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Optimal liquidation with directional views and additional information

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Abstract

We consider the problem of how to optimally close a large asset position in a market with a linear temporary price impact. We take the perspective of an agent with a market opinion that translates into a (linear) drift in asset price dynamics. By appealing to classical stochastic control we derive explicit formulas for the closing strategy that minimizes a sum of execution costs and a quadratic risk functional. We then proceed by comparing agents observing a signal about the asset's future price with agents who do not see the signal. We compute explicitly the expected additional gain due to the signal, and perform a comparative statics analysis.

Introduction

For many companies it is part of day-to-day business to build up and close large asset positions on financial markets. For example, whenever a fund modifies its investment strategy, it will reduce the position of some of its assets, while

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enlarging the holdings of other ones. Energy companies have to unwind long positions of power and buy the commodities they need for the power generation.

Selling or buying a large amount of an asset in short time usually entails a price impact. This is why in practice financial institutions, from now on referred to as agents, frequently unwind large positions by splitting them into smaller parts and closing them successively. Spreading orders over time implies the price impact to be smaller. It entails, however, also price risk. Any liquidation, therefore, involves a trade-off between liquidations costs and risk.

Agents closing a large asset position are often guided by directional views. They may have a particular opinion about the current trend of asset's price. Such an opinion can be the reason for deciding to close the position in the first place. Directional views are often based on assessments of market analysts. Trading houses have teams of analysts constantly observing markets. Analysts provide market assessments or even forecasts that are incorporated in the company's trading decisions. For example, power companies try to estimate, by performing computer-based optimizations, the marginal costs for generating the electricity that will be demanded in future years. By comparing the estimated marginal costs with actual forward market prices, they derive an opinion about the direction power prices will take. The directional opinion guides them in selling power on forward markets.

The first aim of the paper is to analyze the influence of directional views on liquidation strategies. To this end we set up a stylized model of an agent who has to close a single asset position up to a time horizon T . We assume that any transaction has a linear absolute temporary (abbreviated by LAT in the pioneering paper [8]) impact on the asset's price. The fundamental (i.e. non-influenced) price process is assumed to be a Brownian motion, complemented by a linear *drift* that represents the agent's directional view. We suppose that the agent aims at maximizing the expected proceeds (resp. minimizing expected costs) from closing a position, while keeping a quadratic risk functional low.

We characterize optimal execution strategies by appealing to classical stochastic control theory. Profiting from the linear-quadratic model set-up we obtain explicit formulas for the value function and the optimal control. With the optimal position process given in closed form, we can directly study its dependence on the agent's directional view. We observe, for example, a competitive interplay between risk aversion and directional views: the influence of any opinion diminishes as risk aversion increases.

So far the liquidation literature has only briefly analyzed the impact market opinions on trading strategies. Almgren & Chriss [1] calculate optimal *deterministic* liquidation strategies, allowing for directional views. They assume that the agent's objective is to minimize a weighted sum of the mean and the variance of the proceeds. It is remarkable that the optimal strategy from [1] maximizes CARA utility - not only among all deterministic, but even among all *predictable* trajectories. This is shown in [17] for a time continuous version of the Almgren & Chriss model.

The recent preprint [16] also studies the influence of price drifts on optimal liquidation strategies. The agent's objective functional, however, slightly differs from ours. Moreover, a more general semimartingale perspective is taken, leading to a more abstract representation of liquidation strategies in terms of conditional expectations. Within our model the liquidation paths are given explicitly, allowing thus to perform a simple comparative statics analysis.

In the second part of the paper we study the value of a market assessment *before* it is revealed. To put it differently, we look at the *expected additional value* of market expertise. To this end we introduce into our model an expert who obtains a signal about the asset's price at time T . If the expert passes the knowledge on to the agent having to close the position, then we say that the agent is *informed*; else she is *non-informed*. We use the technique of *filtration enlargements* for modeling the information flow of the informed agent.

We aim at comparing the optimal execution strategy of a non-informed agent with the one of an informed agent. We assume that the price signal is the asset price at T disturbed by an independent centered Gaussian noise. We prove an explicit formula for the additional expected gain from the signal. Besides, we perform a comparative statics analysis of the additional gain and show how market frictions limit the value of additional information.

We consider also the case where the signal reveals the asset's *exact* fundamental price at T , i.e. where the signal is not distorted by noise. We show that also in this case the additional gain is finite. The market would admit arbitrage if there were no market frictions. The price impact entailed by any trading implies that the gain from exactly knowing the fundamental price at a future date is only finite.

The value of a price signal has so far been studied mainly within utility maximization models. In [14] the authors calculate, also by employing filtration enlargements, the expected additional *logarithmic utility* of an investor possessing inside information. They do *not* consider market frictions and hence obtain that the additional utility is infinite if the exact asset's price at T is known to the investor. The model of [14] has been put forward in many succeeding paper, e.g. in [2], [4], [5].

The paper is organized as follows : Section 1 is devoted to the presentation of the model, Section 2 studies the case of a linear price drift and obtains a closed form expression of the optimal position for the liquidation problem. In Section 3, we aim at estimating the value of additional information from a risk-neutral agent's perspective before the information is revealed.

1 The model

Consider an agent who has to unwind a position of $X_0 \in \mathbb{R}$ shares of an asset until a time horizon $T > 0$. We assume that the *fundamental asset price* is a drifted Brownian motion satisfying the SDE

$$dS_t = a(t, S_t)dt + \sigma dW_t,$$

where $\sigma > 0$ is a constant volatility, $a : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ a measurable drift function and W a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{[0, T]}, \mathbb{P})$. We assume that there exists $C \in \mathbb{R}_+$ such that $|a(t, s)| \leq C(1 + s)$ for all $t \in [0, T]$ and $s \in \mathbb{R}_+$. We interpret the drift as the agent's directional view about the future price evolution. Moreover, we assume that all prices are *forward prices* so that no discounting is needed.

A *closing strategy* (or simply strategy) of a position $x \in \mathbb{R}$ at time $t \in [0, T)$ is a predictable strategy $\xi = (\xi_u)$ satisfying $\int_t^T \xi_u du = x$. We interpret ξ_t as

the *selling rate* at time $t \in [0, T]$. Given ξ , the total position at time $t \in [0, T]$ is given by

$$X_t = X_0 - \int_0^t \xi_s ds.$$

Notice that $X_T = 0$, i.e. the position is closed at T .

For technical reasons we impose the following integrability condition on the closing strategies: A strategy (ξ_u) , resp. its associated position process (X_t) , is called *admissible* if

(A1) the process ξ is L^2 -integrable, i.e $E(\int_0^T \xi_u^2 du) < \infty$,

(A2) the family $\left(\left(\frac{X_t^*}{T-t} \right)^2 \right)_{0 \leq t \leq T}$ is uniformly integrable and $\lim_{t \rightarrow T} \frac{X_t^2}{T-t} = 0$, a.s.

We denote by $\mathcal{A}(t, x)$ the set of all admissible closing strategies of x at t .

We suppose that any transaction entails a price impact that is linear with respect to the selling rate. Moreover, the impact is assumed to be absolute and only instantaneous. Selling at a rate of ξ_t is thus possible only at the *realized price* of

$$\tilde{S}_t = S_t - \eta \xi_t,$$

where $\eta > 0$ is the price impact parameter.

The final revenues (possibly negative) of the liquidation operation when selling at a rate $(\xi_t)_{t \in [0, T]}$ are given by

$$R_T = \int_0^T \xi_u \tilde{S}_u du.$$

Notice that by the product formula we have

$$\begin{aligned} R_T &= \int_0^T \xi_u S_u du - \eta \int_0^T \xi_u^2 du \\ &= X_0 S_0 + \int_0^T X_u a(u, S_u) du + \int_0^T X_u \sigma dW_u - \eta \int_0^T \xi_u^2 du. \end{aligned} \quad (1)$$

Assumption (A2) guarantees that a position process X , associated to an admissible strategy ξ , is square integrable, and thus taking expectations in (1) we get

$$E(R_T) = X_0 S_0 + E \int_0^T (X_u a(u, S_u) - \eta \xi_u^2) du.$$

We assume that the agent attributes a risk to the open position between t and T measured by $\lambda E \int_t^T X_u^2 du$, where $\lambda \in \mathbb{R}_+$ is the *risk aversion parameter*. The risk term can be interpreted as the time average of the squared value-at-risk of the open position (see the end of Section 1 in [6] for an explanation).

We assume that the agent aims at maximizing the sum of the expected value of the final revenues minus the risk of the open position. More precisely the target function is defined by

$$J(t, x, s, \xi) = E \left[\int_t^T (X_u a(u, S_u) - \eta \xi_u^2 - \lambda X_u^2) du \middle| X_t = x, S_t = s \right],$$

for $(t, x, s) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$. The value function is defined by

$$V(t, x, s) = \sup_{\xi \in \mathcal{A}(t, x)} J(t, x, s, \xi). \quad (2)$$

Remark 1.1. If the impact of the liquidation operation on the price dynamics is not only instantaneous, but lasts in the considered period, one can add to the model a so-called perpetual impact factor, depending on the total amount of the position closed up to time t . The form of the realized price dynamics is then :

$$\tilde{S}_t = S_t - \eta_t - c(X_0 - X_t),$$

where c is the permanent impact factor.

The final revenues in this case are given by

$$R_T = X_0 S_0 + \frac{1}{2} c X_0^2 + \int_0^T X_u a(u, S_u) du + \int_0^T X_u \sigma dW_u - \eta \int_0^T \xi_u^2 du. \quad (3)$$

The only difference to Equation (1) is the constant term $\frac{1}{2} c X_0^2$, and thus the optimization is not changed. The problem is identical.

2 Optimal closure for linear price drifts

In this section we show that if the price drift coefficient is linear in the price, then the optimal position process can be determined in closed form. The value function turns out to be a quadratic form of the position size and the price. Besides, we show that the influence of any view diminishes as risk aversion increases.

2.1 Optimal position process

Notice that the Hamilton-Jacobi-Bellman equation associated to our control problem (2) is given by

$$-V_t - a(t, s)V_s - \frac{1}{2}\sigma^2 V_{ss} - a(t, s)x + \lambda x^2 - \sup_{\xi \in \mathbb{R}} [-\xi V_x - \eta \xi^2] = 0, \quad (4)$$

with the singular terminal condition

$$\lim_{t \uparrow T} V(t, x, s) = \begin{cases} 0, & \text{if } x = 0, \\ -\infty, & \text{if } x \neq 0. \end{cases} \quad (5)$$

The first order condition implies that the supremum on the left hand side of (4) is attained by $\xi^* = -\frac{V_x}{2\eta}$. We obtain a simplified HJB equation

$$-V_t - a(t, s)V_s - \frac{1}{2}\sigma^2 V_{ss} - a(t, s)x + \lambda x^2 - \frac{V_x^2}{4\eta} = 0. \quad (6)$$

From now on, we suppose that the price drift coefficient is an affine linear function

$$a(t, s) = \alpha(t) + \beta(t)s,$$

where α and β are functions on $[0, T]$. The price process is given by

$$S_t = h(0, t)S_0 + \int_0^t h(r, t)\alpha(r)dr + \int_0^t h(r, t)\sigma dW_r,$$

where $h(r, t) = e^{\int_r^t \beta(s)ds}$. In this case the value function is a quadratic function of price and remaining position, as stated in the following theorem:

Theorem 2.1 (Value function for a risk averse agent). *Let $\lambda > 0$ and assume that $a(t, s) = \alpha(t) + \beta(t)s$, where α and β are bounded measurable functions on $[0, T]$. Then the value function is a quadratic function of the position size and the price, more precisely*

$$V(t, x, s) = b(t)x^2 + c(t)xs + d(t)s^2 + e(t)x + f(t)s + g(t), \quad (7)$$

where the coefficients are given by, for all $t \in [0, T]$,

$$\begin{aligned} b(t) &= -\eta\kappa \coth(\kappa(T-t)), \\ c^{hom}(t) &= \frac{\sinh(\kappa T)}{\sinh(\kappa(T-t))} \exp\left(\int_0^t -\beta(u)du\right) \\ c(t) &= c^{hom}(t) \int_t^T \frac{\beta(u)}{c^{hom}(u)} du \\ d^{hom}(t) &= \exp\left(-\int_0^t 2\beta(u)du\right) \\ d(t) &= d^{hom}(t) \int_t^T \frac{c^2(u)}{4\eta d^{hom}(u)} du \\ e^{hom}(t) &= \frac{\sinh(\kappa T)}{\sinh(\kappa(T-t))} \\ e(t) &= e^{hom}(t) \int_t^T (c(u) + 1) \frac{\alpha(u)}{e^{hom}(u)} du \\ f^{hom}(t) &= \exp\left(-\int_0^t \beta(u)du\right) \\ f(t) &= f^{hom}(t) \int_t^T \left(\frac{1}{2\eta}c(u)e(u) + 2\alpha(u)d(u)\right) \frac{1}{f^{hom}(u)} du \\ g(t) &= \int_t^T \left(\frac{e^2(u)}{4\eta} + \alpha(u)f + \sigma^2 d(u)\right) du, \end{aligned}$$

and $\kappa = \sqrt{\frac{\lambda}{\eta}}$. The optimal position process is given by

$$X_t^* = \frac{\sinh(\kappa(T-t))}{\sinh(\kappa T)} \left(X_0 + \frac{1}{2\eta} \int_0^t [c(u)S_u + e(u)] \frac{\sinh(\kappa T)}{\sinh(\kappa(T-u))} du \right), \quad (8)$$

and the optimal control by

$$\xi_t^* = b(t)X_t^* - \frac{1}{2\eta} (c(t)S_t + e(t)). \quad (9)$$

Proof. Let $w(t, x, s) = b(t)x^2 + c(t)xs + d(t)s^2 + e(t)x + f(t)s + g(t)$. We first show that the value function satisfies $V \leq w$. Notice that w is a solution of the HJB Equation (4) and satisfies the terminal condition (5). This follows from the fact that the coefficients satisfy the following ODEs

$$\begin{aligned} -b_t - \frac{1}{\eta}b^2 + \lambda &= 0 \\ -c_t - \frac{1}{\eta}bc - \beta c - \beta &= 0 \\ -d_t - \frac{1}{4\eta}c^2 - 2\beta d &= 0 \\ -e_t - \frac{1}{\eta}be - \alpha c - \alpha &= 0 \\ -f_t - \frac{1}{2\eta}ce - 2\alpha d - \beta f &= 0 \\ -g_t - \frac{1}{4\eta}c^2 - \alpha f - \sigma^2 d &= 0. \end{aligned}$$

Since the functions α and β are bounded, there exists a constant $C \in \mathbb{R}_+$ such that

$$|c(t)| + |d(t)| + |e(t)| + |f(t)| + |g(t)| \leq C(T - t) \quad (10)$$

for all $t \in [0, T]$. Moreover, we have $|b(t)| \leq C \frac{1}{T-t}$.

Let $\xi \in \mathcal{A}(t, x)$ be an arbitrary admissible control and let X be its associated position process. Let $\tau < T$. Itô's formula implies

$$\begin{aligned} w(\tau, X_\tau, S_\tau) &= w(t, x, s) + \int_t^\tau \frac{1}{2}\sigma^2 w_{ss}(u, X_u, S_u)du + M_\tau \\ &\quad + \int_t^\tau [w_t(u, X_u, S_u) - w_x(u, X_u, S_u)\xi_u + a(u, S_u)w_s(u, X_u, S_u)]du, \end{aligned}$$

where $M_s = \int_t^s w_s(u, X_u, S_u)\sigma dW_u$. As $(X_t)_{t \in [0, \tau]}$ is L^2 -bounded and all functions b, c, d, e, f, g and their derivatives are bounded on $[t, \tau]$, M is a strict martingale on $[t, \tau]$. Taking expectations, therefore, leads to

$$\begin{aligned} E(w(\tau, X_\tau, S_\tau)) &= w(t, x, s) + E\left(\int_t^\tau \left(w_t - w_x\xi + aw_s + \frac{1}{2}\sigma^2 w_{ss}\right)(u, X_u, S_u)du\right) \\ &\leq w(t, x, s) + E\left(\int_t^\tau (-a(u, S_u)X_u + \lambda X_u^2 + \eta\xi_u^2)du\right). \end{aligned} \quad (11)$$

As ξ is square integrable (Condition (A1)). This further implies that we have

$$\lim_{\tau \rightarrow T} E\left(\int_t^\tau (-a(u, S_u)X_u + \lambda X_u^2 + \eta\xi_u^2)du\right) = J(t, x, s, \xi).$$

Moreover, since also $\left(\frac{X_t^2}{T-t}\right)_{t \in [0, T]}$ is uniformly integrable and $\lim_{t \rightarrow T} \frac{X_t^2}{T-t} = 0$, we have $\lim_{\tau \rightarrow T} E[w(\tau, X_\tau, S_\tau)] = 0$. Inequality (11), therefore, implies $w(t, x, s) \geq J(t, x, s, \xi)$. Taking the supremum over all admissible controls, one has $V(t, x, s) \leq w(t, x, s)$.

Secondly, we show that the control $(\xi_t^*)_{t \in [0, T]}$ is admissible. Using the majoration (10) on coefficients c, b and e , one can show that there exists a constant C such that

$$|[c(u)S_u + e(u)]| \frac{\sinh(\kappa T)}{\sinh(\kappa(T-u))} \leq C(|S_u| + 1)$$

for all $u \in [0, T]$. With (8) we obtain that $|X_t^*| \leq C(T-t)(1 + \int_0^t |S_u| du)$ and hence Condition (A2) is satisfied.

Condition (A1) is a consequence of $\xi_t^2 \leq C(b(t)^2 X_t^2 + b(t)X_t) \leq C$.

Equality holds in Inequality (11) by choosing $\xi = \xi^*$. This proves that $J(t, x, s, \xi^*) = w(t, s, x)$. Thus the proof is complete. \square

Remark 2.2. The pair (S, X^*) is a Gaussian process with first moments

$$\begin{aligned} E(S_t) &= h(0, t)S_0 + \int_0^t \alpha(u)h(u, t)du, \\ E(X_t^*) &= \frac{T-t}{T} \left(X_0 + \frac{1}{2\eta} \int_0^t \frac{T}{T-u} \left[c(u)h(0, u)S_0 + c(u) \int_0^u \alpha(r)h(r, u)dr + e(u) \right] du \right). \end{aligned}$$

Moreover,

$$\begin{aligned} E(S_t^2) &= \int_0^t \sigma^2 h^2(u, t)du + \left(h(0, t)S_0 + \int_0^t \alpha(u)h(u, t)du \right)^2 \\ E((X_t^*)^2) &= \frac{(T-t)^2}{4\eta^2} \int_0^t \left(\int_r^t \frac{c(u)}{T-u} h(r, u)du \right)^2 \sigma^2 dr + (E(X_t))^2 \\ E(S_t X_t^*) &= \frac{T-t}{T} \frac{1}{2\eta} \int_0^t c(u) \frac{T}{T-u} \text{cov}(S_t, S_u) du + E(S_t)E(X_t), \end{aligned}$$

with $\text{cov}(S_t, S_u) = \int_0^{t \wedge u} \sigma^2 h^{-1}(r, u)h^{-1}(r, t)dr$.

If the agent is risk neutral, i.e. $\lambda = 0$, then we can simplify the coefficients of the value function.

Theorem 2.3 (Value function for a risk neutral agent). *Let $\lambda = 0$ and assume that $a(t, s) = \alpha(t) + \beta(t)s$, where α and β bounded. Then the value function satisfies*

$$V(t, x, s) = b(t)x^2 + c(t)xs + d(t)s^2 + e(t)x + f(t)s + g(t),$$

where

$$\begin{aligned} b(t) &= -\eta \frac{1}{T-t}, \\ c(t) &= \frac{T}{T-t} h^{-1}(0, t) \int_t^T \beta(u) \frac{T-u}{T} h(0, u) du, \\ e(t) &= \frac{T}{T-t} \int_t^T (c(u) + 1) \alpha(u) \frac{T-u}{T} du, \end{aligned}$$

and d, f and g are defined as in Theorem 2.1. The optimal control is given by (9) and the optimal position trajectory by

$$X_t^* = \frac{T-t}{T} \left(X_0 + \frac{1}{2\eta} \int_0^t [c(u)S_u + e(u)] \frac{T}{T-u} du \right). \quad (12)$$

2.2 Directional views versus risk aversion

In this subsection we have a closer look at the interplay between risk aversion and directional views. We show that the influence of a drift on a liquidation decreases as the agent's risk aversion increases. For simplicity we restrict the analysis to the case where the price drift is equal to a constant $\alpha \neq 0$. The price is thus a Brownian motion with drift

$$dS_t = \alpha dt + \sigma dW_t.$$

The value function is obtained as a corollary of Theorem 2.1.

Corollary 2.4. *Assume that the price drift coefficient is constant equal to α . Then the coefficients of the value function (7) satisfy $c = d = f = 0$ and*

$$\begin{aligned} b(t) &= -\eta\kappa \coth(\kappa(T-t)), \\ e(t) &= \frac{\alpha}{\kappa} \tanh\left(\frac{\kappa}{2}(T-t)\right), \\ g(t) &= \int_t^T \left(\frac{e^2(u)}{4\eta}\right) du, \end{aligned}$$

for all $t \in [0, T]$. The optimal position trajectory is given by

$$X_t^* = \frac{\sinh(\kappa(T-t))}{\sinh(\kappa T)} \left(X_0 + \frac{\alpha \sinh(\kappa T)}{2\eta\kappa^2} \left(\tanh\left(\frac{\kappa}{2}T\right) - \tanh\left(\frac{\kappa}{2}(T-t)\right) \right) \right). \quad (13)$$

Proof. The formulas for the coefficients are straightforward to verify. Notice that $e(t) = \frac{1}{\kappa} \frac{\alpha}{\sinh(\kappa(T-t))} (\cosh(\kappa(T-t)) - 1) = \frac{\alpha}{\kappa} \tanh\left(\frac{\kappa}{2}(T-t)\right)$.

Next observe that $\frac{\cosh(\kappa(T-t))-1}{\sinh^2(\kappa(T-t))} = \frac{1}{2 \cosh^2\left(\frac{\kappa}{2}(T-t)\right)}$ and that $\tanh(x)$ is the primitive of $\frac{1}{\cosh^2(x)}$. A straightforward calculation shows that the optimal position trajectory is given by (13). \square

Remark 2.5. a) Observe that the optimal position trajectory (13) is *deterministic*. It coincides with the (time-discrete) optimal position path derived in Section 4.1 of [1].

b) Notice that if α is positive, then $e(t)$ is positive, too. The positive trend entails that the position is closed slower than if there was no trend. If α is negative, then the negative trend implies that the position is closed faster.

We denote \bar{X}^* is the optimal liquidation path for $\alpha = 0$, i.e. from a perspective of an agent without a directional view. The difference to the optimal liquidation process *with* a directional view is given by

$$X_t^* - \bar{X}_t^* = \frac{\alpha}{2\lambda} \sinh(\kappa(T-t)) \left(\tanh\left(\frac{\kappa}{2}T\right) - \tanh\left(\frac{\kappa}{2}(T-t)\right) \right),$$

where we use that $\bar{X}_t^* = \frac{\sinh(\kappa(T-t))}{\sinh(\kappa T)} X_0$. The difference decreases as the risk aversion increases. We have the following asymptotic result:

Proposition 2.6. *Then for any $t \in]0, T[$,*

$$X_t^* - \bar{X}_t^* \sim \frac{\alpha}{2\lambda} \text{ as } \lambda \rightarrow \infty.$$

Proof. Let $t \in]0, T[$. Straightforward calculations show that

$$\lim_{\lambda \rightarrow +\infty} \sinh(\kappa(T-t)) \left(\tanh\left(\frac{\kappa}{2}T\right) - \tanh\left(\frac{\kappa}{2}(T-t)\right) \right) = 1.$$

□

The term $\frac{\alpha}{2\lambda}$ can be interpreted as the asymptotic effect of the directional view on the liquidation, as risk aversion increases. This effect is illustrated in the following figures. Figure 1 shows the optimal position path when risk aversion parameter λ increases (between 1 and 100), for a positive drift α chosen equal to 20% and for a negative drift chosen equal to -20% . Figure 2 plots the convergence of the difference between optimal liquidations processes with and without directional views, as λ goes to infinity. For different values of t , the rate of convergence may vary.

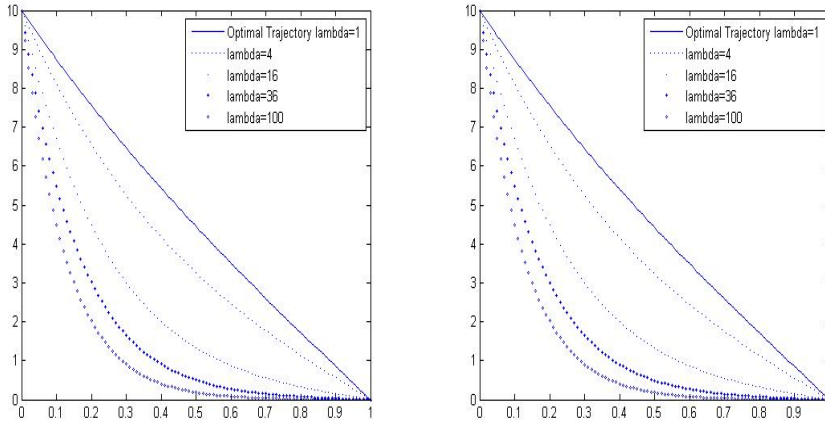


Figure 1: Optimal position for λ from 1 to 100, $T = 1$, $\kappa = 1$, $\alpha = 0.2$ (left) and $\alpha = -0.2$ (right).

3 The value of additional information from a risk-neutral perspective

In this section we aim at estimating the value of additional information from the agent's perspective *before* the information is revealed. For simplicity we suppose that the price process is given by $S_t = \sigma W_t$. This means that S is a martingale with respect to the filtration (\mathcal{F}_t^W) generated by W . We further assume that there is an expert, e.g. a market analyst or an insider, who has obtained a signal about the asset price at time T . We model the signal as a random variable $G = S_T + N$, where N is independent of the price process and normally distributed with mean zero. Since G is Gaussian, it is equivalent to the signal sent from a price $S_{T'}$, where $T' \geq T$. One can interpret the difference $T' - T$ as the variance of the signal's noise.

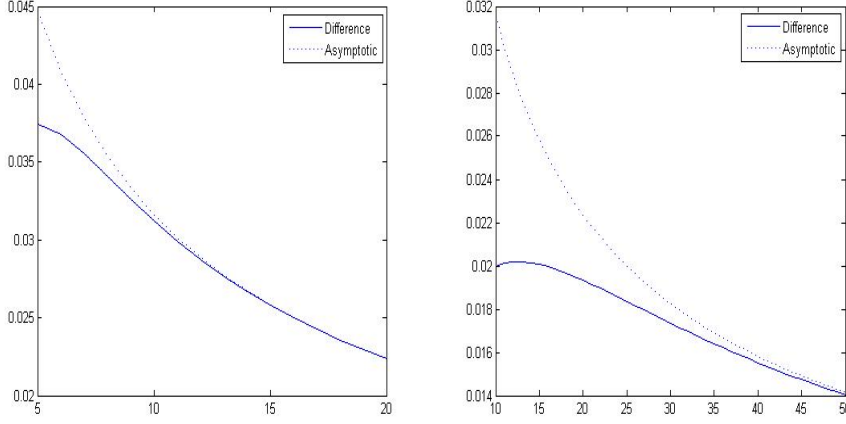


Figure 2: Left : The difference between $X_t^* - \bar{X}_t^*$ for $t = 0.5$, $\eta = 1$ and λ between 25 and 400. Right : $t = 0.9$, $\eta = 1$ and λ between 100 and 2500

If the expert discloses the signal to the agent, then we say that the agent is *informed*. In this case the agent's information flow can be modeled as the following initial enlargement of the Brownian filtration :

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \sigma(S_{T'}), \quad 0 \leq t \leq T.$$

In case the expert does not pass on the signal, we say that the agent is non-informed. The information flow of the agent is then represented by the natural filtration (\mathcal{F}_t^W) .

Throughout this section we assume that the agent, informed or non-informed, is risk-neutral, i.e. that $\lambda = 0$. If the agent is non-informed, then Theorem 2.3 implies that the minimal expected execution costs are given by $V^N(0, x, s) = -\eta \frac{x^2}{T}$ (we use the superscript N to highlight the value function of the non-informed agent).

We can also appeal to Theorem 2.3 to calculate the expected execution costs of an informed agent. To this end we recall that the price dynamics under (\mathcal{G}_t) satisfy

$$dS_t = \sigma dW_t^{\mathcal{G}} + \sigma \frac{S_{T'} - S_t}{T' - t} dt, \quad (14)$$

where $W^{\mathcal{G}}$ is a Brownian motion with respect to (\mathcal{G}_t) (see e.g. [15]).

A strategy (ξ_t) with associated position path (X_t) is called (\mathcal{G}_t) -*admissible* if it is (\mathcal{G}_t) -predictable and satisfies Condition (A1) and (A2). For any $t \in [0, T)$ and $x \in \mathbb{R}$ let $\mathcal{A}^I(t, x)$ denote the set of (\mathcal{G}_t) -admissible strategies satisfying $X_t = x$. For $(t, x, s) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+$ we define the value function, or simply the expected execution costs, of the informed agent by

$$V^I(t, x, s) = \sup_{\xi \in \mathcal{A}^I(t, x)} E \left[\int_t^T (X_u a(u, S_u) - \eta \xi_u^2) du \mid X_t = x, S_t = s \right].$$

Notice that the drift in the (\mathcal{G}_t) -dynamics (14) is linear with $\alpha(t) = \frac{S'_T}{T'-t}$ and $\beta(t) = -\frac{1}{T'-t}$. If $T' > T$, then α and β are bounded and hence Theorem 2.3 applies. The coefficients of the value function, highlighted with the superscript I , are given as follows.

Proposition 3.1. *Let $T' > T$. The coefficients of the value function for the informed agent are given by*

$$\begin{aligned}
b^I(t) &= -\eta \frac{1}{T-t} \\
c^I(t) &= -\frac{1}{2} \frac{T-t}{T'-t} \\
d^I(t) &= \frac{1}{48\eta} \frac{(T-t)^3}{(T'-t)^2} \\
e^I(t) &= \frac{1}{2} \frac{T-t}{T'-t} S_{T'} \\
f^I(t) &= \frac{1}{\eta} \frac{S_{T'}}{T'-t} \left(\frac{1}{8} (T'-T)(T-t) - \frac{1}{24} (T'-t)^2 + \frac{1}{24} \frac{(T'-T)^3}{T'-t} \right) \\
g^I(t) &= \frac{S_{T'}^2}{\eta} \left(\frac{1}{12} (T'-T) - \frac{1}{16} \frac{(T'-T)^2}{T'-t} - \frac{1}{8} (T'-T) \frac{T-t}{T'-t} - \frac{1}{24} (T-t) - \frac{1}{48} \frac{(T'-T)^3}{(T'-t)^2} \right) \\
&\quad + \frac{\sigma^2}{48\eta} \left(\frac{(T'-T)^3}{T'-t} - \frac{3}{2} (T'-T)^2 + 3(T'-T)^2 \ln \left(\frac{T'-t}{T'-T} \right) \right) \\
&\quad + \frac{\sigma^2}{48\eta} \left(-3(T'-T)(T-t) + \frac{1}{2} (T'-t)^2 \right).
\end{aligned}$$

Proof. The result follows from Theorem 2.3 and straightforward calculations. \square

3.1 Revenues from additional information

The value of the additional information can be calculated via several methods. The most intuitive is to compute the expected additional gain of the informed agent. This is the purpose of the following theorem.

Theorem 3.2. *Let $T' > T$. The expected additional gain of the informed agent is given by*

$$E[V^I(0, s, x) - V^N(0, s, x)] = \frac{\sigma^2}{16\eta} \left((T'-T)^2 \ln \left(\frac{T'}{T'-T} \right) - TT' + \frac{3}{2} T^2 \right). \quad (15)$$

Remark 3.3. Notice that the expected additional gain does *not* depend on the price level s and the initial position x , but only on T and T' . In the following we denote the expected additional gain by $\Delta(T, T') = E[V^I(0, s, x) - V^N(0, s, x)]$. Sometimes we write $\Delta(T, T', \sigma, \eta)$ in order to stress its dependence on σ and η .

The following Lemma will be necessary to prove Theorem 3.2.

Lemma 3.4.

$$E \left[\left(\int_0^t \frac{W_T - W_u}{T-u} du \right)^2 \right] = 2t(1 + \ln(T)) - 2T \ln(T) + 2(T-t) \ln(T-t) \quad (16)$$

Proof. The product formula applied to the (\mathcal{G}_t) -semimartingales $X_t = \ln(T-t)$ and $Y_t = W_T - W_t$, $t \in [0, T)$, implies

$$-\int_0^t \frac{W_T - W_u}{T-u} du = \ln(T-t)(W_T - W_t) - \ln(T)W_T - \int_0^t \ln(T-u) dW_u, \quad (17)$$

and hence

$$\begin{aligned} E \left[\left(\int_0^t \frac{W_T - W_u}{T-u} du \right)^2 \right] &= \ln^2(T-t)(T-t) + \ln^2(T)T^2 + \int_0^t \ln^2(T-u) du \\ &\quad - 2\ln(T-t)\ln(T)(T-t) - 2\ln(T) \int_0^t \ln(T-u) du. \end{aligned}$$

A straightforward simplification of the integrals leads to Equation (16). \square

Proof of Theorem 3.2. By Theorem 2.3 the optimal strategy of the informed agent satisfies

$$\xi_t^* = \frac{X_t^*}{T-t} - \frac{1}{4\eta} \frac{T-t}{T'-t} (S_{T'} - S_t),$$

and the optimal position trajectory is given by

$$X_t^* = \frac{T-t}{T} \left(x + \frac{1}{4\eta} T \int_0^t \frac{S_{T'} - S_u}{T'-u} du \right).$$

The martingale property of the price process implies that the value function satisfies

$$\begin{aligned} &V^I(0, x, s) \\ &= -\eta \frac{x^2}{T} + E \int_0^T \left[\frac{3}{2} X_t^* \frac{S_{T'} - S_t}{T'-t} - \frac{1}{16\eta} \left(\int_0^t \frac{S_{T'} - S_u}{T'-u} du \right)^2 - \frac{1}{16\eta} \frac{(T-t)^2}{(T'-t)^2} (S_{T'} - S_t)^2 \right] dt \end{aligned} \quad (18)$$

Observe that

$$\begin{aligned} E(X_t^* (S_{T'} - S_t)) &= (T-t) \frac{1}{4\eta} E \left[(S_{T'} - S_t) \int_0^t \frac{S_{T'} - S_u}{T'-u} du \right] \\ &= (T-t) \frac{1}{4\eta} \int_0^t \frac{E[(S_{T'} - S_t) ((S_{T'} - S_t) + (S_t - S_u))]}{T'-u} du \\ &= \frac{\sigma^2}{4\eta} (T-t)(T'-t) \ln\left(\frac{T'}{T'-t}\right). \end{aligned} \quad (19)$$

Moreover, by Lemma 3.4,

$$E \left[\left(\int_0^t \frac{S_{T'} - S_u}{T'-u} du \right)^2 \right] = \sigma^2 [2t(1 + \ln(T')) - 2T' \ln(T') + 2(T'-t) \ln(T'-t)]. \quad (20)$$

Combining Equation (19) and (20) with (18) yields

$$\begin{aligned} E[V^I(0, x, s)] &= -\eta \frac{x^2}{T} + \frac{3\sigma^2}{8\eta} \int_0^T (T-t) \ln\left(\frac{T'}{T'-t}\right) dt \\ &\quad - \frac{\sigma^2}{8\eta} \int_0^T [t(1 + \ln(T')) - T' \ln(T') + (T'-t) \ln(T'-t)] dt \\ &\quad - \frac{\sigma^2}{16\eta} \int_0^T \frac{(T-t)^2}{(T'-t)} dt. \end{aligned} \quad (21)$$

Notice that

$$\int_0^T (T-t) \ln(T'-t) dt = \frac{1}{2}(T'-T)^2 \ln(T'-T) + TT' \ln(T') - \frac{1}{2}(T')^2 \ln(T') + \frac{1}{2}TT' - \frac{3}{4}T^2$$

and

$$\int_0^T (T'-t) \ln(T'-t) dt = -\frac{1}{2}(T'-T)^2 \ln(T'-T) + \frac{1}{4}(T'-T)^2 + \frac{1}{2}(T')^2 \ln(T') - \frac{1}{4}(T')^2$$

and

$$\int_0^T \frac{(T-t)^2}{(T'-t)} dt = (T'-T)^2 \ln\left(\frac{T'}{T'-T}\right) - 2(T'-T)T + TT' - \frac{1}{2}T^2.$$

A straightforward calculation shows that (21) simplifies to

$$E[V^I(0, x, s)] = -\eta \frac{x^2}{T} + \frac{\sigma^2}{16} \left((T'-T)^2 \ln\left(\frac{T'}{T'-T}\right) - TT' + \frac{3}{2}T^2 \right).$$

□

Remark 3.5. One can alternatively calculate the additional utility by computing the expectation of the coefficients $e^I(0)$, $f^I(0)$ and $g^I(0)$. By simplifying terms one arrives again at formula (15).

The expected additional gain $\Delta(T, T')$ converges to a finite value as $T' \downarrow T$. If $T = T'$, then the market would admit arbitrage if there was no price impact. It has been shown that an informed investor can achieve infinite expected utility in a frictionless market (see e.g. [14] and [11]). In our model, in contrast, the price impact excludes arbitrage and implies that the expected additional gain doesn't become infinite when T' is equal to T .

Notice that if we choose $T' = T$, then the drift in the (\mathcal{G}_t) -price dynamics (14) is not bounded, and hence the assumptions of Theorem 2.1 are technically not satisfied. Nevertheless, one can show that the result applies also to this particular case. To this end one needs to make sure that the candidate for the optimal control is admissible.

Proposition 3.6. *Suppose that $T' = T$. Then the expected additional gain of the informed agent is given by*

$$E[V^I(0, s, x) - V^N(0, s, x)] = \frac{\sigma^2}{32\eta} T^2. \quad (22)$$

The optimal strategy is admissible and the associated position process satisfies
 $X_t^* = \frac{T-t}{T} \left(x + \frac{1}{4\eta} T \int_0^t \frac{S_T - S_u}{T-u} du \right).$

Proof. The first expression is obtained taking the limit in $\Delta(T, T')$ as $T' \downarrow T$. To prove the admissibility, notice first that using Equation (17) for any $p > 2$, there exists C_p such that

$$E \left(\left| \int_0^t \frac{S_T - S_u}{T-u} \right|^p \right) \leq C_p \left\{ \ln^p(T-t)(T-t)^{p/2} + \ln^p(T)T^{p/2} + \left(\int_0^T \ln^2(T-u) du \right)^{p/2} \right\}$$

This shows that $\left(\int_0^t \frac{S_T - S_u}{T-u} du\right)^2$ is uniformly integrable. We further obtain that $\left(\frac{X_t^*}{T-t}\right)^2$, $0 \leq t \leq T$, is uniformly integrable and $\lim_{t \rightarrow T} \frac{X_t^{*2}}{T-t} = 0$, a.s. Moreover the process ξ^* is squared integrable. \square

3.2 Comparative Statics

We next analyze the impact of the model parameters on the additional gain. We start with the dependence on the signal quality.

Sensitivity with respect to the signal noise

If the noise of the signal increases, then the additional revenues of the informed agent decrease. This is indeed confirmed by the next result.

Lemma 3.7. *The expected revenues from additional information decrease as T' increases, i.e. the mapping $f(x) = (x - T)^2 \ln\left(\frac{x}{x-T}\right) - Tx + \frac{3}{2}T^2$ is decreasing on $[T, \infty)$. Moreover, $f(T) = \frac{1}{2}T^2$ and $\lim_{x \rightarrow \infty} f(x) = 0$.*

Proof. Notice that $f'(x) = 2(x - T) \ln\left(\frac{x}{x-T}\right) + \frac{T^2}{x} - 2T$ and

$$f''(x) = -2 \ln\left(1 - \frac{T}{x}\right) - \frac{2T}{x} - \frac{T^2}{x^2}.$$

Since the logarithm is analytic on the open interval $(0, 2)$, we further have for $x > T$

$$f''(x) = 2 \left(\frac{T}{x} + \frac{1}{2} \frac{T^2}{x^2} + \frac{1}{3} \frac{T^3}{x^3} + \dots \right) - \frac{2T}{x} - \frac{T^2}{x^2} = \left(\frac{1}{3} \frac{T^3}{x^3} + \dots \right) \geq 0.$$

Consequently f' is increasing on $[T, \infty)$. Besides observe that $f'(T) = -T$, and

$$\lim_{x \rightarrow \infty} f'(x) = 2(x - T) \left(\frac{T}{x} + \frac{1}{2} \frac{T^2}{x^2} + \frac{1}{3} \frac{T^3}{x^3} + \dots \right) - \frac{2T}{x} - \frac{T^2}{x^2} = 0,$$

which, together with the monotonicity of f' , implies $f' \leq 0$ on $[T, \infty)$. The function f , therefore, is decreasing in x . \square

Sensitivity with respect to the time horizon

The additional gain increases when the time horizon T increases, while T' stays constant. Indeed, a straightforward computation shows that

$\frac{\partial^2 \Delta}{\partial T^2} = 2 \ln\left(\frac{T'}{T'-T}\right) \geq 0$ for all $T \in [0, T']$. Hence the first derivative is increasing. Since $\frac{\partial \Delta}{\partial T}(0, T') = 0$, the first derivative is non-negative and hence Δ is increasing in T .

The increase in expected revenues has three reasons: first the signal becomes more valuable as the difference between T and T' decreases (**information effect**); second there is more time for spreading orders over time and hence one

can reduce trading costs (**liquidity effect**); finally the variance of the price over the trading period increases (**variance effect**).

We next aim at analyzing the three effects separately. The additional revenues depend linearly on the volatility squared. We can thus eliminate the variance effect by making σ^2 inversely proportional to T . We define the variance corrected gain by

$$l(T, x) = \Delta(T, x, \sigma/\sqrt{T}, \eta) = \frac{\sigma^2}{16\eta} \left(\frac{(x-T)^2}{T} \ln \left(\frac{x}{x-T} \right) - x + \frac{3}{2}T \right),$$

for $0 \leq T \leq x$.

We next aim at analyzing the part of revenue increase that goes back to the liquidity effect. To this end we simultaneously change T and T' such that the information content of the signal remains the same. We appeal to the notion of *mutual information* for measuring the information content of the signal.

Recall that the mutual information between two normally distributed random variables X and Y is given by $I(X, Y) = -\frac{1}{2} \ln(1 - \text{corr}^2(X, Y))$ (see e.g. [13]). In particular, for any $\delta > 0$ we have $I(S_T, S_{T+\delta}) = \frac{1}{2} \ln \left(\frac{T+\delta}{T} \right)$.

For $\gamma > 0$ the mutual information $I(S_T, S_{(\gamma+1)T}) = \frac{1}{2} \ln(1 + \gamma)$ does not depend on the time horizon T . We can thus interpret

$$h(T) = l(T, (1 + \gamma)T)$$

as a variance *and* information (v&i) corrected gain function. The next proposition shows that the v&i corrected gain increases as the time horizon increases. The reason is that the additional time for trading allows to reduce liquidity costs and to make more use of the information advantage.

Proposition 3.8 (The Liquidity effect). *Let $\gamma > 0$. The v&i corrected gain function h is linear, increasing and satisfies $h(0) = 0$.*

Proof. Note that

$$\begin{aligned} \frac{16\eta}{\sigma^2} h(T) &= \gamma^2 T \ln \left(\frac{1+\gamma}{\gamma} \right) - \gamma T + \frac{1}{2}T \\ &= \left[\gamma^2 \left(\frac{1}{\gamma} - \frac{1}{2} \frac{1}{\gamma^2} + \frac{1}{3} \frac{1}{\gamma^3} - \frac{1}{4} \frac{1}{\gamma^4} + \dots \right) - \gamma + \frac{1}{2} \right] T \\ &= \gamma^2 \left(\frac{1}{3} \frac{1}{\gamma^3} - \frac{1}{4} \frac{1}{\gamma^4} + \dots \right) T, \end{aligned}$$

which shows that h is non-negative and linearly increasing in T . □

Finally we turn to the information effect. By scaling the volatility with $1/\sqrt{T}$ and the price impact parameter with T , we obtain a variance and liquidity (v&l) corrected gain function

$$k(y) = \Delta(y, T', \sigma/\sqrt{y}, \eta y),$$

defined for all $y \in [0, T']$. The function k describes the gain that exclusively goes back to the additional information, as T approaches T' . It is, as expected, increasing:

Proposition 3.9 (The Information effect). *The v&l corrected gain function k increases superlinearly on $[0, T']$ and satisfies $\lim_{x \uparrow T'} k(x) = \frac{1}{2} \frac{\sigma^2}{16\eta}$.*

Proof. $k''(x) = 2T' \frac{3T'-2x}{x^4} \ln\left(\frac{T'}{T'-x}\right) - \frac{6T'}{x^3} + \frac{1}{x^2}$ is positive and $k'(0) \leq 0$, which implies the first statement. The second is straightforward to show. \square

Conclusion

The paper studies the optimal liquidation problem under directional views and additional information. The kind of additional information chosen here is modeled via an initial enlargement of filtration, sometimes referred to as *strong initial information* (see for example [3], [11], or [10] for an introduction into the subject. See also [12] and [7] for a presentation of other types of additional information). In order to obtain the optimal liquidation strategy in closed form, the price dynamics under the enlargement must have a drift that is linear with respect to the price. For any kind of additional information resp. filtration enlargement under which the drift is linear, one can derive explicitly the additional gain by using Theorem 2.3. For example the additional information studied in the paper of Corcuera et al. [9] (strong noisy information, represented by a signal plus a decreasing noise) leads to a linear drift, and hence fits to our model.

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