A nonparametric stochastic model for non-Gaussian random fields with SO(n,R)-invariance

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ABSTRACT

In this work, we address the construction of a generalized nonparametric probabilistic model for matrix-valued non-Gaussian random fields exhibiting some invariance properties with respect to given orthogonal transformations. Such an issue naturally arises in linear elasticity, where the so-called material symmetries are defined in terms of some $\mathrm{SO}(n, \mathbb{R})$-invariance. Hereinafter, we denote as $\mathbf{x} \mapsto [\mathbf{C}(\mathbf{x})]$ the elasticity tensor random field, the probability model of which must be constructed. It is defined on a given probability space, indexed by a bounded domain $\Omega \subset \mathbb{R}^d$ and takes its values in a given subset $\mathbb{M}_n^{\mathrm{sym}}(\mathbb{R})$ of the set $\mathbb{M}_n^+(\mathbb{R})$ of all the positive-definite symmetric $(n \times n)$ real matrices. First, we present a probabilistic model, the construction of which relies on the overall methodology introduced in [2] and which extends the results derived in [4] for random elasticity matrices. Specifically, the approach introduces a particular algebraic form for the random field $\mathbf{x} \mapsto [\mathbf{C}(\mathbf{x})]$ and involves two additional random fields $\mathbf{x} \mapsto [\mathbf{S}(\mathbf{x})]$ and $\mathbf{x} \mapsto [\mathbf{A}(\mathbf{x})]$, indexed by Ω and taking their values in $\mathbb{M}_n^{\mathrm{sym}}(\mathbb{R})$ and $\mathbb{M}_n^+(\mathbb{R})$ respectively, such that

$$[\mathbf{C}(\mathbf{x})] = [\mathbf{S}(\mathbf{x})]\, [\mathbf{A}(\mathbf{x})]\, [\mathbf{S}(\mathbf{x})] \tag{1}$$

for all $\mathbf{x}$ in Ω. The representation is therefore based on two independent sources of uncertainties, one preserving almost surely the topological structure in $\mathbb{M}_n^{\mathrm{sym}}(\mathbb{R})$ (namely, the random field $\mathbf{x} \mapsto [\mathbf{S}(\mathbf{x})]$) and the other one acting as an anisotropic stochastic germ. The family of first-order marginal probability distributions is prescribed through a nonlinear mapping acting on a set of normalized Gaussian random fields, the definition of which is obtained by having recourse to the maximum entropy principle [1]. In practice, such a representation allows the level of statistical fluctuations of the random field to be uncoupled from the level of fluctuations associated with a stochastic measure of anisotropy. Next we propose a novel numerical strategy for the random generation of the random field. The algorithm consists in solving a family of Itô stochastic differential equations (see [3]) and turns out to be very efficient as the stochastic dimension increases. It is worth noting that unlike the alternative polynomial-chaos based strategy, which typically involves identification procedures (for the coefficients) based on matching the first-order marginal distributions, the algorithm allows for the preservation of the statistical dependence - between the components of the simulated random variables - at no additional computational cost. A new discretization scheme is also proposed. Finally, the approach and algorithms are exemplified by considering the classes of isotropic and transversely isotropic tensors. For instance, the graphs of the first-order marginal probability density functions of components C_{11} and C_{12} at an arbitrary point are displayed below and shows the nongaussianity of the modeled random field.

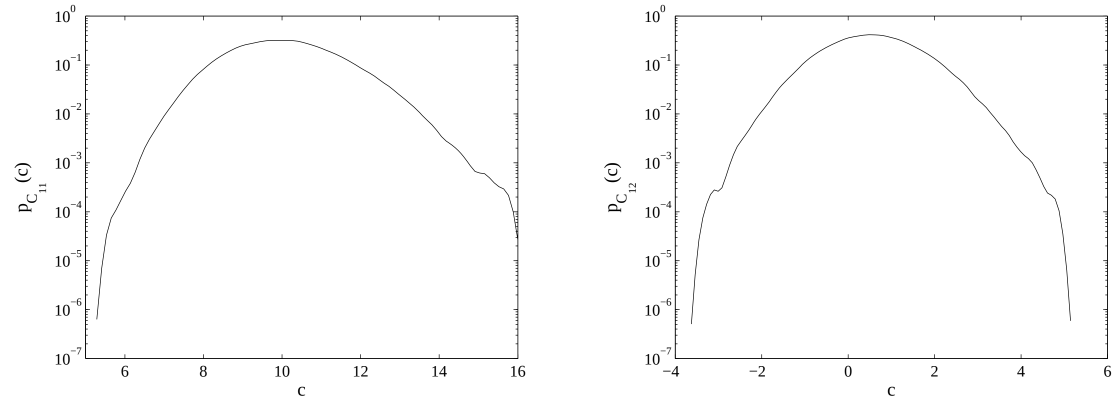

Figure 1: Graph of the probability density function $c \mapsto p_{C_{11}}(c)$ (left) and $c \mapsto p_{C_{12}}(c)$ (right).

References

[1] E.T. Jaynes: Information theory and statistical mechanics. *Phys. Rev.*, 106/108 (1957), 620–630/171–190.

[2] C. Soize: Non-Gaussian positive-definite matrix-valued random fields for elliptic stochastic partial differential operators. *Computer methods in applied mechanics and engineering*, 195 (2006), 26–64.

[3] C. Soize: Construction of probability distributions in high dimension using the maximum entropy principle: Applications to stochastic processes, random fields and random matrices. *International Journal for Numerical Methods in Engineering*, 76 (2008), 1583–1611.

[4] J. Guilleminot, C. Soize: Generalized Stochastic Approach for Constitutive Equation in Linear Elasticity: A Random Matrix Model. *International Journal for Numerical Methods in Engineering*, (2011), accepted for publication on September 18 2011.