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Functional Central Limit Theorem for the Volume of Excursion Sets Generated by Associated Random Fields

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Abstract

We prove a functional central limit theorem for the volume of the excursion sets generated by a stationary and associated random field with smooth realizations.

Keywords: functional central limit theorem, large deviation principle, excursion set, association, random field

2010 MSC: 60F17, 60G60

1. Introduction

Associated random fields form an important class of dependent systems and were first introduced in Esary et al. (1967). Their main advantage compared to mixing systems is that the conditions of limit theorems are easier to verify. A finite collection $X = (X_1, \dots, X_n)$ of real-valued random variables X_k , $k = 1, \dots, n$, is called *associated* if $\text{Cov}(f(X), g(X)) \geq 0$ for any coordinate-wise non-decreasing functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$, whenever the covariance exists. An infinite family of random variables is associated if this is valid for every finite sub-family. In statistical physics, this property is known as the FKG-inequalities, see Fortuin et al. (1971). Associated systems are also encountered in mathematical statistics, reliability theory, random measures and so forth. Starting with Newman (1980), many limit theorems like CLTs, invariance principles, etc. have been proven for associated and related random fields, see Bulinski and Shashkin (2007) and references therein.

The *excursion set* of a random field X at the level $a \in \mathbb{R}$ is the random set $\{t \in \mathbb{R}^d : X_t \geq a\}$. Excursion sets of stationary random fields attract growing attention because of their applications to the modeling of various random structures, see Azaïs and Wschebor (2009), Bulinski et al. (2011). It is natural to consider the volume of excursion sets determined by levels $a \in \mathbb{R}$ in a bounded observation window as a random process and to study its limit behaviour as

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the window size grows. The aim of the present paper is to provide a functional central limit theorem for the volume of excursion sets generated by stationary associated random fields.

First, we introduce some notation. Let X be a real-valued, strictly stationary, square-integrable, associated random field in \mathbb{R}^d with a.s. continuous trajectories such that the components X_t have density bounded by $M < \infty$. Further, assume that for some $c_0 > 0$ and some $\lambda > 9d$ it holds that

$$|\text{Cov}(X_0, X_t)| \leq c_0 (1 + \|t\|_\infty)^{-\lambda}, \quad (1)$$

where $\|\cdot\|_\infty$ denotes the maximum norm. The latter condition concerning the decay of the covariance function is rather technical. For $n \in \mathbb{N}$ and $a \in \mathbb{R}$, denote by

$$Y_n(a) = \frac{1}{n^{d/2}} \left(\nu \left(\left\{ t \in [0, n]^d : X_t \geq a \right\} \right) - n^d \mathbb{P}(X_0 \geq a) \right)$$

the centered and normalized volume of the excursion set of the field X in the set $[0, n]^d$ at the level a , where $\nu(\cdot)$ denotes the Lebesgue measure in \mathbb{R}^d .

2. Main Result and Proof

Theorem 1. *Under the assumptions above, the distributions of the random processes $\{Y_n\}_{n \in \mathbb{N}}$ converge in the Skorokhod space $D(\mathbb{R})$, as $n \rightarrow \infty$, to the distribution of the centered Gaussian process Y with covariance function*

$$\text{Cov}(Y(a), Y(b)) = \int_{\mathbb{R}^d} (\mathbb{P}(X_0 > a, X_t > b) - \mathbb{P}(X_0 > a) \mathbb{P}(X_0 > b)) dt.$$

Remark 1. A Gaussian random field is associated whenever its covariance function is nonnegative (Bulinski and Shashkin, 2007, Theorem 1.2.1). So, the theorem holds for stationary Gaussian fields with exponentially decreasing correlations. Another important example is an autoregression field with positive dependence, i.e. it has a density, though the innovations can be discrete random variables.

Remark 2. In general it is possible that the conditions on the field X are satisfied but the trajectories of Y are not continuous. Consider for example a strictly stationary associated sequence $Z = (Z_n, n \in \mathbb{Z})$ such that Z_0 has bounded density and the covariance function decreases exponentially, for example Z could be Gaussian. For $t \in [n - 1/3, n + 1/3] \setminus \{n\}$, define $Z_t = Z_n$ and on $[n + 1/3, n + 2/3]$, define Z_t by linear interpolation, here $n \in \mathbb{Z}$. Then let $U \sim \mathcal{U}(0, 1)$ be independent of Z . The random process $X_t = Z_{t+U}$ then satisfies the conditions of the theorem but the corresponding trajectories of Y have jumps.

For any $a, b \in \mathbb{R}$ we have $\mathbb{E}(Y(a) - Y(b))^2 \leq K |b - a|^{1-3d/\lambda}$, where the constant K depends on d, c_0, λ , and M . Hence we get the following Large Deviation Principle from Theorem 1 and (Piterbarg, 1996, Theorem 2.8.1).

Corollary 1. *Let $T \subset \mathbb{R}$ with $\nu(T) > 0$ and let X be such that Y is a.s. continuous. Then there exists a constant C (depending only on d, c_0, λ , and M) such that for any $S \subset T$*

$$\mathbb{P} \left(\max_{a \in S} |Y(a)| > u \right) \leq C \nu(T) u^{2d\lambda/(\lambda-3d)} \Psi \left(u / \sup_{a \in S} \sqrt{\text{Var}(Y(a))} \right),$$

as $u \rightarrow \infty$, where Ψ denotes the tail probability function of the standard Gaussian distribution.

If the random field X is Gaussian, even the exact asymptotic behavior of the probability of high excursions of Y can be given due to Theorem 1 and Talagrand (1988).

Corollary 2. *Let $T \subset \mathbb{R}$ with $0 \in T$ and, additionally to the conditions above, let X be Gaussian. Then*

$$\lim_{u \rightarrow \infty} \frac{P(\sup_{a \in T} Y(a) \geq u)}{\Psi(u/\sqrt{\mathbb{E}Y^2(0)})} = 1.$$

Note that the second moment of $Y(0)$ is given by

$$\mathbb{E}Y^2(0) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \arcsin(\text{Corr}(X_0, X_t)) dt,$$

where $\text{Corr}(X_0, X_t)$ denotes the correlation coefficient of X_0 and X_t .

To prove Theorem 1, we start with two auxiliary lemmas. To formulate the first one, let $a, b \in \mathbb{R}$ with $a < b$. Define the function $\varphi_{a,b} : \mathbb{R} \rightarrow \mathbb{R}$ by $\varphi_{a,b}(x) = \mathbb{1}(a < x \leq b) - \mathbb{P}(a < X_0 \leq b)$ where $\mathbb{1} : \mathbb{R} \rightarrow \{0, 1\}$ denotes the indicator function. For $\varepsilon > 0$, let $\varphi_{a,b,\varepsilon} : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous approximation of $\varphi_{a,b}$ which is linear on the intervals $[a - \varepsilon, a]$ and $[b, b + \varepsilon]$ and coincides with $\varphi_{a,b}$ outside these intervals. Clearly, $\varphi_{a,b,\varepsilon}$ has Lipschitz constant $1/\varepsilon$.

Lemma 1. *Let $T = \{t_1, \dots, t_k\}$ be a set of distinct points in \mathbb{R}^d and let $T_1 \cup T_2 = T$ be a partition of T , i.e. $T_1 \neq \emptyset, T_2 \neq \emptyset$ and $T_1 \cap T_2 = \emptyset$. Then it holds that*

$$\text{Cov} \left(\prod_{t \in T_1} \varphi_{a,b}(X_t), \prod_{s \in T_2} \varphi_{a,b}(X_s) \right) \leq \left(\frac{k^2}{4} + 8Mk \right) c_0^{1/3} (1+r)^{-\lambda/3}$$

where $r := \min \{\|t - s\|_\infty, t \in T_1, s \in T_2\}$ is the distance between T_1 and T_2 .

Proof. One has

$$\begin{aligned}
 \text{Cov} \left(\prod_{t \in T_1} \varphi_{a,b}(X_t), \prod_{s \in T_2} \varphi_{a,b}(X_s) \right) &= \text{Cov} \left(\prod_{t \in T_1} \varphi_{a,b,\varepsilon}(X_t), \prod_{s \in T_2} \varphi_{a,b,\varepsilon}(X_s) \right) \\
 &+ \mathbb{E} \left(\prod_{t \in T_1} \varphi_{a,b}(X_t) \left(\prod_{s \in T_2} \varphi_{a,b}(X_s) - \prod_{s \in T_2} \varphi_{a,b,\varepsilon}(X_s) \right) \right) \\
 &+ \mathbb{E} \left(\left(\prod_{t \in T_1} \varphi_{a,b}(X_t) - \prod_{t \in T_1} \varphi_{a,b,\varepsilon}(X_t) \right) \prod_{s \in T_2} \varphi_{a,b,\varepsilon}(X_s) \right) \quad (2) \\
 &+ \mathbb{E} \prod_{t \in T_1} \varphi_{a,b}(X_t) \left(\mathbb{E} \prod_{s \in T_2} \varphi_{a,b,\varepsilon}(X_s) - \mathbb{E} \prod_{s \in T_2} \varphi_{a,b}(X_s) \right) \\
 &+ \left(\mathbb{E} \prod_{t \in T_1} \varphi_{a,b,\varepsilon}(X_t) - \mathbb{E} \prod_{t \in T_1} \varphi_{a,b}(X_t) \right) \mathbb{E} \prod_{s \in T_2} \varphi_{a,b,\varepsilon}(X_s).
 \end{aligned}$$

By the quasi-association inequality (Bulinski and Shashkin, 2007, p. 89) and the covariance condition (1), we have

$$\begin{aligned}
 \text{Cov} \left(\prod_{t \in T_1} \varphi_{a,b,\varepsilon}(X_t), \prod_{s \in T_2} \varphi_{a,b,\varepsilon}(X_s) \right) \\
 \leq \frac{1}{\varepsilon^2} \sum_{t \in T_1} \sum_{s \in T_2} \text{Cov}(X_t, X_s) \leq \frac{k^2 c_0}{4\varepsilon^2} (1+r)^{-\lambda}.
 \end{aligned}$$

Each of the other four summands in (2) is bounded by $2Mk\varepsilon$, since the density of X_t does not exceed M . Hence, the Lemma follows with $\varepsilon = c_0^{1/3} (1+r)^{-\lambda/3}$. \square

In what follows, K is a positive number which depends on d , c_0 , λ , and M only and also may change from line to line.

Lemma 2. *For any $a, b \in \mathbb{R}$ with $0 < a < b < 1$, it holds that*

$$\mathbb{E} (Y_n(a) - Y_n(b))^4 \leq K \left(\frac{(b-a)^{1-9d/\lambda}}{n^d} + (b-a)^{2-6d/\lambda} \right).$$

Proof. To shorten the notation, denote $\mathbf{t} = (t_1, t_2, t_3, t_4) \in [0, n]^{4d}$. By Fubini's theorem, it holds that

$$\mathbb{E} (Y_n(a) - Y_n(b))^4 = \frac{1}{n^{2d}} \int_{[0,n]^{4d}} \mathbb{E} \prod_{i=1}^4 \varphi_{a,b}(X_{t_i}) d\mathbf{t}.$$

This integral is split into three parts and each one is estimated separately. For that, define the function $h : [0, n]^{4d} \rightarrow \mathbb{R}$ by

$$h(\mathbf{t}) = \max_{Q \subset \{1,2,3,4\}} \min_{\substack{i \in Q \\ j \notin Q}} \|t_i - t_j\|_\infty.$$

The distance between two sets Q and \tilde{Q} (in the supremum norm) is denoted by $d_\infty(Q, \tilde{Q})$ where we put $d_\infty(Q, \emptyset) = 0$. Let $c > 1$ be a number to be specified later. The sets

$$A_1 = \left\{ \mathbf{t} \in [0, n]^{4d} : h(\mathbf{t}) \leq c \right\}, A_2 = \left\{ \mathbf{t} \in [0, n]^{4d} : h(\mathbf{t}) \geq c \text{ and } |Q_{\mathbf{t}}^*| = 1 \right\},$$

and $A_3 = [0, n]^{4d} \setminus (A_1 \cup A_2)$ form a partition of $[0, n]^{4d}$, where

$$Q_{\mathbf{t}}^* = \operatorname{argmax}_{Q \subset \{1, 2, 3, 4\}} \min_{\substack{i \in Q \\ j \notin Q}} \|t_i - t_j\|_\infty.$$

We define

$$I_j = \frac{1}{n^{2d}} \int_{A_j} \mathbb{E} \prod_{i=1}^4 \varphi_{a,b}(X_{t_i}) d\mathbf{t}.$$

For A_1 , it holds that $\nu(A_1) \leq n^d (3c)^{3d}$. Hence, by the Cauchy-Schwarz inequality we get $I_1 \leq K(b-a)c^{3d}/n^d$.

For I_2 , w.l.o.g. let $Q_{t_1, t_2, t_3, t_4}^* = \{t_1\}$ in the definition of the set A_2 . Then, by Lemma 1, it holds that

$$I_2 \leq \frac{K}{n^{2d}} \int_{A_2} (1 + d_\infty(\{t_1\}, \{t_2, t_3, t_4\}))^{-\lambda/3} d\mathbf{t}.$$

We can assume in the following that not all distances between pairs of points are equal. Further, w.l.o.g., we assume that

$$d_\infty(\{t_1\}, \{t_2, t_3, t_4\}) = \|t_1 - t_2\|_\infty \leq \min\{\|t_1 - t_3\|_\infty, \|t_1 - t_4\|_\infty\}.$$

One of the points t_3, t_4 belongs to the ball (in supremum norm) with radius $\|t_1 - t_2\|_\infty$ centered at t_2 . Say, it is t_3 , then t_4 in turn belongs to the union of balls of radius $\|t_1 - t_2\|_\infty$ centered at the points t_2 and t_3 . Thus

$$\begin{aligned} I_2 &\leq \frac{K}{n^{2d}} \int \int_{\substack{(t_1, t_2) \in [0, n]^{2d} \\ \|t_1 - t_2\|_\infty \geq c}} (1 + \|t_1 - t_2\|_\infty)^{-\lambda/3} \|t_1 - t_2\|_\infty^{2d} dt_1 dt_2 \\ &\leq \frac{K}{n^d} \int_c^\infty (1+r)^{-\lambda/3} r^{2d} r^{d-1} dr \leq \frac{K}{n^d} c^{3d-\lambda/3}. \end{aligned}$$

For I_3 we consider only points for which $Q_{t_1, t_2, t_3, t_4}^* = \{t_1, t_2\}$ in the definition of A_3 . Then it holds that

$$\begin{aligned} I_3 &= \frac{1}{n^{2d}} \int_{A_3} \operatorname{Cov} \left(\prod_{i=1}^2 \varphi_{a,b}(X_{t_i}), \prod_{i=3}^4 \varphi_{a,b}(X_{t_i}) \right) d\mathbf{t} \\ &\quad + \frac{1}{n^{2d}} \int_{A_3} \mathbb{E} \prod_{i=1}^2 \varphi_{a,b}(X_{t_i}) \mathbb{E} \prod_{i=3}^4 \varphi_{a,b}(X_{t_i}) d\mathbf{t} = J_1 + J_2. \end{aligned}$$

The integral J_1 is estimated in the same way as I_2 yielding $J_1 \leq Kc^{3d-\lambda/3}/n^d$. J_2 is estimated with the help of Lemma 1. By the Cauchy-Schwarz inequality we get

$$J_2 \leq \frac{K}{n^{2d}} \left(\int \int_{(t_1, t_2) \in [0, n]^{2d}} (b-a) \wedge (1 + \|t_1 - t_2\|_\infty)^{-\lambda/3} dt_1 dt_2 \right)^2,$$

where $x \wedge y = \min\{x, y\}$. The double integral can be bounded by

$$\int \int_{(t_1, t_2) \in [0, n]^{2d}} (b-a) \wedge (1 + \|t_1 - t_2\|_\infty)^{-\lambda/3} dt_1 dt_2 \leq (b-a) (2n\gamma)^d + n^d \kappa_d \frac{\gamma^{d-\lambda/3}}{\lambda/3 - d},$$

with some $\gamma > 1$. Choosing $\gamma = (b-a)^{-3/\lambda}$ yields $J_2 \leq K(b-a)^{2-6/\lambda}$. The lemma follows by minimization in c . \square

Finally, we turn to the proof of Theorem 1.

Proof of Theorem 1. The convergence of the finite-dimensional distributions follows from the CLT for associated random fields by standard arguments. It is proved in a recent paper (Bulinski et al., 2011, Theorem 2). As for the tightness, it is enough to prove it for the restriction of the processes Y_n onto an arbitrary segment. W.l.o.g. we use the segment $[0, 1]$. Note that for $a_1, a_2 \in \mathbb{R}$ with $a_1 < a_2$, $b \in (a_1, a_2)$ and $n \in \mathbb{N}$ one has estimates $Y_n(b) \leq Y_n(a_1) + Mn^{d/2}(a_2 - a_1)$ and $Y_n(b) \geq Y_n(a_2) - Mn^{d/2}(a_2 - a_1)$ because X_t has density bounded by M . This bound allows to replace the supremum with a maximum over a finite set of points, plus some additional non-random summand. Namely, for any $a \in [0, 1]$, $k \in \mathbb{N}$ and $\delta > 0$, we have

$$\begin{aligned} \sup_{b \in [a, a+\delta]} |Y_n(b) - Y_n(a)| &\leq \max_{i=1, \dots, \lceil \delta k \rceil} \sup_{b \in [a + (i-1)/k, a + i/k]} |Y_n(b) - Y_n(a)| \\ &\leq \max_{i=1, \dots, \lceil \delta k \rceil} \left| Y_n\left(a + \frac{i}{k}\right) - Y_n(a) \right| + \frac{M}{k} n^{d/2}. \end{aligned}$$

Then, for any $\varepsilon > 0$, it holds that

$$\begin{aligned} \mathbb{P} \left(\sup_{b \in [a, a+\delta]} |Y_n(b) - Y_n(a)| > \varepsilon \right) &\leq \frac{2^3}{\varepsilon^4} \mathbb{E} \max_{i=1, \dots, \lceil \delta k \rceil} \left(Y_n\left(a + \frac{i}{k}\right) - Y_n(a) \right)^4 + \frac{2^3}{\varepsilon^4} \frac{M^4}{k^4} n^{2d}, \end{aligned}$$

where we used Chebyshev's inequality.

In the following, let $\zeta > 0$ be a small number and select $n^{d/2+\zeta} < k = k(n) < n^{\lambda d/(3d+\lambda)}$. With $i_1, i_2 \in \mathbb{N}$, $i_1 < i_2 \leq \delta k$, from Lemma 2 it follows that

$$\mathbb{E} \left(Y_n\left(a + \frac{i_1}{k}\right) - Y_n\left(a + \frac{i_2}{k}\right) \right)^4 < K \left(\frac{i_2 - i_1}{k} \right)^{2-6d/\lambda}$$

By the Móricz theorem (Móricz, 1983, Theorem 2), it holds that

$$\mathbb{E} \max_{j=1, \dots, \lceil \delta k \rceil} (Y_n(a + j/k) - Y_n(a))^4 \leq K \frac{\lceil \delta k \rceil^{2-6d/\lambda}}{k^{2-6d/\lambda}} \leq K \delta^{2-6d/\lambda}.$$

By dividing the unit interval into smaller subintervals of length δ , we get

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{\substack{a, b \in [0, 1] \\ |a-b| < \delta}} |Y_n(a) - Y_n(b)| > 2\varepsilon \right) \\ &= \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\max_{1 \leq m \leq \lfloor 1/\delta \rfloor} \sup_{\substack{a, b \in [(m-1)\delta, m\delta] \\ |a-b| < \delta}} |Y_n(a) - Y_n(b)| > 2\varepsilon \right) \\ &\leq \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} 2\delta^{-1} \left(K \delta^{2-6d/\lambda} + \frac{8}{\varepsilon^4} \frac{M^4}{k^4} n^{2d} \right) = 0, \end{aligned}$$

which is the tightness condition in Billingsley (1968, Theorem 15.5). \square

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