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# The action of pseudo-differential operators on functions harmonic outside a smooth hyper-surface

Louis Boutet de Monvel\* & Yves Colin de Verdière†

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The goal of this note is to describe the action of pseudo-differential operators on the space  $\mathcal{H}$  of  $L^2$  functions which are harmonic outside a smooth closed hyper-surface  $Z$  of a compact Riemannian manifold without boundary  $(X, g)$  and whose traces from both sides of  $Z$  coincide. We will represent these  $L^2$  harmonic functions as harmonic extensions of functions in the Sobolev space  $H^{-1/2}(Z)$  by a Poisson operator  $\mathcal{P}$ . The main result says that, if  $A$  is a pseudo-differential operator of degree  $d < 3$ , the operator

$$B = \mathcal{P}^* \circ A \circ \mathcal{P}$$

is a pseudo-differential operator on  $Z$  of degree  $d - 1$  whose principal symbol of degree  $d - 1$  can be computed by integration of the principal symbol of  $A$  on the co-normal bundle of  $Z$ .

These “bilateral” extensions are simpler (at least for the Laplace operator) than the “unilateral” ones whose study is the theory of pseudo-differential operators on manifolds with boundary (see [1, 2, 3, 4, 6]).

## 1 Symbols

The following classes of symbols are defined in the books [4], sec. 7.1, and in [5], sec. 18.1. A *symbol of degree  $d$*  on  $U_x \times \mathbb{R}_\xi^n$  where  $U$  is an open set in  $\mathbb{R}^N$  is a smooth complex valued function  $a(x, \xi)$  on  $U \times \mathbb{R}^n$  which satisfies the following estimates: for any multi-indices  $(\alpha, \beta)$ , there exists a constant  $C_{\alpha, \beta}$  so that

$$|D_x^\alpha D_\xi^\beta a(x; \xi)| \leq C_{\alpha, \beta} (1 + \|\xi\|)^{d - |\beta|} .$$

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The symbol  $a$  is called *classical* if  $a$  admits an expansion  $a \sim \sum_{l=0}^{\infty} a_{d-l}$  where  $a_j$  is homogeneous of degree  $j$  ( $j$  an integer) for  $\xi \in \mathbb{R}^n$  large enough; more precisely, for any  $J \in \mathbb{N}$ ,  $a - \sum_{j=0}^J a_{d-j}$  is a symbol of degree  $d - J - 1$ .

We will need the

**Lemma 1** *If  $a(x; \xi, \eta)$  is a symbol of degree  $d < -1$  defined on  $U_x \times (\mathbb{R}_\xi^n \times \mathbb{R}_\eta)$ ,  $b(x; \xi) = \int_{\mathbb{R}} a(x; \xi, \eta) d\eta$  is a symbol of degree  $d+1$  defined on  $U_x \times \mathbb{R}_\xi^n$ . Moreover, if  $a$  is classical,  $b$  is also classical and the homogeneous components of  $b$  are given for  $l \leq d+1$ , by  $b_l(x; \xi) = \int_{\mathbb{R}} a_{l-1}(x; \xi, \eta) d\eta$*

## 2 A general reduction Theorem for pseudo-differential operators

We choose local coordinates in some neighborhood of a point in  $Z$  denoted  $x = (z, y) \in \mathbb{R}^{d-1} \times \mathbb{R}$ , so that  $Z = \{y = 0\}$ . We denote by  $(\Omega_j, j = 1, \dots, N)$  a finite cover of  $Z$  by such charts and denote by  $\Omega_0$  an open set disjoint from  $Z$  so that  $X = \cup_{j=0}^N \Omega_j$ . We choose the charts  $\Omega_j$  so that the densities  $|dz|$  and  $|dx|$  are the Lebesgue measures.

If  $X$  is a smooth manifold, we denote by  $\mathcal{D}'(X)$  the space of generalized functions on  $X$  of which the space of smooth functions on  $X$  is a dense subspace. We assume that  $X$  and  $Z$  are equipped with smooth densities  $|dx|$  and  $|dz|$ . This allows to identify generalized functions with Schwartz distributions, i.e. linear functionals on test functions; this duality extending the  $L^2$  product is denoted by  $\langle | \rangle$ . We introduce the extension operator  $\mathcal{E} : \mathcal{D}'(Z) \rightarrow \mathcal{D}'(X)$  sending the distribution  $f$  to the distribution  $f\delta(y=0)$  defined

$$\langle f\delta(y=0) | \phi(z, y) \rangle = \langle f | \phi(z, 0) \rangle$$

and its adjoint, the trace  $\mathcal{T} : C^\infty(X) \rightarrow C^\infty(Z)$  defined by  $\phi \rightarrow \phi|_Z$ . Let  $A$  be a pseudo-differential operator on  $X$ : let us call  $A_j$  the restriction of  $A$  to test functions compactly supported in  $\Omega_j$ . We will work with one of the  $A_j$ 's given by the following ‘‘quantization’’ rule

$$A_j u(z, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i((z-z')\zeta + (y-y')\eta)} a_j(z, y; \zeta, \eta) u(z', y') dz' dy' d\zeta d\eta .$$

So we have formally, using the facts that the densities on  $X$  and  $Z$  are given by the Lebesgue measures in these local coordinates:

$$\mathcal{T} \circ A_j \circ \mathcal{E} v(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d-1}} e^{i(z-z')\zeta} a_j(z, 0; \zeta, \eta) v(z') dz' d\zeta d\eta ,$$

which we can rewrite

$$\mathcal{T} \circ A_j \circ \mathcal{E} v(z) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2d}} e^{i(z-z')\zeta} b_j(z; \zeta) v(z') dz' d\zeta ,$$

with

$$b_j(z; \zeta) = \frac{1}{2\pi} \int_{\mathbb{R}} a_j(z, 0; \zeta, \eta) d\eta . \quad (1)$$

We have the

**Theorem 1** *If  $A$  is a pseudo-differential operator on  $X$  of degree  $m < -1$  whose full symbol in the chart  $\Omega_j$  is  $a_j$ , then the operator  $\mathcal{T} \circ A \circ \mathcal{E}$  is a pseudo-differential operator on  $Z$  of degree  $m + 1$  whose symbol is given in the charts  $\Omega_j \cap Z$  by Equation (1).*

This is proved by looking at the actions on test functions compactly supported in the chart  $\Omega_j$ ,  $j \geq 1$ : then we use Lemma 1.

**Remark 1** *The principal symbol can be described in a more intrinsic way: let  $z \in Z$  be given, from the smooth densities on  $T_z X$  and on  $T_z Z$  given by  $|dx|$  and  $|dz|$ , we get, using the Liouville densities, densities on the dual bundles  $T_z^* Z$  and  $T_z^* X$ . Let us denote by  $\Omega^1(E)$  the 1-dimensional space of densities on the vector space  $E$ . From the exact sequence*

$$0 \rightarrow N_z^* Z \rightarrow T_z^* X \rightarrow T_z^* Z \rightarrow 0 ,$$

we deduce

$$\Omega^1(T^* X) \equiv \Omega^1(N^* Z) \otimes \Omega^1(T^* Z)$$

and a canonical density  $dm(z)$  in  $\Omega^1(N_z^* Z)$ . The principal symbol of  $B = \mathcal{T} \circ A \circ \mathcal{E}$  is given in coordinates by  $b(z, \zeta) = (1/2\pi) \int_{N_z^* Z} a(z; \zeta, \eta) dm(\eta)$ .

### 3 The “bilateral” Dirichlet-to-Neumann operator

We will assume that the local coordinates  $x = (z, y)$  along  $Z$  are chosen so that  $g(z, 0) = h(dz) + dy^2$  and the Riemannian volume along  $Z$  is  $|dx|_g = |dz|_h |dy|$ . We will choose the associated densities on  $X$  and  $Z$ . We will denote by  $\Delta_g$  the Laplace-Beltrami operator on  $(X, g)$  as defined by Riemannian geometers (i.e. with a minus sign in front of the second order derivatives).

If  $f$  is given on  $Z$ , let us denote by  $\mathcal{DN}(f)$  minus the sum of the interior normal derivatives on both sides of  $Z$  of the harmonic extension  $F$  of  $f$ ; this always makes sense, even if the normal bundle of  $Z$  is not orientable. We have the

**Lemma 2** *The distributional Laplacian of the harmonic extension  $F$  of a smooth function  $f$  on  $Z$  is  $\Delta_g F = \mathcal{E}(\mathcal{DN}(f))$ .*

*Proof.* –

The proof is a simple application of the Green's formula: by definition of the action of the Laplacian on distributions, if  $\phi$  is a test function on  $X$ ,  $\langle \Delta_g F | \phi \rangle := \langle F | \Delta_g \phi \rangle$ . We can compute the righthandside integral as an integral on  $X \setminus Z$  using Green's formula.

$$\int_{X \setminus Z} (F \Delta_g \phi - \phi \Delta_g F) |dx|_g = \int_Z (F \delta \phi - \phi \delta F) |dz|_h$$

where  $\delta$  is the sum of the interior normal derivatives from both sides of  $Z$ . Using the fact that  $\Delta_g F = 0$  in  $X \setminus Z$  and  $\delta \phi = 0$ , we get the result. □

Denoting by  $\Delta_g^{-1}$  the “quasi-inverse” of  $\Delta_g$  defined by  $\Delta_g^{-1} \phi_j = \lambda_j^{-1} \phi_j$  for the eigenfunctions  $\phi_j$  of  $\Delta_g$  with non-zero eigenvalue  $\lambda_j$  and  $\Delta_g^{-1} 1 = 0$ , we have  $f = (\mathcal{T} \circ \Delta_g^{-1} \circ \mathcal{E}) \circ \mathcal{DN}(f)$  (mod constants). By Theorem 1, the operator  $B = \mathcal{T} \circ \Delta_g^{-1} \circ \mathcal{E}$  is an elliptic self-adjoint pseudo-differential operator on  $Z$ . The operator  $\mathcal{DN}$  is a right inverse of  $B$  modulo smoothing operators and hence also a left inverse modulo smoothing operators. So that  $\mathcal{DN} = B^{-1}$  is an elliptic self-adjoint of principal symbol the inverse

$$\frac{1}{2\pi} \int_{\mathbb{R}} (\|\zeta\|_h^2 + \eta^2)^{-1} d\eta = \frac{1}{2\|\zeta\|_h},$$

namely  $2\|\zeta\|_h$ . Hence

**Theorem 2** *The bilateral Dirichlet-to-Neumann  $\mathcal{DN}$  is a self-adjoint elliptic pseudo-differential operator of degree 1 on  $L^2(Z, |dz|)$  and of principal symbol  $2\|\zeta\|_h$ . The kernel of  $\mathcal{DN}$  is the space of constant functions.*

The full symbol of  $\mathcal{DN}$  can be computed in a similar way from the full symbol of the resolvent  $\Delta_g^{-1}$  along  $Z$ .

## 4 The Poisson operator

Let  $A$  be an pseudo-differential operator on  $X$  of principal symbol  $a$ . We are interested to the restriction to the space  $\mathcal{H}$  of the quadratic form  $Q_A(F) = \langle AF | F \rangle$  associated to  $A$ . We will parametrize  $\mathcal{H}$  as harmonic extensions of functions which are in  $H^{-\frac{1}{2}}(Z)$  by the so-called Poisson operator denoted by  $\mathcal{P}$ ; the pull-back  $R_A$  of  $Q_A$  on  $L^2(Z)$  is defined by

$$R_A(f) = \langle \mathcal{AP}f | \mathcal{P}f \rangle = \langle \mathcal{P}^* \mathcal{AP}f | f \rangle.$$

The goal of this section is to compute the operator  $B = \mathcal{P}^* \mathcal{AP}$  associated to the quadratic form  $R_A$ .

From Lemma 2, we have, modulo smoothing operators,

$$\mathcal{P} = \Delta_g^{-1} \circ \mathcal{E} \circ \mathcal{DN} .$$

Hence

$$B = \mathcal{DN} \circ [\mathcal{T} \circ (\Delta_g^{-1} \circ A \circ \Delta_g^{-1}) \circ \mathcal{E}] \circ \mathcal{DN} .$$

The operator  $\Delta_g^{-1} \circ A \circ \Delta_g^{-1}$  is a pseudo-differential operator of principal symbol  $a/(\|\zeta\|_h^2 + \eta^2)^2$  near  $Z$ .

Applying Theorem 1 to the inner bracket and Theorem 2, we get the:

**Theorem 3** *If  $A$  is a pseudo-differential operator of degree  $d < 3$  on  $X$  and  $\mathcal{P}$  the Poisson operator associated to  $Z$ , the operator  $B = \mathcal{P}^* A \mathcal{P}$  is a pseudo-differential operator of degree  $d - 1$  on  $Z$  of principal symbol*

$$b(z, \zeta) = \frac{2}{\pi} \|\zeta\|_h^2 \int_{\mathbb{R}} \frac{a(z, 0; \zeta, \eta)}{(\|\zeta\|_h^2 + \eta^2)^2} d\eta .$$

**Remark 2** *Note that if  $A$  is a pseudo-differential operator without the transmission property, the operator  $A \circ \mathcal{P}$  may be ill-behaved and have disagreeable singularities along  $Z$ ; however  $\mathcal{P}^* A \mathcal{P}$  is always a good pseudo-differential operator on  $Z$ .*

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