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Algebraic polynomials and moments of stochastic integrals

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Abstract: We propose an algebraic method for proving estimates on moments of stochastic integrals. The method uses qualitative properties of roots of algebraic polynomials from certain general classes. As an application, we give a new proof of a variation of the Burkholder-Davis-Gundy inequality for the case of stochastic integrals with respect to real locally square integrable martingales. Further possible applications and extensions of the method are outlined.

MSC 2000 Subject Classifications: Primary 60H05, 60E15; secondary 60G44, 33D45, 26C10, 62E20.

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1. Introduction

Connections between special algebraic polynomials and stochastic integrals have a long history (see Wiener [1938], Itô [1951]), and received considerable attention in stochastic analysis (Ikeda and Watanabe [1989], Carlen and Krée [1991], Borodin and Salminen [2002], Nualart and Schoutens [2000]). Fruitful applications of special polynomials have been found in the theory of Markov processes (Kendall [1959], Karlin and McGregor [1957]), financial mathematics

(Schoutens [2000]), statistics (Diaconis and Zabell [1991]). The book Schoutens [2000] contains an extensive overview of this field of stochastic analysis and its applications.

In this paper, we study a different type of application of polynomials to stochastic integration. We show that not only properties of special systems of orthogonal polynomials can be used in stochastic analysis, but also that elementary properties of certain general classes of polynomials can be effectively utilized. In particular, we give a new proof of the Burkholder-Davis-Gundy inequality in Section 2 as an illustration of our approach. Some possible extensions and further applications of our results are described in Section 3, where we also prove the B-D-G inequality for the case of stochastic integrals with respect to locally square integrable martingales.

2. The main idea

We propose an algebraic proof for the following classic variation of the B-D-G inequality.

Theorem 1. *Let $b(s)$, $s \in [0, t]$, be a progressively measurable process, $b \in \mathcal{L}_2[0, T]$, $t \leq T$. Then for every $n \geq 2$ there exists constants $C_1 > 0$, $C_2 > 0$ such that*

$$C_1 \mathbb{E} \left(\int_0^t b^2(s) ds \right)^n \leq \mathbb{E} \left(\int_0^t b(s) dW(s) \right)^{2n} \leq C_2 \mathbb{E} \left(\int_0^t b^2(s) ds \right)^n. \quad (1)$$

The constants C_1 and C_2 depend on n , but not on the process b .

The key feature of our approach is that we use general qualitative properties of roots of algebraic polynomials. The idea of proving the Burkholder-Davis-Gundy inequality via the use of Hermitian polynomials has been already explored by different authors (see, for example, McKean [1969], pp.40-41 or Ikeda and Watanabe [1989]). However, the previous proofs used properties of polynomials in a different way and worked only for $n \leq 4$. Our proof is valid for general

n and, moreover, can be extended to give estimates for more general types of stochastic integrals.

The following two algebraic lemmas are in the core of our method.

Lemma 1. *Consider real polynomials*

$$P_1(z) = \sum_{k=0}^{2m} b_k z^{2k} \quad \text{and} \quad P_2(z) = z \sum_{i=0}^{2m_1} c_i z^{2i}, \quad (2)$$

where $m_1 < m$ is integer and nonnegative, $b_k \geq 0$ for all k , $b_0 > 0$, $b_m > 0$, $c_i \geq 0$ for all i . Then there exists $0 < d_1 < d_2$ such that only for $z \in [d_1, d_2]$ one can have $P_1(z) \leq P_2(z)$, but for $z \notin [d_1, d_2]$ one has $P_1(z) > P_2(z)$.

Proof. (Lemma 1) Note first that $P_1(z)$ is symmetric, $P_1(z) \geq b_0$ for all $z \in \mathbb{R}$, and $P_1(z) \sim b_m z^{2m}$ as $z \rightarrow \infty$. Furthermore, $P_2(-z) = -P_2(z)$, and for $z \geq 0$ one has $P_2(z) \geq 0$, $P_2(0) = 0$, and $\deg P_1(z) < \deg P_2(z)$, where $\deg P_i$ denotes the degree of the polynomial P_i .

This implies that for $z < 0$ one has $P_1(z) > 0 > P_2(z)$. At $z = 0$ it holds that $P_1(0) = b_0 > 0 = P_2(0)$. This shows that all possible solutions of the inequality $P_1(z) \leq P_2(z)$ are positive, i.e. bounded from below by a positive number d_1 .

Since $P_1(z)/P_2(z) \rightarrow \infty$ as $z \rightarrow \infty$, it follows that for sufficiently large $z \geq z_0$ always $P_1(z) > P_2(z)$. Therefore, all possible solutions of the inequality $P_1(z) \leq P_2(z)$ lays in some interval $[d_1, d_2]$ with $d_1 > 0$ and $d_2 > 0$.

□

Lemma 2. *Consider real polynomials*

$$P_1(z) = b_0 z^{2m+1} \quad \text{and} \quad P_2(z) = z \sum_{i=0}^{m_1} c_i z^{2i}, \quad (3)$$

where $m_1 < m$ is integer and nonnegative, $b_0 > 0$, $c_i \geq 0$ for all i , $c_0 > 0$, $c_{m_1} > 0$. Then there exists $d_2 > 0$ such that only for $z \in [-\infty, d_2]$ one can have $P_1(z) \leq P_2(z)$, but for $z > d_2$ one always has $P_1(z) > P_2(z)$.

Proof. (Lemma 2) The proof is analogous to the one of Lemma 1.

□

Now we are able to prove the main theorem.

Proof. (Theorem 1). In the proof below we can assume that b is bounded, since the general case follows by the usual truncation argument.

We denote for brevity

$$\int_0^t b^2(s) ds = \int b^2 ds, \quad \int_0^t b(s) dW(s) = \int b dW.$$

Let us write for $n \geq 1$

$$\begin{aligned} \rho_{2n}(t) &= H_{2n}\left(\int b^2 ds, \int b dW\right) \\ &= \sum_{0 \leq k \leq n} (-1)^k a_k \left(\int b dW\right)^{2n-2k} \left(\int b^2 ds\right)^k, \end{aligned} \quad (4)$$

where H_{2n} stands for the $2n$ -th Hermitian polynomial defined as

$$H_{2n}(x, y) := \frac{(-x)^n}{n!} e^{y^2/2x} \frac{d^n}{dy^n} e^{-y^2/2x} = \sum_{0 \leq k \leq n} (-1)^k a_k x^k y^{2n-2k},$$

and it is known that

$$a_k = \frac{1}{2^k k! (2n - 2k)!}.$$

Taking in (4) the expectation of both sides and noting that $\mathbb{E} \rho_{2n} = 0$ (see McKean [1969], pp. 37-38 or Ikeda and Watanabe [1989], pp. 150-152), we get

$$\sum_{0 \leq k \leq n} (-1)^k a_k \mathbb{E} \left\{ \left(\int b dW \right)^{2n-2k} \left(\int b^2 ds \right)^k \right\} = 0. \quad (5)$$

By Hölder's inequality, for all $1 \leq k \leq n$

$$\begin{aligned} \mathbb{E} \left\{ \left(\int b dW \right)^{2n-2k} \left(\int b^2 ds \right)^k \right\} &\leq \\ &\leq \mathbb{E}^{\frac{n-k}{n}} \left(\int b dW \right)^{2n} \mathbb{E}^{\frac{k}{n}} \left(\int b^2 ds \right)^n. \end{aligned} \quad (6)$$

Part I. Consider first the case of even n , and let $n = 2m$ in (1). Since $a_k \geq 0$ for all k , and also

$$\mathbb{E}\left\{\left(\int b dW\right)^{2n-2k}\left(\int b^2 ds\right)^k\right\} \geq 0$$

for all k , after throwing out from (5) all the summands with even k , except for $k = 0$ and $k = n$, we get

$$\begin{aligned} a_0 \mathbb{E}\left(\int b dW\right)^{2n} - \sum_{0 \leq 2l+1 \leq n} a_{2l+1} \mathbb{E}\left\{\left(\int b dW\right)^{2n-2k(l)}\left(\int b^2 ds\right)^{k(l)}\right\} \\ + a_n \mathbb{E}\left(\int b^2 ds\right)^n \leq 0, \end{aligned} \quad (7)$$

where for integer $l \geq 0$ we denoted $k(l) = 2l + 1$.

Applying inequality (6) to (7), we get

$$\begin{aligned} a_0 \mathbb{E}\left(\int b dW\right)^{2n} - \sum_{0 \leq 2l+1 \leq n} a_{2l+1} \mathbb{E}^{\frac{n-k(l)}{n}}\left(\int b dW\right)^{2n} \mathbb{E}^{\frac{k(l)}{n}}\left(\int b^2 ds\right)^n \\ + a_n \mathbb{E}\left(\int b^2 ds\right)^n \leq 0. \end{aligned} \quad (8)$$

Divide both parts of (8) by $\mathbb{E}\left(\int b^2 ds\right)^n$ and put

$$z := \frac{\mathbb{E}^{1/n}\left(\int b dW\right)^{2n}}{\mathbb{E}^{1/n}\left(\int b^2 ds\right)^n}, \quad (9)$$

then we obtain

$$a_0 z^n - \sum_{0 \leq 2l+1 \leq n} a_{2l+1} z^{n-k(l)} + a_n \leq 0,$$

or equivalently

$$a_0 z^n + a_n \leq \sum_{0 \leq 2l+1 \leq n} a_{2l+1} z^{n-k(l)}. \quad (10)$$

Let us now put in (10) $P_1(z) = a_0 z^n + a_n$, $P_2(z) = \sum_{0 \leq 2l+1 \leq n} a_{2l+1} z^{n-k(l)}$.

By Lemma 1, there exists positive constants C_1, C_2 such that $0 < C_1 \leq z \leq C_2$,

i.e. $C_1^n \leq z^n \leq C_2^n$, and this proves (1) for the case of $n = 2m$.

Part II. Consider now the case of odd n , and let $n = 2m + 1$ in (1). Throwing away from (5) all the summands with even k , except for $k = 0$, we get

$$a_0 \mathbb{E} \left(\int b dW \right)^{2n} - \sum_{0 \leq 2l+1 \leq n} a_{2l+1} \mathbb{E} \left\{ \left(\int b dW \right)^{2n-2k(l)} \left(\int b^2 ds \right)^{k(l)} \right\} \leq 0, \quad (11)$$

and analogously to (10) we derive

$$a_0 z^n \leq \sum_{0 \leq 2l+1 \leq n} a_{2l+1} z^{n-k(l)}, \quad (12)$$

where z is defined by (9).

After applying Lemma 2 to $P_1(z) = a_0 z^n$ and $P_2(z) = \sum_{0 \leq 2l+1 \leq n} a_{2l+1} z^{n-k(l)}$ in (12), we obtain from (12) that $z \leq d_2$ for some positive d_2 . Since n is odd, this implies $z^n \leq d_2^n$ and the upper bound in (1) follows.

It remains only to prove the lower bound in (1) for $n = 2m + 1$. In this case, we leave in (5) only the summands with even k and $k = n$, thus getting

$$\sum_{0 \leq 2k < n} a_{2k} \mathbb{E} \left\{ \left(\int b dW \right)^{2n-4k} \left(\int b^2 ds \right)^{2k} \right\} - a_n \mathbb{E} \left\{ \left(\int b^2 ds \right)^n \right\} \geq 0. \quad (13)$$

Analogously to our previous derivations, this implies the inequality

$$\sum_{0 \leq 2k < n} a_{2k} z^{n-2k} - a_n \geq 0, \quad \text{i.e.}$$

$$P(z) := \sum_{0 \leq 2k < n} a_{2k} z^{n-2k} \geq a_n, \quad (14)$$

where z is again as in (9). Since $P(z)$ is a polynomial of the form $\sum_{i=1}^m b_i z^{2i+1}$, it easily follows that (14) is equivalent to $z \geq C_1$ for some constant $C_1 = C_1(n) > 0$. Therefore, $z^n \geq C_1^n$, and the lower bound in (1) is proved for $n = 2m + 1$. \square

3. Possible extensions

As an immediate extension of the above result, we prove the following more general theorem. This will not be the most general framework where the B-D-G inequality holds or where our approach could work; we decided to stick to the stochastic integrals version for uniformity of presentation.

Theorem 2. *Let a real process $(M(s))_{s \geq 0}$ be a locally square integrable martingale and $\langle M \rangle$ be its quadratic variation process. Let $b(s)$, $s \in [0, t]$, be a bounded progressively measurable process, $b \in \mathcal{L}_2^{loc}[0, T]$, $t \leq T$. Then for every $n \geq 2$ there exists constants $C_1 > 0$, $C_2 > 0$ such that*

$$C_1 \mathbb{E} \left(\int_0^t b^2(s) d\langle M \rangle(s) \right)^n \leq \mathbb{E} \left(\int_0^t b(s) dM(s) \right)^{2n} \leq C_2 \mathbb{E} \left(\int_0^t b^2(s) d\langle M \rangle(s) \right)^n. \quad (15)$$

The constants C_1 and C_2 depend on n , but not on the processes b and M .

Proof. (Theorem 2). Let us write again

$$\begin{aligned} \rho_{2n}(t) &= H_{2n} \left(\int_0^t b^2(s) d\langle M \rangle(s), \int_0^t b(s) dM(s) \right) \\ &= \sum_{0 \leq k \leq n} (-1)^k a_k \left(\int_0^t b dM \right)^{2n-2k} \left(\int_0^t b^2 d\langle M \rangle \right)^k, \end{aligned} \quad (16)$$

where H_{2n} is the $2n$ -th Hermitian polynomial. By Theorem 29, pp. 75-76 of Protter [2004], we have

$$\left\langle \int b dM \right\rangle = \int b^2 d\langle M \rangle.$$

From Theorem 5.1 on p. 152 of Ikeda and Watanabe [1989], we again see that $\mathbb{E} \rho_{2n} = 0$. Therefore, the proof of Theorem 1 can be directly transferred to the present situation. □

In the above proof we have used only some elementary and entirely qualitative facts about certain general types of polynomials, together with such a crude technique as simple throwing out every second term in the martingale identity (5). Nonetheless, we were able to prove a rather general Burkholder-Davis-Gundy theorem. This suggests that our approach can lead to substantially stronger results in estimation of stochastic integrals with respect to Brownian motion, if we could incorporate in our method the existing quantitative estimates for roots of Hermitian polynomials. Indeed, as was shown in Davis [1976], for $p = 2n$, where n is an integer, the best values for C_1 and C_2 in Theorem 1 are l_{2n}^{2n} and r_{2n}^{2n} respectively, where l_{2n}^{2n} and r_{2n}^{2n} are the smallest and the largest positive roots of the Hermite polynomial H_{2n} .

Moreover, equation (4) represents only one of the many polynomial-type martingales that can be composed from stochastic integrals. Other important examples are given by the connection between integrals with respect to the Poisson process and Charlier polynomials (see Bertoin [1996]), the connection between integrals with respect to the Gamma process and Laguerre polynomials, and the link between Sheffer polynomials and several classes of Lévy processes. Many examples of such martingales can be found in Schoutens [2000]. It seems plausible that one could modify the method of the present paper and prove estimates for more general stochastic integrals.

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