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The action of pseudo-differential operators on functions harmonic outside a smooth hyper-surface

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The goal of this note is to describe the action of pseudo-differential operators on the space $\mathcal{H}$ of $L^2$ functions which are harmonic outside a smooth closed hyper-surface $Z$ of a compact Riemannian manifold without boundary $(X, g)$ and whose traces from both sides of $Z$ coincide. We will represent these $L^2$ harmonic functions as harmonic extensions of functions in the Sobolev space $H^{-1/2}(Z)$ by a Poisson operator $P$. The main result says that, if $A$ is a pseudo-differential operator of degree $d < 3$, the operator

$$B = P^* \circ A \circ P$$

is a pseudo-differential operator on $Z$ of degree $d - 1$ whose principal symbol of degree $d - 1$ can be computed by integration of the principal symbol of $A$ on the co-normal bundle of $Z$.

These “bilateral” extensions are simpler (at least for the Laplace operator) than the “unilateral” ones whose study is the theory of pseudo-differential operators on manifolds with boundary (see [1, 2, 3, 4, 6]).

1 Symbols

The following classes of symbols are defined in the books [4], sec. 7.1, and in [5], sec. 18.1. A symbol of degree $d$ on $U_x \times \mathbb{R}^n_x$ where $U$ is an open set in $\mathbb{R}^N$ is a smooth complex valued function $a(x, \xi)$ on $U \times \mathbb{R}^n$ which satisfies the following estimates: for any multi-indices $(\alpha, \beta)$, there exists a constant $C_{\alpha, \beta}$ so that

$$|D_x^\alpha D_\xi^\beta a(x; \xi)| \leq C_{\alpha, \beta}(1 + \|\xi\|)^{d-|\beta|}.$$
The symbol $a$ is called **classical** if $a$ admits an expansion $a \sim \sum_{d=0}^{\infty} a_d$ where $a_d$ is homogeneous of degree $d$ ($j$ an integer) for $\xi \in \mathbb{R}^n$ large enough; more precisely, for any $J \in \mathbb{N}$, $a - \sum_{j=0}^{J} a_d$ is a symbol of degree $d - J - 1$.

We will need the

**Lemma 1** If $a(x; \xi, \eta)$ is a symbol of degree $d < -1$ defined on $U \times (\mathbb{R}^n \times \mathbb{R})$, $b(x; \xi) = \int_{\mathbb{R}} a(x; \xi, \eta) d\eta$ is a symbol of degree $d + 1$ defined on $U \times \mathbb{R}^n$. Moreover, if $a$ is classical, $b$ is also classical and the homogeneous components of $b$ are given for $l \leq d + 1$, by $b_l(x; \xi) = \int_{\mathbb{R}} a_{l-1}(x; \xi, \eta) d\eta$.

## 2 A general reduction Theorem for pseudo-differential operators

We choose local coordinates in some neighborhood of a point in $Z$ denoted $x = (z, y) \in \mathbb{R}^{d-1} \times \mathbb{R}$, so that $Z = \{y = 0\}$. We denote by $(\Omega_j, j = 1, \cdots N)$ a finite cover of $Z$ by such charts and denote by $\Omega_0$ an open set disjoint from $Z$ so that $X = \bigcup_{j=1}^{N} \Omega_j$. We choose the charts $\Omega_j$ so that the densities $|dz|$ and $|dx|$ are the Lebesgue measures.

If $X$ is a smooth manifold, we denote by $\mathcal{D}'(X)$ the space of generalized functions on $X$ of which the space of smooth functions on $X$ is a dense subspace. We assume that $X$ and $Z$ are equipped with smooth densities $|dx|$ and $|dz|$. This allows to identify generalized functions with Schwartz distributions, i.e. linear functionals on test functions; this duality extending the $L^2$ product is denoted by $\langle \rangle$. We introduce the extension operator $E : \mathcal{D}'(Z) \to \mathcal{D}'(X)$ sending the distribution $f$ to the distribution $f\delta(y = 0)$ defined

$$
\langle f\delta(y = 0)|\phi(z, y)\rangle = \langle f|\phi(z, 0)\rangle
$$

and its adjoint, the trace $T : C^\infty(X) \to C^\infty(Z)$ defined by $\phi \to \phi|_{Z}$. Let $A$ be a pseudo-differential operator on $X$: let us call $A_j$ the restriction of $A$ to test functions compactly supported in $\Omega_j$. We will work with one of the $A_j$’s given by the following “quantization” rule

$$
A_j u(z, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i((z'-z)^{(\zeta})+(y-y')\eta)} a_j(z, y; \zeta, \eta) u(z', y') dz' dy' d\zeta d\eta.
$$

So we have formally, using the facts that the densities on $X$ and $Z$ are given by the Lebesgue measures in these local coordinates:

$$
T \circ A_j \circ E v(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d-1}} e^{i(z'-z)^{(\zeta})} a_j(z, 0; \zeta, \eta) v(z') dz' d\zeta d\eta,
$$

which we can rewrite

$$
T \circ A_j \circ E v(z) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2d}} e^{i(z'-z)^{(\zeta})} b_j(z; \zeta) v(z') dz' d\zeta.
$$
with
\[ b_j(z; \zeta) = \frac{1}{2\pi} \int_{\mathbb{R}} a_j(z, 0; \zeta, \eta) d\eta. \] (1)

We have the

**Theorem 1** If \( A \) is a pseudo-differential operator on \( X \) of degree \( m < -1 \) whose full symbol in the chart \( \Omega_j \) is \( a_j \), then the operator \( T \circ A \circ E \) is a pseudo-differential operator on \( Z \) of degree \( m + 1 \) whose symbol is given in the charts \( \Omega_j \cap Z \) by Equation (1).

This is proved by looking at the actions on test functions compactly supported in the chart \( \Omega_j \), \( j \geq 1 \): then we use Lemma 1.

**Remark 1** The principal symbol can be described in a more intrinsic way: let \( z \in Z \) be given, from the smooth densities on \( T_z^*X \) and on \( T_z^*Z \) given by \( |dx| \) and \( |dz| \), we get, using the Liouville densities, densities on the dual bundles \( T_z^*Z \) and \( T_z^*X \). Let us denote by \( \Omega^1(E) \) the 1-dimensional space of densities on the vector space \( E \). From the exact sequence

\[ 0 \to N_z^*Z \to T_z^*X \to T_z^*Z \to 0, \]

we deduce

\[ \Omega^1(T^*X) \cong \Omega^1(N_z^*Z) \otimes \Omega^1(T_z^*Z) \]

and a canonical density \( dm(z) \) in \( \Omega^1(N_z^*Z) \). The principal symbol of \( B = T \circ A \circ E \) is given in coordinates by \( b(z, \zeta) = (1/2\pi) \int_{N_z^*Z} a(z; \zeta, \eta) dm(\eta). \)

3 The “bilateral” Dirichlet-to-Neumann operator

We will assume that the local coordinates \( x = (z, y) \) along \( Z \) are chosen so that \( g(z, 0) = h(dz) + dy^2 \) and the Riemannian volume along \( Z \) is \( |dx|_g = |dz|h|dy| \).

We will choose the associated densities on \( X \) and \( Z \). We will denote by \( \Delta_g \) the Laplace-Beltrami operator on \( (X, g) \) as defined by Riemannian geometers (i.e. with a minus sign in front of the second order derivatives).

If \( f \) is given on \( Z \), let us denote by \( \mathcal{DN}(f) \) minus the sum of the interior normal derivatives on both sides of \( Z \) of the harmonic extension \( F \) of \( f \); this always makes sense, even if the normal bundle of \( Z \) is not orientable. We have the

**Lemma 2** The distributional Laplacian of the harmonic extension \( F \) of a smooth function \( f \) on \( Z \) is \( \Delta_g F = \mathcal{E}(\mathcal{DN}(f)) \).

**Proof.**
The proof is a simple application of the Green’s formula: by definition of the action of the Laplacian on distributions, if \( \phi \) is a test function on \( X \), 
\[
\langle \Delta g F | \phi \rangle := \langle F | \Delta g \phi \rangle.
\]
We can compute the right-hand-side integral as an integral on \( X \setminus Z \) using Green’s formula.

\[
\int_{X \setminus Z} (F \Delta g \phi - \phi \Delta g F) dx|_g = \int_{Z} (F \delta \phi - \phi \delta F) dz|_h
\]

where \( \delta \) is the sum of the interior normal derivatives from both sides of \( Z \). Using the fact that \( \Delta g F = 0 \) in \( X \setminus Z \) and \( \delta \phi = 0 \), we get the result.

Denoting by \( \Delta g^{-1} \) the “quasi-inverse” of \( \Delta g \) defined by \( \Delta g^{-1} \phi_j = \lambda_j^{-1} \phi_j \) for the eigenfunctions \( \phi_j \) of \( \Delta g \) with non-zero eigenvalue \( \lambda_j \) and \( \Delta g^{-1}1 = 0 \), we have \( f = (\mathcal{T} \circ \Delta g^{-1} \circ \mathcal{E}) \circ \mathcal{D} \mathcal{N}(f) \) (mod constants). By Theorem 1, the operator \( B = \mathcal{T} \circ \Delta g^{-1} \circ \mathcal{E} \) is an elliptic self-adjoint pseudo-differential operator on \( Z \). The operator \( \mathcal{D} \mathcal{N} \) is a right inverse of \( B \) modulo smoothing operators and hence also a left inverse modulo smoothing operators. So that \( \mathcal{D} \mathcal{N} = B^{-1} \) is an elliptic self-adjoint of principal symbol the inverse

\[
\frac{1}{2\pi} \int_\mathbb{R} (\| \zeta \|^2_h + \eta^2)^{-1} d\eta = \frac{1}{2\| \zeta \|^2_h},
\]

namely \( 2\| \zeta \|_h \). Hence

**Theorem 2** The bilateral Dirichlet-to-Neumann \( \mathcal{D} \mathcal{N} \) is a self-adjoint elliptic pseudo-differential operator of degree 1 on \( L^2(Z, |dz|) \) and of principal symbol \( 2\| \zeta \|_h \). The kernel of \( \mathcal{D} \mathcal{N} \) is the space of constant functions.

The full symbol of \( \mathcal{D} \mathcal{N} \) can be computed in a similar way from the full symbol of the resolvent \( \Delta g^{-1} \) along \( Z \).

### 4 The Poisson operator

Let \( A \) be an pseudo-differential operator on \( X \) of principal symbol \( a \). We are interested to the restriction to the space \( \mathcal{H} \) of the quadratic form \( Q_A(F) = \langle AF | F \rangle \) associated to \( A \). We will parametrize \( \mathcal{H} \) as harmonic extensions of functions which are in \( H^{-\frac{1}{2}}(Z) \) by the so-called Poisson operator denoted by \( \mathcal{P} \); the pull-back \( R_A \) of \( Q_A \) on \( L^2(Z) \) is defined by

\[
R_A(f) = \langle A \mathcal{P} f | \mathcal{P} f \rangle = \langle \mathcal{P}^* A \mathcal{P} f | f \rangle.
\]

The goal of this section is to compute the operator \( B = \mathcal{P}^* A \mathcal{P} \) associated to the quadratic form \( R_A \).
From Lemma 2, we have, modulo smoothing operators,

$$\mathcal{P} = \Delta_g^{-1} \circ \mathcal{E} \circ \mathcal{DN}.$$ 

Hence

$$B = \mathcal{DN} \circ \left[ \mathcal{T} \circ (\Delta_g^{-1} \circ A \circ \Delta_g^{-1}) \circ \mathcal{E} \right] \circ \mathcal{DN}.$$ 

The operator $\Delta_g^{-1} \circ A \circ \Delta_g^{-1}$ is a pseudo-differential operator of principal symbol $a/(\|\zeta\|^2_h + \eta^2)^2$ near $Z$.

Applying Theorem 1 to the inner bracket and Theorem 2, we get the:

**Theorem 3** If $A$ is a pseudo-differential operator of degree $d < 3$ on $X$ and $\mathcal{P}$ the Poisson operator associated to $Z$, the operator $B = \mathcal{P}^* A \mathcal{P}$ is a pseudo-differential operator of degree $d - 1$ on $Z$ of principal symbol

$$b(z, \zeta) = \frac{2}{\pi} \|\zeta\|^2_h \int_{\mathbb{R}} \frac{a(z, 0; \zeta, \eta)}{(\|\zeta\|^2_h + \eta^2)^2} d\eta.$$ 

**Remark 2** Note that if $A$ is a pseudo-differential operator without the transmission property, the operator $A \circ \mathcal{P}$ may be ill-behaved and have disagreeable singularities along $Z$; however $\mathcal{P}^* A \mathcal{P}$ is always a good pseudo-differential operator on $Z$.

**References**


