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AMBIGUITY OF ω-LANGUAGES OF TURING MACHINES

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Abstract. An ω-language is a set of infinite words over a finite alphabet X. We consider the class of recursive ω-languages, i.e. the class of ω-languages accepted by Turing machines with a Büchi acceptance condition, which is also the class Σ^1_1 of (effective) analytic subsets of X^ω for some finite alphabet X. We investigate here the notion of ambiguity for recursive ω-languages with regard to acceptance by Büchi Turing machines. We first present in detail essentials on the literature on ω-languages accepted by Turing Machines. Then we give a complete and broad view on the notion of ambiguity and unambiguity of Büchi Turing machines and of the ω-languages they accept. To obtain our new results, we make use of results and methods of effective descriptive set theory.

1. Introduction

Languages of infinite words, also called ω-languages, accepted by finite automata were first studied by Büchi to prove the decidability of the monadic second order theory of one successor over the integers. Since then regular ω-languages have been much studied and many applications have been found for specification and verification of non-terminating systems, see [Tho90, Sta97, PP04] for many results and references. Other finite machines, like pushdown automata, multicounter automata, Petri nets, have also been considered for reading of infinite words, see [Sta97, EH93, Fin06].

Turing invented in 1937 what we now call Turing machines. This way he made a unique impact on the history of computing, computer science, and the mathematical theory of computability. Recall that the year 2012 was the Centenary of Alan Turing's birth and that many scientific events have commemorated this year Turing's life and work.

The acceptance of infinite words by Turing machines via several acceptance conditions, like the Büchi or Muller ones, was studied by Staiger and Wagner in [SW77, SW78] and by Cohen and Gold in [CG78]. It turned out that the classes of ω-languages accepted by non-deterministic Turing machines with Büchi or Muller acceptance conditions were the same class, the class of effective analytic sets [SW77, CG78, Sta99].

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We consider in this paper the class of recursive ω-languages, i.e. the class of ω-languages accepted by non-deterministic Turing machines with a Büchi acceptance condition, which is also the class Σ^1_1 of (effective) analytic subsets of X^ω for some finite alphabet X.

The notion of ambiguity is very important in formal language and automata theory and has been much studied for instance in the case of context-free finitary languages accepted by pushdown automata or generated by context-free grammars, [ABB97], and in the case of context-free ω-languages, [Fin03, FS03]. In the case of Turing machines reading finite words, it is easy to see that every Turing machine is equivalent to a deterministic, hence also unambiguous, Turing machine. Thus every recursive finitary language is accepted by an unambiguous Turing machine.

We investigate here the notion of ambiguity for recursive ω-languages with regard to acceptance by Büchi Turing machines. We first present in detail essentials on the literature on ω-languages accepted by Turing Machines. In particular, we describe the different ways of acceptance in which Turing machines (and also other devices) might be used to accept ω-languages. Then we give a complete and broad view on the notion of ambiguity and unambiguity of Buchi Turing machines and of the ω-languages they accept. To obtain our new results, we make use of results and methods of effective descriptive set theory, sometimes already used in other contexts for the study of other classes of ω-languages.

Notice that this study may first seem to be of no practical interest, but in fact non-deterministic Turing machines over infinite data seem to be relevant to real-life algorithmics over streams, where non-determinism may appear either by choice or because of physical constraints and perturbation.

We first show that the class of unambiguous recursive ω-languages is the class Δ^1_1 of hyperarithmetical sets. On the other hand, Arnold studied Büchi transition systems in [Arn83]. In particular, he proved that the analytic subsets of X^ω are the subsets of X^ω which are accepted by finitely branching Büchi transition systems, and that the Borel subsets of X^ω are the subsets of X^ω which are accepted by unambiguous finitely branching Büchi transition systems. Some effective versions of Büchi transition systems were studied by Staiger in [Sta93]. In particular, he proved that the subsets of X^ω which are accepted by strictly recursive finitely branching Büchi transition systems are the effective analytic subsets of X^ω. We obtain also here that the Δ^1_1-subsets of X^ω are the subsets of X^ω which are accepted by strictly recursive unambiguous finitely branching Büchi transition systems. This provides an effective analogue to the above cited result of Arnold.

Next, we prove that recursive ω-languages satisfy the following dichotomy property. A recursive ω-language $L \subseteq X^\omega$ is either unambiguous or has a great degree of ambiguity: for every Büchi Turing machine $\mathcal{T}$ accepting L, there exist infinitely many ω-words which have $2^{\aleph_0}$ accepting runs by $\mathcal{T}$.

We also show that if $L \subseteq X^\omega$ is accepted by a Büchi Turing machine $\mathcal{T}$ and L is an analytic but non-Borel set, then the set of ω-words, which have $2^{\aleph_0}$ accepting runs by $\mathcal{T}$, has cardinality $2^{\aleph_0}$. This extends a similar result of [FS03] in the case of context-free ω-languages and infinitary rational relations. In that case we say that the recursive ω-language L has the maximum degree of ambiguity.

Castro and Cucker studied decision problems for ω-languages of Turing machines in [CC89]. They gave the (high) degrees of many classical decision problems like the emptiness, the finiteness, the cofiniteness, the universality, the equality, and the inclusion problems. In [Fin09b] we obtained many new undecidability results about context-free ω-languages and infinitary rational relations. We prove here new undecidability results about ambiguity of

recursive ω-languages: it is Π^1_2-complete to determine whether a given recursive ω-language is unambiguous and it is Σ^1_2-complete to determine whether a given recursive ω-language has the maximum degree of ambiguity.

Then, using some recent results from [Fin09a] and some results of set theory, we prove that it is equiconsistent with the axiomatic system ZFC that there exists a recursive ω-language in the Borel class $\mathbf{\Pi}^0_2$, hence of low Borel rank, which has also the maximum degree of ambiguity.

The paper is organized as follows. We recall some known notions in Section 2. We study unambiguous recursive ω-languages in Section 3 and inherently ambiguous recursive ω-languages in Section 4. Some concluding remarks are given in Section 5.

2. Reminder of some well-known notions

We assume the reader to be familiar with the theory of formal (ω-)languages [Sta97, PP04]. We recall the usual notations of formal language theory.

If Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \dots a_k$, where $a_i \in \Sigma$ for $i = 1, \dots, k$, and k is an integer ≥ 1. The *length* of x is k, denoted by $|x|$. The *empty word* has no letter and is denoted by ε; its length is 0. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ. A (finitary) *language* V over an alphabet Σ is a subset of $\Sigma^\star$.

The *first infinite ordinal* is ω. An ω-*word* (or infinite word) over Σ is an ω-sequence $a_1 \dots a_n \dots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When $\sigma = a_1 \dots a_n \dots$ is an ω-word over Σ, we write $\sigma(n) = a_n$, $\sigma[n] = \sigma(1)\sigma(2)\dots\sigma(n)$ for all $n \geq 1$ and $\sigma[0] = \varepsilon$.

The concatenation of two finite words u and v is denoted $u \cdot v$ (and sometimes just uv). This operation is extended to the product of a finite word u and an ω-word v: the infinite word $u \cdot v$ is then the ω-word such that:

$(u \cdot v)(k) = u(k)$ if $k \leq |u|$, and $(u \cdot v)(k) = v(k - |u|)$ if $k > |u|$.

The *set of* ω-*words* over an alphabet Σ is denoted by Σ^ω. An ω-*language* V over an alphabet Σ is a subset of Σ^ω, and its complement (in Σ^ω) is $\Sigma^\omega - V$, denoted V^-.

We assume the reader to be familiar with basic notions of topology, which may be found in [Kec95, LT94, Sta97, PP04]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ containing at least two letters, which is called the *prefix metric*, and is defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\mathrm{pref}(u,v)}}$ where $l_{\mathrm{pref}(u,v)}$ is the first integer n such that the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v. This metric induces on Σ^ω the usual Cantor topology in which the *open subsets* of Σ^ω are of the form $W \cdot \Sigma^\omega$, for $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set.

We now recall the definition of the *Borel Hierarchy* of subsets of X^ω.

Definition 2.1. For a non-null countable ordinal α, the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of the Borel Hierarchy on the topological space X^ω are defined as follows: $\mathbf{\Sigma}^0_1$ is the class of open subsets of X^ω, $\mathbf{\Pi}^0_1$ is the class of closed subsets of X^ω, and for any countable ordinal $\alpha \geq 2$:
$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of X^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of X^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Sigma}^0_\gamma$.

A set $L \subseteq X^\omega$ is Borel iff it is in the union $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$, where ω_1 is the first uncountable ordinal.

For a countable ordinal α, a set $L \subseteq X^\omega$ is a Borel set of *rank* α iff it is in $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma}^0_\gamma \cup \mathbf{\Pi}^0_\gamma)$.

There are also some subsets of X^ω which are not Borel. In particular, the class of Borel subsets of X^ω is strictly included in the class $\mathbf{\Sigma}^1_1$ of *analytic sets* which are obtained by projection of Borel sets. The *co-analytic sets* are the complements of analytic sets.

For two alphabets X and Y and two infinite words $x \in X^\omega$ and $y \in Y^\omega$, we denote (x, y) the infinite word over the alphabet $X \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 1$.

Definition 2.2. A subset A of X^ω is in the class $\mathbf{\Sigma}^1_1$ of *analytic* sets iff there exist a finite alphabet Y and a Borel subset B of $(X \times Y)^\omega$ such that: $[x \in A] \Longleftrightarrow [\exists y \in Y^\omega\ (x, y) \in B]$.

We now define completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq X^\omega$ is said to be a $\mathbf{\Sigma}^0_\alpha$ (respectively, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_1$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma}^0_\alpha$ (respectively, $E \in \mathbf{\Pi}^0_\alpha$, $E \in \mathbf{\Sigma}^1_1$) iff there exists a continuous function $f : Y^\omega \to X^\omega$ such that $E = f^{-1}(F)$, i.e. such that ($x \in E$ iff $f(x) \in F$).

We now recall the definition of the arithmetical hierarchy of ω-languages which form the effective analogue to the hierarchy of Borel sets of finite ranks, see [Sta97].

Let X be a finite alphabet. An ω-language $L \subseteq X^\omega$ belongs to the class Σ_n if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times X^\star$ such that

$$L = \{\sigma \in X^\omega \mid \exists k_1 \ldots Q_n k_n \quad (k_1, \ldots, k_{n-1}, \sigma[k_n + 1]) \in R_L\}$$

where Q_i is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An ω-language $L \subseteq X^\omega$ belongs to the class Π_n if and only if its complement $X^\omega - L$ belongs to the class Σ_n. The inclusion relations that hold between the classes Σ_n and Π_n are the same as for the corresponding classes of the Borel hierarchy. The classes Σ_n and Π_n are included in the respective classes $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$ of the Borel hierarchy, and cardinality arguments suffice to show that these inclusions are strict.

As in the case of the Borel hierarchy, projections of arithmetical sets lead beyond the arithmetical hierarchy, to the analytical hierarchy of ω-languages. The first class of this hierarchy is the (lightface) class Σ^1_1 of *effective analytic sets* which are obtained by projection of arithmetical sets. In fact an ω-language $L \subseteq X^\omega$ is in the class Σ^1_1 iff it is the projection of an ω-language over the alphabet $X \times \{0, 1\}$ which is in the class Π_2. The (lightface) class Π^1_1 of *effective co-analytic sets* is simply the class of complements of effective analytic sets. We denote as usual $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.

The (lightface) class Σ^1_1 of effective analytic sets is strictly included into the (boldface) class $\mathbf{\Sigma}^1_1$ of analytic sets.

We assume the reader to be familiar with the arithmetical and analytical hierarchies on subsets of $\mathbb{N}$, these notions may be found in the textbooks on computability theory [Rog67] [Odi89, Odi99]. Notice that we will not have to consider subsets of $\mathbb{N}$ of ranks greater than 2 in the analytical hierarchy, so the most complex subsets of $\mathbb{N}$ occcuring in this paper will be Σ^1_2-sets or Π^1_2-sets.

We shall consider in the sequel some Σ^1_1 or Π^1_1 subsets of product spaces like $X^\omega \times Y^\omega$ or $\mathbb{N} \times Y^\omega$. Moreover, in effective descriptive set theory one often considers the notion of relativized class $\Sigma^1_1(w)$: for $w \in X^\omega$, a set $L \subseteq Y^\omega$ is a $\Sigma^1_1(w)$-set iff there exists a Σ^1_1-set

$T \subseteq X^\omega \times Y^\omega$ such that $L = \{y \in Y^\omega \mid (w, y) \in T\}$. A set $L \subseteq Y^\omega$ is a $\Pi^1_1(w)$-set iff its complement is a $\Sigma^1_1(w)$-set. A set $L \subseteq Y^\omega$ is a $\Delta^1_1(w)$-set iff it is in the class $\Sigma^1_1(w) \cap \Pi^1_1(w)$. We say that $y \in Y^\omega$ is in the class Δ^1_1 (respectively, $\Delta^1_1(w)$) iff the singleton $\{y\}$ is a Δ^1_1-set (respectively, $\Delta^1_1(w)$-set).

Recall now the notion of acceptance of infinite words by Turing machines considered by Cohen and Gold in [CG78].

Definition 2.3. A non-deterministic Turing machine $\mathcal{M}$ is a 5-tuple $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$, where Q is a finite set of states, Σ is a finite input alphabet, Γ is a finite tape alphabet satisfying $\Sigma \subseteq \Gamma$, q_0 is the initial state, and δ is a mapping from $Q \times \Gamma$ to subsets of $Q \times \Gamma \times \{L, R, S\}$. A configuration of $\mathcal{M}$ is a triplet (q, σ, i), where $q \in Q$, $\sigma \in \Gamma^\omega$ and $i \in \mathbb{N}$. An infinite sequence of configurations $r = (q_i, \alpha_i, j_i)_{i \geq 1}$ is called a run of $\mathcal{M}$ on $w \in \Sigma^\omega$ iff:
(a) $(q_1, \alpha_1, j_1) = (q_0, w, 1)$, and
(b) for each $i \geq 1$, $(q_i, \alpha_i, j_i) \vdash (q_{i+1}, \alpha_{i+1}, j_{i+1})$,
where $\vdash$ is the transition relation of $\mathcal{M}$ defined as usual. The run r is said to be complete if every position is visited, i.e. if $(\forall n \geq 1)(\exists k \geq 1)(j_k \geq n)$. The run r is said to be oscillating if some position is visited infinitely often, i.e. if $(\exists k \geq 1)(\forall n \geq 1)(\exists m \geq n)(j_m = k)$.

Definition 2.4. Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ be a non-deterministic Turing machine and $F \subseteq Q$, $\mathcal{F} \subseteq 2^Q$. The ω-language $1'$-accepted (respectively, 2-accepted) by $(\mathcal{M}, F)$ is the set of ω-words $\sigma \in \Sigma^\omega$ such that there exists a complete non-oscillating run $r = (q_i, \alpha_i, j_i)_{i \geq 1}$ of $\mathcal{M}$ on σ such that, for all $i, q_i \in F$ (respectively, for infinitely many $i, q_i \in F$). The ω-language 3-accepted by $(\mathcal{M}, \mathcal{F})$ is the set of ω-words $\sigma \in \Sigma^\omega$ such that there exists a complete non-oscillating run r of $\mathcal{M}$ on σ such that the set of states appearing infinitely often during the run r is an element of $\mathcal{F}$.

The $1'$-acceptance condition is also considered by Castro and Cucker in [CC89]. The 2-acceptance and 3-acceptance conditions are now usually called Büchi and Muller acceptance conditions. Cohen and Gold proved the following result in [CG78, Theorem 8.2].

Theorem 2.5 (Cohen and Gold [CG78])**.** *An ω-language is accepted by a non-deterministic Turing machine with $1'$-acceptance condition iff it is accepted by a non-deterministic Turing machine with Büchi (or Muller) acceptance condition.*

Notice that this result holds because Cohen and Gold's Turing machines accept infinite words via *complete non-oscillating runs*, while $1'$, Büchi or Muller acceptance conditions refer to the sequence of states entered during an infinite run.

There are actually three types of a required behaviour on the input tape which have been considered in the literature. We now recall the classification of these three types given in [Sta99, Sta00].

Type 1. This is the type considered in [SW77, SW78, Sta97]. Here we do not take into consideration the behaviour of the Turing machine on the input tape. Thus the acceptance depends only on the infinite sequence of states entered by the machine during the infinite computation. In particular, the machine may not read the whole input tape.

Type 2. This is the appoach of [EH93]. Here one requires that the machine reads the whole infinite tape (i.e. that the run is complete).

Type 3. This is the type which is considered by Cohen and Gold in [CG78]; the acceptance of infinite words is defined via *complete non-oscillating runs*.

We refer to [SW78, Sta99, FS00, Sta00] for a study of these different approaches. They are in particular explicitely investigated for deterministic Turing machines in [SW78], [Sta99], and [FS00].

Notice that "reading the whole input tape" solely is covered by the Büchi acceptance condition.

In this paper, we shall consider Turing machines accepting ω-words via acceptance by runs reading the whole input tape (i.e., not necessarily non-oscillating). By [Sta99, Theorem 16] (see also [Sta00, Theorem 5.2]) we have the following characterization of the class of ω-languages accepted by these non-deterministic Turing machines.

Theorem 2.6 ([Sta99])**.** *The class of ω-languages accepted by non-deterministic Turing machines with $1'$ (respectively, Büchi, Muller) acceptance condition is the class Σ^1_1 of effective analytic sets.*

In the sequel we shall also restrict our study to the Büchi acceptance condition. But one can easily see that all the results of this paper are true for any other acceptance condition leading to the class Σ^1_1 of effective analytic sets. For instance it follows from [CG78, Note 2 page 12] and from Theorem 2.6 that the class of ω-languages accepted by Cohen's and Gold's non-deterministic Turing machines with $1'$ (respectively, Büchi, Muller) acceptance condition is the class Σ^1_1. Moreover the class Σ^1_1 is also the class of ω-languages accepted by Turing machines with Büchi acceptance condition if we do not require that the Turing machine reads the whole infinite tape but only that it runs forever, [Sta97].

Due to the above results, we shall say, as in [Sta97], that an ω-language is *recursive* iff it belongs to the class Σ^1_1. Notice that in another presentation, as in [Rog67], the recursive ω-languages are those which are in the class $\Sigma_1 \cap \Pi_1$, see also [LT94].

On the other hand, we mention that ω-languages of deterministic Turing machines form the class of boolean combinations of arithmetical Π^0_2-sets, [Sta97]. Selivanov gave a very fine topological classification of these languages, based on the Wadge hierarchy of Borel sets, in [Sel03a, Sel03b].

3. Unambiguous recursive ω-languages

We have said in the preceding section that we shall restrict our study to the Büchi acceptance condition and to acceptance via runs reading the whole input tape.

We now briefly justify the restriction to Type 2 acceptance for the study of ambiguity of recursive ω-languages, by showing that the three types defined in the preceding section, along with the Büchi acceptance condition, give the same class of ω-languages accepted by unambiguous Turing machines.

We first give the two definitions.

Definition 3.1. A Büchi Turing machine $\mathcal{M}$ with Type i acceptance, reading ω-words over an alphabet Σ, is said to be unambiguous iff for every ω-word $x \in \Sigma^\omega$ the machine $\mathcal{M}$ has at most one accepting run over x.

Definition 3.2. Let Σ be a finite alphabet. A recursive ω-language $L \subseteq \Sigma^\omega$ is said to be unambiguous of Type i iff it is accepted by (at least) one unambiguous Büchi Turing machine with Type i acceptance. Otherwise the recursive ω-language L is said to be inherently ambiguous of Type i.

We now informally explain why an ω-language is unambiguous for Type 1 acceptance iff it is unambiguous for Type 2 acceptance iff it is unambiguous for Type 3 acceptance.

(Type 1 unambiguity)$\Rightarrow$ (Type 2 unambiguity).
Let L be an ω-language which is accepted by an unambiguous Büchi Turing machine $\mathcal{M}$ for Type 1 acceptance. Using the "Folding process" described by Cohen and Gold in [CG78, pages 11-12], we can construct another Turing machine $\mathcal{M}'$ which simulates the machine $\mathcal{M}$ and accepts the same language but which has only complete and non-oscillating runs. Notice that each infinite run of $\mathcal{M}$ provides a unique run of $\mathcal{M}'$ thus the ω-language L is accepted unambiguously by the Büchi Turing machine $\mathcal{M}'$ for Type 2 (and also Type 3) acceptance.

(Type 2 unambiguity)$\Rightarrow$ (Type 3 unambiguity).
Let L be an ω-language which is accepted by an unambiguous Büchi Turing machine $\mathcal{M}$ for Type 2 acceptance. Using the fact that "reading the whole input tape" solely is covered by the Büchi acceptance condition, one can construct an equivalent Type 2 Büchi Turing machine $\mathcal{M}'$ which is still unambiguous and accepts the same language with the following additional property: any run which is not complete does not satisfy the Büchi condition. Next we can use again the "Folding process" (see [CG78, Note 2 page 12]) and obtain an unambiguous Büchi Turing machine $\mathcal{M}''$ for Type 3 acceptance which accepts the same ω-language L.

(Type 3 unambiguity)$\Rightarrow$ (Type 1 unambiguity).
Let L be an ω-language which is accepted by an unambiguous Büchi Turing machine $\mathcal{M}$ for Type 3 acceptance. Then every ω-word x which is accepted by the machine $\mathcal{M}$ has a unique accepting run. But there may exist some non-complete, or oscillating, runs of $\mathcal{M}$ over x which satisfy the Büchi acceptance condition. Intuitively we can transform the machine $\mathcal{M}$ to obtain a new machine $\mathcal{M}'$ which has essentially the same runs but in such a way that non-complete, or oscillating, runs of $\mathcal{M}'$ will no longer satisfy the Büchi acceptance condition. Then the new Büchi Turing machine $\mathcal{M}'$ accepts the same ω-language L but for Type 1 acceptance and the machine $\mathcal{M}'$ is unambiguous.

From now on in this paper a Büchi Turing machine will be a Turing machine reading ω-words and accepting ω-words with a Büchi acceptance condition via runs reading the whole input tape. And we shall say that a recursive ω-language is unambiguous iff it is unambiguous of Type 2 (iff it is unambiguous of Type 1 or 3).

We can now state our first result.

Proposition 3.3. *If Σ is a finite alphabet and $L \subseteq \Sigma^\omega$ is an unambiguous recursive ω-language then L belongs to the (effective) class Δ^1_1.*

Proof. Let $L \subseteq \Sigma^\omega$ be an ω-language accepted by an unambiguous Büchi Turing machine $(\mathcal{M}, F)$, where $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ is a Turing machine and $F \subseteq Q$. Recall that a configuration of the Turing machine $\mathcal{M}$ is a triple (q, σ, i), where $q \in Q$, $\sigma \in \Gamma^\omega$ and $i \in \mathbb{N}$. It can be coded by the infinite word $q^i \cdot \sigma$ over the alphabet $Q \cup \Gamma$, where we have assumed without loss of generality that Q and Γ are disjoint. Then a run of $\mathcal{M}$ on $w \in \Sigma^\omega$ is an infinite sequence of configurations $r = (q_i, \alpha_i, j_i)_{i \geq 1}$ which is then coded by an infinite sequence of ω-words $(r_i)_{i \geq 1} = (q_i^{j_i} \cdot \alpha_i)_{i \geq 1}$ over $Q \cup \Gamma$. Using now a recursive bijection $b : (\mathbb{N} \setminus \{0\})^2 \to \mathbb{N} \setminus \{0\}$ and its inverse b^{-1} we can effectively code the sequence $(r_i)_{i \geq 1}$ by a single infinite word $r' \in (Q \cup \Gamma)^\omega$ defined by: for every integer $j \geq 1$ such that $b^{-1}(j) = (i_1, i_2)$, $r'(j) = r_{i_1}(i_2)$. Moreover the infinite word $r' \in (Q \cup \Gamma)^\omega$ can be coded in a recursive manner by an infinite word over the alphabet $\{0, 1\}$. We can then identify r with its code $\bar{r} \in \{0, 1\}^\omega$ and this will be often done in the sequel. Let now R be defined by:

$$R = \{(w, r) \mid w \in \Sigma^\omega \text{ and } r \in \{0, 1\}^\omega \text{ is an accepting run of } (\mathcal{M}, F) \text{ on the } \omega\text{-word } w\}.$$

The set R is a Δ^1_1-set, and even an arithmetical set: it is easy to see that it is accepted by a *deterministic* Muller Turing machine and thus it is a Δ^0_3-subset of the space $(\Sigma \times \{0, 1\})^\omega$, see [Sta97].

Consider now the projection $\mathrm{PROJ}_{\Sigma^\omega} : \Sigma^\omega \times \{0, 1\}^\omega \to \Sigma^\omega$ defined by $\mathrm{PROJ}_{\Sigma^\omega}(w, r) = w$ for all $(w, r) \in \Sigma^\omega \times \{0, 1\}^\omega$. This projection is a recursive function, i.e. "there is an algorithm which given sufficiently close approximations to (w, r) produces arbitrarily accurate approximations to $\mathrm{PROJ}_{\Sigma^\omega}(w, r)$", see [Mos09]. Moreover it is *injective* on the Δ^1_1-set R because the Büchi Turing machine $(\mathcal{M}, F)$ is unambiguous. But the image of a Δ^1_1-set by an injective recursive function is a Δ^1_1-set, see [Mos09, page 169] and thus the recursive ω-language $L = \mathrm{PROJ}_{\Sigma^\omega}(R)$ is a Δ^1_1-subset of Σ^ω. □

In order to prove a converse statement we now first recall the notion of Büchi transition system.

Definition 3.4. A Büchi transition system is a tuple $\mathcal{T} = (\Sigma, Q, \delta, q_0, Q_f)$, where Σ is a finite input alphabet, Q is a countable set of states, $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation, $q_0 \in Q$ is the initial state, and $Q_f \subseteq Q$ is the set of final states. A run of $\mathcal{T}$ over an infinite word $\sigma \in \Sigma^\omega$ is an infinite sequence of states $(t_i)_{i \geq 0}$, such that $t_0 = q_0$, and for each $i \geq 0$, $(t_i, \sigma(i+1), t_{i+1}) \in \delta$. The run is said to be accepting iff there are infinitely many integers i such that t_i is in Q_f. An ω-word $\sigma \in \Sigma^\omega$ is accepted by $\mathcal{T}$ iff there is (at least) one accepting run of $\mathcal{T}$ over σ. The ω-language $L(\mathcal{T})$ accepted by $\mathcal{T}$ is the set of ω-words accepted by $\mathcal{T}$. The transition system is said to be *unambiguous* if each infinite word $\sigma \in \Sigma^\omega$ has at most one accepting run by $\mathcal{T}$. The transition system is said to be *finitely branching* if for each state $q \in Q$ and each $a \in \Sigma$, there are only finitely many states q' such that $(q, a, q') \in \delta$.

Arnold proved the following theorem in [Arn83].

Theorem 3.5. *Let Σ be an alphabet having at least two letters.*

(1) *The analytic subsets of Σ^ω are the subsets of Σ^ω which are accepted by finitely branching Büchi transition systems.*

(2) *The Borel subsets of Σ^ω are the subsets of Σ^ω which are accepted by unambiguous finitely branching Büchi transition systems.*

It is also very natural to consider effective versions of Büchi transition systems where the sets Q, δ, and Q_f are recursive. Such transition systems are studied by Staiger in [Sta93] where Q is actually either the set $\mathbb{N}$ of natural numbers or a finite segment of it, and they are called *strictly recursive*. It is proved by Staiger that the subsets of Σ^ω which are accepted by strictly recursive finitely branching Büchi transition systems are the effective analytic subsets of Σ^ω.

On the other hand, the Büchi transition systems are considered by Finkel and Lecomte in [FL09] where they are used in the study of topological properties of ω-powers. Using an effective version of a theorem of Kuratowski, it is proved in [FL09] that every Δ^1_1-subset of $\{0,1\}^\omega$ is actually accepted by an unambiguous strictly recursive finitely branching Büchi transition system (where the degree of branching of the transition system is actually equal to 2). Using an easy coding this is easily extended to the case of any Δ^1_1-subset of Σ^ω, where Σ is a finite alphabet.

Theorem 3.6. *Let Σ be an alphabet having at least two letters. An ω-language $L \subseteq \Sigma^\omega$ is an unambiguous recursive ω-language iff L belongs to the (effective) class Δ^1_1.*

Proof. The implication from left to right is given by Proposition 3.3. We now prove the reverse implication. Using the fact that every recursive set of finite words over a finite alphabet Γ is accepted by a *deterministic* hence also unambiguous Turing machine reading finite words, we can easily see that every ω-language which is accepted by an unambiguous strictly recursive finitely branching Büchi transition system is also accepted by an unambiguous Büchi Turing machine. □

Notice that we have also the effective analogue to Arnold's Theorem 3.5.

Theorem 3.7. *Let Σ be an alphabet having at least two letters.*

(1) *The effective analytic subsets of Σ^ω are the subsets of Σ^ω which are accepted by strictly recursive finitely branching Büchi transition systems.*
(2) *The Δ^1_1-subsets of Σ^ω are the subsets of Σ^ω which are accepted by strictly recursive unambiguous finitely branching Büchi transition systems.*

Proof. Item 1 is proved in [Sta93]. To prove that every ω-language which is accepted by a strictly recursive unambiguous finitely branching Büchi transition system is a Δ^1_1-set we can reason as in the case of Turing machines (see the proof of Proposition 3.3). As said above, the converse statement is proved in [FL09]. □

4. Inherently ambiguous recursive ω-languages

The notion of ambiguity for context-free ω-languages has been studied in [Fin03, FS03]. In particular it was proved in [FS03] that every context-free ω-language which is non-Borel has a maximum degree of ambiguity. This was proved by stating firstly a lemma, using a theorem of Lusin and Novikov. We now recall this lemma and its proof.

Lemma 4.1 ([FS03])**.** *Let Σ and X be two finite alphabets having at least two letters and B be a Borel subset of $\Sigma^\omega \times X^\omega$ such that $\mathrm{PROJ}_{\Sigma^\omega}(B)$ is not a Borel subset of Σ^ω. Then there*

are $2^{\aleph_0}$ *ω-words* $\alpha \in \Sigma^\omega$ *such that the section* $B_\alpha = \{\beta \in X^\omega \mid (\alpha, \beta) \in B\}$ *has cardinality* $2^{\aleph_0}$.

Proof. Let Σ and X be two finite alphabets having at least two letters and B be a Borel subset of $\Sigma^\omega \times X^\omega$ such that $\mathrm{PROJ}_{\Sigma^\omega}(B)$ is not Borel. In a first step we prove that there are uncountably many $\alpha \in \Sigma^\omega$ such that the section B_α is uncountable. Recall that by a Theorem of Lusin and Novikov, see [Kec95, page 123], if for all $\alpha \in \Sigma^\omega$, the section B_α of the Borel set B was countable, then $\mathrm{PROJ}_{\Sigma^\omega}(B)$ would be a Borel subset of Σ^ω. Thus there exists at least one $\alpha \in \Sigma^\omega$ such that B_α is uncountable. In fact we can prove that the set $U = \{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable }\}$ is uncountable, otherwise $U = \{\alpha_0, \alpha_1, \ldots \alpha_n, \ldots\}$ would be Borel as the countable union of the closed sets $\{\alpha_i\}$, $i \geq 0$. Notice that for $\alpha \in \Sigma^\omega$ we have $\{\alpha\} \times B_\alpha = B \cap [\{\alpha\} \times X^\omega]$ so the set $\{\alpha\} \times B_\alpha$ is Borel as intersection of two Borel sets. Thus for each $n \geq 0$ the set $\{\alpha_n\} \times B_{\alpha_n}$ would be Borel, and $C = \cup_{n \in \omega} \{\alpha_n\} \times B_{\alpha_n}$ would be Borel as a countable union of Borel sets. So $D = B - C$ would be borel too. But all sections of D would be countable thus $\mathrm{PROJ}_{\Sigma^\omega}(D)$ would be Borel by Lusin and Novikov's Theorem. Then $\mathrm{PROJ}_{\Sigma^\omega}(B) = U \cup \mathrm{PROJ}_{\Sigma^\omega}(D)$ would be also Borel as union of two Borel sets, and this would lead to a contradiction. So we have proved that the set $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable }\}$ is uncountable.

On the other hand we know from another Theorem of Descriptive Set Theory that the set $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is countable }\}$ is a $\mathbf{\Pi^1_1}$-subset of Σ^ω, see [Kec95, page 123]. Thus its complement $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable }\}$ is analytic. But by Suslin's Theorem an analytic subset of Σ^ω is either countable or has cardinality $2^{\aleph_0}$, [Kec95, p. 88]. Therefore the set $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable }\}$ has cardinality $2^{\aleph_0}$. Recall now that we have already seen that, for each $\alpha \in \Sigma^\omega$, the set $\{\alpha\} \times B_\alpha$ is Borel. Thus B_α itself is Borel and by Suslin's Theorem B_α is either countable or has cardinality $2^{\aleph_0}$. From this we deduce that $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable }\} = \{\alpha \in \Sigma^\omega \mid B_\alpha \text{ has cardinality } 2^{\aleph_0}\}$ has cardinality $2^{\aleph_0}$. □

We can now apply this lemma to the study of ambiguity of Turing machines, in a similar way as in [FS03] for context-free ω-languages. We can now state the following result.

Theorem 4.2. *Let* $L \subseteq \Sigma^\omega$ *be an ω-language accepted by a Büchi Turing machine* $(\mathcal{M}, F)$ *such that* L *is an analytic but non-Borel set. The set of ω-words, which have* $2^{\aleph_0}$ *accepting runs by* $(\mathcal{M}, F)$*, has cardinality* $2^{\aleph_0}$.

Proof. Let $L \subseteq \Sigma^\omega$ be an analytic but non-Borel ω-language accepted by a Büchi Turing machine $(\mathcal{M}, F)$, where $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ is a Turing machine and $F \subseteq Q$. As in the proof of Proposition 3.3 we consider the set R defined by:

$$R = \{(w, r) \mid w \in \Sigma^\omega \text{ and } r \in \{0,1\}^\omega \text{ is an accepting run of } (\mathcal{M}, F) \text{ on the } \omega\text{-word } w\}.$$

The set R is a Δ^1_1-set, and thus it is a Borel subset of $\Sigma^\omega \times \{0,1\}^\omega$. But by hypothesis the set $\mathrm{PROJ}_{\Sigma^\omega}(R) = L$ is not Borel. Thus it follows from Lemma 4.1 that the set of ω-words, which have $2^{\aleph_0}$ accepting runs by $(\mathcal{M}, F)$, has cardinality $2^{\aleph_0}$. □

We now know that every recursive ω-language which is non-Borel has a maximum degree of ambiguity. On the other hand Proposition 3.3 states that every recursive ω-language which does not belong to the (effective) class Δ^1_1 is actually inherently ambiguous. In fact we can prove a stronger result, using the following effective version of a theorem of Lusin and Novikov:

Theorem 4.3 (see 4.F.16 page 195 of [Mos09])**.** *Let Σ and X be two finite alphabets having at least two letters and B be a Δ^1_1-subset of $\Sigma^\omega \times X^\omega$ such that for all $\alpha \in \Sigma^\omega$ the section $B_\alpha = \{\beta \in X^\omega \mid (\alpha, \beta) \in B\}$ is countable. Then the set $\mathrm{PROJ}_{\Sigma^\omega}(B)$ is also a Δ^1_1-subset of Σ^ω.*

We can now state the following result.

Theorem 4.4. *Let $L \subseteq \Sigma^\omega$ be an ω-language accepted by a Büchi Turing machine $(\mathcal{M}, F)$ such that L is not a Δ^1_1-set. Then there exist infinitely many ω-words which have $2^{\aleph_0}$ accepting runs by $(\mathcal{M}, F)$.*

Proof. Let $L \subseteq \Sigma^\omega$ be an ω-language which is not a Δ^1_1-set and which is accepted by a Büchi Turing machine $(\mathcal{M}, F)$, where $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ is a Turing machine and $F \subseteq Q$. As in the proof of Proposition 3.3 we consider the set R defined by:

$$R = \{(w, r) \mid w \in \Sigma^\omega \text{ and } r \in \{0,1\}^\omega \text{ is an accepting run of } (\mathcal{M}, F) \text{ on the } \omega\text{-word } w\}.$$

The set of accepting runs of $(\mathcal{M}, F)$ on an ω-word $w \in \Sigma^\omega$ is the section

$$R_w = \{r \in \{0,1\}^\omega \mid (w, r) \in R\}.$$

We have seen that the set R is a Δ^1_1-set hence also a Σ^1_1-set, and thus for each ω-word $w \in \Sigma^\omega$ the set R_w is in the relativized class $\Sigma^1_1(w)$. On the other hand it is known that a $\Sigma^1_1(w)$-set is countable if and only if all of its members are in the class $\Delta^1_1(w)$, see [Mos09, page 184]. Therefore the set R_w is countable iff for all $r \in R_w$ $r \in \Delta^1_1(w)$. Notice also that R_w is an analytic set thus it is either countable or has the cardinality $2^{\aleph_0}$ of the continuum.

Recall that Harrington, Kechris and Louveau obtained a coding of Δ^1_1-subsets of $\{0,1\}^\omega$ in [HKL90]. Notice that in the same way they obtained also a coding of the $\Delta^1_1(w)$-subsets of $\{0,1\}^\omega$ which we now recall.

For each $w \in \Sigma^\omega$ there esists a $\Pi^1_1(w)$-set $W(w) \subseteq \mathbb{N}$ and a $\Pi^1_1(w)$-set $C(w) \subseteq \mathbb{N} \times \{0,1\}^\omega$ such that, if we denote $C_n(w) = \{x \in \{0,1\}^\omega \mid (n, x) \in C(w)\}$, then $\{(n, \alpha) \in \mathbb{N} \times \{0,1\}^\omega \mid n \in W(w) \text{ and } \alpha \notin C_n(w)\}$ is a $\Pi^1_1(w)$-subset of the product space $\mathbb{N} \times \{0,1\}^\omega$ and the $\Delta^1_1(w)$-subsets of $\{0,1\}^\omega$ are the sets of the form $C_n(w)$ for $n \in W(w)$.

We can now express $[(\exists n \in W(w))\ C_n(w) = \{x\}]$ by the sentence $\phi(x, w)$:

$$\begin{aligned} &\exists n\ [\ n \in W(w) \text{ and } (n, x) \in C(w) \text{ and} \\ &\qquad \forall y \in \{0,1\}^\omega\ [(n \in W(w) \text{ and } (n, y) \notin C(w)) \text{ or } (y = x)]] \end{aligned}$$

But we know that $C(w)$ is a $\Pi^1_1(w)$-set and that $\{(n, \alpha) \in \mathbb{N} \times \{0,1\}^\omega \mid n \in W(w) \text{ and } \alpha \notin C_n(w)\}$ is a $\Pi^1_1(w)$-subset of $\mathbb{N} \times \{0,1\}^\omega$. Moreover the quantification $\exists n$ in the above formula is a first-order quantification therefore the above formula $\phi(x, w)$ is a Π^1_1-formula. We can now express that R_w is countable by the sentence $\psi(w)$:

$$\forall x \in \{0,1\}^\omega\ [(x \notin R_w) \text{ or } (\exists n \in W(w)\ C_n(w) = \{x\})]$$

that is,

$$\forall x \in \{0,1\}^\omega\ [(x \notin R_w) \text{ or } \phi(x, w)]$$

This is a Π^1_1-formula thus R_w is uncountable is expressed by a Σ^1_1-formula and thus the set

$$D = \{w \mid w \in \Sigma^\omega \text{ and there are uncountably many accepting runs of } (\mathcal{M}, F) \text{ on } w\}.$$

is a Σ^1_1-set.

Towards a contradiction, assume now that the set D is finite. Then for every $x \in D$ the singleton $\{x\}$ is a Δ^1_1-subset of $\{0,1\}^\omega$ because D is a countable Σ^1_1-set. But D is finite so it would be the union of a *finite* set of Δ^1_1-sets and thus it would be also a Δ^1_1-set. Consider now the set $R' = R \setminus (D \times \{0,1\}^\omega)$. This set would be also a Δ^1_1-set and $\mathrm{PROJ}_{\Sigma^\omega}(R') = L \setminus D$ would not be in the class Δ^1_1 because by hypothesis L is not a Δ^1_1-set. But then we could infer from Theorem 4.3 that there would exist an ω-word $w \in L \setminus D$ having uncountably many accepting runs by the Büchi Turing machine $(\mathcal{M}, F)$. This is impossible by definition of D and thus we can conclude that D is infinite, i.e. that there exist infinitely many ω-words which have uncountably many, or equivalently $2^{\aleph_0}$, accepting runs by $(\mathcal{M}, F)$. $\square$

Remark 4.5. We can not obtain a stronger result like "there exist $2^{\aleph_0}$ ω-words which have $2^{\aleph_0}$ accepting runs by $(\mathcal{M}, F)$" in the conclusion of the above Theorem 4.4 because there are some countable subsets of Σ^ω which are in the class $\Sigma^1_1 \setminus \Delta^1_1$.

Remark 4.6. The result given by Theorem 4.4 is a dichotomy result for recursive ω-languages. A recursive ω-language L is either unambiguous or has a great degree of ambiguity: for every Büchi Turing machine $(\mathcal{M}, F)$ accepting it there exist infinitely many ω-words which have $2^{\aleph_0}$ accepting runs by $(\mathcal{M}, F)$. This could be compared to the case of context-free ω-languages accepted by Büchi pushdown automata: it is proved in [Fin03] that there exist some context-free ω-languages which are inherently ambiguous of every finite degree $n \geq 2$ (and also some others of infinite degree).

There are many examples of recursive ω-languages which are Borel and inherently ambiguous of great degree since there are some sets which are $(\Sigma^1_1 \setminus \Delta^1_1)$-sets in every Borel class $\mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$. On the other hand recall that Kechris, Marker and Sami proved in [KMS89] that the supremum of the set of Borel ranks of (effective) Σ^1_1-sets is the ordinal γ^1_2. This ordinal is precisely defined in [KMS89] where it is proved to be strictly greater than the ordinal δ^1_2 which is the first non-Δ^1_2 ordinal. In particular it holds that $\omega_1^{\mathrm{CK}} < \gamma^1_2$, where ω_1^{CK} is the first non-recursive ordinal. On the other hand it is known that the ordinals $\gamma < \omega_1^{\mathrm{CK}}$ are the Borel ranks of (effective) Δ^1_1-sets. Thus we can state the following result.

Proposition 4.7. *If Σ is a finite alphabet and $L \subseteq \Sigma^\omega$ is a recursive ω-language which is Borel of rank α greater than or equal to the ordinal ω_1^{CK} then for every Büchi Turing machine $(\mathcal{M}, F)$ accepting it there exist infinitely many ω-words which have $2^{\aleph_0}$ accepting runs by $(\mathcal{M}, F)$.*

Notice that this can be applied in a similar way to context-free ω-languages accepted by Büchi pushdown automata and to infinitary rational relations accepted by Büchi 2-tape automata, where ambiguity refers here to acceptance by these less powerful accepting devices, see [Fin03, FS03]. If $L \subseteq \Sigma^\omega$ is a context-free ω-language (respectively, $L \subseteq \Sigma^\omega \times \Gamma^\omega$ is an infinitary rational relation) which is Borel of rank α greater than or equal to the ordinal ω_1^{CK} then L is an inherently ambiguous context-free ω-language (respectively, infinitary rational relation) of degree $2^{\aleph_0}$ as defined in [Fin03, FS03].

We have established in Theorem 4.2 that if $L \subseteq \Sigma^\omega$ is an ω-language accepted by a Büchi Turing machine $(\mathcal{M}, F)$ such that L is an analytic but non-Borel set, then the set of ω-words, which have $2^{\aleph_0}$ accepting runs by $(\mathcal{M}, F)$, has cardinality $2^{\aleph_0}$. It is then very natural to ask whether this very strong ambiguity property is characteristic of *non-Borel*

recursive ω-languages or if some *Borel* recursive ω-languages could also have this strongest degree of ambiguity. We first formally define this notion.

Definition 4.8. Let Σ be a finite alphabet and $L \subseteq \Sigma^\omega$ be a recursive ω-language. Then the ω-language L is said to have the maximum degree of ambiguity if, for every Büchi Turing machine $(\mathcal{M}, F)$ accepting L, the set of ω-words, which have $2^{\aleph_0}$ accepting runs by $(\mathcal{M}, F)$, has cardinality $2^{\aleph_0}$. The set of recursive ω-languages having the maximum degree of ambiguity is denoted Max-Amb.

We are firstly going to state some undecidability properties. Recall that a Büchi Turing machine has a finite description and thus one can associate in a recursive and injective manner a positive integer z to each Büchi Turing machine $\mathcal{T}$. The integer z is then called the index of the machine $\mathcal{T}$. In the sequel we consider we have fixed such a Gödel numbering of the Büchi Turing machines, as in [Fin09a, Fin09b], and the Büchi Turing machine of index z, reading words over the alphabet $\Gamma = \{a, b\}$, will be denoted $\mathcal{T}_z$.

We recall the notions of 1-reduction and of Σ^1_n-completeness (respectively, Π^1_n-completeness) for subsets of $\mathbb{N}$ (or of $\mathbb{N}^l$ for some integer $l \geq 2$). Given two sets $A, B \subseteq \mathbb{N}$ we say A is 1-reducible to B and write $A \leq_1 B$ if there exists a total computable injective function f from $\mathbb{N}$ to $\mathbb{N}$ with $A = f^{-1}[B]$. A set $A \subseteq \mathbb{N}$ is said to be Σ^1_n-complete (respectively, Π^1_n-complete) iff A is a Σ^1_n-set (respectively, Π^1_n-set) and for each Σ^1_n-set (respectively, Π^1_n-set) $B \subseteq \mathbb{N}$ it holds that $B \leq_1 A$. It is known that, for each integer $n \geq 1$, there exist some Σ^1_n-complete and some Π^1_n-complete subsets of $\mathbb{N}$; some examples of such sets are described in [Rog67, CC89].

Theorem 4.9. *The unambiguity problem for ω-languages of Büchi Turing machines is Π^1_2-complete, i.e. : The set $\{z \in \mathbb{N} \mid L(\mathcal{T}_z)$ is non-ambiguous $\}$ is Π^1_2-complete.*

Proof. We can first express "$\mathcal{T}_z$ is non-ambiguous" by :

$$\text{“}\forall x \in \Gamma^\omega\ \forall r, r' \in \{0,1\}^\omega [(r \text{ and } r' \text{ are accepting runs of } \mathcal{T}_z \text{ on } x) \rightarrow r = r']\text{”}$$

which is a Π^1_1-formula. Then "$L(\mathcal{T}_z)$ is non-ambiguous" can be expressed by the following formula: "$\exists y[L(\mathcal{T}_z) = L(\mathcal{T}_y)$ and ($\mathcal{T}_y$ is non-ambiguous)]". This is a Π^1_2-formula because "$L(\mathcal{T}_z) = L(\mathcal{T}_y)$" can be expressed by the Π^1_2-formula

$$\text{“}\forall x \in \Gamma^\omega\ [(x \in L(\mathcal{T}_z) \text{ and } x \in L(\mathcal{T}_y)) \text{ or } (x \notin L(\mathcal{T}_z) \text{ and } x \notin L(\mathcal{T}_y))]\text{”},$$

and the quantification $\exists y$ is a first-order quantification bearing on integers. Thus the set $\{z \in \mathbb{N} \mid L(\mathcal{T}_z)$ is non-ambiguous $\}$ is a Π^1_2-set.

To prove completeness we use a construction we already used in [Fin09b]. We first define the following operation on ω-languages. For $x, x' \in \Gamma^\omega$ the ω-word $x \otimes x'$ is defined by: for every integer $n \geq 1$ $(x \otimes x')(2n-1) = x(n)$ and $(x \otimes x')(2n) = x'(n)$. For two ω-languages $L, L' \subseteq \Gamma^\omega$, the ω-language $L \otimes L'$ is defined by $L \otimes L' = \{x \otimes x' \mid x \in L \text{ and } x' \in L'\}$.

We know that there is a simple example of $\mathbf{\Sigma}^1_1$-complete set $L \subseteq \Gamma^\omega$ accepted by a Büchi Turing machine. It is then easy to define an injective computable function θ from $\mathbb{N}$ into $\mathbb{N}$ such that, for every integer $z \in \mathbb{N}$, it holds that $L(\mathcal{T}_{\theta(z)}) = (L \otimes \Gamma^\omega) \cup (\Gamma^\omega \otimes L(\mathcal{T}_z))$. There are now two cases.

First case. $L(\mathcal{T}_z) = \Gamma^\omega$. Then $L(\mathcal{T}_{\theta(z)}) = \Gamma^\omega$ and $L(\mathcal{T}_{\theta(z)})$ is unambiguous.

Second case. $L(\mathcal{T}_z) \neq \Gamma^\omega$. Then there is an ω-word $x \in \Gamma^\omega$ such that $x \notin L(\mathcal{T}_z)$. But $L(\mathcal{T}_{\theta(z)}) = (L \otimes \Gamma^\omega) \cup (\Gamma^\omega \otimes L(\mathcal{T}_z))$ thus $\{\sigma \in \Gamma^\omega \mid \sigma \otimes x \in L(\mathcal{T}_{\theta(z)})\} = L$ is a $\mathbf{\Sigma}^1_1$-complete set.

Thus $L(\mathcal{T}_{\theta(z)})$ is not Borel and this implies, by Theorem 4.2, that $L(\mathcal{T}_{\theta(z)})$ is in Max-Amb and in particular that $L(\mathcal{T}_{\theta(z)})$ is inherently ambiguous.

We have proved, using the reduction θ, that :

$$\{z \in \mathbb{N} \mid L(\mathcal{T}_z) = \Gamma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{T}_z) \text{ is non-ambiguous }\}$$

Thus this latter set is Π^1_2-complete because the universality problem for ω-languages of Turing machines is itself Π^1_2-complete, see [CC89, Fin09b]. □

Theorem 4.10. *The set* $\{z \in \mathbb{N} \mid L(\mathcal{T}_z) \in \text{Max-Amb}\}$ *is* Σ^1_2*-complete.*

Proof. We first show that the set $\{z \in \mathbb{N} \mid L(\mathcal{T}_z) \in \text{Max-Amb}\}$ is in the class Σ^1_2. In a similar way as in the proof of Proposition 3.3 we consider the set R_z defined by:

$$R_z = \{(w, r) \mid w \in \Gamma^\omega \text{ and } r \in \{0,1\}^\omega \text{ is an accepting run of } \mathcal{T}_z \text{ on the } \omega\text{-word } w\}.$$

This set R_z is a Δ^1_1-subset of $\Gamma^\omega \times \{0,1\}^\omega$. Notice that the set of accepting runs of $\mathcal{T}_z$ on an ω-word $w \in \Gamma^\omega$ is the section

$$R_{z,w} = \{r \in \{0,1\}^\omega \mid (w, r) \in R_z\}.$$

It is a set in the relativized class $\Sigma^1_1(w)$ and thus it is uncountable iff it contains a point r_0 such that $\{r_0\}$ is not a $\Delta^1_1(w)$-subset of $\{0,1\}^\omega$. Moreover we have already seen that the set

$$D_z = \{w \mid w \in \Gamma^\omega \text{ and there are uncountably many accepting runs of } \mathcal{T}_z \text{ on } w\}.$$

is a Σ^1_1-set. Thus it is uncountable iff it contains a member which is not in class Δ^1_1. Recall now that Harrington, Kechris and Louveau obtained a coding of Δ^1_1-subsets (respectively, of $\Delta^1_1(w)$-subsets) of $\{0,1\}^\omega$ in [HKL90], (see the proof of the above Theorem 4.4). Then there is a Π^1_1-formula $\Theta_1(w)$ such that for every $w \in \Gamma^\omega$ it holds that $\{w\}$ is in the class Δ^1_1 iff $\Theta_1(w)$ holds. And there is a Π^1_1-formula $\Theta_2(w, r)$ such that for every $w \in \Gamma^\omega$ and $r \in \{0,1\}^\omega$ it holds that $\{r\}$ is in the class $\Delta^1_1(w)$ iff $\Theta_2(w, r)$ holds. We can now express the sentence "the set of ω-words, which have $2^{\aleph_0}$ accepting runs by $\mathcal{T}_z$, has cardinality $2^{\aleph_0}$" by the following formula $\Omega(z)$:

$$\exists w \exists r [\neg\Theta_1(w) \wedge \neg\Theta_2(w, r) \wedge (w, r) \in R_z]$$

This formula $\Omega(z)$ is clearly a Σ^1_1-formula. We can now express the sentence "$L(\mathcal{T}_z) \in$ Max-Amb" by the following sentence:

$$\forall z' \in \mathbb{N}[L(\mathcal{T}_z) \neq L(\mathcal{T}_{z'}) \text{ or } \Omega(z')]$$

This is a Σ^1_2-formula because "$L(\mathcal{T}_z) \neq L(\mathcal{T}_{z'})$" is easily espressed by a Σ^1_2-formula (see the proof of Theorem 4.9), the formula $\Omega(z)$ is a Σ^1_1-formula, and the first-order quantification $\forall z'$ bears on integers. Thus we have proved that the set $\{z \in \mathbb{N} \mid L(\mathcal{T}_z) \in \text{Max-Amb}\}$ is in the class Σ^1_2.

To prove the completeness part of the theorem we can use the same reduction θ as in the proof of the preceding theorem. Recall that we know that there is a simple example of $\mathbf{\Sigma}^1_1$-complete set $L \subseteq \Gamma^\omega$ accepted by a Büchi Turing machine. We have defined, in the proof of the preceding theorem, an injective computable function θ from $\mathbb{N}$ into $\mathbb{N}$ such that, for every integer $z \in \mathbb{N}$, it holds that $L(\mathcal{T}_{\theta(z)}) = (L \otimes \Gamma^\omega) \cup (\Gamma^\omega \otimes L(\mathcal{T}_z))$. We have seen that there are two cases.

First case. $L(\mathcal{T}_z) = \Gamma^\omega$. Then $L(\mathcal{T}_{\theta(z)}) = \Gamma^\omega$ and $L(\mathcal{T}_{\theta(z)})$ is unambiguous.
Second case. $L(\mathcal{T}_z) \neq \Gamma^\omega$. Then $L(\mathcal{T}_{\theta(z)})$ is in Max-Amb.

Thus we have proved, using the reduction θ, that :

$$\{z \in \mathbb{N} \mid L(\mathcal{T}_z) \neq \Gamma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{T}_z) \text{ is in Max-Amb }\}$$

Thus this latter set is Σ^1_2-complete because the universality problem for ω-languages of Turing machines is itself Π^1_2-complete, see [CC89, Fin09b], so $\{z \in \mathbb{N} \mid L(\mathcal{T}_z) \neq \Gamma^\omega\}$ is Σ^1_2-complete. □

We now briefly recall some notions of set theory which will be useful for the next result and refer the reader to a textbook like [Jec02] for more background on set theory.

The usual axiomatic system ZFC is Zermelo-Fraenkel system ZF plus the axiom of choice AC. The axioms of ZFC express some natural facts that we consider to hold in the universe of sets. A model $(\mathbf{V}, \in)$ of an arbitrary set of axioms $\mathbb{A}$ is a collection $\mathbf{V}$ of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of $\mathbb{A}$. We often say " the model $\mathbf{V}$" instead of "the model $(\mathbf{V}, \in)$".

We say that two sets A and B have same cardinality iff there is a bijection from A onto B and we denote this by $A \approx B$. The relation $\approx$ is an equivalence relation. Using the axiom of choice AC, one can prove that any set A can be well-ordered so there is an ordinal γ such that $A \approx \gamma$. In set theory the cardinality of the set A is then formally defined as the smallest such ordinal γ. Such ordinals γ are also called cardinal numbers, or simply cardinals. The infinite cardinals are usually denoted by $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$ The continuum hypothesis CH says that the first uncountable cardinal $\aleph_1$ is equal to $2^{\aleph_0}$ which is the cardinal of the continuum.

If $\mathbf{V}$ is a model of ZF and $\mathbf{L}$ is the class of *constructible sets* of $\mathbf{V}$, then the class $\mathbf{L}$ is a model of ZFC + CH. Notice that the axiom V=L, which means "every set is constructible", is consistent with ZFC because $\mathbf{L}$ is a model of ZFC + V=L, see [Jec02, pages 175-200].

Consider now a model $\mathbf{V}$ of ZFC and the class of its constructible sets $\mathbf{L} \subseteq \mathbf{V}$ which is another model of ZFC. It is known that the ordinals of $\mathbf{L}$ are also the ordinals of $\mathbf{V}$, but the cardinals in $\mathbf{V}$ may be different from the cardinals in $\mathbf{L}$. In particular, the first uncountable cardinal in $\mathbf{L}$ is denoted $\aleph_1^{\mathbf{L}}$, and it is in fact an ordinal of $\mathbf{V}$ which is denoted $\omega_1^{\mathbf{L}}$. It is well-known that this ordinal satisfies the inequality $\omega_1^{\mathbf{L}} \leq \omega_1$. In a model $\mathbf{V}$ of the axiomatic system ZFC + V=L the equality $\omega_1^{\mathbf{L}} = \omega_1$ holds, but in some other models of ZFC the inequality may be strict and then $\omega_1^{\mathbf{L}} < \omega_1$.

The following result was proved in [Fin09a].

Theorem 4.11. *There exists a real-time* 1*-counter Büchi automaton* $\mathcal{A}$*, which can be effectively constructed, such that the topological complexity of the* ω*-language* $L(\mathcal{A})$ *is not determined by the axiomatic system* ZFC*. Indeed it holds that :*

(1) (ZFC + V=L)*. The* ω*-language* $L(\mathcal{A})$ *is an analytic but non-Borel set.*
(2) (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$)*. The* ω*-language* $L(\mathcal{A})$ *is a* $\mathbf{\Pi}^0_2$*-set.*

We can now show that it is consistent with ZFC that some recursive ω-languages in the Borel class $\mathbf{\Pi}^0_2$, hence of a low Borel rank, have the maximum degree of ambiguity.

Theorem 4.12. (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$). *There exists an ω-language accepted by a real-time 1-counter Büchi automaton which belongs to the Borel class $\mathbf{\Pi}_2^0$ and which has the maximum degree of ambiguity with regard to acceptance by Turing machines, i.e. which belongs to the class* Max-Amb.

Proof. Consider the real-time 1-counter Büchi automaton $\mathcal{A}$ given by Theorem 4.11. It may be seen as a Turing machine which has an index z_0 so that $L(\mathcal{A}) = L(\mathcal{T}_{z_0})$. Let now $\mathbf{V}$ be a model of (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$). In this model $L(\mathcal{A})$ is a Borel set in the class $\mathbf{\Pi}_2^0$. We are going to show that it is also in the class Max-Amb.

Consider the model $\mathbf{L}$ which is the class of *constructible sets* of $\mathbf{V}$. The class $\mathbf{L}$ is a model of (ZFC + V=L) and thus by Theorem 4.11 the ω-language $L(\mathcal{A})$ is an analytic but non-Borel set in $\mathbf{L}$. Then it follows from Theorem 4.2 that in $\mathbf{L}$ the ω-language $L(\mathcal{T}_{z_0})$ is in the class Max-Amb. On the other hand, the set $\{z \in \mathbb{N} \mid L(\mathcal{T}_z) \in \text{Max-Amb}\}$ is a Σ_2^1-set by Theorem 4.10. Thus by the Shoenfield's Absoluteness Theorem (see [Jec02, page 490]) this set is the same in the model $\mathbf{V}$ *and* in the model $\mathbf{L}$. This implies that the ω-language $L(\mathcal{A}) = L(\mathcal{T}_{z_0})$ has the maximum degree of ambiguity with regard to acceptance by Turing machines in the model $\mathbf{V}$ too. □

Remark 4.13. In order to prove Theorem 4.12 we do not need to use any large cardinal axiom or even the consistency of such an axiom, because it is known that (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$) is equiconsistent with ZFC. However it is known that the existence of a measurable cardinal (or even of a larger one), or the axiom of analytic determinacy, imply the strict inequality $\omega_1^{\mathbf{L}} < \omega_1$ and thus the existence of the ω-language in the class Max-Amb given by Theorem 4.12.

5. Concluding remarks

We have investigated the notion of ambiguity for recursive ω-languages. In particular Theorem 4.4 gives a remarkable dichotomy result for recursive ω-languages: a recursive ω-language L is either unambiguous or has a great degree of ambiguity.

On the other hand, Theorem 4.12 states that it is consistent with ZFC that there exists a recursive ω-language which belongs to the Borel class $\mathbf{\Pi}_2^0$ and which has the maximum degree of ambiguity. The following question now naturally arises: "Does there exist such a recursive ω-language in *every model of* ZFC ?"

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