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Marcinkiewicz-Zygmund inequality for nonnegative N-demimartingales and related results

Milto Hadjikyriakou*

Abstract

In this paper the Marcinkiewicz-Zygmund inequality for mean zero independent random variables is extended to the case of nonnegative N-demimartingales. This result is instrumental for obtaining large deviation inequalities for nonnegative N-demimartingales as well as related complete convergence results.

Key words and phrases: N-demimartingales, Marcinkiewicz-Zygmund inequality, large deviation inequalities, complete convergence.

AMS 2000 subject classification: 60E15, 60G48

1 Introduction

The concepts of positive and negative association introduced by Esary et al. (1967) and Joag-Dev and Proschan (1983) respectively, have been studied extensively because of their wide applications in probability theory and statistical inference. Throughout the years various results are presented in the literature and among them one can find extensions and generalizations. In particular, Newman and Wright (1982) introduced the concept of a demimartingale and a demisubmartingale as a generalization of the notion of martingales and submartingales. The definition serves, among other things, the purpose of studying in a more general way the behavior of the partial sum of mean zero associated random variables.

Definition 1 A sequence of L^1 random variables $\{S_n, n \in \mathbb{N}\}$ is called a demimartingale if for all $j = 1, 2, \dots$

$$E[(S_{j+1} - S_j)f(S_1, \dots, S_j)] \geq 0$$

for all componentwise nondecreasing functions f whenever the expectation is defined. Moreover, if f is assumed to be nonnegative, the sequence $\{S_n, n \in \mathbb{N}\}$ is called a demisubmartingale.

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Definition 2 A finite collection of random variables X_1, \dots, X_n is said to be (positively) associated if

$$\text{Cov}(f(X_1, \dots, X_n), g(X_1, \dots, X_n)) \geq 0$$

for any componentwise nondecreasing functions f, g on \mathbb{R}^n such that the covariance is defined. An infinite collection is associated if every finite subcollection is associated.

Newman and Wright (1982) proved that the partial sum of mean zero positively associated random variables is a demimartingale.

The idea of a similar generalization for negatively associated random variables leads to the introduction of the concept of the so called N-demimartingales and N-demisupermartingales.

Definition 3 A sequence of L^1 random variables $\{S_n, n \in \mathbb{N}\}$ is called an N-demimartingale if for all $j = 1, 2, \dots$

$$E[(S_{j+1} - S_j)f(S_1, \dots, S_j)] \leq 0$$

for all componentwise nondecreasing functions f provided the expectation is defined. Moreover, if f is assumed to be nonnegative, the sequence $\{S_n, n \in \mathbb{N}\}$ is called an N-demisupermartingale.

Various results and examples of N-demimartingales and N-demisupermartingales can be found in Christofides (2003), Prakasa Rao (2004, 2007) and Hadjikyriakou (2010).

Let us now recall the following definition of negative association.

Definition 4 A finite collection of random variables X_1, \dots, X_n is said to be negatively associated if

$$\text{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$$

for any disjoint subsets A and B of $\{1, 2, \dots, n\}$ and for any two componentwise nondecreasing functions f, g on $\mathbb{R}^{|A|}$ and $\mathbb{R}^{|B|}$ respectively, provided that the covariance is defined. An infinite collection is negatively associated if every finite subcollection is negatively associated.

It is straightforward to verify that the partial sum of mean zero negatively associated random variables is an N-demimartingale.

It is worth mentioning that a martingale with the natural choice of σ -algebras satisfies both definitions of demimartingales and N-demimartingales. Furthermore, it can be verified that a submartingale is a demisubmartingale, and a supermartingale is an N-demisupermartingale. This relation was the motivation to study whether known martingale results can be extended to the case of N-demimartingales and demimartingales.

The Marcinkiewicz-Zygmund inequalities were first stated for mean zero independent random variables. Since martingales are generalizations of sums of independent random variables with mean zero, Burkholder (1966) extended the Marcinkiewicz-Zygmund inequalities to the case of martingale sequences. Lesigne and Volny (1999) used Burkholder's result to obtain the following large deviation inequality for martingales.

Theorem 5 *Let $\{X_n, n \in \mathbb{N}\}$ be a finite sequence of martingale differences where $X_i \in L^p$, $2 \leq p < \infty$, $\|X_i\|_p = [E(|X|^p)]^{\frac{1}{p}} < M < \infty$ for all i . Let $x > 0$. Then*

$$P(|S_n| > nx) \leq (18pq^{\frac{1}{2}})^p \frac{M^p}{x^p n^{\frac{p}{2}}} \quad (1)$$

where q is a real number for which $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 6 *In Theorem 11 of the following Section 2, a similar inequality is presented for nonnegative N -demimartingales. Given that a martingale with the natural choice of σ -algebras is always an N -demimartingale, an application of that inequality to the case of a nonnegative martingale provides a bound which can be compared to the bound of (1). In particular, if $\{S_n, n \in \mathbb{N}\}$ is a sequence of nonnegative martingales such that $\|S_{j+1} - S_j\|_p < M < \infty$ and $p > 2$, then for all $\varepsilon > 0$*

$$P(S_n > n\varepsilon) \leq \frac{(p-1)^{p/2} M^p}{n^{p/2} \varepsilon^p}.$$

This result provides a much sharper bound than the bound provided by (1).

Christofides and Hadjikyriakou (2009) presented the following Azuma-type inequality for a sequence of N -demimartingales.

Theorem 7 *Let $\{S_n, n \in \mathbb{N}\}$ (with $S_0 \equiv 0$) be an N -demimartingale and assume that*

$$|S_i - S_{i-1}| \leq c_i < \infty \quad i = 1, 2, \dots,$$

where c_1, c_2, \dots are positive real numbers. Then for every $\varepsilon > 0$,

$$P(S_n - E(S_n) \geq n\varepsilon) \leq \exp \left\{ \frac{-n^2 \varepsilon^2}{2 \sum_{i=1}^n c_i^2} \right\} \quad (2)$$

and

$$P(|S_n - E(S_n)| \geq n\varepsilon) \leq 2 \exp \left\{ \frac{-n^2 \varepsilon^2}{2 \sum_{i=1}^n c_i^2} \right\}. \quad (3)$$

Theorem 7 provides exponential inequalities for the tail probability $P(S_n \geq n\varepsilon)$ under the assumption of bounded N -demimartingale differences. It is of interest to study the case where this assumption is replaced by

$$\|S_{n+1} - S_n\|_p < M < \infty \text{ for } p > 1.$$

The answer is given through the so called Marcinkiewicz-Zygmund inequality for N-demimartingales.

In Section 2, a Marcinkiewicz-Zygmund type inequality for nonnegative N-demimartingales is established. The result provides an upper bound for the quantity $\|S_n\|_p$ in terms of the N-demimartingale differences of order p , where p is considered to be a real number greater or equal to 1. In Section 3 we present applications of the Marcinkiewicz-Zygmund inequality and in particular a large deviation inequality. In addition, complete convergence results are obtained.

2 Marcinkiewicz-Zygmund inequality for nonnegative N-demimartingales

The next lemmas are essential for the proof of the main result of this section. The proof of the first result follows by standard arguments and it is therefore omitted.

Lemma 8 *Let a, b be real numbers and let $p \in (1, 2]$.*

Then

$$|a + b|^p \leq |a|^p + p|a|^{p-1}\text{sign}(a)b + 2^{2-p}|b|^p. \quad (4)$$

Lemma 9 *Let $p > 2$. Then for $x \geq c > 0$*

$$\sqrt{x^{2-\frac{2}{p}} - c^{2-\frac{2}{p}}} \leq x^{1-\frac{2}{p}} \sqrt{(p-1) \left(x^{\frac{2}{p}} - c^{\frac{2}{p}}\right)}. \quad (5)$$

Proof. The result follows by applying Lemma 2.1 of Rio (2009) for $y = \frac{x}{c}$. ■

The next result is a Marcinkiewicz-Zygmund inequality for nonnegative N-demimartingales.

Theorem 10 *Let $\{S_n, n \in \mathbb{N}\}$ (with $S_0 \equiv 0$) be a nonnegative N-demimartingale.*

Then

(i) For $p \in (1, 2]$,

$$\begin{aligned} \|S_n\|_p^p &\leq \|d_1\|_p^p + 2^{2-p} \sum_{j=1}^{n-1} \|d_{j+1}\|_p^p \\ &\leq 2^{2-p} \sum_{j=1}^n \|d_j\|_p^p. \end{aligned}$$

(ii) For $p > 2$,

$$\begin{aligned} \|S_n\|_p^2 &\leq \|d_1\|_p^2 + (p-1) \sum_{j=1}^{n-1} \|d_{j+1}\|_p^2 \\ &\leq (p-1) \sum_{j=1}^n \|d_j\|_p^2, \end{aligned}$$

where $d_j = S_j - S_{j-1}$, $j = 1, 2, \dots, n$.

Proof. Let $p \in (1, 2]$. By applying Lemma 8 for $a = S_j$, $b = S_{j+1} - S_j$ we have that:

$$\begin{aligned} ES_{j+1}^p &\leq ES_j^p + pE[S_j^{p-1}(S_{j+1} - S_j)] + 2^{2-p}E|d_{j+1}|^p \\ &\leq ES_j^p + 2^{2-p}E|d_{j+1}|^p \end{aligned}$$

where the last inequality follows by the N-demimartingale property.

By induction we have that,

$$\begin{aligned} ES_n^p &\leq Ed_1^p + 2^{2-p} \sum_{j=2}^n E|d_j|^p \\ &= \|d_1\|_p^p + 2^{2-p} \sum_{j=2}^n \|d_j\|_p^p \\ &\leq 2^{2-p} \sum_{j=1}^n \|d_j\|_p^p. \end{aligned}$$

Let $p > 2$.

We assume that $\|S_j\|_p > 0$, for all j , otherwise the result is trivially true.

Following Rio (2009), we define the function φ on $[0, \infty)$ by

$$\varphi(t) = E(S_j + t(S_{j+1} - S_j))^p.$$

We will first prove that,

$$(\varphi(t))^{\frac{2}{p}} \leq \|S_j\|_p^2 + (p-1)\|S_{j+1} - S_j\|_p^2 t^2. \quad (6)$$

By applying Taylor's integral formula of order two for the function φ :

$$\begin{aligned} \|S_j + t(S_{j+1} - S_j)\|_p^p &= \|S_j\|_p^p + ptE[(S_{j+1} - S_j)S_j^{p-1}] \\ &\quad + p(p-1) \int_0^t (t-s)E[(S_{j+1} - S_j)^2|S_j + s(S_{j+1} - S_j)|^{p-2}] ds. \end{aligned}$$

By Hölder's inequality and the N-demimartingale property we have that,

$$\varphi(t) \leq \|S_j\|_p^p + p(p-1)\|d_{j+1}\|_p^2 \int_0^t (t-s)[\varphi(s)]^{1-\frac{2}{p}} ds := \psi(t).$$

For the first two derivatives of the function $\psi(t)$ we may write,

$$\psi'(t) = p(p-1)\|d_{j+1}\|_p^2 \int_0^t [\varphi(s)]^{1-\frac{2}{p}} ds$$

and

$$\psi''(t) = p(p-1)\|d_{j+1}\|_p^2 [\varphi(t)]^{1-\frac{2}{p}} \leq p(p-1)\|d_{j+1}\|_p^2 [\psi(t)]^{1-\frac{2}{p}}.$$

By multiplying the above inequality by $2\psi'$ and integrating between 0 and x we have,

$$\int_0^x 2\psi'(t)\psi''(t)dt \leq 2p(p-1)\|d_{j+1}\|_p^2 \int_0^x \psi'(t)[\psi(t)]^{1-\frac{2}{p}} dt$$

$$\psi'(x) \leq p\|d_{j+1}\|_p \sqrt{[\psi(x)]^{2-\frac{2}{p}} - c^{2-\frac{2}{p}}}$$

where $c = \|S_j\|_p^p$.

Using Lemma 9 we can write

$$\psi'(t) \leq p\sqrt{p-1}\|d_{j+1}\|_p [\psi(t)]^{1-\frac{2}{p}} \sqrt{[\psi(t)]^{\frac{2}{p}} - c^{\frac{2}{p}}}. \quad (7)$$

By defining $z(t) = (\psi(t))^{\frac{2}{p}}$, inequality (7) can be rewritten as

$$z'(t)(z(t) - c^{\frac{2}{p}})^{-\frac{1}{2}} \leq 2\sqrt{p-1}\|d_{j+1}\|_p.$$

By solving this differential inequality for $z(t)$ we get

$$z(t) \leq (p-1)\|d_{j+1}\|_p^2 t^2 + c^{\frac{2}{p}}.$$

Since $\varphi \leq \psi$, then $\varphi^{\frac{2}{p}} \leq z$ and (6) is established. Observe that since $\varphi(1) = \|S_{j+1}\|_p^p$ the desired inequality follows by induction. ■

3 Applications

Although important by itself, the result of Theorem 10 is used in the theorem that follows to establish large deviation inequalities for nonnegative N-demimartingales. Large deviation inequalities serve, among other things, the purpose of studying the asymptotic behavior of statistical functions, and in particular those of estimators.

Theorem 11 *Let $\{S_n, n \in \mathbb{N}\}$ be a nonnegative N-demimartingale such that for $p > 1$*

$$\|S_{j+1} - S_j\|_p < M_{j+1} < \infty, \text{ for } j = 1, 2, \dots$$

Then for every $\varepsilon > 0$,

(i)

$$P(S_n > n\varepsilon) \leq \frac{4}{(2n\varepsilon)^p} \sum_{j=1}^n M_j^p, \quad 1 < p \leq 2$$

(ii)

$$P(S_n > n\varepsilon) \leq \frac{(p-1)^{\frac{p}{2}}}{n^p \varepsilon^p} \left(\sum_{j=1}^n M_j^2 \right)^{\frac{p}{2}}, \quad p > 2.$$

Proof. Let $p \in (1, 2]$. By Lemma 10 and since $\|S_{j+1} - S_j\|_p < M_{j+1}$ we have that

$$ES_n^p \leq 2^{2-p} \sum_{j=1}^n \|d_j\|_p^p \leq 2^{2-p} \sum_{j=1}^n M_j^p.$$

Therefore for every $\varepsilon > 0$,

$$P(S_n > n\varepsilon) \leq \frac{ES_n^p}{n^p \varepsilon^p} \leq \frac{4}{(2n\varepsilon)^p} \sum_{j=1}^n M_j^p.$$

Let $p > 2$. Using Lemma 10 we can write

$$ES_n^p \leq (p-1)^{\frac{p}{2}} \left(\sum_{j=1}^n M_j^2 \right)^{\frac{p}{2}}.$$

Then for every $\varepsilon > 0$,

$$P(S_n > n\varepsilon) \leq \frac{ES_n^p}{n^p \varepsilon^p} \leq \frac{(p-1)^{\frac{p}{2}}}{n^p \varepsilon^p} \left(\sum_{j=1}^n M_j^2 \right)^{\frac{p}{2}}.$$

■

Complete convergence results for nonnegative N-demimartingales can be established by using the large deviation inequality of Theorem 11.

Theorem 12 Let $\{S_n, n \in \mathbb{N}\}$ be a nonnegative N-demimartingale such that for $p \geq 1$

$$\|S_{j+1} - S_j\|_p < M_{j+1} < \infty.$$

Let us consider the following conditions for $p \in (1, 2]$:

- (i) $\sum_{j=1}^{\infty} M_j^p < \infty$ and let r be a positive number such that $pr > 1$,
- (ii) $\sum_{j=1}^n M_j^p = O(n^\alpha)$ where α is a positive number and let r be a positive number such that $pr - 1 > \alpha$,
- (iii) $\sum_{j=2}^{\infty} \frac{M_j^p}{(j-1)^{pr-1}} < \infty$ and let r be a positive number such that $pr > 1$.

For $p > 2$ let the following conditions:

- (iv) $\sum_{j=1}^{\infty} M_j^2 < \infty$ and let r be a positive number such that $pr > 1$,
 (v) $\sum_{j=1}^n M_j^2 = O(n^\alpha)$ where α is a positive number and let r be a positive number such that $pr - \frac{\alpha p}{2} > 1$.

If anyone of the above conditions is true, then

$$n^{-r} S_n \rightarrow 0, \text{ completely.}$$

Proof. Let $p \in (1, 2]$. Assume that (i) is true. Then by using condition (i) and by applying Theorem 11 we have that

$$\begin{aligned} \sum_{n=1}^{\infty} P(S_n > n^r \varepsilon) &\leq \frac{4}{(2\varepsilon)^p} \sum_{n=1}^{\infty} \frac{(\sum_{j=1}^n M_j^p)}{n^{pr}} \\ &\leq \frac{4}{(2\varepsilon)^p} \sum_{n=1}^{\infty} \frac{(\sum_{j=1}^{\infty} M_j^p)}{n^{pr}} < \infty. \end{aligned}$$

Assume now that condition (ii) is valid. By applying Theorem 11 we can write

$$P(S_n > n^r \varepsilon) \leq \frac{4}{(2\varepsilon)^p} \frac{1}{n^{pr}} O(n^\alpha) = \frac{4}{(2\varepsilon)^p} O(n^{\alpha-pr}).$$

Therefore

$$\sum_{n=1}^{\infty} P(S_n > n^r \varepsilon) \leq \frac{4}{(2\varepsilon)^p} \sum_{n=1}^{\infty} O(n^{\alpha-pr}) < \infty$$

for $pr - 1 > \alpha$.

Finally for the case $p \in (1, 2]$ assume that condition (iii) holds. By applying Theorem 11 we arrive at the following inequality

$$\begin{aligned} \sum_{n=1}^{\infty} P(S_n \geq n^r \varepsilon) &\leq \frac{4}{(2\varepsilon)^p} \sum_{n=1}^{\infty} \frac{1}{n^{pr}} \sum_{j=1}^n M_j^p \\ &= \frac{4}{(2\varepsilon)^p} \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \frac{M_j^p}{n^{pr}} \\ &= \frac{4}{(2\varepsilon)^p} \left\{ \sum_{n=1}^{\infty} \frac{M_1^p}{n^{pr}} + \sum_{j=2}^{\infty} \sum_{n=j}^{\infty} \frac{M_j^p}{n^{pr}} \right\}. \end{aligned} \quad (8)$$

It can easily be verified that

$$\sum_{n=j}^{\infty} \frac{1}{n^{pr}} \leq \int_{j-1}^{\infty} x^{-pr} dx = \frac{1}{pr-1} (j-1)^{1-pr}. \quad (9)$$

We can write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{M_1^p}{n^{pr}} &= M_1^p + M_1^p \sum_{n=2}^{\infty} \frac{1}{n^{pr}} \\ &\leq \frac{pr M_1^p}{pr-1} \end{aligned}$$

where the last inequality follows from (9).

Therefore by applying (9) to the second term of (8) we finally have

$$\sum_{n=1}^{\infty} P(S_n \geq n^r \varepsilon) \leq \frac{4}{(2\varepsilon)^p} \left\{ \frac{pr M_1^p}{pr-1} + \frac{1}{pr-1} \sum_{j=2}^{\infty} \frac{M_j^p}{(j-1)^{pr-1}} \right\} < \infty.$$

Let $p > 2$. Assume that condition (iv) holds. By following the same steps as above we can verify that

$$\sum_{n=1}^{\infty} P(S_n > n^r \varepsilon) \leq \frac{(p-1)^{\frac{p}{2}}}{\varepsilon^p} \sum_{n=1}^{\infty} \frac{1}{n^{pr}} \left(\sum_{j=1}^{\infty} M_j^2 \right)^{\frac{p}{2}} < \infty.$$

Finally under the assumption that condition (v) is valid,

$$\sum_{n=1}^{\infty} P(S_n > n^r \varepsilon) \leq \frac{(p-1)^{\frac{p}{2}}}{\varepsilon^p} \sum_{n=1}^{\infty} O\left(\frac{1}{n^{pr-\frac{ap}{2}}}\right) < \infty.$$

for $pr - \frac{ap}{2} > 1$. ■

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