\sigma^2}{16} \left((T'-T)^2 \ln\left(\frac{T'}{T'-T}\right) - TT' + \frac{3}{2} T^2 \right).$$

□

Remark 2.5. One can alternatively calculate the additional utility by computing the expectation of the coefficients $e^I(0)$, $f^I(0)$ and $g^I(0)$. By simplifying terms one arrives again at formula (8).

The expected additional gain $\Delta(T, T')$ converges to a finite value as $T' \downarrow T$. If $T = T'$, then the market would admit arbitrage if there was no price impact. It has been shown that an informed investor can achieve infinite expected utility in a frictionless market (see e.g. [13] and [10]). In our model, in contrast, the price impact excludes arbitrage and implies that the expected additional gain doesn't become infinite when T' is equal to T .

Notice that if we choose $T' = T$, then the drift in the (\mathcal{G}_t) -price dynamics (5) is not bounded, and hence the assumptions of Theorem 3.4 are technically not satisfied. Nevertheless, one can show that the result applies also to this particular case. To this end one needs to make sure that the candidate for the optimal control is admissible.

Proposition 2.6. *Suppose that $T' = T$. Then the expected additional gain of the informed agent is given by*

$$E[V^I(0, s, x) - V^N(0, s, x)] = \frac{\sigma^2}{32\eta} T^2. \quad (15)$$

The optimal strategy is admissible and the associated position process satisfies $X_t^* = \frac{T-t}{T} \left(x + \frac{1}{4\eta} T \int_0^t \frac{S_T - S_u}{T-u} du \right)$.

Proof. The first expression is obtained taking the limit in $\Delta(T, T')$ as $T' \downarrow T$. To prove the admissibility, notice first that using Equation (10) for any $p > 2$, there exists C_p such that

$$E \left(\left| \int_0^t \frac{S_T - S_u}{T-u} \right|^p \right) \leq C_p \left\{ \ln^p(T-t)(T-t)^{p/2} + \ln^p(T)T^{p/2} + \left(\int_0^T \ln^2(T-u) du \right)^{p/2} \right\}$$

This shows that $\left(\int_0^t \frac{S_T - S_u}{T-u} du \right)^2$ is uniformly integrable. We further obtain that $\left(\frac{X_t^*}{T-t} \right)^2$, $0 \leq t \leq T$, is uniformly integrable and $\lim_{t \rightarrow T} \frac{X_t^{*2}}{T-t} = 0$, a.s. Moreover the process ξ^* is squared integrable. □

3 Comparative Statics

We next analyze the impact of the model parameters on the additional gain. We start with the dependence on the signal quality.

3.1 Sensitivity with respect to the signal noise

If the noise of the signal increases, then the additional revenues of the informed agent decrease. This is indeed confirmed by the next result.

Lemma 3.1. *The expected revenues from additional information decrease as T' increases, i.e. the mapping $f(x) = (x - T)^2 \ln \left(\frac{x}{x-T} \right) - Tx + \frac{3}{2}T^2$ is decreasing on $[T, \infty)$. Moreover, $f(T) = \frac{1}{2}T^2$ and $\lim_{x \rightarrow \infty} f(x) = 0$.*

Proof. Notice that $f'(x) = 2(x - T) \ln\left(\frac{x}{x-T}\right) + \frac{T^2}{x} - 2T$ and

$$f''(x) = -2 \ln\left(1 - \frac{T}{x}\right) - \frac{2T}{x} - \frac{T^2}{x^2}.$$

Since the logarithm is analytic on the open interval $(0, 2)$, we further have for $x > T$

$$f''(x) = 2 \left(\frac{T}{x} + \frac{1}{2} \frac{T^2}{x^2} + \frac{1}{3} \frac{T^3}{x^3} + \dots \right) - \frac{2T}{x} - \frac{T^2}{x^2} = \left(\frac{1}{3} \frac{T^3}{x^3} + \dots \right) \geq 0.$$

Consequently f' is increasing on $[T, \infty)$. Besides observe that $f'(T) = -T$, and

$$\lim_{x \rightarrow \infty} f'(x) = 2(x - T) \left(\frac{T}{x} + \frac{1}{2} \frac{T^2}{x^2} + \frac{1}{3} \frac{T^3}{x^3} + \dots \right) - \frac{2T}{x} - \frac{T^2}{x^2} = 0,$$

which, together with the monotonicity of f' , implies $f' \leq 0$ on $[T, \infty)$. The function f , therefore, is decreasing in x . \square

3.2 Sensitivity with respect to the time horizon

The additional gain increases when the time horizon T increases, while T' stays constant. Indeed, a straightforward computation shows that

$\frac{\partial^2 \Delta}{\partial T^2} = 2 \ln\left(\frac{T'}{T'-T}\right) \geq 0$ for all $T \in [0, T']$. Hence the first derivative is increasing. Since $\frac{\partial \Delta}{\partial T}(0, T') = 0$, the first derivative is non-negative and hence Δ is increasing in T .

The increase in expected revenues has three reasons: first the signal becomes more valuable as the difference between T and T' decreases (**information effect**); second there is more time for spreading orders over time and hence one can reduce trading costs (**liquidity effect**); finally the variance of the price over the trading period increases (**variance effect**).

We next aim at analyzing the three effects separately. The additional revenues depend linearly on the volatility squared. We can thus eliminate the variance effect by making σ^2 inversely proportional to T . We define the variance corrected gain by

$$l(T, x) = \Delta(T, x, \sigma/\sqrt{T}, \eta) = \frac{\sigma^2}{16\eta} \left(\frac{(x-T)^2}{T} \ln\left(\frac{x}{x-T}\right) - x + \frac{3}{2}T \right),$$

for $0 \leq T \leq x$.

We next aim at analyzing the part of revenue increase that goes back to the liquidity effect. To this end we simultaneously change T and T' such that the information content of the signal remains the same. We appeal to the notion of *mutual information* for measuring the information content of the signal.

Recall that the mutual information between two normally distributed random variables X and Y is given by $I(X, Y) = -\frac{1}{2} \ln(1 - \text{corr}^2(X, Y))$ (see e.g. [12]). In particular, for any $\delta > 0$ we have $I(S_T, S_{T+\delta}) = \frac{1}{2} \ln\left(\frac{T+\delta}{T}\right)$.

For $\gamma > 0$ the mutual information $I(S_T, S_{(\gamma+1)T}) = \frac{1}{2} \ln(1 + \gamma)$ does not depend on the time horizon T . We can thus interpret

$$h(T) = l(T, (1 + \gamma)T)$$

as a variance *and* information (v&i) corrected gain function. The next proposition shows that the v&i corrected gain increases as the time horizon increases. The reason is that the additional time for trading allows to reduce liquidity costs and to make more use of the information advantage.

Proposition 3.2 (The Liquidity effect). *Let $\gamma > 0$. The v&i corrected gain function h is linear, increasing and satisfies $h(0) = 0$.*

Proof. Note that

$$\begin{aligned} \frac{16\eta}{\sigma^2}h(T) &= \gamma^2 T \ln\left(\frac{1+\gamma}{\gamma}\right) - \gamma T + \frac{1}{2}T \\ &= \left[\gamma^2\left(\frac{1}{\gamma} - \frac{1}{2}\frac{1}{\gamma^2} + \frac{1}{3}\frac{1}{\gamma^3} - \frac{1}{4}\frac{1}{\gamma^4} + \dots\right) - \gamma + \frac{1}{2}\right]T \\ &= \gamma^2\left(\frac{1}{3}\frac{1}{\gamma^3} - \frac{1}{4}\frac{1}{\gamma^4} + \dots\right)T, \end{aligned}$$

which shows that h is non-negative and linearly increasing in T . □

Finally we turn to the information effect. By scaling the volatility with $1/\sqrt{T}$ and the price impact parameter with T , we obtain a variance and liquidity (v&l) corrected gain function

$$k(y) = \Delta(y, T', \sigma/\sqrt{y}, \eta y),$$

defined for all $y \in [0, T']$. The function k describes the gain that exclusively goes back to the additional information, as T approaches T' . It is, as expected, increasing:

Proposition 3.3 (The Information effect). *The v&l corrected gain function k increases superlinearly on $[0, T']$ and satisfies $\lim_{x \uparrow T'} k(x) = \frac{1}{2} \frac{\sigma^2}{16\eta}$.*

Proof. $k''(x) = 2T' \frac{3T'-2x}{x^4} \ln\left(\frac{T'}{T'-x}\right) - \frac{6T'}{x^3} + \frac{1}{x^2}$ is positive and $k'(0) \leq 0$, which implies the first statement. The second is straightforward to show. □

Conclusion

The paper studies the optimal liquidation problem under directional views and additional information. The kind of additional information chosen here is modeled via an initial enlargement of filtration, sometimes referred to as *strong initial information* (see for example [3], [10], or [9] for an introduction into the subject. See also [11] and [6] for a presentation of other types of additional information). In order to obtain the optimal liquidation strategy in closed form, the price dynamics under the enlargement must have a drift that is linear with respect to the price. For any kind of additional information resp. filtration enlargement under which the drift is linear, one can derive explicitly the additional gain by using Theorem ???. For example the additional information studied in the paper of Corcuera et al. [8] (strong noisy information, represented by a signal plus a decreasing noise) leads to a linear drift, and hence fits to our model.

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Annex

Theorem 3.4 (Value function for a risk averse agent). *Let $\lambda > 0$ and assume that $a(t, s) = \alpha(t) + \beta(t)s$, where α and β are bounded measurable functions on $[0, T]$. Then the value function is a quadratic function of the position size and the price, more precisely*

$$V(t, x, s) = b(t)x^2 + c(t)xs + d(t)s^2 + e(t)x + f(t)s + g(t), \quad (16)$$

where the coefficients are given by, for all $t \in [0, T]$,

$$\begin{aligned} b(t) &= -\eta\kappa \coth(\kappa(T-t)), \\ c^{hom}(t) &= \frac{\sinh(\kappa T)}{\sinh(\kappa(T-t))} \exp\left(\int_0^t -\beta(u)du\right) \\ c(t) &= c^{hom}(t) \int_t^T \frac{\beta(u)}{c^{hom}(u)} du \\ d^{hom}(t) &= \exp\left(-\int_0^t 2\beta(u)du\right) \\ d(t) &= d^{hom}(t) \int_t^T \frac{c^2(u)}{4\eta d^{hom}(u)} du \\ e^{hom}(t) &= \frac{\sinh(\kappa T)}{\sinh(\kappa(T-t))} \\ e(t) &= e^{hom}(t) \int_t^T (c(u) + 1) \frac{\alpha(u)}{e^{hom}(u)} du \\ f^{hom}(t) &= \exp\left(-\int_0^t \beta(u)du\right) \\ f(t) &= f^{hom}(t) \int_t^T \left(\frac{1}{2\eta} c(u)e(u) + 2\alpha(u)d(u)\right) \frac{1}{f^{hom}(u)} du \\ g(t) &= \int_t^T \left(\frac{e^2(u)}{4\eta} + \alpha(u)f + \sigma^2 d(u)\right) du, \end{aligned}$$

and $\kappa = \sqrt{\frac{\lambda}{\eta}}$. The optimal position process is given by

$$X_t^* = \frac{\sinh(\kappa(T-t))}{\sinh(\kappa T)} \left(X_0 + \frac{1}{2\eta} \int_0^t [c(u)S_u + e(u)] \frac{\sinh(\kappa T)}{\sinh(\kappa(T-u))} du \right), \quad (17)$$

and the optimal control by

$$\xi_t^* = b(t)X_t^* - \frac{1}{2\eta} (c(t)S_t + e(t)). \quad (18)$$

Remark 3.5. This result can be derived from Theorem 2 in [15]. The proof given below is completely different though, using classical verification arguments.

Proof. Let $w(t, x, s) = b(t)x^2 + c(t)xs + d(t)s^2 + e(t)x + f(t)s + g(t)$. We first show that the value function satisfies $V \leq w$. Notice that w is a solution of the HJB Equation (??) and satisfies the terminal condition (??). This follows from the fact that the coefficients satisfy the following ODEs

$$\begin{aligned} -b_t - \frac{1}{\eta}b^2 + \lambda &= 0 \\ -c_t - \frac{1}{\eta}bc - \beta c - \beta &= 0 \\ -d_t - \frac{1}{4\eta}c^2 - 2\beta d &= 0 \\ -e_t - \frac{1}{\eta}be - \alpha c - \alpha &= 0 \\ -f_t - \frac{1}{2\eta}ce - 2\alpha d - \beta f &= 0 \\ -g_t - \frac{1}{4\eta}e^2 - \alpha f - \sigma^2 d &= 0. \end{aligned}$$

Since the functions α and β are bounded, there exists a constant $C \in \mathbb{R}_+$ such that

$$|c(t)| + |d(t)| + |e(t)| + |f(t)| + |g(t)| \leq C(T - t) \quad (19)$$

for all $t \in [0, T]$. Moreover, we have $|b(t)| \leq C \frac{1}{T-t}$.

Let $\xi \in \mathcal{A}(t, x)$ be an arbitrary admissible control and let X be its associated position process. Let $\tau < T$. Itô's formula implies

$$\begin{aligned} w(\tau, X_\tau, S_\tau) &= w(t, x, s) + \int_t^\tau \frac{1}{2}\sigma^2 w_{ss}(u, X_u, S_u) du + M_\tau \\ &\quad + \int_t^\tau [w_t(u, X_u, S_u) - w_x(u, X_u, S_u)\xi_u + a(u, S_u)w_s(u, X_u, S_u)] du, \end{aligned}$$

where $M_s = \int_t^s w_s(u, X_u, S_u)\sigma dW_u$. As $(X_t)_{t \in [0, \tau]}$ is L^2 -bounded and all functions b, c, d, e, f, g and their derivatives are bounded on $[t, \tau]$, M is a strict martingale on $[t, \tau]$. Taking expectations, therefore, leads to

$$\begin{aligned} E(w(\tau, X_\tau, S_\tau)) &= w(t, x, s) + E\left(\int_t^\tau \left(w_t - w_x\xi + aw_s + \frac{1}{2}\sigma^2 w_{ss}\right)(u, X_u, S_u) du\right) \\ &\leq w(t, x, s) + E\left(\int_t^\tau (-a(u, S_u)X_u + \lambda X_u^2 + \eta\xi_u^2) du\right). \end{aligned} \quad (20)$$

As ξ is square integrable (Condition (A1)). This further implies that we have

$$\lim_{\tau \rightarrow T} E\left(\int_t^\tau (-a(u, S_u)X_u + \lambda X_u^2 + \eta\xi_u^2) du\right) = J(t, x, s, \xi).$$

Moreover, since also $\left(\frac{X_t^2}{T-t}\right)_{t \in [0, T]}$ is uniformly integrable and $\lim_{t \rightarrow T} \frac{X_t^2}{T-t} = 0$, we have $\lim_{\tau \rightarrow T} E[w(\tau, X_\tau, S_\tau)] = 0$. Inequality (20), therefore, implies $w(t, x, s) \geq J(t, x, s, \xi)$. Taking the supremum over all admissible controls, one has $V(t, x, s) \leq w(t, x, s)$.

Secondly, we show that the control $(\xi_t^*)_{t \in [0, T]}$ is admissible. Using the majoration (19) on coefficients c, b and e , one can show that there exists a constant C such that

$$|[c(u)S_u + e(u)]| \frac{\sinh(\kappa T)}{\sinh(\kappa(T-u))} \leq C(|S_u| + 1)$$

for all $u \in [0, T]$. With (17) we obtain that $|X_t^*| \leq C(T-t)(1 + \int_0^t |S_u| du)$ and hence Condition (A2) is satisfied.

Condition (A1) is a consequence of $\xi_t^2 \leq C(b(t)^2 X_t^2 + b(t)X_t) \leq C$.

Equality holds in Inequality (20) by choosing $\xi = \xi^*$. This proves that $J(t, x, s, \xi^*) = w(t, s, x)$. Thus the proof is complete. \square