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The lower algebraic K-theory of the braid groups of the sphere

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Abstract

We study K-theoretical aspects of the braid groups $B_n(\mathbb{S}^2)$ on n strings of the 2-sphere, which by results of the second two authors, are known to satisfy the Farrell-Jones fibered isomorphism conjecture [JM]. In light of this, in order to determine the algebraic K-theory of the group ring $\mathbb{Z}[B_n(\mathbb{S}^2)]$, one should first compute that of its virtually cyclic subgroups, which have been classified recently by D. L. Gonçalves and the first author [GG6]. In this paper, we first calculate the Whitehead and K_{-1}-groups of the group rings of many of the finite subgroups (dicyclic and binary polyhedral) of $B_n(\mathbb{S}^2)$. Some new phenomena occur,

such as the appearance of torsion for the K_{-1}-groups. We then go on to study the case $n = 4$ in detail, which is the smallest value of n for which $B_n(\mathbb{S}^2)$ is infinite. We show that $B_4(\mathbb{S}^2)$ is an amalgamated product of two finite groups, from which we are able to determine a universal space on which the family of virtually cyclic subgroups of $B_4(\mathbb{S}^2)$ acts suitably. We also calculate the algebraic K-theory of the infinite virtually cyclic subgroups of $B_4(\mathbb{S}^2)$, including the Nil groups of the quaternion group of order 8. This enables us to determine completely the Whitehead group and the lower algebraic K-theory of $\mathbb{Z}[B_4(\mathbb{S}^2)]$.

1 Introduction

Given a group G, the fibred isomorphism conjecture of F. T. Farrell and L. E. Jones asserts that the algebraic K-theory of its integral group ring $\mathbb{Z}[G]$ may be computed from the knowledge of the algebraic K-theory groups of its virtually cyclic subgroups (see [FJ1] or the Appendix at the end of this paper for the statement of this conjecture). This conjecture has been verified for a number of classes of groups, such as discrete cocompact subgroups of virtually connected Lie groups [FJ1], finitely-generated Fuchsian groups [BJP], Bianchi groups [BFJP], pure braid groups of aspherical surfaces [AFR], braid groups of aspherical surfaces [FR] and for some classes of mapping class groups [BJL]. In [LO3], Lafont and Ortiz presented explicit computations of the lower algebraic K-theory of hyperbolic 3-simplex reflection groups, and then together with Magurn, for that of certain reflection groups [LMO]. Similar calculations were performed for virtually free groups in [JLMP].

Let $n \in \mathbb{N}$, let M be a surface, and let $B_n(M)$ (resp. $P_n(M)$) denote the *n-string braid group* (resp. *n-string pure braid group*) of M [Bi, Han]. The braid groups of the 2-sphere $\mathbb{S}^2$ were first studied by Zariski, and then later by Fadell and Van Buskirk during the 1960's [Z, FVB]. If M either is the 2-sphere $\mathbb{S}^2$ or the projective plane $\mathbb{R}P^2$, its braid groups are excluded from consideration by [AFR, FR], the principal reason being that they possess torsion [FVB, VB]. The second two authors of this paper proved that the Farrell-Jones isomorphism conjecture holds also for the braid groups of these two surfaces, which using the method prescribed by the conjecture, enabled them to carry out complete computations of the Whitehead groups and lower algebraic K-groups for $P_n(\mathbb{S}^2)$ and $P_n(\mathbb{R}P^2)$ [JM]. One necessary ingredient in this process is the knowledge of the virtually cyclic subgroups of $P_n(M)$. For $n \geqslant 4$, $P_n(\mathbb{S}^2)$ has only one non-trivial finite subgroup, that is generated by the 'full twist' braid, which is central and of order 2, and from this, it is straightforward to see that $P_n(\mathbb{S}^2)$ has very few isomorphism classes of virtually cyclic subgroups. The classification of the isomorphism classes of the virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$, which was established in [GG5] and used subsequently in [JM] to compute the K-theory groups of $\mathbb{Z}[P_n(\mathbb{R}P^2)]$, is rather more involved.

Our aim in this paper is to implement similar K-theoretical computations for the group ring $\mathbb{Z}[B_n(\mathbb{S}^2)]$. In order to do so, one must determine initially the virtually cyclic subgroups (finite, and then infinite) of $B_n(\mathbb{S}^2)$, and then compute the K-groups of these subgroups. If $n \leqslant 3$ then $B_n(\mathbb{S}^2)$ is finite, and so we shall assume in much of this paper that $n \geqslant 4$. The torsion of $B_n(\mathbb{S}^2)$ was determined in [GVB], and its finite order elements were classified in [M]. It was shown by D. L. Gonçalves in collaboration with the first

author that up to isomorphism, the finite subgroups of $B_n(\mathbb{S}^2)$ are cyclic, dicyclic or binary polyhedral (see [GG3] or Section 2). As for the corresponding pure braid groups, one must then determine the infinite virtually cyclic subgroups of $B_n(\mathbb{S}^2)$ with the aid of the characterisation due to Epstein and Wall of infinite virtually cyclic groups [JL, E, WaC1]. This was achieved recently, up to isomorphism and with a few exceptions if n is a small even number [GG6]. A taste of the results is given in Theorem 4 when n is odd and in Theorem 28 when $n = 4$.

In the ensuing quest to compute the lower algebraic K-theory of $\mathbb{Z}[B_n(\mathbb{S}^2)]$, we encountered a number of difficulties, among them:

(a) the family of virtually cyclic subgroups of $B_n(\mathbb{S}^2)$ is relatively large, and depends on n, contrasting sharply with the case of the pure braid groups analysed in [JM].
(b) the lower algebraic K-theory of even the finite subgroups of $B_n(\mathbb{S}^2)$ is poorly understood, and the investigation of the K-groups of dicyclic and binary polyhedral groups presents additional technical obstacles compared to that of the dihedral and polyhedral groups that appear in [LO3, LMO] for example.
(c) in order to apply the method of calculation suggested by the fibred isomorphism conjecture, one needs not only to compute the various Nil groups, but also to discover a suitable universal space for the family of virtually cyclic subgroups of $B_n(\mathbb{S}^2)$. In spite of the rich topological and geometric structures of the braid groups and their associated configuration spaces, this space has thus far proved to be elusive for $n \geqslant 5$.

Sections 3–5 are devoted to the second point, that of the computation of the lower K-theory groups corresponding to the finite subgroups of $B_n(\mathbb{S}^2)$. In Sections 3 and 5, we calculate the Whitehead and the K_{-1}-groups respectively of the integral group rings of many of the finite subgroups of $B_n(\mathbb{S}^2)$. To our knowledge, these sections contain a number of original results, as well as some new phenomena, such as the existence of torsion for some K_{-1}-groups, that did not appear in previous work [JLMP, LO3, LMO]. This necessitates alternative techniques, notably the application of results of Yamada to determine local Schur indices [Y1, Y2]. We believe that the methods that we use to calculate these K-groups for dicyclic groups of certain orders may be extended to dicyclic groups of other orders. In Section 4, we recall briefly the work of Swan pertaining to the calculation of the $\widetilde{K}_0$-groups of the group rings of the binary polyhedral groups, and of the dicyclic groups of order $4m$ for $m \leqslant 11$ [Sw]. For the dicyclic groups, the situation is complicated, and little seems to be known about the $\widetilde{K}_0$-groups corresponding to dicyclic groups of higher order.

In Section 6, we describe the ingredients for the corresponding computations in the case of *infinite* virtually cyclic subgroups. In Section 7, we use the results of the previous sections to develop a specific example, namely $B_4(\mathbb{S}^2)$. This group has several striking properties, one being that it contains a finite normal subgroup isomorphic to the quaternion group of order 8, this being a consequence of [GG4, Theorem 1.3(3)], and another that it is itself isomorphic to an amalgamated product of the generalised quaternion group of order 16 and the binary tetrahedral group, the amalgamation being along this normal subgroup (see Proposition 27). These properties aid greatly, not just in the computation of the K-groups of the virtually cyclic subgroups of $B_4(\mathbb{S}^2)$ and

of the corresponding Nil groups, but also to exhibit an appropriate universal space referred to in (c) above. We study the case of $B_4(\mathbb{S}^2)$ in detail and show how the algebraic and geometric features of this group interact, which allows us to construct its Whitehead group and lower K-groups. Our computations show that both $\mathrm{Wh}(B_4(\mathbb{S}^2))$ and $\widetilde{K}_0(\mathbb{Z}[B_4(\mathbb{S}^2)])$ are infinitely-generated Abelian groups. Indeed, each of these groups contains (infinite) direct sums of infinitely-generated Abelian 2-groups. In contrast, we shall see that $K_{-1}(\mathbb{Z}[B_4(\mathbb{S}^2)]) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. The existence of torsion here is once more a new development compared with the families of groups considered in [JLMP, LO3, LMO]. We sum up these results as follows:

Theorem 1. *The group $B_4(\mathbb{S}^2)$ has the following Whitehead and lower algebraic K-groups:*

$$\mathrm{Wh}(B_4(\mathbb{S}^2)) \cong \mathbb{Z} \oplus \mathrm{Nil}_1,$$
$$\widetilde{K}_0(\mathbb{Z}[B_4(\mathbb{S}^2)]) \cong \mathbb{Z}_2 \oplus \mathrm{Nil}_0, \quad \textit{and}$$
$$K_{-1}(\mathbb{Z}[B_4(\mathbb{S}^2)]) \cong \mathbb{Z}_2 \oplus \mathbb{Z},$$

where for $i = 0, 1$,

$$\mathrm{Nil}_i \cong \bigoplus_{\infty} [2(\mathbb{Z}_2)^{\infty} \oplus 2W],$$

$2(\mathbb{Z}_2)^{\infty}$ denotes a direct sum of two copies of a countably-infinite number of copies of $\mathbb{Z}_2$, and W is an infinitely-generated Abelian group of exponent 2 or 4.

For $n \geqslant 5$, we cannot expect the group $B_n(\mathbb{S}^2)$ to enjoy properties similar to those of $B_4(\mathbb{S}^2)$. Furthermore, we have not been able as yet to determine an appropriate model for the universal space for the family of virtually cyclic subgroups of $B_n(\mathbb{S}^2)$. There are some candidates suggested by the theory of Brunnian braids, but the corresponding subgroups are of large index, and do not seem to be terribly useful from a practical viewpoint. On the positive side, for small odd values of n, the family of virtually cyclic subgroups is relatively small, and our techniques enable us to determine the corresponding K-groups. In [GJM], we shall give some partial computations for other values of n, and outline the missing ingredients required to calculate the Whitehead and lower algebraic K-groups of their integral group rings. If we are able to find an appropriate universal space, we hope to be able to determine the K-groups of $B_n(\mathbb{S}^2)$ for other values of n.

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2 Classification of the virtually cyclic subgroups of $B_n(\mathbb{S}^2)$

If G is a group that satisfies the Farrell-Jones fibred isomorphism conjecture, the lower algebraic K-theory of the group ring $\mathbb{Z}[G]$ may be calculated in principle if one knows the lower algebraic K-theory of the group rings of the virtually cyclic subgroups of G (see the Appendix). Recall that a group is said to be *virtually cyclic* if it possesses a cyclic subgroup of finite index. Clearly any finite group is virtually cyclic. By results of Epstein and Wall [E, WaC1], an infinite group is virtually cyclic if and only if it has two ends. This allows us to show that any infinite virtually cyclic group G is isomorphic either to $F \rtimes \mathbb{Z}$ or to $G_1 *_F G_2$, where F is a finite normal subgroup of G, and in the second case, F is of index 2 in both G_1 and G_2. Consequently, in order to determine the virtually cyclic subgroups of G, one must first discover its finite subgroups. Let $G = B_n(\mathbb{S}^2)$, and if $m \geqslant 2$, let Dic_{4m} denote the *dicyclic* group of order $4m$, with presentation:

$$\mathrm{Dic}_{4m} = \left\langle x, y \;\middle|\; x^m = y^2,\ yxy^{-1} = x^{-1} \right\rangle. \tag{1}$$

If m is a power of 2 then we shall also say that Dic_{4m} is a *generalised quaternion* group, and denote it by $\mathcal{Q}_{4m}$. If $n \leqslant 3$, $B_n(\mathbb{S}^2)$ is finite: $B_1(\mathbb{S}^2)$ is trivial; $B_2(\mathbb{S}^2)$ is isomorphic to $\mathbb{Z}_2$; $B_3(\mathbb{S}^2)$ is isomorphic to Dic_{12}, and its subgroups may be obtained easily. So in most of what follows, we shall assume that $n \geqslant 4$, in which case $B_n(\mathbb{S}^2)$ is infinite. The finite subgroups of $B_n(\mathbb{S}^2)$ were classified up to isomorphism in [GG3]:

Theorem 2 ([GG3, Theorem 1.3])**.** *Let $n \geqslant 4$. The maximal finite subgroups of $B_n(\mathbb{S}^2)$ are isomorphic to one of the following groups:*

(a) $\mathbb{Z}_{2(n-1)}$ *if* $n \geqslant 5$,
(b) Dic_{4n},
(c) $Dic_{4(n-2)}$ *if* $n = 5$ *or* $n \geqslant 7$,
(d) the binary tetrahedral group, denoted by T^*, *if* $n \equiv 4 \pmod 6$,
(e) the binary octahedral group, denoted by O^*, *if* $n \equiv 0, 2 \pmod 6$,
(f) the binary icosahedral group, denoted by I^*, *if* $n \equiv 0, 2, 12, 20 \pmod{30}$.

More information on $\mathrm{T}^*, \mathrm{O}^*$ and I^*, to which we refer collectively as the *binary polyhedral groups*, may be found in [AM, CoH, CM, Wo]. It is well known that the proper subgroups of the dicyclic and the binary polyhedral groups are cyclic, dicyclic or binary polyhedral. We recall from [GG3, page 759 and Proposition 1.5] that the finite subgroups of $B_n(\mathbb{S}^2)$ are periodic of period $1, 2$ or 4, and that for each finite subgroup of $B_n(\mathbb{S}^2)$ there is a single conjugacy class, with the exception of $\mathbb{Z}_4$ and Dic_{4r} for n

even and where r divides $n/2$ or $(n-2)/2$, for which there are two conjugacy classes. Consequently, any such subgroup H of $B_n(\mathbb{S}^2)$ satisfies the following three conditions (see [AM] or [T, page 20]):

(a) the p^2-condition: for any prime divisor p of $|H|$ ($|H|$ denotes the order of H), H contains no subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

(b) the $2p$-condition: for any prime divisor p of $|H|$, any subgroup of H of order $2p$ is cyclic.

(c) the Milnor condition: if H has an element of order 2, this element is unique (and so is central in H).

Remarks 3.

(i) The p^2-condition implies that the Sylow p-subgroups of H are either cyclic or generalised quaternion, the latter case occurring only if $p = 2$.

(ii) If G is a dicyclic or binary polyhedral group, the centre $Z(G)$ is generated by the unique element of order 2.

The second step in the process is to classify the infinite virtually cyclic subgroups of $B_n(\mathbb{S}^2)$. Up to isomorphism, and with a finite number of exceptions, this has been achieved recently in [GG6]. The statement of the main result of [GG6] is somewhat long to explain here, but to give a flavour of the results, we state the classification when n is odd, in which case the classification is complete for all values of n. We shall also recall the case $n = 4$ in Theorem 28.

Theorem 4 ([GG6, Theorem 7]). *Let $n \geqslant 3$ be odd. Then up to isomorphism, the virtually cyclic subgroups of $B_n(\mathbb{S}^2)$ are as follows.*

(a) The finite virtually cyclic subgroups of $B_n(\mathbb{S}^2)$ are:

(i) Dic_{4m}, where $m \geqslant 3$ divides n or $n-2$.

(ii) $\mathbb{Z}_m$, where $m \in \mathbb{N}$ divides $2n$, $2(n-1)$ or $2(n-2)$.

(b) If in addition $n \geqslant 5$, then the following groups are the isomorphism classes of the infinite virtually cyclic subgroups of $B_n(\mathbb{S}^2)$.

(I) (a) $\mathbb{Z}_m \rtimes_\theta \mathbb{Z}$, where $\theta(1) \in \{\mathrm{Id}, -\mathrm{Id}\}$, m is a strict divisor of $2(n-i)$, for $i \in \{0,2\}$, and $m \neq n-i$.

(b) $\mathbb{Z}_m \times \mathbb{Z}$, where m is a strict divisor of $2(n-1)$.

(c) $Dic_{4m} \times \mathbb{Z}$, where $m \geqslant 3$ is a strict divisor of $n-i$ for $i \in \{0,2\}$.

(II) (a) $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \mathbb{Z}_{4q}$, where q divides $(n-1)/2$.

(b) $Dic_{4q} *_{\mathbb{Z}_{2q}} Dic_{4q}$, where $q \geqslant 2$ is a strict divisor of $n-i$, and $i \in \{0,2\}$.

We start by discussing the lower algebraic K-theory of the group rings of many of the finite groups appearing in Theorem 2.

3 Whitehead groups of the finite subgroups of $B_n(\mathbb{S}^2)$

If G is a finite group, recall that the *Whitehead group* $\mathrm{Wh}(G)$ of G is a finitely-generated Abelian group, and so may be written in the form:

$$\mathrm{Wh}(G) = \mathbb{Z}^r \oplus \mathrm{SK}_1(\mathbb{Z}[G]), \tag{2}$$

where $\mathrm{SK}_1(\mathbb{Z}[G])$ is isomorphic to the torsion subgroup of $\mathrm{Wh}(G)$ [WaC2]. The group $\mathrm{SK}_1(\mathbb{Z}[G])$ may thus be written as the direct sum of its q-torsion groups:

$$\mathrm{SK}_1(\mathbb{Z}[G]) = \bigoplus_q \mathrm{SK}_1(\mathbb{Z}[G])_{(p)},$$

where q runs over the prime divisors of $|\mathrm{SK}_1(\mathbb{Z}[G])|$. If p is a prime divisor of $|G|$, let $S_p(G)$ denote the p-Sylow subgroup of G. Using [U], we see that $\mathrm{SK}_1(\mathbb{Z}[G])$ is trivial for any finite subgroup G of $B_n(\mathbb{S}^2)$:

Proposition 5. *Let $n \in \mathbb{N}$, and let G be a finite subgroup of $B_n(\mathbb{S}^2)$. Then $\mathrm{SK}_1(\mathbb{Z}[G])$ is trivial.*

Proof. As we mentioned above, any finite subgroup G of $B_n(\mathbb{S}^2)$ is cyclic, dicyclic or binary polyhedral. If G is cyclic, dicyclic of order $8m$, $m \in \mathbb{N}$, or binary polyhedral the result follows from [U, Theorem A, parts (1), (3), (5), (6) and (7)]. The only other possibility is where G is dicyclic of order $4m$, where m is odd. In this case, the Sylow p-subgroups of G are cyclic, and from [T, page 20], G admits a presentation of the Type I groups of [U, Appendix]. The result then follows from [U, Theorem A, part (2)]. □

Remark 6. Proposition 5 may also be proved by applying [O, Theorem 14.2(i) and Example 14.4].

Consequently, if G is a finite subgroup of $B_n(\mathbb{S}^2)$ then $\mathrm{Wh}(G)$ is a free Abelian group by equation (2), and it remains to calculate its rank. This is achieved in the following proposition.

Proposition 7. *Let $n \in \mathbb{N}$, and let G be a finite subgroup of $B_n(\mathbb{S}^2)$. Then $\mathrm{Wh}(G) \cong \mathbb{Z}^r$, where:*

$$r = \begin{cases} \left\lfloor \frac{m}{2} \right\rfloor + 1 - \delta(m) & \text{if } G \cong \mathbb{Z}_m,\ m \in \mathbb{N} \\ m + 1 - \delta(2m) & \text{if } G \cong Dic_{4m},\ m \geqslant 2 \\ 0 & \text{if } G \cong \mathrm{T}^* \\ 1 & \text{if } G \cong \mathrm{O}^* \\ 2 & \text{if } G \cong \mathrm{I}^*, \end{cases}$$

and $\delta(q)$ denotes the number of divisors of $q \in \mathbb{N}$.

Before proving Proposition 7, we introduce some notation and state and prove Lemmas 8 and 9. If G is a finite group, let r_1 denote the number of conjugacy classes of unordered pairs $\{g, g^{-1}\}$ in G (where $g \in G$), and let r_2 be the number of conjugacy classes of cyclic subgroups of G. For each divisor d of $|G|$, let $r_1(d)$ be the number of conjugacy classes of unordered pairs $\{g, g^{-1}\}$ in G, where g runs over the elements of G of order d, and let $r_2(d)$ be the number of conjugacy classes of cyclic subgroups of G of order d. If g, g' are elements of G of the same order d such that $\{g, g^{-1}\}$ is conjugate to $\{g', g'^{-1}\}$ then there exists $h \in G$ such that either $hgh^{-1} = g'$ or $hgh^{-1} = g'^{-1}$, so $h\langle g\rangle h^{-1} = \langle g\rangle = \langle g^{-1}\rangle$, and thus $r_1(d) \geqslant r_2(d)$.

Lemma 8. *Let G be a finite group. Then $r_1(d) = r_2(d)$ for all $d \in \{1, 2, 3, 4, 6\}$.*

Proof of Lemma 8. If $d \in \{1,2\}$ and if $g \in G$ is of order d then the pair $\{g, g^{-1}\}$ reduces to $\{g\}$ and then clearly $r_1(d) = r_2(d)$. So assume that $d \in \{3,4,6\}$. From above, it suffices to show that $r_1(d) \leqslant r_2(d)$. Note that if $g \in G$ is of order d then the elements of $\langle g \rangle$ of order d are precisely g and g^{-1}. If $g' \in G$ is also of order d and $\langle g \rangle$ and $\langle g' \rangle$ are conjugate then g is conjugate to g' or g'^{-1}, so $\{g, g^{-1}\}$ is conjugate to $\{g', g'^{-1}\}$, which completes the proof of the lemma. □

Recall that O^* is generated by the elements X, P, Q and R, subject to the following relations [Wo, page 198]:

$$\begin{cases} X^3 = 1,\ P^2 = Q^2 = R^2,\ PQP^{-1} = Q^{-1}, XPX^{-1} = Q,\ XQX^{-1} = PQ, \\ RXR^{-1} = X^{-1},\ RPR^{-1} = QP,\ RQR^{-1} = Q^{-1}. \end{cases} \tag{3}$$

It follows that O^* contains T^* as an index 2 subgroup generated by X, P and Q that are subject to the relevant relations given by the first line of (3), where the subgroup $\langle P, Q \rangle$ is isomorphic to $\mathcal{Q}_8$, and X is of order 3 and acts by conjugation on $\langle P, Q \rangle$ by permuting P, Q and PQ cyclically, so that $\mathrm{T}^* \cong \mathcal{Q}_8 \rtimes \mathbb{Z}_3$. Further, $\mathrm{O}^* \setminus \mathrm{T}^*$ is comprised of 12 elements each of order 4 and 8. We recall also that $|\mathrm{I}^*| = 120$, that I^* has one element each of order 1 and 2, thirty elements of order 4, twenty elements each of order 3 and 6, and twenty-four elements each of order 5 and 10, and that I^* contains subgroups isomorphic to T^*.

The following lemma gathers some properties of O^* and I^* that we shall use at various points in the paper, notably in the proofs of Propositions 7 and 23.

Lemma 9.

(a) Let G be a dicyclic or binary polyhedral group, and let $g \in G$ be an element of order greater than or equal to 3. Then the centraliser of g in G is cyclic.

(b) In the group O^*:

(i) there are two conjugacy classes of elements of order 8, $r_1(8) = 2$ and $r_2(8) = 1$.

(ii) there are two conjugacy classes of elements of order 4, and $r_1(4) = r_2(4) = 2$.

(c) In the group I^*:

(i) there are two conjugacy classes of elements of order 5, $r_1(5) = 2$ and $r_2(5) = 1$.

(ii) there are two conjugacy classes of elements of order 10, $r_1(10) = 2$ and $r_2(10) = 1$.

Proof.

(a) Let $C_G(g)$ denote the centraliser of g in G. Then $g \in Z(C_G(g))$, so $|Z(C_G(g))| \geqslant 3$. The subgroups of G are cyclic, dicyclic or binary polyhedral (see [GG6, Proposition 85] for the binary polyhedral case). Then $Z(C_G(g))$ is cyclic because the centre of a dicyclic or binary polyhedral group is isomorphic to $\mathbb{Z}_2$ by Remarks 3(ii).

(b) (i) Consider the action of O^* by conjugation on the set W of the twelve elements of O^* of order 8. If $g \in W$, $C_{\mathrm{O}^*}(g)$ contains $\langle g \rangle$, and since O^* is of order 48 but contains no element of order 16, it follows from part (a) that $|C_{\mathrm{O}^*}(g)| = 8$ for all $g \in W$, whence $C_{\mathrm{O}^*}(g) = \langle g \rangle$. The orbit-stabiliser theorem implies that the conjugacy class of g contains 6 elements, and hence O^* possesses exactly two conjugacy classes of elements of order

8. Let H be a Sylow 2-subgroup of O^* that contains g. Then $|H| = 16$, and since H cannot be cyclic, we see from Remarks 3(i) that $H \cong \mathcal{Q}_{16}$. Further, g generates the unique cyclic subgroup of order 8 of H, and the presentation of $\mathcal{Q}_{16}$ given by (1) implies that $ygy^{-1} = g^{-1}$ for all $y \in H \backslash \langle g \rangle$. In particular, g and g^{-1} are conjugate in O^* for all $g \in W$, from which we deduce that $r_1(8) = 2$. Each of the three Sylow 2-subgroups of O^* contains exactly one of the three cyclic subgroups of order 8. The fact that the Sylow 2-subgroups are pairwise conjugate implies that the cyclic subgroups of order 8 are too, and hence $r_2(8) = 1$.

(ii) Consider the presentation of O^* given by equation (3). The group O^* possesses two types of elements of order 4: those belonging to the subgroup $\langle P, Q \rangle$, which in turn is a subgroup of $\langle P, Q, X \rangle$ of O^*, and those belonging to $\mathrm{O}^* \setminus \langle P, Q, X \rangle$. The structure of the group $\langle P, Q, X \rangle \cong \mathrm{T}^*$ implies that the elements of order 4 of $\langle P, Q \rangle$ are pairwise conjugate. Since $\langle P, Q \rangle$ is the unique subgroup of $\langle P, Q, X \rangle$ isomorphic to $\mathcal{Q}_8$ that is normal in O^*, it follows that $\langle P, Q \rangle \lhd \mathrm{O}^*$, and the six elements of order 4 lying in $\langle P, Q \rangle$ form one conjugacy class of elements of order 4 in O^*. So let U denote the set of the twelve elements of $\mathrm{O}^* \setminus \langle P, Q, X \rangle$ of order 4, and consider the action of O^* on U by conjugation. The stabiliser of an element $g \in U$ for this action is $C_{\mathrm{O}^*}(g)$, which is cyclic by part (a). Further, $C_{\mathrm{O}^*}(g)$ contains $\langle g \rangle$, so must be of order $4, 8, 12$ or 16. Since O^* has no cyclic subgroup of order 12 or 16, it follows that $|C_{\mathrm{O}^*}(g)| \in \{4, 8\}$. Suppose that there exists $g \in U$ for which $C_{\mathrm{O}^*}(g) \cong \mathbb{Z}_8$. Then there exists $h \in \mathrm{O}^*$ of order 8 such that $C_{\mathrm{O}^*}(g) = \langle h \rangle$, and up to replacing h by h^{-1}, we may suppose that $g = h^2$. But $\langle P, Q, X \rangle$ is normal in O^* and of index 2, so $h^2 \in \langle P, Q, X \rangle$, which contradicts the fact that $g \notin \langle P, Q, X \rangle$. We thus conclude that $|C_{\mathrm{O}^*}(g)| = 4$ for all $g \in U$, whence $C_{\mathrm{O}^*}(g) = \langle g \rangle$. By the orbit-stabiliser theorem, the conjugacy class of any such g contains all twelve elements of U, from we deduce that O^* possesses exactly two conjugacy classes of elements of order 4, and that $r_2(4) = 2$. The equality $r_1(4) = r_2(4)$ follows from Lemma 8.

(c) (i) Since I^* has twenty-four elements of order 5, it possesses six cyclic subgroups of order 5. Since these subgroups are the Sylow 5-subgroups of I^*, they are pairwise conjugate, and $r_2(5) = 1$. Let $g \in \mathrm{I}^*$ be an element of order 5. If z denotes the unique central element z of I^* of order 2, the centraliser $C_{\mathrm{I}^*}(g)$ of g contains the cyclic subgroup $\langle gz \rangle$ of order 10. By part (a), $C_{\mathrm{I}^*}(g)$ is cyclic, and since I^* possesses no element of order greater than 10, it follows that $C_{\mathrm{I}^*}(g) = \langle gz \rangle$. By applying the orbit-stabiliser theorem to the action by conjugation of I^* on the set of elements of I^* of order 5, it follows that the conjugacy class of g contains twelve elements, so I^* possesses two conjugacy classes C_1 and C_2 of elements of order 5. Since the subgroups of order 5 are pairwise conjugate, each such subgroup contributes two elements to each of C_1 and C_2. Thus if $g \in \mathrm{I}^*$ is of order 5, it is conjugate to exactly one other element h of $\langle g \rangle$. Note that $h \neq g^2$ (resp. $h \neq g^{-2}$) for otherwise g would be conjugate to g^2 (resp. g^{-2}), then g^2 would be conjugate to g^{-1} (resp. to g), and the conjugacy class of g would contain at least three elements of $\langle g \rangle$, which is not possible. Hence g is conjugate to g^{-1} for every element $g \in \mathrm{I}^*$ of order 5, but is not conjugate to g^2 or to g^{-2}. This proves that the conjugacy class of $\{g, g^{-1}\}$ is equal to that of g, and so $r_1(5) = 2$.

(ii) The subgroups of I^* of order 10 are obtained by adjoining z to the subgroups of order 5. The statements then follow from part (i). □

We now prove Proposition 7.

Proof of Proposition 7. Let G be isomorphic to a finite subgroup of $B_n(\mathbb{S}^2)$. We recall once more that G is cyclic, dicyclic or binary polyhedral. By [KL, page 39], the rank r of G is equal to $r_1 - r_2$, and so

$$r = \sum_{d\,||G|} \big(r_1(d) - r_2(d)\big). \tag{4}$$

We treat the possibilities for G separately.

(a) $G \cong \mathbb{Z}_m$, where $m \in \mathbb{N}$. Since G is Abelian, r_1 is just the number of unordered pairs $\{g, g^{-1}\}$ in G, where g runs over the elements of G, and r_2 is the number of cyclic subgroups of G. Since $g = g^{-1}$ if and only if $|\langle g\rangle| \in \{1, 2\}$, we have that $r_1 = \frac{m-1}{2} + 1$ if m is odd, and $r_1 = \frac{m-2}{2} + 2$ if m is even. So $r_1 = \left\lfloor \frac{m}{2} \right\rfloor + 1$. Since the subgroups of $\mathbb{Z}_m$ are in bijection with the divisors of m, we have $r_2 = \delta(m)$, so $r = r_1 - r_2 = \left\lfloor \frac{m}{2} \right\rfloor + 1 - \delta(m)$ as required.
(b) $G \cong \mathrm{Dic}_{4m}$, where $m \geqslant 2$. Let $G = \langle x\rangle \coprod \langle x\rangle y$ be given by equation (1). Since the elements of $\langle x\rangle y$ are of order 4, it follows from Lemma 8 and equation (4) that they do not contribute to r. So we just need to consider the contributions of the elements of $\langle x\rangle$ to r_1 and r_2. Using equation (1), the conjugacy classes of the elements of $\langle x\rangle$ in G are $\{x^i, x^{-i}\}$, where $0 \leqslant i \leqslant n$. Since $\langle x\rangle$ is of order $2m$, as in the cyclic case, its elements contribute $m + 1$ to the r_1-term, and $\delta(2m)$ to the r_2-term, and thus $r = m + 1 - \delta(2m)$.
(c) $G \cong \mathrm{T}^*$. Let G have the presentation given by the first line of (3). Since $\mathrm{T}^* \cong \mathcal{Q}_8 \rtimes \mathbb{Z}_3$, where the action of X permutes cyclically the elements P, Q and PQ of $\langle P, Q\rangle$, T^* is comprised of elements of order $1, 2, 3, 4$ and 6. The fact that $r = 0$ then follows directly from Lemma 8 and equation (4).
(d) $G \cong \mathrm{O}^*$. Recall that G has elements of order $1, 2, 3, 4, 6$ and 8. Again by Lemma 8 and equation (4), the contributions to r_1 and r_2 from the elements of order $1, 2, 3, 4$ and 6 cancel pairwise, and since $r_1(8) - r_2(8) = 1$ by Lemma 9(b)(i), $r = 1$ by equation (4), whence $\mathrm{Wh}(\mathrm{O}^*) \cong \mathbb{Z}$.
(e) $G \cong \mathrm{I}^*$. Then G has elements of order $1, 2, 3, 4, 5, 6$ and 10. Once more, by Lemma 8 and equation (4), the contributions to r of the elements of order $1, 2, 3, 4$ and 6 cancel pairwise, so $r = (r_1(5) - r_2(5)) + (r_1(10) - r_2(10)) = 2$ by Lemma 9(c), whence $\mathrm{Wh}(\mathrm{I}^*) \cong \mathbb{Z}^2$. □

4 $\widetilde{K}_0(\mathbb{Z}[G])$ for the finite subgroups of $B_n(\mathbb{S}^2)$

Let G be a finite group. The calculation of $\widetilde{K}_0(\mathbb{Z}[G])$ is a difficult problem, even when the order of G is small. It is known that $\widetilde{K}_0(\mathbb{Z}[G])$ is isomorphic to the *ideal class group* $\mathrm{Cl}(\mathbb{Z}[G])$ of $\mathbb{Z}[G]$ [CR2, Section 49.11]. If G is Abelian then $\widetilde{K}_0(\mathbb{Z}[G])$ is trivial if and only if G is either cyclic of order n, $n \in \{1, 2, \ldots, 11, 13, 14, 17, 19\}$, or is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ [CR2, Corollary 50.17]. If G is non Abelian and $\widetilde{K}_0(\mathbb{Z}[G]) = 1$ then G is isomorphic to one of Dih_{2q}, $q \geqslant 3$, A_4, S_4 or A_5, Dih_{2q} being the dihedral group of order $2q$ [EH].

The following theorem summarises the results of [Sw] concerning some of the non-Abelian finite subgroups of $B_n(\mathbb{S}^2)$.

Theorem 10 ([Sw, Theorems III and IV, Corollary 10.12])**.**

(a) $\widetilde{K}_0(\mathbb{Z}[Dic_{4m}]) \cong \begin{cases} \mathbb{Z}_2 & \text{if } m \in \{2,3,4,5,7,8,11\} \\ \mathbb{Z}_2^2 & \text{if } m = 9 \\ \mathbb{Z}_2^3 & \text{if } m \in \{6,10\}. \end{cases}$

(b) $\widetilde{K}_0(\mathbb{Z}[\mathrm{T}^*]) \cong \mathbb{Z}_2$, $\widetilde{K}_0(\mathbb{Z}[\mathrm{O}^*]) \cong \mathbb{Z}_2^2$ *and* $\widetilde{K}_0(\mathbb{Z}[\mathrm{I}^*]) \cong \mathbb{Z}_2^3$.

5 $K_{-1}(\mathbb{Z}[G])$ for the finite subgroups of $B_n(\mathbb{S}^2)$

Let G be a finite subgroup of $B_n(\mathbb{S}^2)$. In order to determine $K_{-1}(\mathbb{Z}[G])$, we shall use the following special case of a result of Carter. First we recall that a simple Artinian ring A is isomorphic to $\mathcal{M}_n(D)$ for some positive integer n and some skew field D. Further, D is finite dimensional over its centre E, the dimension being a square $[D : E]$, and the *Schur index* of A is equal to $\sqrt{[D : E]}$ [CR1, Section 27].

Theorem 11 ([Ca, Theorem 1])**.** *Let G be a finite group of order q. Then*

$$K_{-1}(\mathbb{Z}[G]) \cong \mathbb{Z}^r \oplus \mathbb{Z}_2^s, \tag{5}$$

where r is given by

$$r = 1 - r_{\mathbb{Q}} + \sum_{p \,|\, |G|} \left(r_{\mathbb{Q}_p} - r_{\mathbb{F}_p} \right), \tag{6}$$

$r_{\mathbb{Q}}$ (resp. $r_{\mathbb{Q}_p}$, $r_{\mathbb{F}_p}$) denotes the number of isomorphism classes of irreducible $\mathbb{Q}$- (resp. $\mathbb{Q}_p$, $\mathbb{Z}_p$) representations of G, and s is equal to the number of simple components of $\mathbb{Q}[G]$ that have even Schur index m but have odd local Schur indices m_q at every finite prime of the centre which divides q.

So to calculate $K_{-1}(\mathbb{Z}[G])$, we must determine the quantities r_F for the various fields appearing in equation (6), as well as the number s. For the finite subgroups G of $B_n(\mathbb{S}^2)$, we divide this calculation into two parts. In Section 5.1, we determine the torsion of $K_{-1}(\mathbb{Z}[G])$. In Section 5.2, we determine the rank of $K_{-1}(\mathbb{Z}[G])$. Equation (5) will then yield $K_{-1}(\mathbb{Z}[G])$.

5.1 Torsion of $K_{-1}(\mathbb{Z}[G])$ for the finite subgroups of $B_n(\mathbb{S}^2)$

Let G be a finite subgroup of $B_n(\mathbb{S}^2)$, and let s be as defined in equation (5). As remarked in [Ca, page 1928], a consequence of Theorem 11 is that $K_{-1}(\mathbb{Z}[G])$ is torsion free if G is Abelian. In particular, if G is cyclic then $s = 0$. If G is non cyclic, then as we shall see, $K_{-1}(\mathbb{Z}[G])$ may have torsion. Although equation (5) clearly allows for this possibility, this appears to be a new phenomenon, and contrasts with the calculations given in [LO3, LMO] for example. We thus require new techniques to calculate the torsion

of $K_{-1}(\mathbb{Z}[G])$. If G is dicyclic then we make use of results due to Yamada concerning the computation of the (local) Schur indices of the simple components of $\mathbb{Q}[G]$ [Y2]. If G is binary polyhedral then one may apply induction/restriction techniques and the Mackey formula.

Assume first that $G \cong \mathrm{Dic}_{4m}$ is dicyclic, where $m \geqslant 3$ is odd. If m is an odd prime then we determine $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{4m}])$. In principle, our method should apply to any odd value of m, not just for prime values. If m is odd, the Wedderburn decomposition over $\mathbb{Q}$ of the algebra $\mathbb{Q}[\mathrm{Dic}_{4m}]$ is given in [CR1, Example 7.40]:

$$\begin{aligned}\mathbb{Q}[\mathrm{Dic}_{4m}] &\cong \mathbb{Q}[\mathrm{Dih}_{2m}] \oplus \mathbb{Q}(i) \oplus \left(\bigoplus_{d_0 | m,\, d_0 > 1} \mathbb{H}_{2d_0} \right) \\ &\cong \mathbb{Q}^2 \oplus \left(\bigoplus_{d | m,\, d > 2} \mathcal{M}_2 \left(\mathbb{Q}\left(\zeta_d + \zeta_d^{-1} \right) \right) \right) \oplus \mathbb{Q}(i) \oplus \left(\bigoplus_{d_0 | m,\, d_0 > 1} \mathbb{H}_{2d_0} \right), \end{aligned} \tag{7}$$

where ζ_d is a primitive d^{th} root of unity, and

$$\mathbb{H}_d = E_d \oplus E_d i \oplus E_d j \oplus E_d k \tag{8}$$

is the quaternion skew field with centre $E_d = \mathbb{Q}\left(\zeta_d + \zeta_d^{-1} \right)$. In particular, if $m = \mu$ is prime then

$$\mathbb{Q}[\mathrm{Dic}_{4\mu}] \cong \mathbb{Q}^2 \oplus \mathcal{M}_2\left(E_\mu\right) \oplus \mathbb{Q}(i) \oplus \mathbb{H}_{2\mu}. \tag{9}$$

Note that the number of components in equation (9) is equal to the number of conjugacy classes of cyclic subgroups of $\mathrm{Dic}_{4\mu}$, and that the components are in one-to-one correspondence with the irreducible $\mathbb{Q}$-representations of $\mathrm{Dic}_{4\mu}$. The first four components of equation (9) are matrix rings over fields, and so their Schur index is equal to one. By equation (5), the torsion of $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{4\mu}])$ is then either trivial or equal to $\mathbb{Z}_2$ depending on the Schur and local Schur indices of the remaining component $\mathbb{H}_{2\mu}$. We now determine precisely this torsion.

Proposition 12. *Let μ be an odd prime, and let $G \cong Dic_{4\mu}$. Then the torsion of $K_{-1}(\mathbb{Z}[G])$ is trivial if $\mu \equiv 3 \bmod 4$, and is equal to $\mathbb{Z}_2$ if $\mu \equiv 1 \bmod 4$.*

Proof. With the notation of [Y2, Proposition 4], we have $m = 2\mu$, $r = 2\mu - 1$, $s = 2$, $h = \mu$ and u is the order of $2\mu - 1 \bmod^{\times} 2\mu$, so $u = s = 2$. From [Y2, Example 3, Section 6], there are representations of $\mathbb{Q}[\mathrm{Dic}_{4\mu}]$ of the form $U^{(2)}_{\alpha,0}$, where $0 \leqslant \alpha \leqslant 2\mu - 1$. Using [Y2, Proposition 5], $U^{(2)}_{1,0}$ gives rise to an irreducible representation of $\mathbb{Q}[\mathrm{Dic}_{4\mu}]$, and the last part of [Y2, Example 3, Section 6] implies that its Schur index is equal to two. Since the Schur index of each of the first four components of equation (9) is equal to one, it follows that the simple component $\mathbb{H}_{2\mu}$ of $\mathbb{Q}[\mathrm{Dic}_{4\mu}]$ corresponds to $U^{(2)}_{1,0}$.

We now apply [Y2, Proposition 9] to $U^{(2)}_{1,0}$. Within our framework, the enveloping algebra $\mathrm{env}_{\mathbb{Q}}\left(U^{(2)}_{1,0}\right)$ with respect to $\mathbb{Q}$ is isomorphic to the simple component $\mathbb{H}_{2\mu}$, and

the centre $E_{2\mu}$ of $\mathbb{H}_{2\mu}$ is isomorphic to $\mathbb{Q}\left(\chi_{1,0}^{(2)}\right)$, $\chi_{1,0}^{(2)}$ being the character of $U_{1,0}^{(2)}$ [Y1, Introduction]. With the notation of [Y2, Proposition 9], we have $d_1 = \frac{2\mu}{\gcd(2\mu,1)} = 2\mu$ and $v_1 = \frac{2\mu}{\gcd(2\mu,\mu)} = 2$. Let $\mathfrak{p}$ be a finite prime of the centre E_μ of $\mathbb{H}_{2\mu}$ that divides 4μ. We consider two cases.

(a) If $\mathfrak{p}$ does not divide 2μ then the local Schur index $\Lambda_\mathfrak{p}$ is equal to 1 by [Y2, Proposition 9(I)].

(b) Suppose that $\mathfrak{p}$ divides 2μ. Since μ is odd, $\mathfrak{p}$ divides p, where $p \in \{2, \mu\}$. We distinguish these two possibilities, the notation being that of [Y2, Proposition 9].

(i) Suppose that $\mathfrak{p}$ divides 2. Then we have $p = 2$, $b = z = 1$, $a = 1$ and t' is the order of $2\mu - 1 \bmod^\times \mu$, so $t' = 2$. Thus $e_\mathfrak{p} = 1$, and hence $c_\mathfrak{p} = \Lambda_\mathfrak{p} = 1$.

(ii) Suppose that $\mathfrak{p}$ divides μ. Then $p = \mu$, $b = 0$, $z = 2$, $a = 1$, $\left\langle 2\mu - 1 \bmod^\times 2 \right\rangle = \left\langle \mu \bmod^\times 2 \right\rangle = \{1\}$, $f = \widetilde{f} = t' = 1$, $q = \mu$, $e_\mathfrak{p} = 2$, $c_\mathfrak{p} = \gcd(2, \mu - 1) = 2$ and

$$\Lambda_\mathfrak{p} = \frac{2}{\gcd\left(2, \frac{\mu-1}{2}\right)} = \begin{cases} 1 & \text{if } \mu \equiv 1 \bmod 4 \\ 2 & \text{if } \mu \equiv 3 \bmod 4 \end{cases}$$

by [Y2, Proposition 9(II)]. Thus if $\mu \equiv 1 \bmod 4$, the simple component $\mathbb{H}_{2\mu}$ of $\mathbb{Q}[G]$ whose Schur index is equal to two satisfies the property that its local Schur indices at every finite prime of the centre are odd. Hence the integer s of equation (5) is equal to one, so the torsion of $K_{-1}(\mathbb{Z}[G])$ is $\mathbb{Z}_2$. If $\mu \equiv 3 \bmod 4$ then $\Lambda_\mathfrak{p} = 2$ for any finite prime $\mathfrak{p}$ that divides μ, so $s = 0$, and hence $K_{-1}(\mathbb{Z}[G])$ is torsion free. □

As another example, we calculate the torsion of $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{4m}])$ in the case where m is a power of 2 (so Dic_{4m} is a generalised quaternion group).

Proposition 13. *The torsion of $K_{-1}(\mathbb{Z}[\mathcal{Q}_{2^k}])$ is trivial if $k = 3$, and is equal to $\mathbb{Z}_2$ if $k \geqslant 4$.*

Proof. Let $k \geqslant 3$. Then $\mathrm{Dic}_{2^k} = \mathcal{Q}_{2^k}$. From [CR1, Example 7.40, case 1],

$$\mathbb{Q}[\mathcal{Q}_{2^k}] \cong \mathbb{Q}[\mathrm{Dih}_{2^{k-2}}] \oplus \mathbb{H}_{2^{k-1}}, \tag{10}$$

where $\mathbb{H}_{2^{k-1}}$ is the quaternion skew field defined by equation (8). Using [CR1, Example 7.39], each simple component of $\mathbb{Q}[\mathrm{Dih}_{2^{k-2}}]$ is a matrix ring over a field, and so its Schur index is equal to one. As in the proof of Proposition 12, one may show that the Schur index of the remaining simple component $\mathbb{H}_{2^{k-1}}$ of equation (10) is equal to two, and that this component corresponds to the irreducible representation $U_{1,0}^{(2)}$. To study the local Schur indice $\Lambda_\mathfrak{p}$ of each finite prime $\mathfrak{p}$ dividing the centre $E_{2^{k-1}}$ of the simple component $\mathbb{H}_{2^{k-1}}$, we again apply [Y1, Proposition 9]. With the same notation, we have $m = 2^{k-1}$, $r = 2^{k-1} - 1$, $u = s = 2$, $h = 2^{k-2}$, $d_1 = 2^{k-1}$ and $v_1 = 2$. If $\mathfrak{p}$ does not divide 2^{k-1} then $\Lambda_\mathfrak{p} = 1$ by [Y1, Proposition 9(I)]. So suppose that $\mathfrak{p}$ divides 2^{k-1}. With the notation of [Y1, Proposition 9(II)], $b = z = 1$ and $p = 2$. If $k = 3$ then we are in the exceptional case of [Y1, Proposition 9(II)], so $\Lambda_\mathfrak{p} = 2$. Thus there exists a finite prime of the centre $E_{2^{k-1}}$ of $\mathbb{H}_{2^{k-1}}$ dividing 2^k with even local Schur index, and

it follows from Theorem 11 that the torsion of $K_{-1}(\mathbb{Z}[\mathcal{Q}_8])$ is trivial. Assume then that $k \geqslant 4$. So $f = \widetilde{f} = t' = 1$ and $e_{\mathfrak{p}} = q = 2$, thus $c_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} = 1$. Then the simple component $\mathbb{H}_{2^{k-1}}$ of $\mathbb{Q}[\mathcal{Q}_{2^k}]$ whose Schur index is equal to two satisfies the property that its local Schur indices at every finite prime of the centre dividing 2^k are odd. Hence the integer s of equation (5) is equal to one, and thus the torsion of $K_{-1}(\mathbb{Z}[\mathcal{Q}_{2^k}])$ is equal to $\mathbb{Z}_2$ as required. □

Now let G be a binary polyhedral group. We recall that a group is said to be 2-*hyperelementary* if it is a semi-direct product of a cyclic normal subgroup of odd order and a 2-group. Since G is not itself 2-hyperelementary, induction/restriction techniques may be used to calculate the torsion of $K_{-1}(\mathbb{Z}[G])$.

Proposition 14. *The torsion of $K_{-1}(\mathbb{Z}[G])$ is trivial if $G \cong \mathrm{T}^*$, and is equal to $\mathbb{Z}_2$ if $G \cong \mathrm{O}^*$ or $G \cong \mathrm{I}^*$.*

Proof. Let G be a binary polyhedral group. Applying [Ca, Theorem 3(iii) and page 1936], we have the composition

$$\oplus\, K_{-1}(\mathbb{Z}[H]) \xrightarrow{\mathsf{ind}} K_{-1}(\mathbb{Z}[G]) \xhookrightarrow{\mathsf{res}} \oplus K_{-1}(\mathbb{Z}[H]), \tag{11}$$

where ind and res are the usual induction and restriction maps that are surjective and injective respectively, and H runs over the conjugacy classes of the 2-hyperelementary subgroups of G [Ca, Theorem 3(iii) and page 1936]. Restricting to the corresponding torsion subgroups, we see that the torsion of $K_{-1}(\mathbb{Z}[G])$ injects into that of $\oplus K_{-1}(\mathbb{Z}[H])$. The non-trivial 2-hyperelementary subgroups of T^* are $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ and $\mathcal{Q}_8$, those of O^* are $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_8$, Dic_{12}, $\mathcal{Q}_8$ and $\mathcal{Q}_{16}$, and those of I^* are $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_{10}$, $\mathcal{Q}_8$, Dic_{12} and Dic_{20} (see [Sw, Lemma 14.3] and [GG6, Proposition 85]). If $m \in \mathbb{N}$, the group algebra $\mathbb{Q}[\mathbb{Z}_m]$ splits [CR1, Example 7.38], so the torsion of $K_{-1}(\mathbb{Z}[\mathbb{Z}_m])$ is trivial [Ca, page 1928]. Further, by Propositions 12 and 13, the torsion of $K_{-1}(\mathbb{Z}[\mathcal{Q}_8])$ and of $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{12}])$ is also trivial, and setting $L = \mathcal{Q}_{16}$ (resp. $L = \mathrm{Dic}_{20}$) if $G = \mathrm{O}^*$ (resp. $G = \mathrm{I}^*$), the torsion of $K_{-1}(\mathbb{Z}[L])$ is $\mathbb{Z}_2$. The injectivity of res in equation (11) implies that the torsion of $K_{-1}(\mathbb{Z}[\mathrm{T}^*])$ is trivial, which gives the result in this case.

So let $G = \mathrm{O}^*$ or I^*, and let L be as defined above. Now G possesses a single conjugacy class of subgroups isomorphic to L [Sw, Lemma 14.3], and since L is the only subgroup of G for which the torsion of $K_{-1}(\mathbb{Z}[L])$ is non trivial, we need only consider the restriction of equation (11) to the factor $H = L$:

$$K_{-1}(\mathbb{Z}[L]]) \xrightarrow{\mathsf{ind}} K_{-1}(\mathbb{Z}[G]) \xhookrightarrow{\mathsf{res}} K_{-1}(\mathbb{Z}[L]). \tag{12}$$

It thus suffices to show that the restriction of (12) to the corresponding torsion subgroups is the identity. Now $K_{-1}(\cdot)$ is a Mackey functor [O, Theorem 11.2], so we may apply Mackey's formula that describes the composition (12) as the sum of the maps

$$K_{-1}(\mathbb{Z}[L]]) \xrightarrow{\mathsf{ind}} K_{-1}(\mathbb{Z}[x_i^{-1}Lx_i \cap L]) \xrightarrow{c_{x_i}} K_{-1}(\mathbb{Z}[L]), \tag{13}$$

where $G = \coprod Lx_iL$ is a double coset decomposition of G, and the map c_{x_i} is induced by the homomorphism $x_i^{-1}Lx_i \cap L \longrightarrow L$ defined by $y \mapsto x_iyx_i^{-1}$ [O, Section 11a]. Let $N_G(L)$ denote the normaliser of L in G. If $x_i \notin N_G(L)$ then the torsion of $K_{-1}(\mathbb{Z}[x_i^{-1}Lx_i \cap L])$ is trivial, and the corresponding map (13) contributes zero to the restriction of (12) to the torsion subgroups. If on the other hand, $x_i \in N_G(L)$, the corresponding map (13) is an isomorphism. Now $L \subset N_G(L) \subset G$, and since L is not normal in G and G has no proper subgroup that strictly contains L [GG6, Proposition 85], it follows that $N_G(L) = L$. So there is only one double coset representative x_i that belongs to $N_G(L)$, and for this x_i, it follows that the restriction of (12) to the torsion subgroups is equal to the restriction of the isomorphism (13) to the torsion subgroups. Since the torsion of $K_{-1}(\mathbb{Z}[L])$ is $\mathbb{Z}_2$, the same conclusion holds for $K_{-1}(\mathbb{Z}[G])$. □

Remarks 15.

(a) The induction/restriction arguments in the proof of Proposition 14 were inspired by those given in [Sw, Paragraph 14] for the $\widetilde{K}_0$-groups.

(b) Let $G = \mathrm{O}^*$ or I^*. We sketch an alternative proof of the fact that $K_{-1}(\mathbb{Z}[G])$ has non-trivial torsion that uses [Sw, Proposition 4.11], the details of which will appear in [GJM]. The embedding of G in the Hamilton quaternions $\mathbb{H}$ [CoH, Chapter 7] induces an algebra homomorphism $\psi_G\colon \mathbb{Q}[G] \longrightarrow \mathbb{H}$. By [Sw, Proposition 4.11 and its proof], $\psi_G(\mathbb{Z}[G])$ is a maximal order Γ_G that is completely described in [Sw, page 79], from which one may prove that $\mathrm{Im}(\psi_G)$ is equal to $\mathbb{H}_d$, where $d = 8$ (resp. $d = 5$) if $G = \mathrm{O}^*$ (resp. $G = \mathrm{I}^*$), in other words, $\mathbb{H}_d$ appears as a factor in the Wedderburn decomposition of $\mathbb{Q}[G]$. On the other hand, from equation (10) (resp. equation (9)), we know that $\mathbb{H}_d$ also appears in the Wedderburn decomposition of $\mathbb{Q}[\mathcal{Q}_{16}]$ (resp. $\mathbb{Q}[\mathrm{Dic}_{20}]$), and from the proof of Proposition 13 (resp. Proposition 12), that it contributes a $\mathbb{Z}_2$-term to the torsion of $\mathbb{K}_{-1}(\mathbb{Z}[\mathcal{Q}_{16}])$ (resp. $\mathbb{K}_{-1}(\mathbb{Z}[\mathrm{Dic}_{20}])$). It follows then from [Ca, Theorem 1] that $K_{-1}(\mathbb{Z}[G])$ has non-trivial torsion.

(c) Using the `GAP` package `Wedderga` [BKOOR], one may obtain the complete Wedderburn decomposition for the binary polyhedral groups:

$$\mathbb{Q}[\mathrm{T}^*] \cong \mathbb{Q} \oplus \mathbb{Q}(\zeta_3) \oplus \mathcal{M}_3(\mathbb{Q}) \oplus \mathbb{H}_4 \oplus \mathbb{H}(\mathbb{Q}(\zeta_3)) \tag{14}$$

$$\mathbb{Q}[\mathrm{O}^*] \cong 2\mathbb{Q} \oplus \mathcal{M}_2(\mathbb{Q}) \oplus 2\mathcal{M}_3(\mathbb{Q}) \oplus \mathbb{H}_8 \oplus \mathcal{M}_2(\widehat{\mathbb{H}}), \quad \text{and} \tag{15}$$

$$\mathbb{Q}[\mathrm{I}^*] \cong \mathbb{Q} \oplus \mathcal{M}_4(\mathbb{Q}) \oplus \mathbb{H}_5 \oplus \mathcal{M}_2(\widehat{\mathbb{H}}) \oplus \mathcal{M}_5(\mathbb{Q}) \oplus \mathcal{M}_3(\mathbb{H}(\mathbb{Q})) \oplus \mathcal{M}_3(\mathbb{Q}(\sqrt{5})), \tag{16}$$

where $\widehat{\mathbb{H}}$ is the quaternion algebra $(-1,-3)/\mathbb{Q}$. Somewhat surprisingly, we were not able to find these decompositions in the literature.

5.2 The rank of $K_{-1}(\mathbb{Z}[G])$ for the finite subgroups of $B_n(\mathbb{S}^2)$

Let G be a finite subgroup of $B_n(\mathbb{S}^2)$. To calculate the rank of $K_{-1}(\mathbb{Z}[G])$, we shall apply equation (6). In each case, we will thus need to calculate the number r_F of distinct irreducible $F[G]$-modules, where F is equal respectively to $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{F}_p$. Before doing so, we recall the requisite theory (see [CR1, pages 492 and 508] or [O, pages 25–26]).

Let F be a field of characteristic $p \geqslant 0$, where p is prime. If $p > 0$, let G be a finite group of exponent m. Define

$$\widehat{m} = \begin{cases} m & \text{if } p = 0 \\ m/p^a & \text{if } p > 0, \text{ where } a \text{ is the largest power of } p \text{ that divides } m. \end{cases}$$

Let $F(\zeta_{\widehat{m}})$ be a field extension of F by a primitive $\widehat{m}^{\text{th}}$ root of unity, which we denote by $\zeta_{\widehat{m}}$. Then $F(\zeta_{\widehat{m}})$ is a Galois extension of F, whose Galois group, denoted by $\mathrm{Gal}(F(\zeta_{\widehat{m}})/F)$, is given by:

$$\mathrm{Gal}(F(\zeta_{\widehat{m}})/F) = \{\phi\colon\ F(\zeta_m) \longrightarrow F(\zeta_{\widehat{m}}) \mid \phi \text{ is an automorphism and } \phi(z) = z \text{ for all } z \in F\}.$$

Each automorphism $\sigma \in \mathrm{Gal}(F(\zeta_{\widehat{m}})/F)$ is uniquely determined by its action on $\zeta_{\widehat{m}}$, and is given by $\sigma(\zeta_{\widehat{m}}) = \zeta_{\widehat{m}}^t$, where t is an integer that is uniquely defined modulo $\widehat{m}$. Hence t corresponds to an element of the multiplicative group of units $\mathbb{Z}_{\widehat{m}}^*$, and there is an injective group homomorphism:

$$\begin{cases} \phi\colon\ \mathrm{Gal}(F(\zeta_{\widehat{m}})/F) \longrightarrow \mathbb{Z}_{\widehat{m}}^* \\ \qquad\qquad\qquad\qquad \sigma \longmapsto t, \end{cases}$$

defined by $\phi(\sigma) = t$. We now recall the definition of F-conjugacy class.

(a) If f, g are elements of G, we say that they are *F-conjugate* if there exists $t \in \mathrm{Im}(\phi)$ and $\alpha \in G$ such that $f^t = \alpha g \alpha^{-1}$. The F-conjugacy relation is an equivalence relation on G, and the F-equivalence class of f in G will be denoted by $[f]_F$.

(b) Let

$$G'_p = \{g \in G \mid \gcd(p, o(g)) = 1\},$$

be the set of *p-regular elements* of G, where $o(g)$ denotes the order of $g \in G$. An F-conjugacy class of G is said to be *p-regular* if it is contained in G'_p.

If $f \in G$ then we denote its usual conjugacy class by $[f]$.

Remarks 16.

(a) It follows from the definition that

$$[f]_F = \bigcup_{t \in \mathrm{Im}(\phi)} [f^t], \tag{17}$$

in other words, an F-conjugacy class is a union of normal conjugacy classes. In particular, $[f]_F \supset [f]$. Further, the number of F-conjugacy classes of elements of order n is bounded above by the number of usual conjugacy classes of elements of order n.

(b) If $F = \mathbb{Q}$ then ϕ is an isomorphism [O, Theorem 1.5], and $f, g \in G$ are F-conjugate if and only if $\langle f \rangle$ and $\langle g \rangle$ are conjugate subgroups of G.

By the Witt-Berman Theorem, we have:

Theorem 17 ([CR1, Theorems 21.5 and 21.25]). *Let G be a finite group, and let F be a field of characteristic $p \geqslant 0$, where p is prime if $p > 0$.*
(a) If $p = 0$ then r_F is equal to the number of F-conjugacy classes in G.
(b) If $p > 0$ then r_F is equal to the number of p-regular F-conjugacy classes in G.

We also need the following results concerning the structure of the Galois groups.

Theorem 18 ([Se]). *Suppose that n is odd or divisible by 4. Then $\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p$ is a Galois extension of $\mathbb{Q}_p$, and its Galois group G is as follows.*
(a) If p does not divide n then G is cyclic, and there exists an element $\sigma \in G$, the Frobenius element of the extension satisfying $\sigma(\zeta_n) = \zeta_n^p$ that generates G. Further, the order of σ is the order of p considered as an element of $\mathbb{Z}_n^*$.
(b) If $n = p^m$, $m \geqslant 1$, then G is of order $p^{m-1}(p-1)$, and we have a group isomorphism $G \cong \mathbb{Z}_{p^m}^*$. Hence G is cyclic if p is odd or if $p = m = 2$, and is isomorphic to the direct product $\mathbb{Z}_2$ (generated by the class of -1) and $\mathbb{Z}_{p^{m-2}}$ (generated by the class of 5) if $p = 2$ and $m \geqslant 3$.
(c) Suppose that $n = p^m n_1$, where $n_1 \geqslant 2$ and p does not divide n_1. Let ζ_1 be a primitive ${n_1}^{th}$ root of unity, and let ρ be a primitive p^{mth} root of unity. Then $G \simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_1)/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p(\rho)/\mathbb{Q}_p)$.

Theorem 19 ([CoD]). *Let k denote the order of p considered as an element of $\mathbb{Z}_n^*$. Then $\mathrm{Gal}(\mathbb{F}_p(\zeta_n)/\mathbb{F}_p)$ is isomorphic to $\mathbb{Z}_k$.*

We suppose in what follows that G is dicyclic of order $4m$. We first apply the above results in order to determine the rank of $K_{-1}(\mathbb{Z}[G])$ where m is an odd prime. We then go on to to study the case where m is a power of 2.

Theorem 20. *Let m be an odd prime, and let λ be the number of $\mathbb{Q}_2$-conjugacy classes (or equivalently $\mathbb{F}_2$-conjugacy classes) of the elements of Dic_{4m} of order m. Then*

$$K_{-1}(\mathbb{Z}[Dic_{4m}]) \cong \begin{cases} \mathbb{Z}^\lambda \oplus \mathbb{Z}_2 & \text{if } m \equiv 1 \bmod 4 \\ \mathbb{Z}^\lambda & \text{if } m \equiv 3 \bmod 4. \end{cases}$$

Proof. Let $G = \mathrm{Dic}_{4m}$ be given by the presentation (1). By Proposition 12 and equation (5), it suffices to show that the rank of $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{4m}])$ is equal to λ. The group G has one element each (e and x^m respectively) of order 1 and 2, $(m-1)$ elements of order $2m$, of the form x^i, i odd, $1 \leqslant i \leqslant 2m-1$, and $i \neq m$, $(m-1)$ elements of order m, of the form x^i, i even, $2 \leqslant i \leqslant 2m-2$, and $2m$ elements of order 4, of the form $y, xy, \ldots, x^{2m-1}y$. The elements of order 1 and 2 each form a single (usual) conjugacy class, those of order 4 form 2 conjugacy classes, $\{x^i y \mid 0 \leqslant i \leqslant 2m-2, i \text{ even}\}$ and $\{x^i y \mid 1 \leqslant i \leqslant 2m-1, i \text{ odd}\}$, while those of order m and $2m$ form $(m-1)$ conjugacy classes of the form $\{x^i, x^{-i}\}$ for $i = 1, \ldots, m-1$. Since $r_{\mathbb{Q}}$ is equal to the number of simple components in the Wedderburn decomposition of $\mathbb{Q}[\mathrm{Dic}_{4m}]$, it follows from equation (9) that $r_{\mathbb{Q}} = 5$. This may also be obtained by observing that the subgroups of

Dic_{4m} of order 4 are its Sylow 2-subgroups, and so Dic_{4m} possesses a single conjugacy class of subgroups of order 4.

We must thus calculate $r_{\mathbb{Q}_p}$ and $r_{\mathbb{F}_p}$ for $p \in \{2, m\}$. Since there is a unique conjugacy class of elements of order 1 and 2, these elements contribute 1 to each of $r_{\mathbb{Q}_p}$ and $r_{\mathbb{F}_p}$, except in the case of $r_{\mathbb{F}_2}$, where the element 2 is not 2-regular, so contributes zero. We thus focus on the elements of order 4, m and $2m$.

- $\mathbb{Q}_2$-conjugacy classes of the order 4 elements: by Theorem 18(b), the monomorphism $\phi\colon \mathrm{Gal}(\mathbb{Q}_2(\zeta_4)/\mathbb{Q}_2) \longrightarrow \mathbb{Z}_4^*$ is an isomorphism and $\mathrm{Im}(\phi) = \{1, 3\}$. By equation (17), $[y]_{\mathbb{Q}_2} = [y] \cup [y^3] = [y] \cup [x^m y]$ as $y^3 = y^2.y = x^m y$, and so there is a single $\mathbb{Q}_2$-class of order 4 elements because m is odd.
- $\mathbb{Q}_2$-conjugacy classes of the elements of order m and $2m$: by hypothesis, the number of $\mathbb{Q}_2$-conjugacy classes of the elements of order m is equal to λ. Theorem 18(c) implies that the number of $\mathbb{Q}_2$-conjugacy classes of the elements of order $2m$ is also equal to λ.

We conclude that $r_{\mathbb{Q}_2} = 2\lambda + 3$.

- $\mathbb{Q}_m$-conjugacy classes of the order 4 elements: by Theorem 18(a), we have a monomorphism $\phi\colon \mathrm{Gal}(\mathbb{Q}_m(\zeta_4)/\mathbb{Q}_m) \longrightarrow \mathbb{Z}_4^*$, $\mathrm{Gal}(\mathbb{Q}_m(\zeta_4)/\mathbb{Q}_m)$ is cyclic, and its order is equal to that of m considered as an element of $\mathbb{Z}_4^*$. If $m \equiv 3 \bmod 4$ then ϕ is an isomorphism and $\mathrm{Im}(\phi) = \{1, 3\}$. By equation (17), $[y]_{\mathbb{Q}_m} = [y] \cup [y^3] = [y] \cup [x^m y]$ as $y^3 = y^2.y = x^m y$, and so there is a single $\mathbb{Q}_m$-class of order 4 elements since m is odd. If $m \equiv 1 \bmod 4$ then $\mathrm{Gal}(\mathbb{Q}_m(\zeta_4)/\mathbb{Q}_m)$ is trivial and $\mathrm{Im}(\phi) = \{1\}$. In this case, the $\mathbb{Q}_m$-conjugacy classes coincide with the usual conjugacy classes, so there are two $\mathbb{Q}_m$-conjugacy classes of elements of order 4.
- $\mathbb{Q}_m$-conjugacy classes of the elements of order m: by Theorem 18(b), the monomorphism $\phi\colon \mathrm{Gal}(\mathbb{Q}_m(\zeta_m)/\mathbb{Q}_m) \longrightarrow \mathbb{Z}_m^*$ is an isomorphism and $\mathrm{Im}(\phi) = \{1, \ldots, m-1\}$. By equation (17), $[x^2]_{\mathbb{Q}_m} = \bigcup_{i=1}^{m-1} [x^{2i}]$, so there is a single $\mathbb{Q}_m$-class of order m elements.
- $\mathbb{Q}_m$-conjugacy classes of the elements of order $2m$: by Theorem 18(c), we have the monomorphism $\phi\colon \mathrm{Gal}(\mathbb{Q}_m(\zeta_{2m})/\mathbb{Q}_m) \longrightarrow \mathbb{Z}_{2m}^*$, where

$$\mathrm{Gal}(\mathbb{Q}_m(\zeta_{2m})/\mathbb{Q}_m) \cong \mathrm{Gal}(\mathbb{Q}_m(\zeta_m)/\mathbb{Q}_m) \times \mathrm{Gal}(\mathbb{Q}_m(\zeta_2)/\mathbb{Q}_m) \cong \mathrm{Gal}(\mathbb{Q}_m(\zeta_m)/\mathbb{Q}_m) \cong \mathbb{Z}_{m-1},$$

by the previous calculation and the fact that $\mathrm{Gal}(\mathbb{Q}_m(\zeta_2)/\mathbb{Q}_m)$ is trivial by Theorem 18(a). So ϕ is an isomorphism, and as in the previous case, we conclude that there is a single $\mathbb{Q}_m$-class of order $2m$ elements.

It thus follows that $r_{\mathbb{Q}_m} = 6$ if $m \equiv 1 \bmod 4$, and $r_{\mathbb{Q}_m} = 5$ if $m \equiv 3 \bmod 4$.

- 2-regular $\mathbb{F}_2$-conjugacy classes: we have $G_2' = \{e, x^2, x^4, \ldots, x^{2m-2}\}$, which splits as the disjoint union of $(m+1)/2$ (usual) conjugacy classes in Dic_{4m}, comprised of $\{e\}$, and $\left\{x^{2i}, x^{2(m-i)}\right\}$ for $i = 1, \ldots, (m-1)/2$. We thus need to study the $\mathbb{F}_2$-conjugacy classes of the elements of order m. By Theorem 19, we have $\phi\colon \mathrm{Gal}(\mathbb{F}_2(\zeta_m)/\mathbb{F}_2) \longrightarrow \mathbb{Z}_m^*$, where $\mathrm{Gal}(\mathbb{F}_2(\zeta_m)/\mathbb{F}_2)$ is cyclic, of order that of 2 considered as an element of $\mathbb{Z}_m^*$, and $\mathrm{Im}(\phi) = \langle 2 \rangle$.

We return for a moment to the $\mathbb{Q}_2$-conjugacy classes of the elements of order m. Replacing ϕ by ϕ_1 to distinguish it from the monomorphism ϕ of the previous paragraph, by Theorem 18(a), we have $\phi_1\colon\ \mathrm{Gal}(\mathbb{Q}_2(\zeta_m)/\mathbb{Q}_2) \longrightarrow \mathbb{Z}_m^*$, and $\mathrm{Gal}(\mathbb{Q}_2(\zeta_m)/\mathbb{Q}_2)$ is cyclic, of order that of 2 considered as an element of $\mathbb{Z}_m^*$. Thus $\mathrm{Im}(\phi_1) = \langle 2\rangle$ also. In particular, the $\mathbb{F}_2$-conjugacy class of an element of Dic_{4m} of order m is equal to its $\mathbb{Q}_2$-conjugacy class, and thus the number of $\mathbb{F}_2$-conjugacy classes of elements of order m is equal to λ. We deduce that $r_{\mathbb{F}_2} = \lambda + 1$.

• m-regular $\mathbb{F}_m$-conjugacy classes: we have $G'_m = \{e, x^m, y, xy, x^2y, \dots, x^{2m-2}y, x^{2m-1}y\}$, in which the four (usual) conjugacy classes in Dic_{4m} are $\{e\}$, $\{x^m\}$, $\{y, x^m y, \dots, x^{2m-2}y\}$ and $\{xy, x^3y, \dots, x^{2m-1}y\}$. We thus need to study the $\mathbb{F}_m$-conjugacy classes of the latter two classes, which are those of the elements of order 4. By Theorem 19, we have the monomorphism $\phi\colon\ \mathrm{Gal}(\mathbb{F}_m(\zeta_4)/\mathbb{F}_m) \longrightarrow \mathbb{Z}_4^*$, and $\mathrm{Gal}(\mathbb{F}_m(\zeta_4)/\mathbb{F}_m)$ is cyclic, of order that of m considered as an element of $\mathbb{Z}_4^*$. As in the case of the $\mathbb{Q}_m$-conjugacy classes of the order 4 elements, if $m \equiv 3 \bmod 4$, there is a single $\mathbb{F}_m$-class of order 4 elements, while if $m \equiv 1 \bmod 4$, the $\mathbb{F}_m$-conjugacy classes coincide with the usual conjugacy classes, and so there are two $\mathbb{F}_m$-conjugacy classes of order 4 elements. Hence $r_{\mathbb{F}_m} = 4$ if $m \equiv 1 \bmod 4$ and $r_{\mathbb{F}_m} = 3$ if $m \equiv 3 \bmod 4$.

So by equation (6), the rank r of $K_{-1}(\mathbb{Z}[\mathbb{Z}_m \rtimes_{-1} \mathbb{Z}_4])$ is given by:

$$\begin{aligned} r &= 1 - r_{\mathbb{Q}} + (r_{\mathbb{Q}_2} - r_{\mathbb{F}_2}) + (r_{\mathbb{Q}_m} - r_{\mathbb{F}_m}) \\ &= \begin{cases} 1 - 5 + (2\lambda + 3) - (\lambda + 1) + (6 - 4) & \text{if } m \equiv 1 \bmod 4 \\ 1 - 5 + (2\lambda + 3) - (\lambda + 1) + (5 - 3) & \text{if } m \equiv 3 \bmod 4 \end{cases} \\ &= \lambda. \end{aligned}$$

□

If m is an odd prime, the proof of Theorem 20 indicates that the number λ of $\mathbb{Q}_2$-conjugacy classes of the elements of Dic_{4m} of order m is related to the order of the subgroup $\langle 2\rangle$ in $\mathbb{Z}_m^*$. The question of when 2 generates $\mathbb{Z}_m^*$ is open and constitutes a special case of Artin's primitive root conjecture. The following proposition shows that it is also interesting for us to know whether $-1 \in \langle 2\rangle$, and enables us to determine the rank of $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{4m}])$ solely in terms of $|\langle 2\rangle|$.

Proposition 21. *Let m and λ be defined as in the statement of Theorem 20. Then*

$$\lambda = \begin{cases} (m-1)/|\langle 2\rangle| & \text{if } -1 \in \langle 2\rangle \\ (m-1)/2\,|\langle 2\rangle| & \text{if } -1 \notin \langle 2\rangle. \end{cases}$$

Examples.

(a) Suppose that m is a Fermat number, of the form $2^{2^s} + 1$, where $s \in \mathbb{N}$. Then $|\langle 2\rangle| = 2^{s+1}$ and $-1 \in \langle 2\rangle$, so the rank of $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{4m}])$ is equal to $\lambda = 2^{2^s - s - 1}$. For example, if $m = 257$ then $\lambda = 16$ and $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{1\,028}]) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^{16}$.

(b) Suppose that m is a Mersenne prime, of the form $2^p - 1$, where p is prime. Then $|\langle 2\rangle| = p$ and $-1 \notin \langle 2\rangle$, so the rank of $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{4m}])$ is equal to $\lambda = \frac{2^p - 2}{2p} = \frac{2^{p-1} - 1}{p}$. For example, if $m = 127$ then $\lambda = 9$ and $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{508}]) \cong \mathbb{Z}^9$, and if $m = 8\,191$ then $\lambda = 315$ and $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{32\,728}]) \cong \mathbb{Z}^{315}$.

Proof of Proposition 21. Using equation (1), the elements of Dic_{4m} of order m are of the form x^{2i}, $1 \leqslant i \leqslant m-1$, and $[x^{2i}] = \{x^{2i}, x^{-2i}\}$, in particular, they form $(m-1)/2$ distinct (usual) conjugacy classes in Dic_{4m}. Let $1 \leqslant i \leqslant m-1$. Since m is prime, there exist $\lambda, \mu \in \mathbb{Z}$ such that $\lambda i + \mu m = 1$. One may check easily that the maps $[x^2]_{\mathbb{Q}_2} \longrightarrow [x^{2i}]_{\mathbb{Q}_2}$ and $[x^{2i}]_{\mathbb{Q}_2} \longrightarrow [x^2]_{\mathbb{Q}_2}$, defined respectively by $y \longmapsto y^i$ and $z \longmapsto z^\lambda$, are mutual inverses, and hence $[x^{2i}]_{\mathbb{Q}_2}$ has the same number of elements as $[x^2]_{\mathbb{Q}_2}$. Thus the number of $\mathbb{Q}_2$-conjugacy classes of the elements of order m, which is equal to λ, is just $(m-1)$ divided by the cardinal of $[x^2]_{\mathbb{Q}_2}$. Theorem 18(a) and equation (17) imply that:

$$[x^2]_{\mathbb{Q}_2} = \bigcup_{t \in \langle 2 \rangle} [x^{2t}] = \left\{ x^{2i} \;\middle|\; i \in \langle 2 \rangle \right\} \cup \left\{ x^{-2i} \;\middle|\; i \in \langle 2 \rangle \right\} = \left\{ x^{2i} \;\middle|\; i \in \langle 2 \rangle \right\} \cup \left\{ x^{2i} \;\middle|\; i \in -\langle 2 \rangle \right\}.$$

Now $-\langle 2 \rangle$ is the $\langle 2 \rangle$-coset of -1 in $\mathbb{Z}_m^*$, so $\left\{ x^{2i} \;\middle|\; i \in \langle 2 \rangle \right\}$ and $\left\{ x^{2i} \;\middle|\; i \in -\langle 2 \rangle \right\}$ have the same cardinal $|\langle 2 \rangle|$, and are either equal or disjoint. Since $-1 \in -\langle 2 \rangle$, they are equal if and only if $-1 \in \langle 2 \rangle$. This being the case, the cardinal of $[x^2]_{\mathbb{Q}_2}$ is equal to $|\langle 2 \rangle|$, and $\lambda = (m-1)/|\langle 2 \rangle|$. If $-1 \notin \langle 2 \rangle$, the two cosets $\langle 2 \rangle$ and $-\langle 2 \rangle$ are disjoint, thus the cardinal of $[x^2]_{\mathbb{Q}_2}$ is equal to $2|\langle 2 \rangle|$, and $\lambda = (m-1)/2|\langle 2 \rangle|$ as required. □

The methods used above allow us in theory to calculate $K_{-1}(\mathbb{Z}[\mathrm{Dic}_{4m}])$ for any $m \geqslant 2$, not just for m an odd prime. As another example, consider the case where m is a power of 2, so $G \cong \mathcal{Q}_{2^k}$ is the generalised quaternion group of order 2^k, where $k = m+2$.

Proposition 22. *$K_{-1}(\mathbb{Z}[\mathcal{Q}_{2^k}])$ is trivial if $k = 3$, and is isomorphic to $\mathbb{Z}_2$ if $k \geqslant 4$.*

Proof. By Theorem 11 and Proposition 13, it suffices to show that for all $k \geqslant 3$, the rank of $K_{-1}(\mathbb{Z}[\mathcal{Q}_{2^k}])$ is zero. We must calculate $r_{\mathbb{Q}}$, $r_{\mathbb{Q}_2}$ and $r_{\mathbb{F}_2}$. Using the presentation (1) of $\mathcal{Q}_{2^k}$, we see that $\mathcal{Q}_{2^k} = \langle x \rangle \coprod \langle x \rangle y$, and that the elements of $\langle x \rangle y$ are all of order 4. So G_2' consists of the identity element, whence $r_{\mathbb{F}_2} = 1$.

We now determine the number $r_{\mathbb{Q}}$ of $\mathbb{Q}$-conjugacy classes, which by Remark 16(b), is equal to the number of conjugacy classes of cyclic subgroups in $\mathcal{Q}_{2^k}$. The elements of $\mathcal{Q}_{2^k}$ are of order 2^l, $0 \leqslant l \leqslant k-1$, and if $l \neq 2$ then the elements of order 2^l are contained entirely within $\langle x \rangle$. Thus there is just one subgroup of order 2^l for each such l, and so these subgroups contribute $k-1$ to $r_{\mathbb{Q}}$. Suppose then that $l = 2$. Using the relations

$$y(x^i y)y^{-1} = x^{-i}y,\ x(x^i y)x^{-1} = x^{i+2}y \text{ and } (x^i y)^{-1} = x^{i+2^{k-2}}y \tag{18}$$

in $\mathcal{Q}_{2^k}$, we see that there are at most three conjugacy classes of cyclic subgroups of order 4, represented by the subgroups $\left\langle x^{2^{k-3}} \right\rangle$, $\langle y \rangle$ and $\langle xy \rangle$. Since $\left\langle x^{2^{k-3}} \right\rangle$ is contained in the normal subgroup $\langle x \rangle$ of $\mathcal{Q}_{2^k}$, it cannot be conjugate to the two other subgroups, and using relations (18), we see that $\langle y \rangle$ and $\langle xy \rangle$ are non conjugate. We thus conclude that $r_{\mathbb{Q}} = k+2$. This number may also be obtained by counting the number of simple components in the Wedderburn decomposition (7) of $\mathbb{Q}[\mathcal{Q}_{2^k}]$.

Finally we calculate $r_{\mathbb{Q}_2}$. Consider the elements of $\mathcal{Q}_{2^k}$ of order 2^l, where $0 \leqslant l \leqslant k-1$. If $l \in \{0, 1\}$ then there is just one element of order 2^l, and so the contribution to $r_{\mathbb{Q}_2}$ is one in each case. If $l = 2$ then $\mathrm{Gal}(\mathbb{Q}_2(\zeta_{2^2})/\mathbb{Q}_2) \cong \mathbb{Z}_{2^2}^* = \{1, 3\}$ by Theorem 18(b).

Hence for every element z of $\mathcal{Q}_{2^k}$ of order 4, $[z]_{\mathbb{Q}_2} = [z] \cup [z^3] = [z] \cup [z^{-1}] = [z]$ since in $\mathcal{Q}_{2^k}$, every element is conjugate to its inverse. Thus the elements of $\mathcal{Q}_{2^k}$ of order 4 contribute 3 to $r_{\mathbb{Q}_2}$. Suppose then that $l \geqslant 3$. The elements of order 2^l are contained in $\langle x \rangle$, are elements of the subgroup $\left\langle x^{2^{k-l-1}} \right\rangle$ of the form $x^{2^{(k-l-1)}r}$, where $\gcd(r, 2^l) = 1$, and so are of the form $x^{2^{(k-l-1)}r}$, where $r \in \{1, 3, \ldots, 2^l - 1\}$. On the other hand, applying Theorem 18(b), we see that $\mathrm{Gal}(\mathbb{Q}_2(\zeta_{2^l})/\mathbb{Q}_2) \cong \mathbb{Z}_{2^l}^*$. Now $\mathbb{Z}_{2^l}^* = \{1, 3, \ldots, 2^l - 1\}$, and thus

$$\left[x^{2^{k-l-1}}\right]_{\mathbb{Q}_2} = \left[x^{2^{k-l-1}}\right] \cup \left[x^{3(2^{k-l-1})}\right] \cup \cdots \cup \left[x^{(2^l-1)(2^{k-l-1})}\right].$$

From above, this is precisely the set of all elements of order 2^l, and hence for each $3 \leqslant l \leqslant k-1$, the elements of order 2^l contribute one to $r_{\mathbb{Q}_2}$. Summing over all possible values of l yields $r_{\mathbb{Q}_2} = k + 2$, and applying equation (6), we obtain $r = 1 - r_{\mathbb{Q}} + r_{\mathbb{Q}_2} - r_{\mathbb{F}_2} = 0$, which proves the proposition. □

We now turn to the calculation of $K_{-1}(\mathbb{Z}[G])$, where G is a binary polyhedral group.

Proposition 23. *$K_{-1}(\mathbb{Z}[\mathrm{T}^*]) \cong \mathbb{Z}$, $K_{-1}(\mathbb{Z}[\mathrm{O}^*]) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ and $K_{-1}(\mathbb{Z}[\mathrm{I}^*]) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^2$.*

Proof. Let G be a binary polyhedral group. By Proposition 14, it suffices to calculate the rank of $K_{-1}(\mathbb{Z}[G])$. Using Remark 16(b) and the notation of Section 3, $r_{\mathbb{Q}} = \sum_{d \,|\, |G|} r_2(d)$. Clearly $r_2(1) = r_2(2) = 1$. If $G = \mathrm{T}^*$ or O^* (resp. $G = \mathrm{I}^*$) and $d = 3$ (resp. $d \in \{3, 5\}$) then the subgroups of G of order d are its Sylow d-subgroups, and thus $r_2(d) = 1$. By adjoining the central element of order 2 to such subgroups, one sees that $r_2(2d) = 1$ also. We now consider the three possibilities for G.

(i) $G = \mathrm{T}^*$: G has cyclic groups of order $1, 2, 3, 4$ and 6. Using the notation of equation (3), $G = \langle P, Q, X \rangle$, $\langle P, Q \rangle \cong \mathcal{Q}_8$, and the action by conjugation of X permutes P, Q and PQ. The subgroups of G of order 4 are the subgroups of $\langle P, Q \rangle$ of order 4, and they are pairwise conjugate, so $r_2(4) = 1$. It follows from above that $r_{\mathbb{Q}} = 5$.
(ii) $G = \mathrm{I}^*$: G has cyclic groups of order $1, 2, 3, 4, 5, 6$ and 10. The Sylow 2-subgroups of G are isomorphic to $\mathcal{Q}_8$, and since I^* contains a subgroup isomorphic to T^*, they are each contained in a subgroup isomorphic to T^*. It follows once more than the elements of G of order 4 are pairwise conjugate, so $r_2(4) = 1$, and $r_{\mathbb{Q}} = 7$ from above.
(iii) $G = \mathrm{O}^*$: G has cyclic groups of order $1, 2, 3, 4, 6$ and 8. We saw in Lemma 9(b) that $r_2(4) = 2$ and $r_2(8) = 1$, so $r_{\mathbb{Q}} = 7$ from above.

We remark that these values of $r_{\mathbb{Q}}$ may also be obtained from the corresponding Wedderburn decompositions given in equations (14), (15) and (16).

We first calculate the rank of $K_{-1}(\mathbb{Z}[\mathrm{T}^*])$, where a presentation of T^* is given by the first line of equation (3). The complement $\mathrm{T}^* \setminus \langle P, Q \rangle$ consists of 8 elements each of order 3 and 6, so $G_3' = \langle P, Q \rangle$. Using the action of $\langle X \rangle$ on $\langle P, Q \rangle$, the elements of T^* of order 4 are pairwise conjugate. Hence for each $m \in \{1, 2, 4\}$, the elements of order m form a single conjugacy class, and thus form a single $\mathbb{Q}_p$-conjugacy class for $p \in \{2, 3\}$. In particular, $r_{\mathbb{F}_3} = 3$. On the other hand, G_2' consists of the identity and the 8 elements of T^* of order 3. These elements generate four subgroups of T^* isomorphic

to $\mathbb{Z}_3$, which are conjugate because they are the Sylow 3-subgroups of T^*. We now determine the relation between the $\mathbb{F}_2$-conjugacy class of an element of T^* of order 3 and that of its inverse. By Theorem 19, we have an isomorphism $\phi\colon\ \mathrm{Gal}(\mathbb{F}_2(\zeta_3)/\mathbb{F}_2) \longrightarrow \mathbb{Z}_3^*$, with $\mathrm{Im}(\phi) = \{1,2\}$. Thus $[X]_{\mathbb{F}_2} = [X] \cup [X^2] = [X] \cup [X^{-1}]$. It follows that there is a single $\mathbb{F}_2$-conjugacy class of elements of order 3, whence $r_{\mathbb{F}_2} = 2$.

It remains to determine the number of $\mathbb{Q}_p$-conjugacy classes, $p \in \{2,3\}$, of the elements of T^* of order 3 and 6. We first calculate the number of $\mathbb{Q}_p$-conjugacy classes of the elements of order 3. These elements form two (usual) conjugacy classes:

$$\left\{X, P^{-1}X, Q^{-1}X, (PQ)^{-1}X\right\} \text{ and } \left\{X^{-1}, PX^{-1}, QX^{-1}, (PQ)X^{-1}\right\}.$$

By Theorem 18(a) and (b), $\phi\colon\ \mathrm{Gal}(\mathbb{Q}_p(\zeta_3)/\mathbb{Q}_p) \longrightarrow \mathbb{Z}_3^*$ is an isomorphism, $\mathrm{Im}(\phi) = \{1,2\}$, and $[X]_{\mathbb{Q}_p} = [X] \cup [X^2] = [X] \cup [X^{-1}]$, which is the union of the above two (usual) conjugacy classes of elements of order 3. We have the same result for the elements of order 6 of T^*, since they are obtained from those of order 3 by adjoining the central element of T^* of order 2. Thus $r_{\mathbb{Q}_2} = r_{\mathbb{Q}_3} = 5$. By equation (6), the rank of $K_{-1}(\mathbb{Z}[T^*])$ is equal to one.

We now calculate the rank of $\mathbb{K}_{-1}(\mathbb{Z}[O^*])$. The group O^* contains a unique subgroup $\langle P, Q, X\rangle$ isomorphic to T^*, plus twelve elements each of order 4 and 8. So from the previous case, we obtain $\mathbb{F}_2 = 2$, and the elements of O^* of order $1, 2, 3$ and 6 each give rise to a single $\mathbb{Q}_p$-conjugacy class of O^* for $p \in \{2,3\}$. It remains to calculate the number of $\mathbb{Q}_p$-conjugacy classes of elements of order 4 and 8, as well as $r_{\mathbb{F}_3}$. We first consider the case of the elements of order 8. By Lemma 9(b), $r_2(8) = 1$, so the three cyclic subgroups of O^* of order 8 are conjugate, and $r_1(8) = 2$, hence there are exactly two (usual) conjugacy classes of elements of order 8, represented by $[v]$ and $[v^3]$, where $v \in O^*$ is any element of O^* of order 8. We claim that v and v^3 are $\mathbb{Q}_p$-conjugate, since by Theorem 18(a) and (b), $\phi\colon\ \mathrm{Gal}(\mathbb{Q}_p(\zeta_8)/\mathbb{Q}_p) \longrightarrow \mathbb{Z}_8^*$ satisfies $\mathrm{Im}(\phi) \supset \{1,3\}$, hence $[v]_{\mathbb{Q}_p} \supset [v] \cup [v^3]$. So there is a single $\mathbb{Q}_p$-conjugacy class of elements of order 8. We now turn to the elements of order 4. By Theorem 18(a) and (b), $\phi\colon\ \mathrm{Gal}(\mathbb{Q}_p(\zeta_4)/\mathbb{Q}_p) \longrightarrow \mathbb{Z}_4^*$ is an isomorphism, $\mathrm{Im}(\phi) = \{1,3\}$, and $[w]_{\mathbb{Q}_p} = [w] \cup [w^{-1}]$ for every element $w \in O^*$ of order 4. But by Lemma 9(b)(ii) and its proof, $[w] = [w^{-1}]$ and the number of $\mathbb{Q}_p$-conjugacy classes of elements of order 4, which is the number of $\mathbb{Q}$-conjugacy classes of these elements, is equal to 2, from which we conclude that $r_{\mathbb{Q}_p} = 7$.

Finally, we calculate $r_{\mathbb{F}_3}$. With the notation of equation (3), the set of 3-regular elements of O^* is equal to $\langle P, Q\rangle \cup (O^* \setminus \langle P, Q, X\rangle)$. Let $m \in \{4,8\}$. By Theorem 19, $\phi\colon\ \mathrm{Gal}(\mathbb{F}_3(\zeta_m)/\mathbb{F}_3) \longrightarrow \mathbb{Z}_m^*$ satisfies $\mathrm{Im}(\phi) = \{1,3\}$. As in the previous paragraph, the number of $\mathbb{F}_3$-conjugacy classes of elements of order m is just the number of $\mathbb{Q}$-conjugacy classes of these elements. It follows that $r_{\mathbb{F}_3} = 5$, and equation (6) implies that the rank of $K_{-1}(\mathbb{Z}[O^*])$ is equal to one.

Now we determine the rank of $\mathbb{K}_{-1}(\mathbb{Z}[I^*])$. The elements of I^* are of order l, where $l \in \{1,2,3,4,5,6,10\}$. As we saw in (ii) above, $r_2(l) = 1$ for all such l, and so any two cyclic subgroups of I^* of the same order are conjugate. Consequently, to study the various F-conjugacy classes, it suffices to consider the F-conjugacy relation for the elements of a

given cyclic subgroup of each possible order l. Suppose that $p \in \{2,3,5\}$. We first calculate the number of $\mathbb{Q}_p$-conjugacy classes of elements of order l, where $l \in \{3,5\}$. A case-by-case analysis using Theorem 18(a) and (b) implies that $\phi\colon\ \mathrm{Gal}(\mathbb{Q}_p(\zeta_l)/\mathbb{Q}_p) \longrightarrow \mathbb{Z}_l^*$ is an isomorphism, and so the non-trivial elements of each cyclic subgroup of I^* of order l are pairwise $\mathbb{Q}_p$-conjugate. It follows then that there is a single $\mathbb{Q}_p$-conjugacy class of elements of order l in I^* for the given values of p and l. Since the elements of order 6 (resp. 10) of I^* are obtained from those of order 3 (resp. 5) by adjoining the central element of I^* of order 2, there is just one $\mathbb{Q}_p$-conjugacy class of elements of order 6 and of order 10. It remains to study the $\mathbb{Q}_p$-conjugacy classes of the elements of I^* of order 4. By Lemma 8 and (ii) above, $r_1(4) = r_2(4) = 1$, so there is a single (usual) conjugacy class of elements of order 4 in I^*, and thus there is a single $\mathbb{Q}_p$-conjugacy class of elements of order 4. We thus obtain $r_{\mathbb{Q}_p} = 7$ for all $p \in \{2,3,5\}$.

We now turn to the $\mathbb{F}_2$-conjugacy classes of I^*. The set G_2' is the union of the elements of I^* of order 1, 3 and 5. For $m \in \{3,5\}$, $\mathrm{Gal}(\mathbb{F}_2(\zeta_m)/\mathbb{F}_2) \longrightarrow \mathbb{Z}_m^*$ is an isomorphism by Theorem 19, and $\mathrm{Im}(\phi) = \{1,\ldots,m-1\}$. Once more, there is a single $\mathbb{F}_2$-conjugacy class of 2-regular elements of order 3 or 5, so $r_{\mathbb{F}_2} = 3$.

Finally, we calculate the number of $\mathbb{F}_3$- and $\mathbb{F}_5$-conjugacy classes of I^*. The set G_3' (resp. G_5') is the union of the elements of order 5 and 10 (resp. of order 3 and 6) with the elements of order 1, 2 and the (conjugacy class of) elements of order 4. But for $m, p \in \{3,5\}$, where $m \neq p$, $\mathrm{Gal}(\mathbb{F}_p(\zeta_m)/\mathbb{F}_p)$ is isomorphic to $\mathbb{Z}_{m-1}$ by Theorem 19. In particular, $\mathrm{Im}(\phi) = \{1,\ldots,m-1\}$. As before, we conclude that there is a single $\mathbb{F}_p$-conjugacy class of p-regular elements of order m. The same is true for the elements of order $2m$, and so $r_{\mathbb{F}_3} = r_{\mathbb{F}_5} = 5$. Applying equation (6), we see that the rank of $K_{-1}(\mathbb{Z}[\mathrm{I}^*])$ is equal to two as required. □

6 Calculation of the lower K-theory of infinite virtually cyclic groups

In this section, we provide the ingredients needed to compute the Whitehead and lower algebraic K-groups of infinite virtually cyclic groups. As we mentioned at the beginning of Section 2, an infinite virtually cyclic group Γ is isomorphic to one of the following:

(I) a semi-direct product of the form $\Gamma \cong F \rtimes_\alpha \mathbb{Z}$, where F is a finite group and $\alpha \in \mathrm{Hom}(\mathbb{Z}, \mathrm{Aut}(F))$. Such Γ surject onto $\mathbb{Z}$ with finite kernel F.
(II) an amalgam of the form $\Gamma \cong G_1 *_F G_2$ where G_1, G_2 are finite groups containing a common subgroup F of index 2 in both G_1 and G_2. Such Γ surject onto the infinite dihedral group Dih_∞ with finite kernel F.

We shall say that these infinite virtually cyclic groups are of *Type I* or of *Type II* respectively.

For virtually cyclic groups Γ of Type I, the algebraic K-groups of $\mathbb{Z}[\Gamma]$ are described by the Bass-Heller-Swan formula with $\alpha = 1$, which asserts that for a finite group π,

there is a natural decomposition [Ba]:

$$K_i(\mathbb{Z}[\pi \times \mathbb{Z}]) = K_i(\mathbb{Z}[\pi]) \oplus K_{i-1}(\mathbb{Z}[\pi]) \oplus 2\,NK_i(\mathbb{Z}[\pi]) \quad \text{for all } i \in \mathbb{Z}, \tag{19}$$

where the i^{th} *Bass Nil group* of π, denoted by $NK_i(\mathbb{Z}[\pi])$, is defined to be the kernel of the homomorphism in K-groups induced by the evaluation $e\colon\ \mathbb{Z}[\pi][t] \longrightarrow \mathbb{Z}[\pi]$ at $t = 1$. In the reduced version, equation (19) takes the form:

$$\begin{aligned} \mathrm{Wh}(\pi \times \mathbb{Z}) &= \mathrm{Wh}(\pi) \oplus \widetilde{K}_0(\mathbb{Z}[\pi]) \oplus 2\,NK_1(\mathbb{Z}[\pi]), \quad \text{and} \\ \widetilde{K}_0(\mathbb{Z}[\pi \times \mathbb{Z}]) &= \widetilde{K}_0(\mathbb{Z}[\pi]) \oplus K_{-1}(\mathbb{Z}[\pi]) \oplus 2\,NK_0(\mathbb{Z}[\pi]). \end{aligned}$$

When $\alpha \neq 1$, the group ring $\mathbb{Z}[\Gamma]$ is equal to $\mathbb{Z}[F \rtimes_\alpha \mathbb{Z}] \cong \mathbb{Z}[F]_\alpha[t, t^{-1}]$, the latter being the *twisted* Laurent polynomial ring of $\mathbb{Z}[F]$, and the twisting is given by the action of α. In this case, the Bass Nil groups are replaced by the *Farrell-Hsiang Nil groups* $NK_i(\mathbb{Z}[F], \alpha) \oplus NK_i(\mathbb{Z}[F], \alpha^{-1})$ [FH].

For virtually cyclic groups Γ of Type II, the fundamental work by Waldhausen gives rise to an exact sequence [WaF]:

$$\begin{aligned} \cdots \longrightarrow K_n(\mathbb{Z}[F]) \longrightarrow K_n(\mathbb{Z}[G_1]) \oplus K_n(\mathbb{Z}[G_2]) \longrightarrow K_n(\mathbb{Z}[\Gamma])/\,\mathbf{Nil}_n^W \longrightarrow K_{n-1}(\mathbb{Z}[F]) \\ \longrightarrow K_{n-1}(\mathbb{Z}[G_1]) \oplus K_{n-1}(\mathbb{Z}[G_2]) \longrightarrow K_{n-1}(\mathbb{Z}[G])/\,\mathbf{Nil}_{n-1}^W \longrightarrow \cdots, \end{aligned}$$

where $\mathbf{Nil}_n^W$ denotes the *Waldhausen Nil groups* [WaF] defined by:

$$\mathbf{Nil}_n^W = \mathrm{Nil}_n^W(\mathbb{Z}[F]; \mathbb{Z}[G_1 \backslash F], \mathbb{Z}[G_2 \backslash F]).$$

If Γ is an infinite virtually cyclic group of Type II, there is a surjection $f\colon\ \Gamma \twoheadrightarrow \mathrm{Dih}_\infty$ with finite kernel F. Let T be the unique infinite cyclic subgroup of Dih_∞ of index 2. Then the subgroup $\widetilde{\Gamma} = f^{-1}(T) \subset \Gamma$ is an infinite virtually cyclic group of Type I, so $\widetilde{\Gamma}$ is of the form $F \rtimes_\alpha T$. In this situation, it was established recently that the Waldhausen Nil groups may be identified with the Farrell-Hsiang Nil groups as follows [DKR, LO2]:

$$\mathbf{Nil}_n^W = \mathrm{Nil}_n^W(\mathbb{Z}[F]; \mathbb{Z}[G_1 \backslash F], \mathbb{Z}[G_2 \backslash F]) \cong NK_n(\mathbb{Z}[F], \alpha) \oplus NK_n(\mathbb{Z}[F], \alpha^{-1}).$$

On the other hand, since the Nil groups vanish in negative degrees, one has the following:

Theorem 24 ([FJ2, Theorem 2.1])**.** *Let Γ be an infinite virtually cyclic group. Then:*
(a) $K_{-1}(\mathbb{Z}[\Gamma])$ is a finitely-generated Abelian group.
(b) $K_{-1}(\mathbb{Z}[\Gamma])$ is generated by the images of $K_{-1}(\mathbb{Z}[F])$ under the maps induced by the inclusions $F \subset \Gamma$, where F runs over the representatives of the conjugacy classes of finite subgroups of Γ.

So in order to compute the K-groups of an infinite virtually cyclic group, we need to understand the K-groups of the corresponding finite kernel subgroup and the corresponding Bass or Farrell-Hsiang Nil groups. This shall be achieved in the following section for the group $B_4(\mathbb{S}^2)$.

7 Whitehead groups and lower K-theory groups of the group ring $\mathbb{Z}[B_n(\mathbb{S}^2)]$, $n \leqslant 4$

As we mentioned in Section 2, $B_n(\mathbb{S}^2)$ is finite for $n \leqslant 3$. For these values of n, the corresponding K-groups were given in Sections 3–5. This section, which is divided into five subsections is devoted to the explicit computation of the Whitehead and lower K-groups of $\mathbb{Z}[B_n(\mathbb{S}^2)]$ for the special case $n = 4$. The aim is to prove Theorem 1, whose statement we recall here.

Theorem 1. *The group $B_4(\mathbb{S}^2)$ has the following Whitehead and lower algebraic K-groups:*

$$\begin{gathered}\mathrm{Wh}(B_4(\mathbb{S}^2)) \cong \mathbb{Z} \oplus \mathrm{Nil}_1,\\ \widetilde{K}_0(\mathbb{Z}[B_4(\mathbb{S}^2)]) \cong \mathbb{Z}_2 \oplus \mathrm{Nil}_0, \quad \textit{and}\\ K_{-1}(\mathbb{Z}[B_4(\mathbb{S}^2)]) \cong \mathbb{Z}_2 \oplus \mathbb{Z},\end{gathered}$$

where for $i = 0, 1$,

$$\mathrm{Nil}_i \cong \bigoplus_{\infty}[2(\mathbb{Z}_2)^{\infty} \oplus 2W],$$

$2(\mathbb{Z}_2)^{\infty}$ denotes a direct sum of two copies of a countably-infinite number of copies of $\mathbb{Z}_2$, and W is an infinitely-generated Abelian group of exponent 2 or 4.

In Section 7.1, we start by recalling a presentation of $B_4(\mathbb{S}^2)$. We then study the algebraic description of the finite subgroups of $B_4(\mathbb{S}^2)$ given by Theorem 2, which enables us to prove in Proposition 27 that $B_4(\mathbb{S}^2)$ may be expressed as an amalgamated product of T^* and $\mathcal{Q}_{16}$ along their common normal subgroup that is isomorphic to $\mathcal{Q}_8$. This implies in particular that $B_4(\mathbb{S}^2)$ is hyperbolic in the sense of Gromov (as we shall see in Proposition 38, $B_4(\mathbb{S}^2)$ is virtually free). In Section 7.2, we carry out K-theoretical calculations using Section 6 and [JL, JLMP], from which we are able to prove Theorem 1, up to determining the Nil_i factors. The rest of the section is then devoted to determining these factors. In order to do this, in Section 7.3, we study the structure of the maximal virtually cyclic subgroups of $B_4(\mathbb{S}^2)$, our main result being Theorem 30. Using [JLMP], in Section 7.4, we show that the maximal infinite virtually cyclic subgroups of $B_4(\mathbb{S}^2)$ possess an infinite number of conjugacy classes. Finally, in Section 7.5, we put together all of these ingredients to compute the Nil_i factors in Theorem 1.

7.1 Generalities about $B_4(\mathbb{S}^2)$

We recall several results concerning $B_4(\mathbb{S}^2)$. As we shall see, it possesses some very interesting properties that will allow us to calculate its K-groups.

Theorem 25 ([FVB]). *The group $B_4(\mathbb{S}^2)$ admits the following presentation:*
generators: $\sigma_1, \sigma_2, \sigma_3$.
relations:

$$\begin{aligned}\sigma_1\sigma_3 &= \sigma_3\sigma_1\\ \sigma_1\sigma_2\sigma_1 &= \sigma_2\sigma_1\sigma_2\\ \sigma_2\sigma_3\sigma_2 &= \sigma_3\sigma_2\sigma_3\\ \sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 &= 1.\end{aligned} \tag{20}$$

The first three 'braid relations' will be used freely and without further comment in what follows. Since the given generators together with the first three relations constitute a presentation of the Artin braid group B_4, it follows that $B_4(\mathbb{S}^2)$ is a quotient of B_4. We may determine generators of the representatives of the conjugacy classes of the finite subgroups of $B_4(\mathbb{S}^2)$ in terms of the above generators of $B_4(\mathbb{S}^2)$. According to Murasugi [M], any finite order element of $B_4(\mathbb{S}^2)$ is conjugate to a power of one of the following elements:

$$\begin{cases}\alpha_0 = \sigma_1\sigma_2\sigma_3 & \text{(of order 8)}\\ \alpha_1 = \sigma_1\sigma_2\sigma_3^2 & \text{(of order 6)}\\ \alpha_2 = \sigma_1\sigma_2^2 & \text{(of order 4).}\end{cases} \tag{21}$$

Let

$$\Delta_4 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 \tag{22}$$

denote the 'half twist' braid. Then the 'full twist' braid Δ_4^2 generates the centre of $B_4(\mathbb{S}^2)$ and is the unique element of $B_4(\mathbb{S}^2)$ of order 2 (this is true in general, see [GVB]). Let $Q = \langle \alpha_0^2, \Delta_4 \rangle$. By [GG4, Theorem 1.3(3)], Q is isomorphic to $\mathcal{Q}_8$, and is a normal subgroup of $B_4(\mathbb{S}^2)$. Further, it is well known that (see [GG6, Lemma 29] for example):

$$\alpha_0\sigma_i\alpha_0^{-1} = \sigma_{i+1} \text{ for } i = 1, 2, \text{ and } \alpha_0^2\sigma_3\alpha_0^{-2} = \sigma_1, \tag{23}$$

and that:

$$\Delta_4\sigma_i\Delta_4^{-1} = \sigma_{n-i} \quad \text{for all } i = 1, 2, 3 \tag{24}$$

(these two relations in fact hold in B_4).

Remarks 26.

(a) By Theorem 2, the isomorphism classes of the maximal finite subgroups of $B_4(\mathbb{S}^2)$ are T^* and $\mathcal{Q}_{16}$.

(b) Within $B_4(\mathbb{S}^2)$, there is a single conjugacy class of each of T^* and $\mathcal{Q}_{16}$ [GG6, Proposition 1.5(1)]. These subgroups may be realised algebraically as follows:

(i) $\mathcal{Q}_{16}$ may be realised as $\langle \alpha_0, \Delta_4 \rangle$ (α_0 is of order 8 and Δ_4 is of order 4) [GG2]. In particular,

$$\Delta_4\alpha_0\Delta_4^{-1} = \alpha_0^{-1} \quad \text{and} \quad \alpha_0^4 = \Delta_4^2. \tag{25}$$

(ii) By [GG6, Remark 3.2], T^* may be realised as $\left\langle \sigma_1\sigma_3^{-1}, \Delta_4 \right\rangle \rtimes \left\langle \alpha_1^2 \right\rangle \cong \mathcal{Q}_8 \rtimes \mathbb{Z}_3$. Note that the first factor of the semi-direct product is Q, since by equations (21) and (22), we have:

$$\alpha_0^{-2}\Delta_4 = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}.\,\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 = \sigma_1\sigma_3^{-1}. \tag{26}$$

The action is given by $\alpha_1^2\Delta_4\alpha_1^{-2} = \sigma_1\sigma_3^{-1}$ and $\alpha_1^2\sigma_1\sigma_3^{-1}\alpha_1^{-2} = \Delta_4\sigma_1\sigma_3^{-1} = \alpha_0^{-2}$. The only other isomorphism class of finite non-Abelian subgroups of $B_4(\mathbb{S}^2)$ is $\mathcal{Q}_8$: the subgroups $Q = \left\langle \alpha_0^2, \Delta_4 \right\rangle$ and $Q' = \left\langle \alpha_0^2, \alpha_0\Delta_4 \right\rangle$ of $\langle \alpha_0, \Delta_4 \rangle$ are isomorphic to $\mathcal{Q}_8$ and realise the two conjugacy classes of $\mathcal{Q}_8$ in $B_4(\mathbb{S}^2)$ [GG6, Proposition 1.5(2) and Theorem 1.6].
(c) By Theorem 2, the remaining finite subgroups are cyclic, and as we mentioned previously, are realised up to conjugacy by powers of the α_i, $i \in \{0,1,2\}$. For each finite cyclic subgroup, there is a single conjugacy class, with the exception of $\mathbb{Z}_4$, which is realised by both of the non-conjugate subgroups $\left\langle \alpha_0^2 \right\rangle$ and $\langle \alpha_2 \rangle$ [GG6, Proposition 1.5(2) and Theorem 1.6].

As we mentioned above, Q is a normal subgroup of $B_4(\mathbb{S}^2)$. From this, we may obtain the following:

Proposition 27. $B_4(\mathbb{S}^2) \cong \mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathrm{T}^*$.

Proof. Let $\Gamma = B_4(\mathbb{S}^2)/Q$. Since $\sigma_1\sigma_3^{-1} \in Q$ and $\left\langle \sigma_1\sigma_3^{-1} \right\rangle$ is not normal in $B_4(\mathbb{S}^2)$ by Remarks 26(b)(ii), it follows that the normal closure of $\sigma_1\sigma_3^{-1}$ in $B_4(\mathbb{S}^2)$ is Q, and that a presentation of Γ is obtained by adjoining the relation $\sigma_1 = \sigma_3$ to the presentation of $B_4(\mathbb{S}^2)$ given in Theorem 25. We see then that Γ is generated by elements $\overline{\sigma_1}$ and $\overline{\sigma_2}$, subject to the two relations $\overline{\sigma_1}\,\overline{\sigma_2}\,\overline{\sigma_1} = \overline{\sigma_2}\,\overline{\sigma_1}\,\overline{\sigma_2}$ and $(\overline{\sigma_1}\,\overline{\sigma_2}\,\overline{\sigma_1})^2 = 1$. Let

$$\Lambda = \left\langle a, b \;\middle|\; a^2 = b^3 = 1 \right\rangle$$

denote the free product $\mathbb{Z}_2 * \mathbb{Z}_3$, and consider the map $\varphi\colon \Lambda \longrightarrow \Gamma$ defined on the generators of Λ by $\varphi(a) = \overline{\sigma_1}\,\overline{\sigma_2}\,\overline{\sigma_1}$ and $\varphi(b) = \overline{\sigma_1}\,\overline{\sigma_2}$. Since $(\varphi(b))^3 = (\overline{\sigma_1}\,\overline{\sigma_2})^3 = (\overline{\sigma_1}\,\overline{\sigma_2}\,\overline{\sigma_1})^2 = (\varphi(a))^2 = 1$, φ extends to a homomorphism that is surjective since $\varphi(b^{-1}a) = \overline{\sigma_1}$ and $\varphi(a^{-1}b^2) = \overline{\sigma_2}$. Conversely, the map $\psi\colon \Gamma \longrightarrow \Lambda$ defined on the generators of Γ by $\psi(\overline{\sigma_1}) = b^{-1}a$ and $\psi(\overline{\sigma_2}) = a^{-1}b^2$ extends to a homomorphism since:

$$\psi(\overline{\sigma_1})\,\psi(\overline{\sigma_2})\,\psi(\overline{\sigma_1}) = b^{-1}aa^{-1}b^2b^{-1}a = a = a^{-1}b^2b^{-1}aa^{-1}b^2 = \psi(\overline{\sigma_2})\,\psi(\overline{\sigma_1})\,\psi(\overline{\sigma_2}) \quad \text{and}$$
$$(\psi(\overline{\sigma_1})\,\psi(\overline{\sigma_2})\,\psi(\overline{\sigma_1}))^2 = a^2 = 1,$$

and is surjective because $\psi(\overline{\sigma_1}\,\overline{\sigma_2}\,\overline{\sigma_1}) = a$ and $\psi(\overline{\sigma_1}\,\overline{\sigma_2}) = b$. Hence φ is an isomorphism, and $\Gamma \cong \mathbb{Z}_2 * \mathbb{Z}_3$

Let $G = \mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathrm{T}^*$, and consider the following presentation of G with generators u, v, p, q, r that are subject to the relations:

$$\begin{cases} p^2 = q^2,\ qpq^{-1} = p^{-1},\ rpr^{-1} = q,\ rqr^{-1} = pq,\ r^3 = 1 \\ u^4 = v^2,\ vuv^{-1} = u^{-1} \\ u^2 = p,\ v = q, \end{cases}$$

so that $\langle p,q,r\rangle \cong \mathrm{T}^*$, $\langle u,v\rangle \cong \mathcal{Q}_{16}$, and $\langle p,q,r\rangle \cap \langle u,v\rangle = H$, where $H = \langle p,q\rangle \cong \mathcal{Q}_8$. It follows from this presentation that $H \lhd G$ and $G/H \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Let $f\colon\ G \longrightarrow B_4(\mathbb{S}^2)$ be the map defined on the generators of G by $f(u) = \alpha_0^{-1}$, $f(p) = \alpha_0^{-2}$, $f(v) = f(q) = \Delta_4$ and $f(r) = \alpha_1^2$. Using Remarks 26(b), we see that f respects the relations of G, and so extends to a homomorphism that sends H isomorphically onto Q. Further, $\alpha_1^3 = \Delta_4^2$ using equation (21) and the uniqueness of Δ_4^2 as the element of order 2 of $B_4(\mathbb{S}^2)$. Thus $\alpha_1 = \Delta_4^2\alpha_1^{-2}$, and since $B_4(\mathbb{S}^2) = \langle \alpha_0, \alpha_1\rangle$ [GG1, Theorem 3], it follows that f is surjective. We thus obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 1 \\ & & \cong\downarrow{\scriptstyle f|_H} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \widehat{f}} & & \\ 1 & \longrightarrow & Q & \longrightarrow & B_4(\mathbb{S}^2) & \longrightarrow & \Gamma & \longrightarrow & 1, \end{array}$$

where $\widehat{f}$ is the homomorphism induced by f on the quotients. Since f is surjective, $\widehat{f}$ is too, and the isomorphisms $G/H \cong \mathbb{Z}_2 * \mathbb{Z}_3 \cong \Gamma$ and the fact that $\mathbb{Z}_2 * \mathbb{Z}_3$ is Hopfian (see [DN] for example) imply that $\widehat{f}$ is an isomorphism. The result is then a consequence of the 5-Lemma. □

7.2 Preliminary K-theoretical calculations for $\mathbb{Z}[B_4(\mathbb{S}^2)]$

Since $B_4(\mathbb{S}^2)$ is an amalgam of finite groups by Proposition 27, it is a hyperbolic group in the sense of Gromov (we shall see in Proposition 38 that $B_4(\mathbb{S}^2)$ is in fact virtually free). Using the results of [JL] and [JLMP, Example 3.2], we may compute $K_n(\mathbb{Z}[B_4(\mathbb{S}^2)])$, and we obtain:

$$K_n(\mathbb{Z}[B_4(\mathbb{S}^2)]) \cong A_n \oplus B_n \oplus \left(\bigoplus_{V\in\mathcal{V}} \mathrm{Coker}_n\,(V)\right),$$

where

$$A_n = \mathrm{Coker}\big(K_n(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow K_n(\mathbb{Z}[\mathcal{Q}_{16}]) \oplus K_n(\mathbb{Z}[\mathrm{T}^*])\big),$$

and

$$B_n = \mathrm{Ker}\big(K_{n-1}(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow K_{n-1}(\mathbb{Z}[\mathcal{Q}_{16}]) \oplus K_{n-1}(\mathbb{Z}[\mathrm{T}^*])\big),$$

$\mathrm{Coker}_n\,(V)$ corresponds to the various Nil groups described in Section 6 (and will be determined in what follows), and the sum is over the family $\mathcal{V}$ of conjugacy classes of maximal infinite virtually cyclic subgroups of $B_4(\mathbb{S}^2)$. By using the pseudo-isotopy

functor instead of K, we obtain similar formulæ for the Whitehead and $\widetilde{K}_0$-groups:

$$\mathrm{Wh}(B_4(\mathbb{S}^2)) = \begin{cases} \mathrm{Coker}\big(\mathrm{Wh}(\mathcal{Q}_8) \longrightarrow \mathrm{Wh}(\mathcal{Q}_{16}) \oplus \mathrm{Wh}(\mathrm{T}^*)\big) \\ \oplus \\ \mathrm{Ker}\left(\widetilde{K}_0(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\mathcal{Q}_{16}]) \oplus \widetilde{K}_0(\mathbb{Z}[\mathrm{T}^*])\right) \\ \oplus \\ \mathrm{Nil}_1, \end{cases}$$

$$\widetilde{K}_0(\mathbb{Z}[B_4(\mathbb{S}^2)]) = \begin{cases} \mathrm{Coker}\left(\widetilde{K}_0(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\mathcal{Q}_{16}]) \oplus \widetilde{K}_0(\mathbb{Z}[\mathrm{T}^*])\right) \\ \oplus \\ \mathrm{Ker}\left(\widetilde{K}_{-1}(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow \widetilde{K}_{-1}(\mathbb{Z}[\mathcal{Q}_{16}]) \oplus \widetilde{K}_{-1}(\mathbb{Z}[\mathrm{T}^*])\right) \\ \oplus \\ \mathrm{Nil}_0 \end{cases}$$

and the K_1-group:

$$K_{-1}(\mathbb{Z}[B_4(\mathbb{S}^2)]) = \mathrm{Coker}\big(K_{-1}(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow K_{-1}(\mathbb{Z}[\mathcal{Q}_{16}]) \oplus K_{-1}(\mathbb{Z}[\mathrm{T}^*])\big),$$

where for $i = 0, 1$, Nil_i splits as a direct sum of Bass or Farrell-Hsiang Nil groups over representatives of $\mathcal{V}$ (see Section 6). From Theorem 10 and Propositions 7, 22 and 23, we have the following:

$\mathrm{Wh}(\mathcal{Q}_8) = 0$	$\widetilde{K}_0(\mathbb{Z}[\mathcal{Q}_8]) \cong \mathbb{Z}_2$	$K_{-1}(\mathbb{Z}[\mathcal{Q}_8]) = 0$
$\mathrm{Wh}(\mathcal{Q}_{16}) \cong \mathbb{Z}$	$\widetilde{K}_0(\mathbb{Z}[\mathcal{Q}_{16}]) \cong \mathbb{Z}_2$	$K_{-1}(\mathbb{Z}[\mathcal{Q}_{16}]) \cong \mathbb{Z}_2$
$\mathrm{Wh}(\mathrm{T}^*) = 0$	$\widetilde{K}_0(\mathbb{Z}[\mathrm{T}^*]) \cong \mathbb{Z}_2$	$K_{-1}(\mathbb{Z}[\mathrm{T}^*]) \cong \mathbb{Z}$.

Moreover, by [Sw, Lemma 14.6], the induction $\widetilde{K}_0(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\mathcal{Q}_{16}])$ is zero and $\widetilde{K}_0(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\mathrm{T}^*])$ is an isomorphism by hyperelementary induction (cf. [Sw, Theorem 14.1(1)]). Hence we have the following isomorphisms:

$$\mathrm{Wh}(B_4(\mathbb{S}^2)) \cong \mathbb{Z} \oplus \mathrm{Nil}_1,\ \widetilde{K}_0(\mathbb{Z}[B_4(\mathbb{S}^2)]) \cong \mathbb{Z}_2 \oplus \mathrm{Nil}_0 \text{ and } K_{-1}(\mathbb{Z}[B_4(\mathbb{S}^2)]) \cong \mathbb{Z}_2 \oplus \mathbb{Z}, \quad (27)$$

which proves Theorem 1 up to the computation of the Nil_i terms. To complete the proof, we next need to determine the maximal infinite virtually cyclic subgroups of $B_4(\mathbb{S}^2)$, and the number of their conjugacy classes, which we do in the following section.

7.3 Maximal virtually cyclic subgroups of $B_4(\mathbb{S}^2)$

Recall from Remarks 26 that up to isomorphism, the maximal finite subgroups of $B_4(\mathbb{S}^2)$ are $\mathcal{Q}_{16}$ and T^*, and that there exists a single conjugacy class of each. The isomorphism classes of the infinite virtually cyclic subgroups have also been classified:

Theorem 28 ([GG6, Theorem 5]). *The infinite virtually cyclic subgroups of $B_4(\mathbb{S}^2)$ are isomorphic to the following groups:*
(a) subgroups of Type I: $\mathbb{Z}_k \times \mathbb{Z}$, $k \in \{1,2,4\}$; $\mathbb{Z}_4 \rtimes \mathbb{Z}$ *for the non-trivial action;* $\mathcal{Q}_8 \rtimes \mathbb{Z}$ *for all three possible actions (the trivial action, and the actions of order two and three respectively).*
(b) subgroups of Type II: $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$, $\mathbb{Z}_8 *_{\mathbb{Z}_4} \mathbb{Z}_8$, $\mathbb{Z}_8 *_{\mathbb{Z}_4} \mathcal{Q}_8$, $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathcal{Q}_8$, $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$.

For each of the Type II subgroups given in Theorem 28(b), abstractly there is a single isomorphism class, with the exception of $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ for which there are two isomorphism classes [GG6, Proposition 11]. In the latter case, we recall the following results concerning the structure and the realisation in $B_4(\mathbb{S}^2)$ of the two classes.

Proposition 29 ([GG6, Propositions 11 and 78]). *Abstractly, there are exactly two isomorphism classes of the amalgamated product $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, possessing the following presentations:*

$$\Gamma_1 = \left\langle a,b,x,y \;\middle|\; a^4 = b^2,\ x^4 = y^2,\ bab^{-1} = a^{-1},\ yxy^{-1} = x^{-1},\ x^2 = a^2,\ y = b \right\rangle \tag{28}$$

and

$$\Gamma_2 = \left\langle a,b,x,y \;\middle|\; a^4 = b^2,\ x^4 = y^2,\ bab^{-1} = a^{-1},\ yxy^{-1} = x^{-1},\ x^2 = b,\ y = a^2b \right\rangle. \tag{29}$$

Further, for $i \in \{1,2\}$, $B_4(\mathbb{S}^2)$ possesses a subgroup G_i such that $G_i \cong \Gamma_i$.

Since $\mathrm{Out}(\mathcal{Q}_8) \cong S_3$, there are three isomorphism classes of Type I groups of the form $\mathcal{Q}_8 \rtimes \mathbb{Z}$. For $j \in \{1,2,3\}$, let $\mathcal{Q}_8 \rtimes_j \mathbb{Z}$ denote the group for which the action is of order j. We consider the problem of deciding which subgroups of $B_4(\mathbb{S}^2)$ are maximal within the family of virtually cyclic subgroups.

Theorem 30.
(a) Let G be a maximal infinite virtually cyclic subgroup of $B_4(\mathbb{S}^2)$. Then G is isomorphic to one of the following groups: $\mathcal{Q}_8 \rtimes \mathbb{Z}$ for one of the three possible actions, or $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$.
(b) If G is a subgroup of $B_4(\mathbb{S}^2)$ isomorphic to T^* then it is maximal as a virtually cyclic subgroup.
(c) For each $j \in \{1,2,3\}$, there are subgroups of $B_4(\mathbb{S}^2)$ isomorphic to $\mathcal{Q}_8 \rtimes_j \mathbb{Z}$ that are maximal as virtually cyclic subgroups, and others that are non maximal.
(d) There exist subgroups of $B_4(\mathbb{S}^2)$ isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ that are maximal as virtually cyclic subgroups, and others that are non maximal.

We shall prove the theorem in several steps. Since $B_4(\mathbb{S}^2)$ is a hyperbolic group in the sense of Gromov, the following result implies that every infinite virtually cyclic subgroup $B_4(\mathbb{S}^2)$ is contained in a unique maximal virtually cyclic subgroup, in other words, there are no infinite ascending chains (for inclusion) of infinite virtually cyclic subgroups.

Proposition 31 ([JL, Propositions 5,6 and Remark 7]). *Every infinite virtually cyclic subgroup of a Gromov hyperbolic group is contained in a unique maximal virtually cyclic subgroup.*

We start with the proof of parts (a) and (b) of Theorem 30.

Proposition 32.
(a) Let G be a maximal virtually cyclic subgroup of $B_4(\mathbb{S}^2)$. Then G is isomorphic to one of the following groups: T^*, $\mathcal{Q}_8 \rtimes \mathbb{Z}$ for one of the three possible actions, or $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$.
(b) If G is a subgroup of $B_4(\mathbb{S}^2)$ isomorphic to T^* then it is maximal as a virtually cyclic subgroup.

Proof of Proposition 32. Let G be a maximal virtually cyclic subgroup of $B_4(\mathbb{S}^2)$.
(a) First assume that G is finite. Then G is a maximal finite subgroup of $B_4(\mathbb{S}^2)$, so is isomorphic to either $\mathcal{Q}_{16}$ or T^* by Theorem 2. Suppose that $G \cong \mathcal{Q}_{16}$. Since $B_4(\mathbb{S}^2)$ possesses a single conjugacy class of subgroups isomorphic to $\mathcal{Q}_{16}$ by Remarks 26(b), by Proposition 29, there exists a subgroup of $B_4(\mathbb{S}^2)$ isomorphic to the amalgamated product $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, of which one of the factors is G, so G is not maximal as a virtually cyclic subgroup. Thus $G \cong \mathrm{T}^*$.

Now assume that G is infinite and of Type I, so $G \cong F \rtimes \mathbb{Z}$, for some action of $\mathbb{Z}$ on F, where F is finite and is the torsion subgroup of G. Suppose that F is either trivial or isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_4$, and let u be a generator of the $\mathbb{Z}$-factor of G. Up to conjugation, we claim that $F \subset \langle \alpha_0^2 \rangle$. If F is trivial or isomorphic to $\mathbb{Z}_2$ then $F \subset \langle \Delta_4^2 \rangle \subset \langle \alpha_0^2 \rangle$ since $\alpha_0^4 = \Delta_4^2$. So suppose that $F \cong \mathbb{Z}_4$. By Remarks 26(c), $B_4(\mathbb{S}^2)$ admits two conjugacy classes of subgroups isomorphic to $\mathbb{Z}_4$, generated respectively by α_0^2 and α_2. But since u normalises F and the normaliser of $\langle \alpha_2 \rangle$ in $B_4(\mathbb{S}^2)$ is finite [GG6, Proposition 8(b)], it follows that F is conjugate to $\langle \alpha_0^2 \rangle$. This proves the claim, and so conjugating G if necessary, we may suppose that $F \subset \langle \alpha_0^2 \rangle$. Since $Q = \langle \alpha_0^2, \Delta_4 \rangle \lhd B_4(\mathbb{S}^2)$ and $Q \cong \mathcal{Q}_8$, the subgroup $\langle \alpha_0^2, \Delta_4, u \rangle$ is isomorphic to one of the three Type I groups $\mathcal{Q}_8 \rtimes_j \mathbb{Z}$, $j \in \{1,2,3\}$, and admits $G = \langle \alpha_0^2, u \rangle$ as a proper subgroup. Hence G is non maximal as a virtually cyclic subgroup of $B_4(\mathbb{S}^2)$. The result in this case is then a consequence of Theorem 28(a).

Finally suppose that G is a Type II subgroup of $B_4(\mathbb{S}^2)$ that is non isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$. By Theorem 28(b), we may write $G = G_1 *_H G_2$, where either:
(i) G_1 and G_2 are subgroups of $B_4(\mathbb{S}^2)$ isomorphic to $\mathcal{Q}_8$ or $\mathbb{Z}_8$, and $H = G_1 \cap G_2$ is isomorphic to $\mathbb{Z}_4$, or
(ii) G_1 and G_2 are subgroups of $B_4(\mathbb{S}^2)$ isomorphic to $\mathbb{Z}_4$, and $H = G_1 \cap G_2 = \langle \Delta_4^2 \rangle$.
By Remarks 26, in $B_4(\mathbb{S}^2)$, there are two conjugacy classes of subgroups isomorphic to $\mathcal{Q}_8$ represented by Q and Q', one conjugacy class of subgroups isomorphic to $\mathbb{Z}_8$, represented by $\langle \alpha_0 \rangle$, and two conjugacy classes of subgroups isomorphic to $\mathbb{Z}_4$, represented by $\langle \alpha_0^2 \rangle$ and $\langle \alpha_2 \rangle$. Conjugating G if necessary, we may suppose that $G = G_1 *_{H'} G_2'$, where $G_2' = \lambda G_2 \lambda^{-1}$ for some $\lambda \in B_4(\mathbb{S}^2)$, $H' = G_1 \cap G_2'$, and for $i = 1,2$, H_i is equal to Q, Q' or $\langle \alpha_0 \rangle$ in case (i), and to $\langle \alpha_0^2 \rangle$ or $\langle \alpha_2 \rangle$ in case (ii) (since Q is normal in $B_4(\mathbb{S}^2)$ and G is infinite, one may in fact see that $H_i \neq Q$ in case (i) and $H_i \neq \langle \alpha_0^2 \rangle$ in case (ii), but we do not need to use this for the rest of the proof). Let $L_1 = \langle \alpha_0, \Delta_4 \rangle$, and let $L_2 = \lambda L_1 \lambda^{-1}$. Then $L_1 \cong L_2 \cong \mathcal{Q}_{16}$, and $G_i \subset L_i$ for $i = 1,2$. Further, $Q \subset L_1 \cap L_2$ because $Q \lhd B_4(\mathbb{S}^2)$. Let $L = \langle L_1 \cup L_2 \rangle$. Since $G \subset L$, L is infinite, so $L_1 \cap L_2 = Q$.

Thus $L \cong \mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, and G is a non-maximal virtually cyclic subgroup of $B_4(\mathbb{S}^2)$. It then follows from Theorem 28(b) that any maximal virtually cyclic subgroup of $B_4(\mathbb{S}^2)$ of Type II is isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$.
(b) By Theorem 28(b), none of the infinite virtually cyclic subgroups of $B_4(\mathbb{S}^2)$ admit subgroups isomorphic to T^*, so any subgroup of $B_4(\mathbb{S}^2)$ isomorphic to T^* is maximal as a virtually cyclic subgroup. Combined with part (a), this shows in fact that G is finite if and only if $G \cong T^*$. □

Before proving Theorem 30(c), we state and prove the following lemma.

Lemma 33. *Let $\pi\colon B_4(\mathbb{S}^2) \longrightarrow \mathbb{Z}_6$ denote the Abelianisation homomorphism. Then:*
(a) If H is a subgroup of $B_4(\mathbb{S}^2)$ that is isomorphic to either $\mathbb{Z}_8$, $\mathcal{Q}_8$ or $\mathcal{Q}_{16}$ then $\pi(H) \subset \langle \overline{3} \rangle$.
(b) If G is a subgroup of $B_4(\mathbb{S}^2)$ that is isomorphic to an amalgamated product of one of the forms $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathcal{Q}_8$, $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathbb{Z}_8$ or $\mathbb{Z}_8 *_{\mathbb{Z}_4} \mathbb{Z}_8$ then $\pi(G) \subset \langle \overline{3} \rangle$.

Proof of Lemma 33.
(a) Consider the subgroup $K = \langle \alpha_0, \Delta_4 \rangle$. As we mentioned in the proof of Proposition 32, K contains representatives of the conjugacy classes of all subgroups of $B_4(\mathbb{S}^2)$ that are isomorphic to $\mathbb{Z}_8$, $\mathcal{Q}_8$ or $\mathcal{Q}_{16}$. So there exists $\lambda \in B_4(\mathbb{S}^2)$ such that $\lambda H \lambda^{-1} \subset K$. Now $\pi(\alpha_0) = \overline{3}$ and $\pi(\Delta_4) = \overline{0}$, thus $\pi(K) \subset \langle \overline{3} \rangle$, which yields the result.
(b) If G is a subgroup of $B_4(\mathbb{S}^2)$ that is isomorphic to one of the given amalgamated products then by Remarks 26(c), the factors appearing in the amalgamation are subgroups of conjugates of K, and thus $\pi(G) \subset \pi(K) \subset \langle \overline{3} \rangle$ by part (a). □

To prove Theorem 30(c), for each $j \in \{1, 2, 3\}$, we shall exhibit a subgroup of $B_4(\mathbb{S}^2)$ that is isomorphic $\mathcal{Q}_8 \rtimes_j \mathbb{Z}$, and that is maximal (resp. non maximal) as a virtually cyclic subgroup of $B_4(\mathbb{S}^2)$. The proof of the existence of a maximal virtually cyclic subgroup of $B_4(\mathbb{S}^2)$ that is isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$ is long, and will be treated separately. With the exception of this case, we now prove Theorem 30(c) as well as part (d).

Proposition 34.
(a) For each $j \in \{1, 2, 3\}$, there are subgroups of $B_4(\mathbb{S}^2)$ isomorphic to $\mathcal{Q}_8 \rtimes_j \mathbb{Z}$ that are non maximal as virtually cyclic subgroups.
(b) For each $j \in \{2, 3\}$, there are subgroups of $B_4(\mathbb{S}^2)$ isomorphic to $\mathcal{Q}_8 \rtimes_j \mathbb{Z}$ that are maximal as virtually cyclic subgroups.
(c) There exist subgroups of $B_4(\mathbb{S}^2)$ isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ that are maximal as virtually cyclic subgroups, and others that are non maximal.

Proof of Proposition 34.
(a) By Proposition 29, for $i = 1, 2$, $B_4(\mathbb{S}^2)$ possesses a subgroup G_i that is isomorphic to the amalgamated product Γ_i given by equations (28) and (29), and so admits a presentation given by the corresponding equation. The amalgamating subgroup $\Gamma = \langle a^2, b \rangle = \langle x^2, y \rangle$ is isomorphic to $\mathcal{Q}_8$, and the element $a^{-1}x$ is a product of elements chosen alternately from the two sets $\langle a, b \rangle \setminus \langle a^2, b \rangle$ and $\langle x, y \rangle \setminus \langle x^2, y \rangle$, so is of infinite order by standard properties of amalgamated products. Consider the subgroup $H_i = \langle \Gamma_i \cup \{a^{-1}x\} \rangle$

of G_i. One may check that $\langle a^{-1}x\rangle$ acts by conjugation on $\langle a^2, b\rangle$. If $i = 1$ then

$$\begin{cases} a^{-1}x.\,a^2.\,x^{-1}a = a^2 \\ a^{-1}x.\,b.\,x^{-1}a = a^{-1}xyx^{-1}a = a^{-1}xyx^{-1}y^{-1}ya = a^{-1}x^2ya = abab^{-1}b = b \\ a^{-1}x.\,a^2b.\,x^{-1}a = a^2b, \end{cases} \tag{30}$$

and hence $H_1 \cong \mathcal{Q}_8 \times \mathbb{Z}$. If $i = 2$, a similar computation shows that conjugation by $a^{-1}x$ permutes cyclically a^2, b^{-1} and $a^{-2}b^{-1}$, and thus $H_2 \cong \mathcal{Q}_8 \rtimes_3 \mathbb{Z}$. In each case, $H_i \subsetneq G_i$ because $[G_i : H_i] = 2$. Now G_i is isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, and so H_i is non maximal as a virtually cyclic subgroup of $B_4(\mathbb{S}^2)$, which proves the statement for $j \in \{1, 3\}$. It thus remains to treat the case $j = 2$. Note that in G_1, the action by conjugation of $xa^{-1}x$ on Γ is as follows:

$$\begin{cases} xa^{-1}x.\,a^2.\,x^{-1}ax^{-1} = xa^2x^{-1} = a^2 \\ xa^{-1}x.\,b.\,x^{-1}ax^{-1} = xyx^{-1} = xyx^{-1}y^{-1}y = x^2y = a^2b \\ xa^{-1}x.\,a^2b.\,x^{-1}ax^{-1} = a^4b = b^{-1}, \end{cases} \tag{31}$$

using equation (30). Now $xa^{-1}x$ is of infinite order, so we conclude from equation (31) that the subgroup $\langle \Gamma \cup \{xa^{-1}x\}\rangle$ is isomorphic to $\mathcal{Q}_8 \rtimes_2 \mathbb{Z}$. Furthermore, this subgroup is contained (strictly) in G_1, so is non maximal.

(b) Let $j = 2$. Consider the subgroup $H = \langle Q \cup \{\sigma_1\}\rangle$ of $B_4(\mathbb{S}^2)$. By Proposition 31, H is contained in a maximal virtually cyclic subgroup M of $B_4(\mathbb{S}^2)$. Since Q is normal in $B_4(\mathbb{S}^2)$ and σ_1 is of infinite order, H must be isomorphic to a semi-direct product of the form $\mathcal{Q}_8 \rtimes_k \mathbb{Z}$ for some $k \in \{1, 2, 3\}$. To determine k, we study the action by conjugation of σ_1 on Q. Using equation (26), we have:

$$\begin{cases} \sigma_1.\,\alpha_0^2.\,\sigma_1^{-1} = \sigma_1\alpha_0^2\sigma_1^{-1}\alpha_0^{-2}\alpha_0^2 = \sigma_1\sigma_3^{-1}\alpha_0^2 \quad \text{by equation (23)} \\ \qquad\qquad = \alpha_0^{-2}\Delta_4\alpha_0^2 = \alpha_0^{-4}\Delta_4 = \Delta_4^{-1} \quad \text{by equation (25)} \\ \sigma_1.\,\Delta_4^{-1}.\,\sigma_1^{-1} = \sigma_1\Delta_4^{-1}\sigma_1^{-1}\Delta_4\Delta_4^{-1} = \sigma_1\sigma_3^{-1}\Delta_4^{-1} \quad \text{by equation (24)} \\ \qquad\qquad = \alpha_0^{-2} \quad \text{by equation (26)} \\ \sigma_1.\,\alpha_0^2\Delta_4.\,\sigma_1^{-1} = \Delta_4^{-1}\alpha_0^2 = \alpha_0^2\Delta_4. \end{cases} \tag{32}$$

Since σ_1 is of infinite order, H is thus isomorphic to $\mathcal{Q}_8 \rtimes_2 \mathbb{Z}$ because the action fixes the subgroup $\langle \alpha_0^2\Delta_4\rangle$ of order 4 of Q, and exchanges $\langle \alpha_0^2\rangle$ and Δ_4. But $\pi(\sigma_1) = \overline{1} \notin \langle \overline{3}\rangle$, so H is not contained in any subgroup of the form $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathcal{Q}_8$, $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathbb{Z}_8$ or $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ by Lemma 33(b). It cannot be contained either in a subgroup isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$ or $\mathcal{Q}_8 \rtimes_3 \mathbb{Z}$ because the actions on Q are not compatible. This implies that M, which is maximal in $B_4(\mathbb{S}^2)$ as a virtually cyclic subgroup, must also be isomorphic to $\mathcal{Q}_8 \rtimes_2 \mathbb{Z}$.

Now assume that $j = 3$. As in the case $j = 2$, if there exists a subgroup L of $B_4(\mathbb{S}^2)$ that is isomorphic to $\mathcal{Q}_8 \rtimes_3 \mathbb{Z}$, it cannot be contained in a subgroup of $B_4(\mathbb{S}^2)$ isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$ or to $\mathcal{Q}_8 \rtimes_2 \mathbb{Z}$. Moreover, by Lemma 33(b), if $\pi(L) \not\subset \langle \overline{3}\rangle$ then L is not contained in any subgroup of $B_4(\mathbb{S}^2)$ isomorphic to $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathcal{Q}_8$, $\mathbb{Z}_8 *_{\mathbb{Z}_4} \mathbb{Z}_8$, $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathbb{Z}_8$ or

$\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$. As in the previous paragraph, we conclude using Proposition 31 that L is contained in a maximal virtually cyclic subgroup of $B_4(\mathbb{S}^2)$ that must also be isomorphic to $\mathcal{Q}_8 \rtimes_3 \mathbb{Z}$. To prove the result, we exhibit such a subgroup L. Consider the action by conjugation of σ_2 on Q:

$$\begin{cases} \sigma_2.\alpha_0^2.\sigma_2^{-1} = \alpha_0.\alpha_0^{-1}\sigma_2\alpha_0.\alpha_0\sigma_2^{-1}\alpha_0^{-1}.\alpha_0 = \alpha_0\sigma_1\sigma_3^{-1}\alpha_0 \quad \text{by equation (23)} \\ \qquad\qquad = \alpha_0\alpha_0^{-2}\Delta_4\alpha_0 \quad \text{by equation (26)} \\ \qquad\qquad = \alpha_0^{-2}\Delta_4 = (\alpha_0^2\Delta_4)^{-1} \quad \text{by equation (25)} \\ \sigma_2.\Delta_4.\sigma_2^{-1} = \sigma_2\Delta_4\sigma_2^{-1}\Delta_4^{-1}.\Delta_4 = \Delta_4 \\ \sigma_2.\alpha_0^2\Delta_4.\sigma_2^{-1} = \alpha_0^{-2}\Delta_4.\Delta_4 = \alpha_0^2. \end{cases} \tag{33}$$

In particular, $\sigma_2^4 x \sigma_2^{-4} = x$ for all $x \in Q$. This implies that the action by conjugation of $z = \sigma_2^7\sigma_1$ on the elements of Q is the same as that of $\sigma_2^3\sigma_1$. By equations (32) and (33), this action is as follows:

$$\begin{aligned} &\alpha_0^2 \overset{\sigma_1}{\longmapsto} \Delta_4^{-1} \overset{\sigma_2^3}{\longmapsto} \Delta_4^{-1} \\ &\Delta_4^{-1} \overset{\sigma_1}{\longmapsto} \alpha_0^{-2} \overset{\sigma_2^3}{\longmapsto} (\alpha_0^2\Delta_4)^{-1} \\ &(\alpha_0^2\Delta_4)^{-1} \overset{\sigma_2^3\sigma_1}{\longmapsto} \Delta_4^{-1}\alpha_0^{-2}\Delta_4 = \alpha_0^2. \end{aligned}$$

Hence the action by conjugation of z on Q is of order 3. Further, $\pi(z) = \overline{2} \notin \langle\overline{3}\rangle$, which shows that $L = \langle Q \cup \{z\}\rangle$ is not contained in any subgroup isomorphic to an amalgamated product of the form $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathcal{Q}_8$, $\mathbb{Z}_8 *_{\mathbb{Z}_4} \mathbb{Z}_8$, $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathbb{Z}_8$ or $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ by Lemma 33(b). To prove that $L \cong \mathcal{Q}_8 \rtimes_3 \mathbb{Z}$, it remains to show that z is of infinite order. To achieve this, we shall write z^3, which is an element of $P_4(\mathbb{S}^2)$, in terms of the basis $(A_{1,4}, A_{2,4})$ of the free group $\pi_1(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}, x_4)$, which we identify with the kernel of the homomorphism $P_4(\mathbb{S}^2) \longrightarrow P_3(\mathbb{S}^2)$ given geometrically by forgetting the last string. We recall that for $1 \leqslant i < j \leqslant 4$, the elements

$$A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1},$$

form a generating set of $P_4(\mathbb{S}^2)$. Thus $A_{i,i+1} = \sigma_i^2$, and the generators $A_{i,j}$ satisfy the following relations (the relations are not complete):

$$A_{1,2}A_{1,3}A_{1,4} = 1 \tag{34}$$
$$A_{1,2}A_{2,3}A_{2,4} = 1 \tag{35}$$
$$A_{1,3}A_{2,3}A_{3,4} = 1 \tag{36}$$
$$A_{1,4}A_{2,4}A_{3,4} = 1 \tag{37}$$
$$A_{1,2}A_{1,3}A_{1,4}A_{2,3}A_{2,4}A_{3,4} = \Delta_4^2. \tag{38}$$

It follows from relations (34), (37) and (38) that

$$A_{2,3} = \Delta_4^2 A_{3,4}^{-1}A_{2,4}^{-1} = \Delta_4^2 A_{1,4}, \tag{39}$$

from relations (36), (37) and (39) that

$$A_{1,3} = A_{3,4}^{-1}A_{2,3}^{-1} = A_{1,4}A_{2,4}A_{1,4}^{-1}\Delta_4^2, \tag{40}$$

and from relations (35) and (39) that

$$A_{1,2} = A_{2,4}^{-1}A_{2,3}^{-1} = A_{2,4}^{-1}A_{1,4}^{-1}\Delta_4^2. \tag{41}$$

Applying the relation $\sigma_i\sigma_{i+1}^k\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i^k\sigma_{i+1}$ for all $i \in \{1,2\}$ and $k \in \mathbb{Z}$, we obtain:

$$\begin{aligned}
z^3 &= (\sigma_2^7\sigma_1)^3 = A_{2,3}^3\sigma_2\sigma_1\sigma_2^7\sigma_1^{-1}.\,\sigma_1^2\sigma_2^7\sigma_1 = A_{2,3}^3\sigma_1^7\sigma_2\sigma_1^2\sigma_2^7\sigma_1 = A_{2,3}^3A_{1,2}^3\sigma_1\sigma_2\sigma_1^2\sigma_2^7\sigma_1\\
&= A_{2,3}^3A_{1,2}^3\sigma_1.\,\sigma_2\sigma_1^2\sigma_2^{-1}.\,\sigma_2^8\sigma_1 = A_{2,3}^3A_{1,2}^3A_{2,3}\sigma_1\sigma_2^8\sigma_1 = A_{2,3}^3A_{1,2}^3A_{2,3}A_{1,2}\sigma_1^{-1}\sigma_2^8\sigma_1\\
&= A_{2,3}^3A_{1,2}^3A_{2,3}A_{1,2}\sigma_2\sigma_1^8\sigma_2^{-1} = A_{2,3}^3A_{1,2}^3A_{2,3}A_{1,2}A_{1,3}^4\\
&= (\Delta_4^2A_{1,4})^3(A_{2,4}^{-1}A_{1,4}^{-1}\Delta_4^2)^3\Delta_4^2A_{1,4}A_{2,4}^{-1}A_{1,4}^{-1}\Delta_4^2(A_{1,4}A_{2,4}A_{1,4}^{-1}\Delta_4^2)^4\\
&= A_{1,4}^3A_{2,4}^{-1}A_{1,4}^{-1}A_{2,4}^{-1}A_{1,4}^{-1}A_{2,4}^2A_{1,4}^{-1}
\end{aligned}$$

by equations (39), (40) and (41). But $(A_{1,4}, A_{2,4})$ is a basis of $\pi_1(\mathbb{S}^2\setminus\{x_1,x_2,x_3\},x_4)$, so $z^3 \neq 1$, and since $z^3 \in \langle A_{1,4}, A_{2,4}\rangle$, it is of infinite order. We conclude that $L \cong \mathcal{Q}_8 \rtimes_3 \mathbb{Z}$, which completes the proof in this case.
(c) The existence of subgroups of $B_4(\mathbb{S}^2)$ isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ that are non maximal as virtually cyclic subgroups is actually a consequence of the structure of the amalgamated product. Indeed, consider the following short exact sequence

$$1 \longrightarrow \mathcal{Q}_8 \longrightarrow \mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16} \xrightarrow{p} \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow 1.$$

Now $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to the infinite dihedral group $\mathrm{Dih}_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$. Then for all $n \in \mathbb{N}$, $n \geqslant 2$, the subgroup $n\mathbb{Z} \rtimes \mathbb{Z}$ is abstractly isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_2$ while being a proper subgroup (in other words, it is non co-Hopfian). Thus $p^{-1}(n\mathbb{Z} \rtimes \mathbb{Z})$ is isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ while being a proper subgroup (of index n). In particular, since $B_4(\mathbb{S}^2)$ contains a subgroup Γ that is isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, Γ admits proper subgroups that are also isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, and any one of these subgroups is a non-maximal virtually cyclic subgroup that is isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$. Conversely, let G be a subgroup of $B_4(\mathbb{S}^2)$ that is isomorphic to $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$. By Proposition 31, G is a contained in a subgroup M of $B_4(\mathbb{S}^2)$ that is maximal as a virtually cyclic subgroup. But Theorem 28 implies that the only isomorphism class of infinite virtually cyclic subgroups of $B_4(\mathbb{S}^2)$ that contains $\mathcal{Q}_{16}$ is $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, and so we conclude that $M \cong \mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, which completes the proof. □

We now complete the proof of Theorem 30(c) by proving the existence of maximal virtually cyclic subgroups of $B_4(\mathbb{S}^2)$ that are isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$.

Proposition 35. *The group $B_4(\mathbb{S}^2)$ contains maximal virtually cyclic subgroups that are isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$.*

Proof. As before, let Q denote the normal subgroup $\langle \alpha_0^2, \Delta_4 \rangle$ of $B_4(\mathbb{S}^2)$, and let $H_1 = \langle \Delta_4 \rangle$, $H_2 = \langle \alpha_0^2 \rangle$ and $H_3 = \langle \alpha_0^2 \Delta_4 \rangle$ be the three subgroups of Q isomorphic to $\mathbb{Z}_4$. Then $B_4(\mathbb{S}^2)$ acts transitively on the set $\mathcal{H} = \{H_1, H_2, H_3\}$ by conjugation, and this action gives rise to the permutation representation $\psi\colon\ B_4(\mathbb{S}^2) \longrightarrow S_3$ that satisfies the following relation:

$$\text{for all } 1 \leqslant i, j \leqslant 3, \text{ and for all } \beta \in B_4(\mathbb{S}^2),\ \big(\beta H_i \beta^{-1} = H_j\big) \iff \big(\psi(\beta)(i) = j\big).$$

Note that the homomorphism ψ is surjective, that $\psi(\sigma_1) = (1,2)$ by equation (32), and that $\psi(\sigma_2) = (2,3)$ by equation (33). Since $\sigma_1 \sigma_3^{-1} \in Q$ by equation (26), and the action of the elements of Q on $\mathcal{H}$ is trivial, it follows that $\psi(\sigma_3) = \psi(\sigma_1)$. If β is of infinite order then $\langle Q \cup \{\beta\} \rangle \cong \mathcal{Q}_8 \rtimes \mathbb{Z}$, and the order of the action of $\mathbb{Z}$ on $\mathcal{Q}_8$ is that of the element $\psi(\beta)$. The first step is to describe $\mathrm{Ker}\,(\psi)$ whose elements of infinite order will give rise to subgroups of $B_4(\mathbb{S}^2)$ isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$.

Lemma 36. *$\mathrm{Ker}\,(\psi)$ is isomorphic to the direct product of Q with a free group $\mathbb{F}_2(x,y)$ of rank 2, for which a basis (x,y) is given by:*

$$x = \alpha_0^2 \Delta_4 \sigma_1^2 \text{ and } y = \Delta_4 \sigma_2^2. \tag{42}$$

Proof of Lemma 36. By Remarks 26(b) and Proposition 27, $B_4(\mathbb{S}^2) \cong \mathrm{T}^* *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, where the T^*-factor G_1 of $B_4(\mathbb{S}^2)$ is generated by Q and α_1^2, and the $\mathcal{Q}_{16}$-factor G_2 of $B_4(\mathbb{S}^2)$ is generated by Q and α_0, so $G_1 \cap G_2 = Q$. Consider the canonical projection

$$\rho\colon\ B_4(\mathbb{S}^2) \longrightarrow B_4(\mathbb{S}^2)/Q.$$

As in the proof of Proposition 27, we identify the quotient $B_4(\mathbb{S}^2)/Q$ with the free product $\mathbb{Z}_3 * \mathbb{Z}_2$, the $\mathbb{Z}_3$- (resp. $\mathbb{Z}_2$-) factor being generated by $a = \rho(\alpha_1)$ (resp. $b = \rho(\alpha_0)$). Consider the surjective homomorphism $\widehat{\psi}\colon\ \mathbb{Z}_3 * \mathbb{Z}_2 \longrightarrow S_3$ defined by $\widehat{\psi}(a) = (1,3,2)$ and $\widehat{\psi}(b) = (1,3)$. Since $\psi(\alpha_0) = \psi(\sigma_1\sigma_2\sigma_3) = (1,2)(2,3)(1,2) = (1,3)$, $\psi(\alpha_1) = \psi(\sigma_1\sigma_2\sigma_3^2) = (1,2)(2,3) = (1,3,2)$ and $B_4(\mathbb{S}^2) = \langle \alpha_0, \alpha_1 \rangle$ by [GG1, Theorem 3], it follows that $\widehat{\psi} \circ \rho = \psi$, so ρ induces a homomorphism $\widehat{\rho}\colon\ \mathrm{Ker}\,(\psi) \longrightarrow \mathrm{Ker}\big(\widehat{\psi}\big)$ of the respective kernels. We thus obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Ker}\,(\widehat{\rho}) & \longrightarrow & Q & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathrm{Ker}\,(\psi) & \longrightarrow & B_4(\mathbb{S}^2) & \xrightarrow{\psi} & S_3 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \widehat{\rho}} & & \downarrow{\scriptstyle \rho} & & \| & & \\
1 & \longrightarrow & \mathrm{Ker}\big(\widehat{\psi}\big) & \longrightarrow & \mathbb{Z}_3 * \mathbb{Z}_2 & \xrightarrow{\widehat{\psi}} & S_3 & \longrightarrow & 1, \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{43}$$

as well as the equality $\mathrm{Ker}\,(\widehat{\rho}) = Q$. Taking the Schreier transversal $\{1, a, a^2, b, ab, a^2b\}$ for $\widehat{\psi}$ and applying the Reidemeister-Schreier rewriting process [J], we see that $\mathrm{Ker}\big(\widehat{\psi}\big)$ is a free group of rank 2 with basis $\big((ab)^2, (ba)^2\big)$, which implies that $\mathrm{Ker}\,(\psi) \cong \mathcal{Q}_8 \rtimes \mathbb{F}_2$ by the commutative diagram (43). To determine the action of $\mathrm{Ker}\big(\widehat{\psi}\big)$ on Q, note by equations (32) and (33) that σ_1^2 and σ_2^2 belong to $\mathrm{Ker}\,(\psi)$, and that:

$$\rho(\sigma_1^2) = \rho(\sigma_3^2) = \big(\rho(\alpha_0^{-1}\alpha_1)\big)^2 = (ba)^2$$
$$\rho(\sigma_2^2) = \rho(\alpha_0\sigma_1^2\alpha_0^{-1}) = (ab)^2,$$

so $\widehat{\rho}(\sigma_1^2) = (ba)^2$ and $\widehat{\rho}(\sigma_2^2) = (ab)^2$. The same equations imply that the actions by conjugation of σ_1^2 and σ_2^2 on Q yield elements of $\mathrm{Inn}\,(Q)$, namely conjugation by $\alpha_0^2\Delta_4$ and by Δ_4 respectively. Let $s\colon\ \mathrm{Ker}\big(\widehat{\psi}\big) \longrightarrow \mathrm{Ker}\,(\psi)$ be the section defined on the basis of $\mathrm{Ker}\big(\widehat{\psi}\big)$ by $s\left((ba)^2\right) = x$ and $s\left((ab)^2\right) = y$. The action of these two elements on Q is thus trivial, which shows that $\mathrm{Ker}\,(\psi) \cong \mathcal{Q}_8 \times \mathbb{F}_2$ as required. □

Using the definition of ψ, a transversal of $\mathrm{Ker}\,(\psi)$ in $B_4(\mathbb{S}^2)$ is seen to be:

$$\mathcal{T} = \{e, \sigma_1, \sigma_2, \sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2, \sigma_2\sigma_1\}\,. \tag{44}$$

Before going any further with the proof of Proposition 35, we determine the action by conjugation of these coset representatives on x and y.

Lemma 37. *Let $\tau \in \mathcal{T}\setminus\{e\}$. Then*

$$\tau x\tau^{-1} = \begin{cases} x & \text{if } \tau = \sigma_1 \\ \Delta_4^2x^{-1}y^{-1} & \text{if } \tau = \sigma_2 \\ \Delta_4^2y & \text{if } \tau = \sigma_1\sigma_2\sigma_1 \\ \Delta_4^2y & \text{if } \tau = \sigma_1\sigma_2 \\ \Delta_4^2x^{-1}y^{-1} & \text{if } \tau = \sigma_2\sigma_1 \end{cases} \quad \text{and} \quad \tau y\tau^{-1} = \begin{cases} y^{-1}x^{-1} & \text{if } \tau = \sigma_1 \\ y & \text{if } \tau = \sigma_2 \\ \Delta_4^2x & \text{if } \tau = \sigma_1\sigma_2\sigma_1 \\ y^{-1}x^{-1} & \text{if } \tau = \sigma_1\sigma_2 \\ \Delta_4^2x & \text{if } \tau = \sigma_2\sigma_1. \end{cases}$$

Proof of Lemma 37. The action by conjugation of σ_1 and σ_2 on σ_1^2 and σ_2^2 is given by:

$$\begin{aligned}
\sigma_1\sigma_1^2\sigma_1^{-1} &= \sigma_1^2 \quad \text{and} \quad \sigma_2\sigma_2^2\sigma_2^{-1} = \sigma_2^2 \\
\sigma_1\sigma_2^2\sigma_1^{-1} &= \sigma_2^{-1}\sigma_1^2\sigma_2 = \sigma_2^{-2}.\,\sigma_2\sigma_1^2\sigma_2 = \sigma_2^{-2}\sigma_3^{-2} \quad \text{by equation (20)} \\
&= \sigma_2^{-2}\alpha_0^2\sigma_1^{-2}\alpha_0^{-2} \quad \text{by equation (23)} \\
&= \sigma_2^{-2}\sigma_1^{-2}.\,\sigma_1^2\alpha_0^2\sigma_1^{-2}\alpha_0^{-2} = \Delta_4^2\sigma_2^{-2}\sigma_1^{-2} \quad \text{by equations (25) and (32)} \\
\sigma_2\sigma_1^2\sigma_2^{-1} &= \sigma_2\sigma_1^2\sigma_2^{-1} = \sigma_2\sigma_1^2\sigma_2.\,\sigma_2^{-2} = \Delta_4^2\sigma_1^{-2}\sigma_2^{-2} \quad \text{in a similar manner.}
\end{aligned}$$

Using also equations (32) and (33) as well as the fact that x and y commute with the elements of Q, we see that:

$$\begin{aligned}
\sigma_1x\sigma_1^{-1} &= \sigma_1\alpha_0^2\Delta_4\sigma_1^2\sigma_1^{-1} = \alpha_0^2\Delta_4\sigma_1^2 = x \\
\sigma_1y\sigma_1^{-1} &= \sigma_1\Delta_4\sigma_2^2\sigma_1^{-1} = \alpha_0^2\Delta_4^2\sigma_2^{-2}\sigma_1^{-2} = \alpha_0^2\Delta_4^2y^{-1}\Delta_4x^{-1}\alpha_0^2\Delta_4 = y^{-1}x^{-1} \\
\sigma_2x\sigma_2^{-1} &= \sigma_2\alpha_0^2\Delta_4\sigma_1^2\sigma_2^{-1} = \alpha_0^2\Delta_4^2\sigma_1^{-2}\sigma_2^{-2} = \alpha_0^2\Delta_4^2x^{-1}\alpha_0^2\Delta_4y^{-1}\Delta_4 = \Delta_4^2x^{-1}y^{-1} \\
\sigma_2y\sigma_2^{-1} &= \sigma_2\Delta_4\sigma_2^2\sigma_2^{-1} = \alpha_0^2\Delta_4^2\sigma_2^{-2}\sigma_1^{-2} = \Delta_4^2\sigma_2^{-2} = y,
\end{aligned}$$

from which we deduce that:

$$(\sigma_1\sigma_2\sigma_1)x(\sigma_1\sigma_2\sigma_1)^{-1} = \Delta_4^2 y, \quad (\sigma_1\sigma_2)x\sigma_2^{-1}\sigma_1^{-1} = \Delta_4^2 y, \qquad (\sigma_2\sigma_1)x\sigma_1^{-1}\sigma_2^{-1} = \Delta_4^2 x^{-1}y^{-1},$$
$$(\sigma_1\sigma_2\sigma_1)y(\sigma_1\sigma_2\sigma_1)^{-1} = \Delta_4^2 x, \quad (\sigma_1\sigma_2)y\sigma_2^{-1}\sigma_1^{-1} = y^{-1}x^{-1}, \quad (\sigma_2\sigma_1)y\sigma_1^{-1}\sigma_2^{-1} = \Delta_4^2 x,$$

as required. □

We continue with the proof of Proposition 35. Let $z \in B_4(\mathbb{S}^2)$ be an element of infinite order, and suppose that $\Gamma = \langle Q \cup \{z\}\rangle \cong \mathcal{Q}_8 \rtimes_j \mathbb{Z}$, where $j \in \{2,3\}$. Our aim is to obtain necessary conditions on the generators of the infinite cyclic factor of those subgroups of Γ that are isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$. Thus will enable us to construct subgroups of $B_4(\mathbb{S}^2)$ that are isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$ but are not contained in any subgroup isomorphic to $\mathcal{Q}_8 \rtimes_j \mathbb{Z}$, where $j \in \{2,3\}$. With this in mind, let Δ be a subgroup of Γ that is isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$. Since the finite-order elements of Γ are precisely the elements of Q, the subgroup of Δ that is isomorphic to $\mathcal{Q}_8$ is Q. The remaining elements of Γ, of the form $q.z^k$, where $q \in Q$ and $k \in \mathbb{Z}\setminus\{0\}$, are of infinite order. In order that such an element belong to the centraliser of Q (and thus form a subgroup isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$), the fact that the action of z on Q is of order j implies that k must be a multiple of j, and thus $\Delta = \langle Q \cup \{q.z^{\lambda j}\}\rangle = \langle Q \cup \{z^{\lambda j}\}\rangle \subset \langle Q \cup \{z^j\}\rangle$ for some $\lambda \in \mathbb{Z}\setminus\{0\}$. In particular, $\langle Q \cup \{z^j\}\rangle$ is the maximal subgroup of Γ that is isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$.

Since the action by conjugation of z on Q is of order j, it follows from the definition of ψ that z belongs to one of the cosets $\tau.\mathrm{Ker}\,(\psi)$ of $B_4(\mathbb{S}^2)$ where $\tau \in \mathcal{T}\setminus\{e\}$, $\mathcal{T}$ being the transversal of equation (44). More precisely, $z \in \tau.\mathrm{Ker}\,(\psi)$, where $\tau \in \{\sigma_1, \sigma_2, \sigma_1\sigma_2\sigma_1\}$ if $j = 2$, and $\tau \in \{\sigma_1\sigma_2, \sigma_2\sigma_1\}$ if $j = 3$. Further, by Lemma 36 there exist $v \in \mathrm{Ker}\,(\psi)$, $u \in \mathbb{F}_2(x,y)$ and $q_1 \in Q$ such that $z = \tau v$ and $v = q_1 u$. Let us write $u = u(x,y)$ as a freely reduced word in $\mathbb{F}_2(x,y)$:

$$u = x^{\varepsilon_1}y^{\delta_1}\cdots x^{\varepsilon_r}y^{\delta_r},$$

where $\varepsilon_i, \delta_i \in \mathbb{Z}$ for all $i = 1,\ldots,r$, and $\delta_1, \varepsilon_2, \ldots, \delta_{r-1}, \varepsilon_r$ are non-zero. If $v \in \mathrm{Ker}\,(\psi)$, let $\overline{v}$ denote the image of v under projection onto the $\mathbb{F}_2(x,y)$-factor, followed by Abelianisation of $\mathbb{F}_2(x,y)$. We now compute $\overline{z^j}$. We have that:

$$z^j = (\tau q_1 u)^j = \begin{cases} \underbrace{\tau q_1 \tau^{-1}}_{\in Q}\,\underbrace{\tau u \tau q_1 \tau^{-1} u^{-1}}_{\in Q}(\tau u)^2 & \text{if } j = 2 \\ \underbrace{\tau q_1 \tau^{-1}}_{\in Q}\,\underbrace{\tau u \tau q_1 \tau^{-1} u^{-1} \tau^{-1}}_{\in Q}\,\underbrace{\tau u \tau u \tau q_1 \tau^{-1} u^{-1} \tau^{-1} u^{-1} \tau^{-1}}_{\in Q}(\tau u)^3 & \text{if } j = 3 \end{cases}$$
$$= q'(\tau u)^j, \quad \text{where } q' \in Q.$$

Now

$$(\tau u)^j = \begin{cases} \tau u \tau^{-1}.\tau^2.u & \text{if } j = 2 \\ (\tau u \tau^{-1})(\tau^2 u \tau^{-2}).\tau^3.u & \text{if } j = 3. \end{cases}$$

Applying Lemma 37, and using equation (42) as well as the fact that x and y commute with the elements of Q, it follows that there exists $q'' \in Q$ such that $(\tau u)^j = q'^{-1}q''w$,

where $w \in \mathbb{F}_2(x,y)$ is given by:

$$\begin{cases} x^{\varepsilon_1}(y^{-1}x^{-1})^{\delta_1}\cdots x^{\varepsilon_r}(y^{-1}x^{-1})^{\delta_r}xx^{\varepsilon_1}y^{\delta_1}\cdots x^{\varepsilon_r}y^{\delta_r} & \text{if } \tau=\sigma_1 \\ (x^{-1}y^{-1})^{\varepsilon_1}y^{\delta_1}\cdots(x^{-1}y^{-1})^{\varepsilon_r}y^{\delta_r}yx^{\varepsilon_1}y^{\delta_1}\cdots x^{\varepsilon_r}y^{\delta_r} & \text{if } \tau=\sigma_2 \\ y^{\varepsilon_1}x^{\delta_1}\cdots y^{\varepsilon_r}x^{\delta_r}x^{\varepsilon_1}y^{\delta_1}\cdots x^{\varepsilon_r}y^{\delta_r} & \text{if } \tau=\sigma_1\sigma_2\sigma_1 \\ y^{\varepsilon_1}(y^{-1}x^{-1})^{\delta_1}\cdots y^{\varepsilon_r}(y^{-1}x^{-1})^{\delta_r}(y^{-1}x^{-1})^{\varepsilon_1}x^{\delta_1}\cdots(y^{-1}x^{-1})^{\varepsilon_r}x^{\delta_r}x^{\varepsilon_1}y^{\delta_1}\cdots x^{\varepsilon_r}y^{\delta_r} & \text{if } \tau=\sigma_1\sigma_2 \\ (x^{-1}y^{-1})^{\varepsilon_1}x^{\delta_1}\cdots(x^{-1}y^{-1})^{\varepsilon_r}x^{\delta_r}y^{\varepsilon_1}(x^{-1}y^{-1})^{\delta_1}\cdots y^{\varepsilon_r}(x^{-1}y^{-1})^{\delta_r}x^{\varepsilon_1}y^{\delta_1}\cdots x^{\varepsilon_r}y^{\delta_r} & \text{if } \tau=\sigma_2\sigma_1. \end{cases}$$

We have also used the fact that:

$$(\sigma_1\sigma_2\sigma_1)^2=(\sigma_1\sigma_2)^3=(\sigma_2\sigma_1)^3=(\sigma_1\sigma_3^{-1})^2=\Delta_4^2 \quad \text{by equations (20), (25) and (26).}$$

Since $z^j = q''w$, and q'' commutes with w, relative to the basis $(\overline{x},\overline{y})$ of the Abelianisation $\mathbb{Z}^2$ of $\mathbb{F}_2(x,y)$, we obtain:

$$\overline{z^{\lambda j}} = \begin{cases} \lambda.\left(2(\varepsilon_1+\cdots+\varepsilon_r)-(\delta_1+\cdots+\delta_r)+1,0\right) & \text{if } \tau=\sigma_1 \\ \lambda.\left(0,2(\delta_1+\cdots+\delta_r)-(\varepsilon_1+\cdots+\varepsilon_r)+1\right) & \text{if } \tau=\sigma_2 \\ \lambda(\varepsilon_1+\cdots+\varepsilon_r+\delta_1+\cdots+\delta_r).(1,1) & \text{if } \tau=\sigma_1\sigma_2\sigma_1 \\ (0,0) & \text{if } \tau=\sigma_1\sigma_2 \text{ or } \tau=\sigma_2\sigma_1, \end{cases}$$

for all $\lambda \in \mathbb{Z}\setminus\{0\}$. We conclude that if Δ is a subgroup of $\Gamma = \langle Q \cup \{z\}\rangle \cong \mathcal{Q}_8 \rtimes_j \mathbb{Z}$ that is isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$ then $\Delta = \left\langle Q \cup \left\{q.z^{\lambda j}\right\}\right\rangle$, where $\overline{z^{\lambda j}} \in \left\{(a,b)\in\mathbb{Z}^2 \;\middle|\; ab(a-b)=0\right\}$ relative to the basis $(\overline{x},\overline{y})$ of the Abelianisation $\mathbb{Z}^2$ of $\mathbb{F}_2(x,y)$.

To complete the proof of Proposition 35, we shall exhibit an element $w \in B_4(\mathbb{S}^2)$ for which:

(a) w is a non-trivial element of $\mathbb{F}_2(x,y)$ such that $\overline{w}=(c,d)$, where $cd(c-d)\neq 0$.
(b) $\pi(w) \notin \langle\overline{3}\rangle$.

Since $\mathbb{F}_2(x,y) \subset \mathrm{Ker}\,(\psi)$, the first condition implies that such an element w is a suitable generator of the $\mathbb{Z}$-factor of a subgroup of $B_4(\mathbb{S}^2)$ that is isomorphic to $\mathcal{Q}_8\times\mathbb{Z}$, but which from the above discussion, is not contained in any subgroup that is isomorphic to $\mathcal{Q}_8\rtimes_j\mathbb{Z}$ for $j\in\{2,3\}$. By Lemma 33(b), the second condition implies that $\langle Q\cup\{w\}\rangle$ is not contained in any subgroup of $B_4(\mathbb{S}^2)$ that is isomorphic to $\mathcal{Q}_{16}*_{\mathcal{Q}_8}\mathcal{Q}_{16}$, $\mathcal{Q}_8*_{\mathbb{Z}_4}\mathcal{Q}_8$, $\mathbb{Z}_8*_{\mathbb{Z}_4}\mathbb{Z}_8$ or $\mathcal{Q}_8*_{\mathbb{Z}_4}\mathbb{Z}_8$.

Take $w=xy^3$, and let $\Delta=\langle Q\cup\{w\}\rangle$. Then $\Delta\cong\mathcal{Q}_8\times\mathbb{Z}$ since $w\in\mathrm{Ker}\,(\psi)$ is an element of infinite order, and by Proposition 31, there exists a maximal infinite virtually cyclic subgroup M of $B_4(\mathbb{S}^2)$ that contains Δ. Clearly condition (a) above holds, and equation (42) implies that condition (b) is also satisfied. It follows from the previous paragraph and Theorem 28(b) that $M\cong\mathcal{Q}_8\times\mathbb{Z}$, which completes the proof of Proposition 35. In conjunction with Proposition 34, this also proves parts (c) and (d) of Theorem 30. □

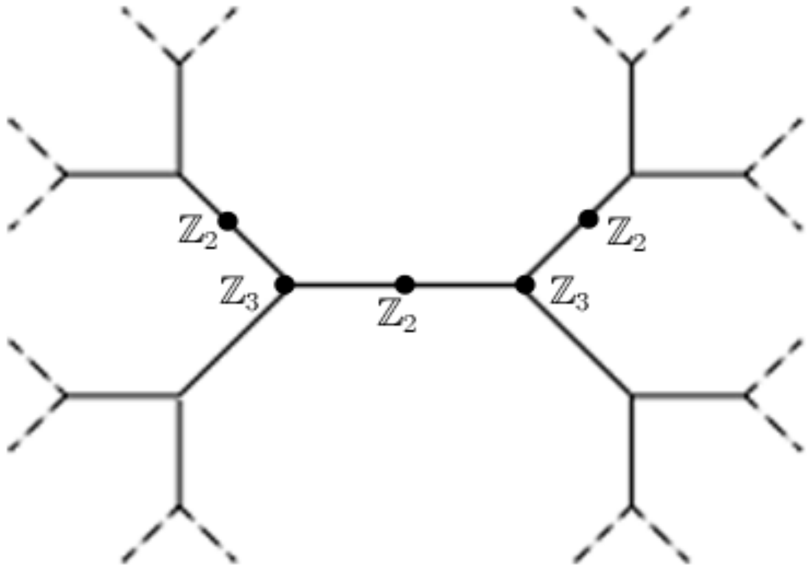

Figure 1: The tree T, showing the edge and vertex stabilisers under the action of $\mathrm{PSL}(2,\mathbb{Z})$.

7.4 Conjugacy classes of maximal infinite virtually cyclic subgroups in $B_4(\mathbb{S}^2)$

In order to determine the number of conjugacy classes of maximal infinite virtually cyclic subgroups, we follow the procedure given in [JLMP, Section 2.5] based on the action of $B_4(\mathbb{S}^2)$ on a suitable tree. Using the proof of Proposition 27, we identify $B_4(\mathbb{S}^2)$ with $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathrm{T}^*$, and the quotient group $B_4(\mathbb{S}^2)/Q$ with the modular group $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = b^3 = 1 \rangle$. Thus we have the following short exact sequence:

$$1 \longrightarrow Q \longrightarrow B_4(\mathbb{S}^2) \xrightarrow{\rho} \mathbb{Z}_2 * \mathbb{Z}_3 \longrightarrow 1, \tag{45}$$

ρ being the quotient map as in the proof of Proposition 27. There is a well-known action of $\mathrm{PSL}(2,\mathbb{Z})$ on the tree T of Figure 1, the edge and vertex stabilisers being $\mathbb{Z}_2$ and $\mathbb{Z}_3$. The quotient of T by this action is the graph:

$$\mathbb{Z}_3 \bullet\!\!\longrightarrow\!\!\bullet\, \mathbb{Z}_2.$$

It follows from the short exact sequence (45) that $B_4(\mathbb{S}^2)$ acts on T via ρ, and since $\mathrm{Ker}\,(\rho) \cong \mathcal{Q}_8$, the quotient graph of this action is:

$$\mathcal{Q}_{16} \bullet\!\!\underset{\mathcal{Q}_8}{\longrightarrow}\!\!\bullet\, \mathrm{T}^*\,.$$

We now apply the Reidemeister-Schreier rewriting process to the Abelianisation homomorphism $\widetilde{\pi}\colon \mathbb{Z}_2 * \mathbb{Z}_3 \longrightarrow \mathbb{Z}_6$. A computation similar to that given in the proof of

Lemma 36 shows that $\Gamma_2(\mathbb{Z}_2 * \mathbb{Z}_3)$ is a free group, that we denote by $\mathbf{F}_2$, of rank two with basis $\big([a,b],[a,b^2]\big)$.

Proposition 38. *Let $\widetilde{\mathbf{F}} = \rho^{-1}(\mathbf{F}_2)$. Then there exists a (free) subgroup $\widetilde{\mathbf{F}}_k$ of $\widetilde{\mathbf{F}}$ of rank $k \geqslant 2$ that is normal and of finite index in $B_4(\mathbb{S}^2)$.*

Remark 39. The above construction gives rise to the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Q & \longrightarrow & \widetilde{\mathbf{F}} & \xrightarrow{\rho|_{\widetilde{\mathbf{F}}}} & \mathbf{F}_2 & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Q & \longrightarrow & B_4(\mathbb{S}^2) & \xrightarrow{\rho} & \mathbb{Z}_2 * \mathbb{Z}_3 & \longrightarrow & 1. \\
 & & & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} & & \\
 & & & & \mathbb{Z}_6 & = & \mathbb{Z}_6 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

We see that $\Gamma_2(B_4(\mathbb{S}^2)) = \widetilde{\mathbf{F}} \cong \mathcal{Q}_8 \rtimes \mathbf{F}_2$, which yields an alternative proof of the decomposition given in [GG4, Theorem 1.3(3)].

Proof of Proposition 38. By Remark 39, $\widetilde{\mathbf{F}}$ is isomorphic to a semi-direct product of the form $\mathcal{Q}_8 \rtimes \mathbf{F}_2$. Let $s\colon\ \mathbf{F}_2 \longrightarrow \widetilde{\mathbf{F}}$ be a section for $\rho\,\big|_{\widetilde{\mathbf{F}}}$. Since $s(\mathbf{F}_2)$ is of finite index in $B_4(\mathbb{S}^2)$, it suffices to take $\widetilde{\mathbf{F}}_k$ to be the intersection of the conjugates of $s(\mathbf{F}_2)$ in $B_4(\mathbb{S}^2)$. □

The group $\mathbf{F}_2$ acts freely on T, the resulting quotient space being a graph Γ_1 that is homotopy equivalent to a wedge of two circles. The group $\widetilde{\mathbf{F}}_k$ also acts freely on T in the same way as its image $\rho(\widetilde{\mathbf{F}}_k)$ in $\mathbb{Z}_2 * \mathbb{Z}_3$, the quotient graph $\Gamma = T/\widetilde{\mathbf{F}}_k$ being a finite-sheeted covering space of Γ_1.

According to [JLMP, Section 2.5], in order to determine the number of conjugacy classes of the maximal infinite virtually cyclic subgroups of $B_4(\mathbb{S}^2)$, we need to analyse the closed geodesics in the graph Γ and their stabilisers under the action of $B_4(\mathbb{S}^2)/\widetilde{\mathbf{F}}_k$. In fact, from the results in the previous sections, we need only study the closed geodesics that are fixed pointwise by the image of Q in the quotient $B_4(\mathbb{S}^2)/\widetilde{\mathbf{F}}_k$. The pointwise fixed graph of the image of Q is all of Γ and its fundamental group has rank at least 2. It follows that there are infinitely many conjugacy classes of maximal infinite virtually cyclic subgroups of the form $\mathcal{Q}_8 \rtimes_j \mathbb{Z}$ for $j \in \{1,2,3\}$ and of $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$.

7.5 Nil group computations

In this section, we compute the Bass Nil groups $NK_i(\mathbb{Z}[\mathcal{Q}_8])$ for $i = 0, 1$, as well as the twisted versions. In the non-twisted case, we show that:

Proposition 40. *For $i = 0, 1$, the groups $NK_i(\mathbb{Z}[\mathcal{Q}_8])$ are isomorphic to a countably-infinite direct sum of copies of $\mathbb{Z}_2$.*

In order to prove this proposition, we shall first consider the ring R of Lipschitz quaternions of the form $a + bi + cj + dk$ where $a, b, c, d \in \mathbb{Z}$, and show that its Nil groups are trivial. Recall that the ring S of Hurwitz quaternions consists of the quaternions of the form $a + bi + cj + dk$ where a, b, c, d are either all integers or all half integers. It is known that S is a non-commutative integral domain [Ch], and so is a regular ring. In this section, we denote the cyclic group of order n by C_n.

Theorem 41. *For $i = 0, 1, 2$, $NK_i(R) = 0$.*

Proof. Clearly $R \subset S$. By [Ch], $M = (1 + i)R$ is the R-conductor of R in S, and both R/M and S/M are fields. Consider the following commutative square:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/M & \longrightarrow & S/M, \end{array}$$

The associated Mayer-Vietoris sequence and the long exact sequence of the pairs (S, M) and (R, M) yield $NK_i(R) \cong NK_i(S) = 0$ for $i = 0, 1, 2$. □

We now prove Proposition 40.

Proof of Proposition 40. We first consider the case $i = 0$. Let $\mathcal{Q}_8$ have the presentation:

$$\mathcal{Q}_8 = \left\langle x, y \;\middle|\; x^2 = y^2, yxy^{-1} = x^{-1} \right\rangle.$$

The group ring $\mathbb{Z}[\mathcal{Q}_8]$ fits in a Cartesian square (see [CR2, Theorem 50.31, page 266]):

$$\begin{array}{ccc} \mathbb{Z}[\mathcal{Q}_8] & \xrightarrow{f} & \mathbb{Z}[C_2 \times C_2] \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p} \\ R & \longrightarrow & \mathbb{F}_2[C_2 \times C_2], \end{array} \tag{46}$$

where g is defined on the generators of $\mathcal{Q}_8$ by $g(x) = i$ and $g(y) = j$, and f is induced from the homomorphism $\mathcal{Q}_8 \longrightarrow C_2 \times C_2$ given by taking the quotient of $\mathcal{Q}_8$ by its centre. This Cartesian square gives rise to the following Mayer-Vietoris sequence:

$$NK_2(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow NK_2(R) \oplus NK_2(\mathbb{Z}[C_2 \times C_2]) \longrightarrow NK_2(\mathbb{F}_2[C_2 \times C_2]) \longrightarrow NK_1(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow \cdots.$$

By [LO1, Lemmas 5.3 and 5.4] and [We, Theorem 1.3], we have:

$$\begin{aligned} &NK_1(\mathbb{Z}[C_2\times C_2]) \cong \Omega_{\mathbb{F}_2} \cong \mathbb{F}_2[x]\,\mathrm{d}x, && NK_0(\mathbb{Z}[C_2\times C_2]) \cong V = x\mathbb{F}_2[x] \\ &NK_1(\mathbb{F}_2[C_2]) \cong (1+x\varepsilon\mathbb{F}_2[x])^\times \cong V, \quad \text{and} && NK_0(\mathbb{F}_2[C_2]) = 0. \end{aligned} \tag{47}$$

As Abelian groups, $\Omega_{\mathbb{F}_2}$ and V are both countable infinite direct sums of copies of $\mathbb{Z}_2$. On the other hand, the ring R has trivial Nil groups by Theorem 41. The above Mayer-Vietoris sequence thus reduces to:

$$\begin{gathered} NK_2(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow NK_2(\mathbb{Z}[C_2\times C_2]) \longrightarrow NK_2(\mathbb{F}_2[C_2\times C_2]) \xrightarrow{\delta} NK_1(\mathbb{Z}[\mathcal{Q}_8]) \xrightarrow{f_{*1}} \\ NK_1(\mathbb{Z}[C_2\times C_2]) \xrightarrow{p_*} NK_1(\mathbb{F}_2[C_2\times C_2]) \xrightarrow{\tau} NK_0(\mathbb{Z}[\mathcal{Q}_8]) \xrightarrow{f_{*0}} \\ NK_0(\mathbb{Z}[C_2\times C_2]) \longrightarrow NK_0(\mathbb{F}_2[C_2\times C_2]) \longrightarrow 0 \end{gathered} \tag{48}$$

(the labelled homomorphisms are discussed below). The components of the homomorphism $p_*\colon\ NK_1(\mathbb{Z}[C_2\times C_2]) \longrightarrow NK_1(\mathbb{F}_2[C_2\times C_2])$ are induced by the projections

$$\begin{array}{ccc} \mathbb{Z}[C_2\times C_2] & \xrightarrow{\quad p_i \quad} & \mathbb{F}_2[C_2\times C_2] \\ & \searrow \quad \nearrow & \\ & \mathbb{Z}[C_2] & \end{array} \tag{49}$$

where for $i = 1,2,3$, $p_i\colon\ C_2\times C_2 \longrightarrow C_2$ are the non-trivial homomorphisms of $C_2\times C_2$ defined by $p_1((1,0)) = p_2((0,1)) = p_3((1,1)) = 0$. The commutative diagram (49) thus induces the following commutative diagram:

$$\begin{array}{ccc} NK_1(\mathbb{Z}[C_2\times C_2]) & \xrightarrow{\quad p_{i*} \quad} & NK_1(\mathbb{F}_2[C_2\times C_2]) \\ & \searrow \quad \nearrow & \\ & NK_1(\mathbb{Z}[C_2]). & \end{array}$$

But since $NK_1(\mathbb{Z}[C_2]) = 0$ [Har], it follows that $p_* = 0$, and the homomorphism

$$\tau\colon\ NK_1(\mathbb{F}_2[C_2\times C_2]) \longrightarrow NK_0(\mathbb{Z}[\mathcal{Q}_8])$$

is a monomorphism. On the other hand, we claim that the homomorphism

$$f_{*0}\colon\ NK_0(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow NK_0(\mathbb{Z}[C_2\times C_2])$$

is zero. To see this, let $w \in NK_0(\mathbb{Z}[\mathcal{Q}_8])$. Then $f_{*0}(w) \in NK_0(\mathbb{Z}[C_2\times C_2]) \cong NK_1(\mathbb{F}_2[\varepsilon])$. Identifying $NK_1(\mathbb{F}_2[\varepsilon])$ with the corresponding summand in $NK_1(\mathbb{F}_2[C_2\times C_2]) \cong V^3$ given by $(1+\overline{v}x\mathbb{F}[x])^\times$ of $NK_1(\mathbb{F}_2[\varepsilon])$ (see [We, Remark before Theorem 1.3]), we have that $\tau(w') = w$ for some $w' \in NK_1(\mathbb{F}_2[\varepsilon])$. Exactness of the sequence (48) implies that $f_{*0}(w) = 0$, so τ is an isomorphism, and hence $NK_0(\mathbb{Z}[\mathcal{Q}_8]) \cong \bigoplus_\infty \mathbb{Z}_2$, a countably-infinite direct sum of copies of $\mathbb{Z}_2$.

We now turn to the case $i = 1$. Consider the following homomorphism:

$$f_{*1}\colon\ NK_1(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow NK_1(\mathbb{Z}[C_2 \times C_2]).$$

Exactness of the sequence (48) and the fact that τ is a monomorphism implies that f_{*1} is surjective, which gives rise to the following short exact sequence:

$$0 \longrightarrow \mathrm{Im}(\delta) \longrightarrow NK_1(\mathbb{Z}[\mathcal{Q}_8]) \xrightarrow{f_{*1}} NK_1(\mathbb{Z}[C_2 \times C_2]) \longrightarrow 0,$$

where $\delta\colon\ NK_2(\mathbb{Z}[C_2 \times C_2]) \longrightarrow NK_1(\mathbb{Z}[\mathcal{Q}_8])$ is the homomorphism that appears in the sequence (48). Recall from (47) that

$$NK_1(\mathbb{Z}[C_2 \times C_2]) \cong \Omega_{\mathbb{F}_2[x]} = \mathrm{Coker}\left(NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{F}_2[C_2])\right).$$

It follows that elements of $NK_1(\mathbb{Z}[C_2 \times C_2])$ lift to $NK_1(\mathbb{Z}[\mathcal{Q}_8])$. This therefore defines a section for f_{*1} and hence:

$$NK_1(\mathbb{Z}[\mathcal{Q}_8]) \cong NK_1(\mathbb{Z}[C_2 \times C_2]) \oplus \mathrm{Im}(\delta).$$

Since $\mathrm{Im}(\delta)$ is an Abelian group of exponent 2, it follows that $NK_1(\mathbb{Z}[\mathcal{Q}_8]) \cong \bigoplus_{\infty} \mathbb{Z}_2$. □

Remark 42. In the rest of the manuscript, we shall write $NK_i^{\alpha}(\mathbb{Z}[\mathcal{Q}_8]) = NK_i(\mathbb{Z}[\mathcal{Q}_8], \alpha)$.

In order to complete the proof of Theorem 1, it remains to determine the twisted Nil groups of $\mathcal{Q}_8$. Recall from Section 7.3 that up to isomorphism, there are two non-trivial semi-direct products of the form $\mathcal{Q}_8 \rtimes \mathbb{Z}$, namely $\mathcal{Q}_8 \rtimes_j \mathbb{Z}$, where $j \in \{2, 3\}$.

Proposition 43. *Let $i \in \{0, 1\}$.*
(a) For the action α of $\mathbb{Z}$ on $\mathcal{Q}_8$ of order 3,

$$NK_i^{\alpha}(\mathbb{Z}[\mathcal{Q}_8]) \cong NK_i^{\alpha^{-1}}(\mathbb{Z}[\mathcal{Q}_8]) \cong NK_i(\mathbb{Z}[\mathcal{Q}_8]) \cong \bigoplus_{\infty} \mathbb{Z}_2. \tag{50}$$

(b) For the action α of $\mathbb{Z}$ on $\mathcal{Q}_8$ of order 2, the twisted Nil groups are infinitely-generated Abelian groups of exponent 2 or 4.

Proof.
(a) Since the action of $\mathbb{Z}$ on $\mathcal{Q}_8$ is of order three, there is a surjective homomorphism $\varphi\colon\ \mathcal{Q}_8 \rtimes_3 \mathbb{Z} \twoheadrightarrow \mathcal{Q}_8 \rtimes \mathbb{Z}_3$ onto the group $\mathcal{Q}_8 \rtimes \mathbb{Z}_3 \cong \mathrm{T}^*$ defined by taking the $\mathbb{Z}$-factor modulo 3. We use the technique of induction on hyperelementary subgroups [FH, proof of Theorem 3.2] that asserts that:

$$K_i(\mathbb{Z}[\mathcal{Q}_8 \rtimes_3 \mathbb{Z}]) \cong \lim_{H \in \mathrm{Hyp}} K_i(\mathbb{Z}[\varphi^{-1}(H)]),$$

where Hyp denotes the set of hyperelementary subgroups of $\mathcal{Q}_8 \rtimes \mathbb{Z}_3$, and the limit is with respect to the morphisms induced by conjugation and inclusion in the category Hyp. Using the proof of Proposition 14, we see that the hyperelementary subgroups

of $\mathcal{Q}_8 \rtimes \mathbb{Z}_3$ are isomorphic to one of $\mathbb{Z}_6, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_4$ or $\mathcal{Q}_8$, and their inverse images by φ are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}, \mathbb{Z}, \mathbb{Z}_2 \times \mathbb{Z}, \mathbb{Z}_4 \times \mathbb{Z}$ and $\mathcal{Q}_8 \times \mathbb{Z}$ respectively. With the exception of the last two, the corresponding group rings of these groups have trivial Nil groups. Further, the subgroups of $\mathcal{Q}_8 \rtimes_3 \mathbb{Z}$ that are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}$ are pairwise conjugate, and there is only one maximal element of the form $\mathcal{Q}_8 \times \mathbb{Z}$ in the limit. We thus obtain equation (50) using Proposition 40.

(b) Consider the action α of $\mathbb{Z}$ on $\mathcal{Q}_8$ that is of order 2 given by exchanging the generators x and y of $\mathcal{Q}_8$. Comparing with the Cartesian square (46), we observe that this action may be transposed in all the rings of (46), thus giving rise to the following Cartesian square of twisted polynomial rings:

$$\begin{array}{ccc} \mathbb{Z}[\mathcal{Q}_8]_\alpha[t] & \xrightarrow{f} & \mathbb{Z}[C_2 \times C_2]_\alpha[t] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ R_\alpha[t] & \longrightarrow & \mathbb{F}_2[C_2 \times C_2]_\alpha[t], \end{array}$$

where the induced action of α exchanges the generators in all group rings, and exchanges i and j in R. By [F, Theorem 1.6], the Farrell-Hsiang Nil groups of R also vanish, and hence we obtain the following long exact sequence:

$$\begin{gathered} NK_2^\alpha(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow NK_2^\alpha(\mathbb{Z}[C_2 \times C_2]) \longrightarrow NK_2^\alpha(\mathbb{F}_2[C_2 \times C_2]) \xrightarrow{\delta} NK_1^\alpha(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow \\ NK_1^\alpha(\mathbb{Z}[C_2 \times C_2]) \xrightarrow{p_*} NK_1^\alpha(\mathbb{F}_2[C_2 \times C_2]) \longrightarrow NK_0^\alpha(\mathbb{Z}[\mathcal{Q}_8]) \longrightarrow \\ NK_0^\alpha(\mathbb{Z}[C_2 \times C_2]) \longrightarrow NK_0^\alpha(\mathbb{F}_2[C_2 \times C_2]) \longrightarrow 0, \end{gathered}$$

where the labelled homomorphisms will be analysed in the rest of the proof. By an argument similar to that given in the proof of Proposition 40, we observe that the homomorphism $p_*\colon\ NK_1^\alpha(\mathbb{Z}[C_2 \times C_2]) \longrightarrow NK_1^\alpha(\mathbb{F}_2[C_2 \times C_2])$ is zero, and thus the homomorphism $NK_1^\alpha(\mathbb{F}_2[C_2 \times C_2]) \hookrightarrow NK_0^\alpha(\mathbb{Z}[\mathcal{Q}_8])$ is injective by exactness. We now apply the following proposition.

Proposition 44 ([R]). *Let $i_*\colon\ K_i(R_\alpha[t^2]) \longrightarrow K_i(R_\alpha[t])$ and $i^*\colon\ K_i(R_\alpha[t]) \longrightarrow K_i(R_\alpha[t^2])$ be the induction and transfer homomorphisms respectively. Then $i^*i_*(x) = x + \alpha(x)$ for $i = 1, 2$.*

The element $x = 1 + \bar{u}t^2$ belongs to the Nil part of $K_1(\mathbb{F}_2[C_2 \times C_2]_\alpha[t^2])$, is non trivial, and $i^*i_*(x) = 1 + (\bar{u} + \bar{v})t^2$, which is also non trivial in the Nil part of $K_1(\mathbb{F}_2[C_2 \times C_2]_\alpha[t^2])$. But the restriction of i^*i_* to the Nil part factors through $NK_1^\alpha(\mathbb{F}_2[C_2 \times C_2])$, hence this latter group is non trivial, and since it injects into $NK_0^\alpha(\mathbb{Z}[\mathcal{Q}_8])$, it follows that $NK_0^\alpha(\mathbb{Z}[\mathcal{Q}_8])$ is non trivial as well.

We now analyse the homomorphism $\delta\colon\ NK_2^\alpha(\mathbb{F}_2[C_2 \times C_2]) \longrightarrow NK_1^\alpha(\mathbb{Z}[\mathcal{Q}_8])$ that appears in the above exact sequence. Consider the non-trivial elements (as Dennis-Stein symbols: see, for example [KS]) $\langle x^{2n}\bar{u}\bar{v}, \bar{v}\rangle \in NK_2^\alpha(\mathbb{F}_2[C_2 \times C_2])$. Since these elements do not lift to $NK_2(\mathbb{Z}[C_2 \times C_2])$ and $i^*i_*\left(\langle x^{2n}\bar{u}\bar{v}, \bar{v}\rangle\right) = \langle x^{2n}\bar{u}\bar{v}, \bar{v}\rangle\langle x^{2n}\bar{u}\bar{v}, \bar{u}\rangle$, which is also non trivial in the Nil part of $K_2(\mathbb{F}_2[C_2 \times C_2]_\alpha[t^2])$, it follows that

$$0 \neq \delta\left(\langle x^{2n}\bar{u}\bar{v}, \bar{v}\rangle\langle x^{2n}\bar{u}\bar{v}, \bar{u}\rangle\right) \in NK_1^\alpha(\mathbb{Z}[\mathcal{Q}_8]).$$

Thus $NK_0^\alpha(\mathbb{Z}[\mathcal{Q}_8])$ and $NK_1^\alpha(\mathbb{Z}[\mathcal{Q}_8])$ are non-trivial infinitely-generated Abelian groups, of exponent 2 or 4. This proves Proposition 43. □

In summary, Propositions 40 and 43 give us the Nil_i summands in equation (27), and the decompositions in the statement of Theorem 1 then follow. This completes the proof of that theorem.

Appendix: The Fibred Isomorphism Conjecture

The setup

Let $\mathcal{S}\colon$ TOP $\longrightarrow \Omega$-SPECTRA be a covariant homotopy functor. Let $\mathbf{F}$ be the category of continuous surjective maps: objects in $\mathbf{F}$ are continuous surjective maps $p\colon E \longrightarrow B$, where E, B are objects in TOP, and morphisms between pairs of maps $p_1\colon E_1 \longrightarrow B_1$ and $p_2\colon E_2 \longrightarrow B_2$ consist of continuous maps $f\colon E_1 \longrightarrow E_2$ and $g\colon B_1 \longrightarrow B_2$ that make the following diagram commute:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{g} & B_2. \end{array} \tag{51}$$

Within this framework, Quinn constructs a functor from $\mathbf{F}$ to Ω-SPECTRA [Q]. The value of this Ω-spectrum at the object $(p\colon E \longrightarrow B)$ is denoted by $\mathbb{H}(B; \mathcal{S}(p))$. The value at the object $(E \longrightarrow *)$ is $\mathcal{S}(E)$. The map of spectra $\mathbb{A}\colon \mathbb{H}(B_1; \mathcal{S}(p_1)) \longrightarrow \mathbb{H}(B_2; \mathcal{S}(p_2))$ associated to the commutative diagram (51) is known as the *Quinn* assembly map. Other ingredients for the Fibred Isomorphism Conjecture may be found in [FJ1].

The conjecture

Given a discrete group Γ, let $E_{\mathcal{VC}}\Gamma$ be a universal Γ-space for the family of virtually cyclic subgroups of Γ, let $\mathcal{B}_{\mathcal{VC}}\Gamma$ denote the orbit space $E_{\mathcal{VC}}\Gamma/\Gamma$, and let X be a space on which Γ acts freely and properly discontinuously. If (f, g) is the following morphism in $\mathbf{F}$:

$$\begin{array}{ccc} E_{\mathcal{VC}}\Gamma \times_\Gamma X & \xrightarrow{f} & X/\Gamma \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathcal{B}_{\mathcal{VC}}\Gamma & \xrightarrow{g} & * \end{array}$$

then the *Fibred Isomorphism Conjecture* for the functor $\mathcal{S}$ and the group Γ is the assertion that

$$\mathbb{A}\colon \mathbb{H}(\mathcal{B}_{\mathcal{VC}}\Gamma; \mathcal{S}(p_1)) \longrightarrow \mathcal{S}(X/\Gamma)$$

is a homotopy equivalence, and hence the induced map

$$\mathbb{A}_*\colon \pi_n(\mathbb{H}(\mathcal{B}_{\mathcal{VC}}\Gamma; \mathcal{S}(p_1)) \longrightarrow \pi_n(\mathcal{S}(X/\Gamma))$$

is an isomorphism for all $n \in \mathbb{Z}$. This conjecture was stated in [FJ1] for the functors $\mathcal{S} = \mathcal{P}_*(\cdot)$, $\mathcal{K}(\cdot)$ and $\mathcal{L}^{-\infty}$, the pseudoisotopy, algebraic K-theory and $\mathcal{L}^{-\infty}$-theory functors respectively. In this paper, we use the functor $\mathcal{S} = \mathcal{P}_*(\cdot)$. The relation between $\mathcal{P}_*(\cdot)$ and lower algebraic K-theory is described in the work of Anderson and Hsiang, who proved the following [AH]:

$$\pi_j(\mathcal{P}_*(\mathbf{X})) = \begin{cases} \mathrm{Wh}(\mathbb{Z}\pi_1(\mathbf{X})) & \text{if } j = -1 \\ \widetilde{K}_0(\mathbb{Z}\pi_1(\mathbf{X})) & \text{if } j = -2 \\ K_{j+2}(\mathbb{Z}\pi_1(\mathbf{X})) & \text{if } j \leqslant -3. \end{cases}$$

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