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# Functional co-monotony of processes with an application to peacocks

GILLES PAGÈS \*

## Abstract

We show that several general classes of stochastic processes satisfy a functional co-monotony principle, including processes with independent increments, Brownian diffusions, Liouville processes. As a first application, we recover some recent results about peacock processes obtained by Hirsch et al. in [11] (see also [3]) which were themselves motivated by a former work of Carr et al. in [4] about the sensitivity of Asian options with respect to their volatility and residual maturity (seniority).

**Keywords** Co-monotony; antithetic simulation method; processes with independent increments; Liouville processes; fractional Brownian motion; Asian options, sensitivity.

## 1 Introduction

The aim of this paper is to show that the classical co-monotony principle for real-valued random variables also holds for large classes of stochastic processes like Brownian diffusion processes, Processes with independent increments, Liouville processes, fractional Brownian processes, etc, if one considers the natural partial order on the space of real-valued functions defined on an interval. real-valued functions. We also provide few examples of application, with a special emphasis on peacocks (English quasi-acronym for “processus croissants pour l’ordre convexe”) inspired by recent works by Hirsch et al. in [11], which find themselves their original motivation in [4] by Carr et al. about the sensitivities of Asian options in a Black-Scholes Model.

The starting point of what can be called *co-monotony principle* finds its origin in the following classical proposition dealing with one-dimensional real-valued random variables.

**Proposition 1.1** (One dimensional co-monotony principle). *Let  $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathbb{R}$  be a random variable and let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be two monotone functions sharing the same (opposite) monotony property.*

(a) *If  $f(X), g(X), f(X)g(X) \in L^1(\mathbb{P})$ , then  $\text{Cov}(f(X), g(X)) \geq 0$  ( $\leq 0$  respectively) i.e.*

$$\mathbb{E} f(X)g(X) \geq \mathbb{E} f(X)\mathbb{E} g(X) \quad (\leq \text{ respectively}).$$

*Furthermore, the inequality holds in a strict sense if and only if  $f(X)$  or  $g(X)$  is not  $\mathbb{P}$ -a.s. constant.*

(b) *If  $f$  and  $g$  have the same constant sign, then integrability is no longer requested. As a consequence, if  $f$  and  $g$  have opposite monotony, then*

$$\|f(X)g(X)\|_1 = \mathbb{E} f(X)g(X) \leq \mathbb{E} f(X)\mathbb{E} g(X) = \|f(X)\|_1 \|g(X)\|_1.$$

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The inequalities are straightforward consequences of Fubini's Theorem applied on  $(\mathbb{R} \times \mathbb{R}, \mathcal{B}or(\mathbb{R})^{\otimes 2}, \mathbb{P}_X^{\otimes 2})$  to the function  $(x, x') \mapsto (f(x) - f(x'))(g(x) - g(x'))$  where  $\mathbb{P}_X$  denotes the distribution of  $X$ .

Typical applications of this scalar co-monotony principle are, among others, the antithetic simulation method for variance reduction and more recently *a priori* sign results for the sensitivity of derivatives in Finance.

▷ *Antithetic simulation.* Let  $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathbb{R}$  be a random variable and let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a non-increasing function such that  $X \stackrel{\mathcal{L}}{\sim} \varphi(X)$ . Then for every monotone function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(X) \in L^2(\mathbb{P})$  and  $\mathbb{P}(f(X) \neq \mathbb{E}f(X)) > 0$ , we have

$$\text{Var}\left(\frac{f(X) + f \circ \varphi(X)}{2}\right) = \frac{2(\text{Var}(f(X)) + \text{Cov}(f(X), f \circ \varphi(X)))}{4} < \frac{\text{Var}(f(X))}{2}$$

since  $\text{Cov}(f(X), f \circ \varphi(X)) < 0$ .

The variance is reduced by more than a 2 factor whereas the complexity of the simulation of  $\frac{f(X) + f \circ \varphi(X)}{2}$  is only twice higher than that of  $f(X)$  (if one neglects the additional cost of the computation of  $\varphi(x)$  compared to that of  $x$ ).

▷ *Sensitivity (vega of an option).* Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be a convex function with (at most) polynomial growth at 0 and  $+\infty$  in the sense that there exists a real constant  $C > 0$  such that

$$\forall x \in (0, +\infty), \quad |f(x)| \leq C(x^r + x^{-r})$$

and let  $Z : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathbb{R}$  be an  $\mathcal{N}(0; 1)$ -distributed random variable. Set for every  $\sigma > 0$

$$\varphi(\sigma) = \mathbb{E} f\left(e^{\sigma Z - \frac{\sigma^2}{2}}\right).$$

Although it does not appear as a straightforward consequence of its definition, one easily derives from the above proposition that  $\varphi$  is a non-decreasing function of  $\sigma$  on  $(0, \infty)$ . In fact  $f$  is differentiable outside an at most countable subset of  $(0, +\infty)$  (where its right and left derivatives differ) and its derivative  $f'$  is non-decreasing, with polynomial growth as well since

$$|f'(x)| \leq \max(|f(x+1) - f(x)|, 2x^{-1}|f(x) - f(x/2)|), \quad x \in (0, +\infty).$$

Since  $Z$  has no atom, one easily checks that one can interchange derivative and expectation to establish that  $\varphi$  is differentiable with derivative

$$\varphi'(\sigma) = \mathbb{E}\left(f'\left(e^{\sigma Z - \frac{\sigma^2}{2}}\right)e^{\sigma Z - \frac{\sigma^2}{2}}(Z - \sigma)\right), \quad \sigma > 0.$$

A Cameron-Martin change of variable then yields

$$\varphi'(\sigma) = \mathbb{E}\left(f'\left(e^{\sigma Z + \frac{\sigma^2}{2}}\right)Z\right)$$

so that, applying the co-monotony principle to the two non-decreasing (square integrable) functions  $z \mapsto f'\left(e^{\sigma z - \frac{\sigma^2}{2}}\right)$  and  $z \mapsto z$ , implies

$$\varphi'(\sigma) \geq \mathbb{E}\left(f'\left(e^{\sigma Z + \frac{\sigma^2}{2}}\right)\right)\mathbb{E}(Z) = \mathbb{E}\left(f'\left(e^{\sigma Z + \frac{\sigma^2}{2}}\right)\right) \times 0 = 0.$$

Extensions of the above co-monotony principle to functions on  $\mathbb{R}^d$ ,  $d \geq 2$ , are almost as classical as the one dimensional case. They can be established by induction when both functions  $f(x_1, \dots, x_d)$  and

$g(x_1, \dots, x_d)$  defined on  $\mathbb{R}^d$  are co-monotone in each variable  $x_i$  (i.e. having the same or an opposite monotony property not depending on  $i$ ) and when the  $\mathbb{R}^d$ -valued random vector  $X$  has independent marginals.

Our aim in this paper is to show that this co-monotony principle can be again extended into a *functional co-monotony principle* satisfied by various classes of stochastic processes  $X = (X_t)_{t \in [0, T]}$  whose paths lie in a sub-space  $E$  of the vector space  $\mathcal{F}([0, T], \mathbb{R})$  of real valued functions defined on the interval  $[0, T]$ ,  $T > 0$ , equipped with the pointwise (*partial*) order on functions, defined by

$$\forall \alpha, \beta \in \mathcal{F}([0, T], \mathbb{R}), \alpha \leq \beta \text{ if } \forall t \in [0, T], \alpha(t) \leq \beta(t).$$

Then a functional  $F : E \rightarrow \mathbb{R}$  is said to be non-decreasing if

$$\forall \alpha, \beta \in E, \quad \alpha \leq \beta \implies F(\alpha) \leq F(\beta).$$

The choice of  $E$  will be motivated by the pathwise regularity of the process  $X$ . The space  $E$  will also be endowed with a metric topology (and its Borel  $\sigma$ -field) so that  $X$  can be seen as an  $E$ -valued random vector. The functionals  $F$  and  $G$  involved in the co-monotony principle will be assumed to be continuous on  $E$  (at least  $\mathbb{P}_X$ -a.s.). Typical choices for  $E$  will be  $E = \mathcal{C}([0, T], \mathbb{R})$ ,  $\mathcal{C}([0, T], \mathbb{R}^d)$ ,  $\mathcal{ID}([0, T], \mathbb{R})$  or  $\mathcal{ID}([0, T], \mathbb{R}^d)$  and occasionally  $L_{\mathbb{R}^d}^p([0, T], dt)$  (in this case we will switch to the *dt*-a.e. pointwise order instead of the pointwise order). Then by co-monotony principle, we mean that for every non-decreasing functionals  $F$  and  $G$  defined on  $E$ ,  $\mathbb{P}_X$ -a.s. continuous,

$$\mathbb{E} F(X)G(X) \geq \mathbb{E} F(X)\mathbb{E} G(X).$$

(The case of non-increasing functionals follows by considering the opposite functionals and the opposite monotony case by considering the opposite of only one of the functionals). Among the (classes of) processes of interest, we will consider continuous Gaussian processes with nonnegative covariance function (like the standard and the fractional Brownian motion, “nonnegative” Liouville processes), the Markov processes with monotony preserving transitions (which includes of course Brownian diffusions), processes with independent increments, etc.

The main problem comes from the fact that the naive point wise order on functional spaces is not total so that the formal one-dimensional proof based on Fubini’s theorem no longer applies.

As applications of such functional results, we will be able to extend the above sign property for the *vega* of a “vanilla” option (whose payoff function is a function of the risky asset  $S_T$  at the maturity  $T$ ) to “exotic” options. By “exotic”, we classically mean that their payoff is typically a path-dependent functional  $F((S_t)_{t \in [0, T]})$  of the risky asset  $(S_t)_{t \in [0, T]}$ . The dynamics of this risky asset is still a Black-Scholes model where  $S_t^\sigma = s_0 e^{\sigma W_t + (r - \frac{\sigma^2}{2})t}$ ,  $s_0 > 0$ . Doing so we will retrieve Carr et al. results about the sensitivity of Asian type options in a Black-Scholes model with respect to the volatility (see [4]). We will also emphasize the close connection between co-monotony and the theory of *peacocks* <sup>(1)</sup> characterized by Kellerer in [17] and recently put back into light in the book [11] (see also the references therein). Let us briefly recall that an integrable process  $(X_\lambda)_{\lambda \geq 0}$  is a peacock if for every convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\mathbb{E}|f(X_\lambda)| < +\infty$ ,  $\lambda \geq 0$ , the function  $\lambda \mapsto \mathbb{E} f(X_\lambda)$  is non-decreasing. Kellerer’s characterization theorem says that a process is a peacock if and only if there exists a martingale  $(M_\lambda)_{\lambda \geq 0}$  such that  $X_\lambda \stackrel{\mathcal{L}}{\sim} M_\lambda$ ,  $\lambda \in \mathbb{R}_+$ . Moreover, the process  $(M_\lambda)_{\lambda \geq 0}$  can be chosen to be Markovian. This proof being non-constructive, it does not help at all establishing whether or not a process is a peacock. See also a new proof of Kellerer’s Theorem due to Hirsch and

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<sup>1</sup>stands for the French acronym PCOC (Processus Croissant pour l’Ordre Convexe).

Roynette in [13]. By contrast, one can find in [11] a huge number of peacocks with an explicit marginal martingale representation characterized through various tools from stochastic processes theory.

More generally, when applied in its “opposite” version, the co-monotony principle between non-negative function simply provides a significant improvement of the Hölder inequality since it makes the  $L^1$ -norm sub-multiplicative. It can be used to produce less conservative bounds in various fields of applied probability, like recently in [18, 25] where to provide bounds depending on functionals of a Brownian diffusion process, in the spirit of the inequalities proposed in Section 5.3 for barrier options.

The paper is organized as follows: Section 2 is devoted to the finite-dimensional co-monotony principle, Section 3 to the functional co-monotony principle for continuous processes, Section 4 to càdlàg processes. Section 5 deals with examples of applications, to peacocks and to exotic options for which we establish universal bounds (among price dynamics sharing the functional co-monotony principle).

NOTATION: •  $x_{0:n} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ ,  $x_{1:n} = (x_1, \dots, x_n) \in \mathbb{R}^n$ , etc.  $(x|y) = \sum_{0 \leq k \leq n} x_k y_k$  denotes the canonical inner product on  $\mathbb{R}^{n+1}$ .

- We denote by  $\leq$  the componentwise order on  $\mathbb{R}^{n+1}$  defined by  $x_{0:n} \leq x'_{0:n}$  if  $x_i \leq x'_i$ ,  $i = 0, \dots, n$ .
- $\mathcal{M}(d, r)$  denotes the vector space of matrices with  $d$  rows and  $r$  columns.  $M^*$  denotes the transpose of matrix  $M$ .
- $\perp\!\!\!\perp$  will emphasize in formulas the independence between two processes.
- $\|\alpha\|_{\text{sup}} = \sup_{t \in [0, T]} |\alpha(t)|$  for any function  $\alpha : [0, T] \rightarrow \mathbb{R}$ .
- $u_+$  denotes the positive part of the real number  $u$ .  $\lambda_d$  denotes the Lebesgue measure on  $(\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d))$ .
- $X \stackrel{\mathcal{L}}{\sim} \mu$  means that the random vector  $X$  has distribution  $\mu$ .

## 2 Finite-dimensional co-monotony principle

### 2.1 Definition and main results

**Definition 2.1.** (a) Let  $(P(x, dy))_{x \in \mathbb{R}}$  be a transition probability, i.e. a family of probability measures such that for every  $x \in \mathbb{R}$ ,  $P(x, dy)$  is a probability measure on  $(\mathbb{R}, \mathcal{B}or(\mathbb{R}))$  and for every  $B \in \mathcal{B}or(\mathbb{R})$ ,  $x \mapsto P(x, B)$  is a Borel function. The transition  $(P(x, dy))_{x \in \mathbb{R}}$  is monotony preserving if, for every bounded or nonnegative monotone function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the function  $Pf$  defined for every real number  $x \in \mathbb{R}$  by  $Pf(x) = \int f(y)P(x, dy)$  is monotone with the same monotony.

(b) Two Borel functions  $\Phi, \Psi : \mathbb{R}^d \rightarrow \mathbb{R}$  are componentwise co-monotone if, for every  $i \in \{1, \dots, d\}$ , and every  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in \mathbb{R}^{d-1}$ , both section functions  $x_i \mapsto \Phi(x_1, \dots, x_i, \dots, x_d)$  and  $x_i \mapsto \Psi(x_1, \dots, x_i, \dots, x_d)$  have the same monotony not depending on  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$  nor on  $i$ .

(c) If  $\Phi$  and  $-\Psi$  are co-monotone,  $\Phi$  and  $\Psi$  are said to be anti-monotone.

In finite dimension, the main result is the following.

**Proposition 2.1.** (a) Let  $X = (X_k)_{0 \leq k \leq n}$  be an  $\mathbb{R}$ -valued Markov chain defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  having a (regular) version of its transitions

$$P_{k-1, k}(x, dy) = \mathbb{P}(X_k \in dy | X_{k-1} = x), \quad k = 1, \dots, n$$

which are monotony preserving in the above sense. Then, for every pair of co-monotone functions  $\Phi, \Psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  such that  $\Phi(X), \Psi(X), \Phi(X)\Psi(X) \in L^1(\mathbb{P})$ ,

$$\mathbb{E} \Phi(X)\Psi(X) \geq \mathbb{E} \Phi(X)\mathbb{E} \Psi(X). \quad (2.1)$$

If  $\Phi$  and  $\Psi$  are monotone with opposite monotony, the reverse inequality holds. In both cases, if  $\Phi$  and  $\Psi$  both take values in  $\mathbb{R}_+$  or  $\mathbb{R}_-$  then the inequalities remain true (in  $\overline{\mathbb{R}}$ ) without integrability assumption.

(b) If the random variables  $X_0, \dots, X_n$  are independent, the conclusion remains true under the following weak co-monotony assumption: there exists a permutation  $\tau$  of the index set  $\{0, \dots, n\}$  such that for every  $i \in \{0, \dots, n\}$  and every  $(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^n$ ,  $x_i \mapsto \Phi(x_0, \dots, x_i, \dots, x_n)$  and  $x_i \mapsto \Psi(x_0, \dots, x_i, \dots, x_n)$  have the same monotony, possibly depending on  $(x_{\tau(0)}, \dots, x_{\tau(i-1)})$ . Then the same conclusion as in (a) holds true.

**Proof.** (a) One proceeds by induction on  $n \in \mathbb{N}$ . If  $n = 0$ , the result follows from the scalar co-monotony principle applied to  $X_0$  (with distribution  $\mu_0$ ).

(n)  $\implies$  (n+1): Assume  $\Phi(X_{0:n+1})$  and  $\Psi(X_{0:n+1})$  are square integrable. We may assume by changing if necessary the functionals into their opposite that both  $\Phi$  and  $\Psi$  are componentwise non-decreasing. Put  $\mathcal{F}_k^X = \sigma(X_0, \dots, X_k)$ ,  $k = 0, \dots, n$ . It follows from the Markov property that

$$\mathbb{E}\left(\Phi(X_{0:n+1}) \mid \mathcal{F}_n^X\right) = \Phi^{(n)}(X_{0:n})$$

where

$$\Phi^{(n)}(x_{0:n}) = P_{n,n+1}(\Phi(x_{0:n}, \cdot))(x_n).$$

In particular, we have  $\mathbb{E}(\Phi(X_{0:n+1})) = \mathbb{E}(\Phi^{(n)}(X_{0:n}))$ . We have accordingly

$$\mathbb{E}\left(\Phi(X_{0:n+1})\Psi(X_{0:n+1}) \mid \mathcal{F}_n^X\right) = \left(P_{n,n+1}(\Phi(x_{0:n}, \cdot)\Psi(x_{0:n}, \cdot))(x_n)\right)_{|x_k=X_k, k=0, \dots, n}.$$

Applying the one dimensional co-monotony principle with  $P_{n,n+1}(x_n, dy)$  to  $\Phi(x_{0:n}, \cdot)$  and  $\Psi(x_{0:n}, \cdot)$  we get for every  $(x_{0:n}) \in \mathbb{R}^{n+1}$ ,

$$\begin{aligned} P_{n,n+1}(\Phi(x_{0:n}, \cdot)\Psi(x_{0:n}, \cdot))(x_n) &\geq P_{n,n+1}(\Phi(x_{0:n}, \cdot))(x_n)P_{n,n+1}(\Psi(x_{0:n}, \cdot))(x_n) \\ &= \Phi^{(n)}(x_{0:n})\Psi^{(n)}(x_{0:n}) \end{aligned} \tag{2.2}$$

so that, considering  $X_{0:n+1}$  and taking expectation, we get

$$\mathbb{E}\Phi(X_{0:n+1})\Psi(X_{0:n+1}) \geq \mathbb{E}\left(\Phi^{(n)}(X_{0:n})\Psi^{(n)}(X_{0:n})\right).$$

It is clear that for every  $i \in \{0, \dots, n-1\}$ ,  $x_i \mapsto \Phi^{(n)}(x_0, \dots, x_n)$  is non-decreasing since the transition  $P_{n,n+1}$  is a nonnegative operator. Now let  $x_n, x'_n \in \mathbb{R}$ ,  $x_n \leq x'_n$ .

$$P_{n,n+1}(\Phi(x_0, \dots, x_n, \cdot))(x_n) \leq P_{n,n+1}(\Phi(x_0, \dots, x'_n, \cdot))(x_n) \leq P_{n,n+1}(\Phi(x_0, \dots, x'_n, \cdot))(x'_n)$$

where the first inequality follows from the non-negativity of the operator  $P_{n,n+1}$  and the second follows from its monotony preserving property since  $x_{n+1} \mapsto \Phi(x_{0:n+1})$  is non-decreasing. The function  $\Psi^{(n)}$ , defined likewise, shares the same properties.

An induction assumption applied to the Markov chain  $(X_k)_{0 \leq k \leq n}$  completes the proof since

$$\begin{aligned} \mathbb{E}\left(\Phi^{(n)}(X_{0:n})\Psi^{(n)}(X_{0:n})\right) &\geq \mathbb{E}\Phi^{(n)}(X_{0:n})\mathbb{E}\Psi^{(n)}(X_{0:n}) \\ &= \mathbb{E}\Phi(X_{0:n+1})\mathbb{E}\Psi(X_{0:n+1}). \quad \square \end{aligned}$$

(b) By renumbering the  $(n+1)$ -tuple  $(X_0, \dots, X_n)$  we may assume  $\tau = id$ . Then the transition  $P_{k-1,k}(x_{k-1}, dy) = \mathbb{P}_{X_k}(dy)$  does not depend upon  $x_{k-1}$  so that  $P_{k-1,k}f$  is a constant function.

Then (2.2) holds as an equality and the monotony of  $\Phi^{(n)}$  in each of its variable  $x_0, \dots, x_n$  is that of  $\Phi$  for the same variables. A careful inspection of the proof of claim (a) then shows that the weak co-monotony is enough to conclude.  $\square$

As a straightforward consequence of the fact that the functions  $x_{0:n} \mapsto x_k$  and  $x_{0:n} \mapsto x_\ell$  are co-monotone, we derive the following corollary which provides a necessary condition for the co-monotony property.

**Corollary 2.1.** *If  $X = (X_k)_{0 \leq k \leq n}$  is a monotony preserving Markov chain in the sense of (a) in the above proposition and if  $X_k \in L^2$ ,  $k = 0, \dots, n$ , then,*

$$\forall k, \ell \in \{0, \dots, n\}, \quad \text{Cov}(X_k, X_\ell) = \mathbb{E} X_k X_\ell - \mathbb{E} X_k \mathbb{E} X_\ell \geq 0$$

*i.e. the covariance matrix of the chain has nonnegative entries.*

EXAMPLE. Let  $A, B \in \mathcal{B}or(\mathbb{R}^{n+1})$  be two Borel sets such that, for every  $x = x_{0:n} \in A$ ,  $x + te_i \in A$  for every  $t \in \mathbb{R}_+$  and every  $i \in \{0, \dots, n\}$  (where  $e_i$  denotes the  $i^{\text{th}}$  vector of the canonical basis of  $\mathbb{R}^{n+1}$ ), idem for  $B$ . Then for any  $\mathbb{R}^d$ -Markov chain  $X = (X_k)_{0 \leq k \leq n}$ , having monotony preserving transitions (in the sense of Proposition 2.1(a)), we have

$$\mathbb{P}((X_0, \dots, X_n) \in A \cap B) \geq \mathbb{P}((X_0, \dots, X_n) \in A) \mathbb{P}((X_0, \dots, X_n) \in B).$$

The monotony preserving property of the transitions  $P_{k-1,k}$  cannot be relaxed as emphasized by the following easy counter-example.

COUNTER-EXAMPLE. Let  $X = (X_0, X_1)$  be a Gaussian bi-variate random vector with distribution  $\mathcal{N}\left(0; \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$  where the correlation  $\rho \in (-1, 0)$ . One checks that the transition  $P_{0,1}(x_0, dx_1)$  reads on bounded or nonnegative Borel functions

$$P_{0,1}(f)(x_0) := \mathbb{E}(f(X_1) | X_0 = x_0) = \mathbb{E}f\left(\rho x_0 + \sqrt{1 - \rho^2} Z\right), \quad Z \stackrel{\mathcal{L}}{\sim} \mathcal{N}(0; 1).$$

This shows that  $P_{0,1}$  is monotony... *inverting*. In particular we have  $\mathbb{E} X_0 X_1 = \rho < 0 = \mathbb{E} X_0 \mathbb{E} X_1$ . In fact it is clear that  $(X_0, X_1)$  shares the co-monotony principle if and only if  $\rho \geq 0$ . In the next section we extend this result to higher dimensional Gaussian vectors .

## 2.2 More on the Gaussian case.

Let  $X = (X_0, \dots, X_n)$  be a centered Gaussian vector with covariance matrix  $\Sigma = [\sigma_{ij}]_{0 \leq i, j \leq n}$ . This covariance matrix characterizes the distribution of  $\Sigma$  so it characterizes as well whether or not  $X$  shares a co-monotony property in the sense of (2.1). But can we *read easily* this property on  $\Sigma$ ?

As mentioned above, a necessary condition for co-monotony is obviously that

$$\forall i, j \in \{0, \dots, n\}, \quad \sigma_{ij} = \text{Cov}(X_i, X_j) \geq 0.$$

In fact this simple condition does characterize co-monotony: this result, due to L. Pitt, is established in [22].

**Theorem 2.1** (Pitt, 1982). *A Gaussian random vector  $X$  with covariance matrix  $\Sigma = [\sigma_{ij}]_{0 \leq i, j \leq n}$  shares a co-monotony property (2.1) if and only if*

$$\forall i, j \in \{0, \dots, n\}, \quad \sigma_{ij} \geq 0.$$

**Remarks.** • Extensions have been proved in [15]. Typically, if  $Z \sim \mathcal{N}(0; I_{n+1})$ , under appropriate regularity and integrability assumptions on a function  $h : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , one has

$$\left( \forall x \in \mathbb{R}^{n+1}, \frac{\partial^2 h}{\partial x_i \partial x_j}(x) \geq 0 \right) \implies \left( \sigma_{ij} \mapsto \mathbb{E}(h(\sqrt{\Sigma}Z)) \text{ is non-decreasing} \right).$$

• Another natural criterion for co-monotony – theoretically straightforward although not easy to “read” in practice on the covariance matrix itself – is to make the assumption that there exists a matrix  $A = [a_{ij}]_{0 \leq i \leq n, 1 \leq j \leq r}$ ,  $r \in \mathbb{N}^*$ , with *nonnegative entries*  $a_{ij} \geq 0$  such that  $\Sigma = AA^*$ . Then  $X \stackrel{\mathcal{L}}{\sim} AZ$ ,  $Z \stackrel{\mathcal{L}}{\sim} \mathcal{N}(0; I_r)$ . So every component is a linear combination with nonnegative coefficients of the components of  $Z$  and Proposition 2.1(b) straightforwardly implies that  $X$  shares the co-monotony property (2.1).

However, surprisingly, this criterion is not a characterization in general: if  $n + 1 \leq 4$ , symmetric matrices  $\Sigma$  with nonnegative entries can always be decomposed as  $\Sigma = AA^*$  where  $A$  has non negative entries. But if  $n + 1 \geq 5$ , this is no longer true. The negative answer is inspired by a former counter-example – originally due to Horn – when  $n + 1 = r \geq 5$ , reported and justified in [10] (see Equations (15.39) and (15.53) and the lines that follow, see also [6] for an equivalent formulation). To be precise, the nonnegative symmetric  $5 \times 5$  matrix  $\Sigma$  (with rank 4) defined by

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 3/4 & 0 & 1/2 \\ 0 & 3/4 & 1 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 1 & 0 \\ 1/2 & 1/2 & 0 & 0 & 1 \end{bmatrix}$$

has nonnegative entries but cannot be written  $AA^*$  where  $A$  has nonnegative entries. Another reference of interest about this question is [1].

### 2.3 Application to the Euler scheme.

The Euler scheme of a diffusion is an important example of Markov chain to which one may wish to apply the co-monotony principle. Let  $X = (X_t)_{t \in [0, T]}$  be a Brownian diffusion assumed to be solution to the *SDE*

$$dX_t^x = b(t, X_t^x)dt + \sigma(t, X_t^x)dW_t, \quad t \in [0, T], \quad X_0 = x.$$

Its Euler scheme with step  $h = T/n$  and Brownian increments is entirely characterized by its transitions

$$P_{k, k+1}(f)(x) = \mathbb{E} f \left( x + hb(t_k^n, x) + \sigma(t_k^n, x)\sqrt{h}Z \right), \quad Z \stackrel{\mathcal{L}}{\sim} \mathcal{N}(0; 1), \quad k = 0, \dots, n-1,$$

where  $t_k^n = \frac{k}{n}T$ ,  $k = 0, \dots, n$ . One easily checks that if the function  $b$  is Lipschitz continuous in  $x$  uniformly in  $t \in [0, T]$  and if  $\sigma(t, x) = \sigma(t)$  is deterministic and lies in  $L^2([0, T], dt)$ , then, for  $n$  large enough, the Euler transition  $P$  is monotony preserving.

This follows from the fact that  $x \mapsto x + hb(t, x)$  is non-decreasing provided  $h \in (0, \frac{1}{[b]_{\text{Lip}}})$ , where  $[b]_{\text{Lip}}$  is the uniform Lipschitz coefficient of  $b$ .

## 3 Functional co-monotony principle

The aim of this section is to extend the above co-monotony principle to continuous time processes relying on the above multi-dimensional co-monotony result. To do so, we will view processes as

random variables taking values in a path vector subspace  $E \subset \mathcal{F}([0, T], \mathbb{R})$  endowed with the (trace of the) Borel  $\sigma$ -field of pointwise convergence topology on  $\mathcal{F}([0, T], \mathbb{R})$ , namely  $\sigma(\pi_t, t \in [0, T])$  where  $\pi_t(\alpha) = \alpha(t)$ ,  $\alpha \in E$ . Consequently, a process  $X$  having  $E$ -valued paths can be seen as an  $E$ -valued random variable if and only if for every  $t \in [0, T]$ ,  $X_t$  is an  $\mathbb{R}$ -valued random variable (which is in some sense a tautology since it is the lightest definition of a stochastic process).

We consider on  $E$  the (partial) order induced by the natural partial “pointwise order” on  $\mathcal{F}([0, T], \mathbb{R})$  defined by

$$\forall \alpha, \beta \in \mathcal{F}([0, T], \mathbb{R}), \alpha \leq \beta \quad \text{if } \forall t \in [0, T], \alpha(t) \leq \beta(t).$$

**Definition 3.1.** (a) A measurable functional  $F : E \rightarrow \mathbb{R}$  is monotone if it is either non-decreasing or non-increasing with respect to the order on  $E$ .

(b) A pair of measurable functionals are co-monotone if they are both monotone, with the same monotony.

Then, in order to establish a functional co-monotony principle (see the the definition below) our approach will be transfer a finite dimensional co-monotony principle satisfied by appropriate converging (time) discretizations of the process  $X$  of interest. Doing so we will need to equip  $E$  with a topology ensuring the above convergence for the widest class of ( $\mathbb{P}_X$ -a.s. continuous) functionals. That is why we will consider as often as we can the sup-norm topology, not only on  $\mathcal{C}([0, T], \mathbb{R})$  but also on the Skorokhod space  $\mathcal{D}([0, T], \mathbb{R})$  of càdlàg (French acronym for right continuous left limited) functions defined on  $[0, T]$  since there are more continuous functionals for this topology than for the classical Skorokhod (or  $J_1$ ) topology. We recall that  $\mathcal{D}_T := \sigma(\pi_t, t \in [0, T])$  is the Borel  $\sigma$ -field related to both the  $\|\cdot\|_{\text{sup}}$ -norm and the  $J_1$ -topology on the Skorokhod space.

We will also consider (see SectionCadlagMarkov) the space  $L^p([0, T], \mu)$ ,  $0 < p < +\infty$  with its usual  $L^p(\mu)$ -norm where  $\mu$  is a finite measure on  $[0, T]$ ). In the latter case (which is not – strictly speaking – a set of functions), we will consider the “ $\mu$ -a.e.” (partial) order

$$\alpha \leq_{\mu} \beta \quad \text{if } \alpha(t) \leq \beta(t) \quad \mu(dt)\text{-a.e.}$$

A functional  $F$  monotone for the order  $\leq_{\mu}$  is called  $\mu$ -monotone. The definition of  $\mu$ -co-monotony follows likewise.

**Definition 3.2.** A process  $X$  whose paths take values in a normed vector space  $(E, \|\cdot\|_E)$  satisfies a co-monotony principle if, for every bounded, co-monotone,  $\mathbb{P}_X$ -a.s. continuous measurable functionals  $F, G : E \rightarrow \mathbb{R}$ ,

$$\mathbb{E} F(X)G(X) \geq \mathbb{E} F(X)\mathbb{E} G(X).$$

EXTENSIONS. • The extension to square integrable or nonnegative continuous functionals is canonical by a standard truncation procedure: replace  $F$  by  $F_K := (-K) \vee (F \wedge K)$ ,  $K > 0$ , and let  $K$  go to infinity.

• Furthermore, the inequality also holds for pairs of co-monotone functionals  $F, G$  whose truncations  $F_K$  and  $G_K$  are limits in  $L^2(\mathbb{P}_X)$  of  $\mathbb{P}_X$ -a.s. continuous co-monotone functionals.

We will often rely on the following technical lemma.

**Proposition 3.1.** (a) Let  $(X_n)_{n \geq 1}$  be a sequence of independent  $E$ -valued random vectors defined on  $(\Omega, \mathcal{A}, \mathbb{P})$  where  $(E, \|\cdot\|_E, \leq)$  is a partially ordered normed vector space <sup>(2)</sup>. Assume that, for every  $n \geq 1$ ,  $X_n$  satisfies a co-monotony principle i.e. for every pair of  $\mathbb{P}_{X_n}$ -a.s. continuous bounded Borel measurable functionals  $F, G : E \rightarrow \mathbb{R}$

$$\mathbb{E} F(X_n)G(X_n) \geq \mathbb{E} F(X_n)\mathbb{E} G(X_n).$$

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<sup>2</sup>For every  $\alpha, \beta, \gamma \in E$  and every  $\lambda \in \mathbb{R}_+$ ,  $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$  and  $\lambda \geq 0 \Rightarrow \lambda\alpha \leq \lambda\beta$ .

Let  $(a_n)_{n \geq 1}$  be a sequence of real numbers such that the series  $X = \sum_{n \geq 1} a_n X_n$  converges a.s. for the norm  $\|\cdot\|_E$ . Then  $X$  satisfies a co-monotony principle.

(b) Assume furthermore that  $(E, \|\cdot\|_E)$  is a Banach space with an unconditional norm, that the  $X_n$  are nonnegative processes for the order on  $E$  and that  $\sum_{n \geq 1} X_n$  converges in  $L_E^1(\mathbb{P})$ . Then, for every sequence of independent random variables  $(A_n)_{n \geq 1}$  taking values in a fixed compact interval of  $\mathbb{R}$  and independent of  $(X_n)_{n \geq 1}$ , the series  $X = \sum_{n \geq 1} A_n X_n$  satisfies a co-monotony principle in the sense of Definition 3.2.

**Remark.** When  $E = \mathcal{C}([0, T], \mathbb{R})$ , Lévy-Itô-Nisio's Theorem (see e.g. [19], Theorem 6.1, p.151) shows to some extent the equivalence between functional convergence in distribution and a.s. convergence for series of independent processes as above.

**Proof.** We may assume without loss of generality that the two functionals  $F$  and  $G$  are non-decreasing. We first show the result for the sum of two independent processes, i.e. we assume  $a_k = 0$ ,  $k \geq 3$  (and  $a_1 a_2 \neq 0$ ). By Fubini's Theorem

$$\mathbb{E} F(a_1 X_1 + a_2 X_2) G(a_1 X_1 + a_2 X_2) = \mathbb{E} \left( \left[ \mathbb{E} (F(a_1 X_1 + a_2 \alpha) G(a_1 X_1 + a_2 \alpha)) \right]_{|\alpha=X_2} \right).$$

Let  $\text{Cont}(F)$  denote the set of elements of  $E$  at which  $F$  is continuous. It follows, still from Fubini's Theorem, that

$$1 = \mathbb{P}(a_1 X_1 + a_2 X_2 \in \text{Cont}(F)) = \int \mathbb{P}_{X_2}(d\alpha_2) \mathbb{P}_{X_1}(F(a_1 \cdot + a_2 \alpha_2))$$

so that  $\mathbb{P}_{X_2}(d\alpha_2)$ -a.s.  $\alpha_1 \mapsto F(a_1 \alpha_1 + a_2 \alpha_2)$  and  $\alpha_1 \mapsto G(a_1 \alpha_1 + a_2 \alpha_2)$  are  $\mathbb{P}_{X_1}$ -a.s. continuous. Noting that these functionals are co-monotone (non-decreasing if  $a_1 \geq 0$ , non-increasing if  $a_1 \leq 0$ ), this implies

$$\mathbb{E}(F(a_1 X_1 + a_2 \alpha_2) G(a_1 X_1 + a_2 \alpha_2)) \geq \mathbb{E}(F(a_1 X_1 + a_2 \alpha_2)) \mathbb{E}(G(a_1 X_1 + a_2 \alpha_2)).$$

Now, both

$$\alpha_2 \mapsto \mathbb{E} F(a_1 X_1 + a_2 \alpha_2) = \int \mathbb{P}_{X_1}(d\alpha_1) F(a_1 \alpha_1 + a_2 \alpha_2)$$

and

$$\alpha_2 \mapsto \mathbb{E} G(a_1 X_1 + a_2 \alpha_2) = \int \mathbb{P}_{X_1}(d\alpha_1) G(a_1 \alpha_1 + a_2 \alpha_2)$$

are co-monotone (non-decreasing if  $a_2 \geq 0$ , non-increasing if  $a_2 \leq 0$ ) and one checks that both are  $\mathbb{P}_{X_2}(d\alpha_2)$ -a.s. continuous which implies in turn that

$$\begin{aligned} & \mathbb{E} \left( \left[ \mathbb{E} F(a_1 X_1 + a_2 \alpha) \right]_{|\alpha=X_2} \left[ \mathbb{E} G(a_1 X_1 + a_2 \alpha) \right]_{|\alpha=X_2} \right) \\ & \geq \mathbb{E} \left( \left[ \mathbb{E} F(a_1 X_1 + a_2 \alpha) \right]_{|\alpha=X_2} \right) \mathbb{E} \left( \left[ \mathbb{E} G(a_1 X_1 + a_2 \alpha) \right]_{|\alpha=X_2} \right) \\ & = \mathbb{E} F(a_1 X_1 + a_2 X_2) \mathbb{E} G(a_1 X_1 + a_2 X_2) \end{aligned}$$

where we used again Fubini's Theorem in the second line.

One extends this result by induction to the case where  $X = X_1 + \dots + X_n$ .

To make  $n$  go to infinity, we proceed as follows: let  $\mathcal{G}_n = \sigma(X_k, k \geq n+1)$ . By the reverse martingale convergence theorem, we know that for any bounded measurable functional  $\Phi : E \rightarrow \mathbb{R}$ ,

$$\mathbb{E}(\Phi(X) | \mathcal{G}_n) = \left[ \mathbb{E} \Phi(X_1 + \dots + X_n + \tilde{\alpha}_n) \right]_{\tilde{\alpha}_n = \tilde{X}_n}$$

where  $\tilde{X}_n = X - (X_1 + \dots + X_n)$ . We know from the above case  $n = 2$  that, for  $\Phi = F, G \mathbb{P}_{\tilde{X}_n}(d\tilde{\alpha}_n)$ -a.s.,  $\alpha \mapsto \Phi(\alpha + \tilde{\alpha}_n)$  is  $\mathbb{P}_{X_1 + \dots + X_n}(d\alpha)$ -continuous so that

$$\mathbb{E} F G(X_1 + \dots + X_n + \tilde{\alpha}_n) \geq \mathbb{E} F(X_1 + \dots + X_n + \tilde{\alpha}_n) \mathbb{E} G(X_1 + \dots + X_n + \tilde{\alpha}_n).$$

This equality also reads

$$\mathbb{E}(F G(X) | \mathcal{G}_n) \geq \mathbb{E}(F(X) | \mathcal{G}_n) \mathbb{E}(G(X) | \mathcal{G}_n)$$

which in turn implies by letting  $n \rightarrow \infty$

$$\mathbb{E} F(X) G(X) \geq \mathbb{E} F(X) \mathbb{E} G(X)$$

owing to the reverse martingale convergence theorem.

(b) For every bounded sequence  $(a_n)_{n \geq 1}$ , it follows from the unconditionality of the norm  $\|\cdot\|_E$  that  $\sum_{n \geq 1} a_n X_n$  a.s. converges in  $L_E^1(\mathbb{P})$ . Then it follows from (a) that, for every  $n \geq 1$ ,

$$\mathbb{E} \left( F \left( \sum_{k=1}^n a_k X_k \right) G \left( \sum_{k=1}^n a_k X_k \right) \right) \geq \mathbb{E} F \left( \sum_{k=1}^n a_k X_k \right) \mathbb{E} G \left( \sum_{k=1}^n a_k X_k \right).$$

Now for every  $k \in \{1, \dots, n\}$ , the function defined on the real line by  $a_k \mapsto \mathbb{E} F(\sum_{i=1}^n a_i X_i)$  has the same monotony as  $F$  since  $X_k \geq 0$  and is bounded. Consequently for any pair  $F, G$  of bounded co-monotone Borel functionals,

$$\begin{aligned} \mathbb{E} \left( \left[ \mathbb{E} F \left( \sum_{k=1}^n a_k X_k \right) \right]_{|a_{1:n}=A_{1:n}} \left[ \mathbb{E} G \left( \sum_{k=1}^n a_k X_k \right) \right]_{|a_{1:n}=A_{1:n}} \right) \\ \geq \mathbb{E} \left( \left[ \mathbb{E} F \left( \sum_{k=1}^n a_k X_k \right) \right]_{|a_{1:n}=A_{1:n}} \right) \times \mathbb{E} \left( \left[ \mathbb{E} G \left( \sum_{k=1}^n a_k X_k \right) \right]_{|a_{1:n}=A_{1:n}} \right). \end{aligned}$$

The conclusion follows for a fixed  $n \geq 1$  by preconditioning. One concludes by letting  $n$  go to infinity since  $F$  and  $G$  are continuous.  $\square$

**A first application to Gaussian processes.** Let  $X = (X_t)_{t \in [0, T]}$  be a continuous centered Gaussian process with a covariance operator  $C_X$  defined on the Hilbert space  $L_T^2 := L^2([0, T], dt)$  into itself by

$$C_X(f) = \mathbb{E} \langle f, X \rangle_{L_T^2} X = \int_0^T \mathbb{E}(X_s X_\cdot) f(s) ds \in L_T^2.$$

The process  $X$  can be seen as a random vector taking values in the separable Banach space  $\mathcal{C}([0, T], \mathbb{R})$  (equipped with the sup-norm). Assume that  $C_X$  admits a decomposition as follows

$$C_X = AA^*, \quad A : (K, |\cdot|_K) \longrightarrow \mathcal{C}([0, T], \mathbb{R}), \text{ } A \text{ continuous linear mapping,}$$

where  $(K, |\cdot|_K)$  is a separable Hilbert space.

Then, we know from Proposition 1 (and Theorem 1) in [20], that for any orthonormal basis (or even any *Parseval frame*, see [20])  $(e_n)_{n \geq 1}$  of  $K$  that the sequence  $(A(e_n))_{n \geq 1}$  is *admissible* for the process  $X$  in the following sense: for any i.i.d. sequence  $(\xi_n)_{n \geq 1}$  of normally distributed random variables defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$

$$\left\{ \begin{array}{l} (i) \quad \sum_{n \geq 1} \xi_n A(e_n) \text{ a.s. converges in } (\mathcal{C}([0, T], \mathbb{R}), \|\cdot\|_{\text{sup}}) \\ (ii) \quad \sum_{n \geq 1} \xi_n A(e_n) \stackrel{\mathcal{L}}{\sim} X. \end{array} \right.$$

Assume furthermore that *all the continuous functions*  $A(e_n)$  are nonnegative. Then, for every  $n \geq 1$ , the continuous stochastic process  $X_n = \xi_n A(e_n)$  satisfies a co-monotony principle (for the natural pointwise partial order on  $\mathcal{C}([0, T], \mathbb{R})$ ). This makes up a sequence of independent random elements of  $\mathcal{C}([0, T], \mathbb{R})$ . It follows from Proposition 3.1(a) that the process  $X$  satisfies a co-monotony principle.

EXAMPLE: Let us consider the standard Brownian motion  $W$  with covariance function  $\mathbb{E} W_t W_s = s \wedge t$ . One checks that  $C_W = AA^*$  where  $A : L_T^2 \rightarrow \mathcal{C}([0, T], \mathbb{R})$  is defined by

$$Af \equiv \left( t \mapsto \int_0^t f(s) ds \right) \in \mathcal{C}([0, T], \mathbb{R}).$$

Applied to the orthonormal basis  $e_n(t) = \sqrt{\frac{2}{T}} \sin(\pi n \frac{t}{T})$ ,  $n \geq 1$ , we get

$$A(e_n)(t) = \sqrt{2T} \frac{1 - \cos(\frac{\pi n t}{T})}{\pi n} \geq 0, \quad t \in [0, T], \quad n \geq 1,$$

so that

$$\widetilde{W} = \sqrt{2T} \sum_{n \geq 1} \frac{\xi_n}{\pi n} \left( 1 - \cos(\pi n \frac{\cdot}{T}) \right), \quad (\xi_n)_{n \geq 1} \text{ i.i.d.}, \quad \xi_1 \stackrel{\mathcal{L}}{\sim} \mathcal{N}(0; 1),$$

is an *a.s.* converging series for the sup-norm which defines a standard Brownian motion. As a consequence, the standard Brownian motion satisfies a co-monotony principle (in the sense of Definition 3.2). <sup>(3)</sup>

The above approach is clearly neither the most elementary way nor the most straightforward to establish the co-monotony principle for the Wiener process as we will see further on. Furthermore, like in finite dimension, it is not true that there is an equivalence between the fact that a continuous process  $X$  satisfies a co-monotony principle and the existence of a decomposition  $C_X = AA^*$  and of an orthonormal basis (or Parseval frame) whose image by  $A$  is made of nonnegative functions. Thus, no such decomposition is known to us for the fractional Brownian motion (with Hurst constant  $H \neq \frac{1}{2}$ ) although it satisfies a co-monotony principle (see Section 3.3.3 further on).

### 3.1 From $[0, T]$ to $\mathbb{R}_+$

We state our results for processes defined on a finite interval  $[0, T]$ . However they can be extended canonically on  $\mathbb{R}_+$ , provided that there exists a sequence of positive real constants  $T_N \uparrow +\infty$  such that  $\mathbb{P}_X(d\alpha)$ -*a.s.* on  $E \subset \mathcal{F}(\mathbb{R}_+, \mathbb{R})$ ,  $\alpha^{T_n} \equiv (t \mapsto \alpha(\cdot \wedge T_n))$  converges in  $E$  toward  $\alpha$  for the topology on  $E$ . Such a sequence does exist the topology of convergence on compact sets but also for the Skorokhod topology on the positive real line. Then the transfer of co- and anti-monotony property (if any) from the stopped process  $X^{T_n}$  to  $X$  is straightforward for bounded functionals. The extension to square integrable or nonnegative functionals follows by the usual truncation arguments.

### 3.2 Continuous Markov Processes

We use implicitly that the Borel  $\sigma$ -field of  $(\mathcal{C}([0, T], \mathbb{R}), \|\cdot\|_{\text{sup}})$  is  $\sigma(\pi_t, t \in [0, T])$ , where  $\pi_t(\alpha) = \alpha(t)$  for every  $\alpha \in \mathcal{C}([0, T], \mathbb{R})$  and every  $t \in [0, T]$ .

---

<sup>3</sup>The fact that  $A(L_T^2)$  is the Cameron-Martin space *i.e.* the reproducing space of the covariance operator, which is obvious here, is a general fact for any such decomposition (see [20]).

**Proposition 3.2.** *Let  $X = (X_t)_{t \in [0, T]}$  be a continuous Markov process defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  with transition operators  $(P_{s,t})_{0 \leq s \leq t \leq T}$  satisfying the monotony preserving property. Let  $F, G : \mathcal{C}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be two co-monotone functionals,  $\mathbb{P}_X$ -a.s. continuous with respect to the sup norm topology. If  $F(X), G(X), F(X)G(X)$  are integrable or  $F$  and  $G$  have  $\mathbb{P}_X$ -a.s. a constant common sign, then*

$$\mathbb{E} F(X)G(X) \geq \mathbb{E} F(X)\mathbb{E} G(X).$$

**Proof.** We may assume that  $F$  and  $G$  are both non-decreasing for the natural order on  $\mathcal{C}([0, T], \mathbb{R})$ . Let  $n \in \mathbb{N}$ ,  $n \geq 1$ . We introduce the uniform mesh  $t_k^n = \frac{kT}{n}$ ,  $k = 0, \dots, n$  and, for every function  $\alpha \in \mathcal{C}([0, T], \mathbb{R})$ , the canonical linear interpolation approximation

$$\alpha^{(n)}(t) = \frac{t_{k+1}^n - t}{t_{k+1}^n - t_k^n} \alpha(t_k^n) + \frac{t - t_k^n}{t_{k+1}^n - t_k^n} \alpha(t_{k+1}^n), \quad t \in [t_k^n, t_{k+1}^n], \quad k = 0, \dots, n-1.$$

One checks that  $\|\alpha - \alpha^{(n)}\|_{\text{sup}} \leq w(\alpha, T/n)$  goes to 0 as  $n \rightarrow \infty$  where  $w(\alpha, \cdot)$  denotes the uniform continuity modulus of  $\alpha$ . As a consequence,  $X$  having a.s. continuous paths by assumption, the sequence  $(X^{(n)})_{n \geq 1}$  of interpolations of  $X$  a.s. uniformly converges toward  $X$ .

Then set for every  $n \geq 1$  and every  $x = x_{0:n} \in \mathbb{R}^{n+1}$ ,

$$\chi^n(x, t) = \frac{t_{k+1}^n - t}{t_{k+1}^n - t_k^n} x_k + \frac{t - t_k^n}{t_{k+1}^n - t_k^n} x_{k+1}, \quad t \in [t_k^n, t_{k+1}^n], \quad k = 0, \dots, n-1$$

and

$$F_n(x) = F(\chi^n(x, \cdot)).$$

It is clear that if  $x \leq x'$  in  $\mathbb{R}^{n+1}$  (in the componentwise sense) then  $F_n(x) \leq F_n(x')$  since  $\chi^n(x, \cdot) \leq \chi^n(x', \cdot)$  as functions. This is equivalent to the fact that  $F_n$  is non-decreasing in each of its variables.

On the other hand  $X^{(n)} = \chi((X_{t_k^n})_{0 \leq k \leq n}, \cdot)$  so that  $F_n((X_{t_k^n})_{0 \leq k \leq n}) = F(X^{(n)})$ . The sequence  $(X_{t_k^n})_{0 \leq k \leq n}$  is a discrete time Markov chain whose transition operators  $P_{t_k^n, t_{k+1}^n}$ ,  $k = 0, \dots, n-1$  satisfy the monotony preserving property. As a consequence, it follows from Proposition 2.1 that if  $F$  and  $G$  are bounded, for every  $n \in \mathbb{N}$ ,

$$\mathbb{E} F(X^{(n)})G(X^{(n)}) \geq \mathbb{E} F(X^{(n)})\mathbb{E} G(X^{(n)}).$$

One derives the expected inequality by letting  $n$  go to infinity since  $F$  and  $G$  are continuous with respect to the sup norm. The extension to unbounded functionals  $F$  or  $G$  follows by the usual truncation arguments.  $\square$

**EXAMPLES.** (a) *Wiener integrals.* Let  $X_t = \int_0^t f(s) dW_s$ ,  $t \in [0, T]$ , where  $f \in L^2([0, T], dt)$  and  $W = (W_t)_{t \in [0, T]}$  is a standard Wiener process. Then  $X$  admits a continuous modification which is a process with independent increments, hence a continuous Markov process whose transitions  $P_{s,t}(x, dy) = \mathcal{N}(0; \int_s^t f^2(u) du)$  are monotony preserving since they do not depend on  $x \in \mathbb{R}$ .

See also Section 3.3 for a simpler proof based on a Gaussian argument.

(b) *Brownian diffusions.* We consider the Brownian diffusion

$$SDE \equiv dX_t^x = b(t, X_t^x) dt + \sigma(t, X_t^x) dW_t, \quad X_0^x = x$$

where  $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous in  $x$ , uniformly in  $t \in [0, T]$  and  $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  (is continuous) with linear growth in  $x$ , uniformly in  $t \in [0, T]$  and satisfies

$$\forall x, y \in \mathbb{R}, \forall t \in [0, T], \quad |\sigma(t, x) - \sigma(t, y)| \leq \rho(|x - y|),$$

where

$$\rho : \mathbb{R} \rightarrow \mathbb{R} \text{ is increasing, } \rho(0) = 0 \text{ and } \int_{0+} \frac{du}{\rho^2(u)} = +\infty.$$

Then the equation (*SDE*) satisfies a weak existence property since  $b$  and  $\sigma$  are continuous with linear growth: its Euler scheme weakly functionally converges to a weak solution of (*SDE*) as its step  $T/n$  goes to 0 (see Theorem 5.3 in [14]). It also satisfies a strong uniqueness property (see Proposition 2.13 in [16]) hence a strong existence-uniqueness property. This implies the existence of (Feller) Markov transitions  $(P_{s,t}(x, dy))_{t \geq s \geq 0}$  such that, *a.s.* for every  $x \in \mathbb{R}$ ,  $P_{s,t}(f)(x) = \mathbb{E}(f(X_t) | X_s = x)$  (see *e.g.* Theorem 1.9 in [23]). Furthermore the flow  $(X_t^x)_{x \in \mathbb{R}, t \in [0, T]}$  satisfies a comparison principle (Yamada-Watanabe's Theorem) *i.e.* for every  $x, x' \in \mathbb{R}$ ,  $x \leq x'$ ,  $\mathbb{P}$ -*a.s.*, for every  $t \in [0, T]$ ,  $X_t^x \leq X_t^{x'}$ . The functional co-monotony principle follows immediately.

**Remarks.** • When  $\sigma(x, t) = \sigma(t)$ ,  $\sigma$  continuous, the result can be derived from the co-monotony principle for the Euler scheme (see Section 2) since this scheme functionally weakly converges toward  $X^x$  as its step  $T/n$  goes to 0.

• This result is strongly related to strong uniqueness of solutions of (*SDE*). The above conditions are not minimal, see *e.g.* [7] for more insights on these aspects.

**Application to Brownian bridge.** Up to a canonical space-time rescaling and a translation by a deterministic function, we may focus on the standard Brownian bridge on the unit interval defined by  $X_t = W_t - tW_1$ ,  $t \in [0, 1]$ . We know, *e.g.* from [23], that  $X$  is solution to an *SDE*

$$dX_t = -\frac{X_t}{1-t}dt + dB_t, \quad X_0 = 0,$$

where  $B = (B_t)_{t \in [0, 1]}$  is a standard Brownian motion. Let  $\eta \in (0, 1)$ . The process  $(X_t)_{t \in [0, 1-\eta]}$  satisfies a co-monotony principle since  $b(t, x) = -\frac{x}{1-t}$  and  $\sigma(t, x) = 1$  are continuous on  $[0, 1-\eta] \times \mathbb{R}$  and Lipschitz continuous in  $x$  uniformly in  $t \in [0, 1-\eta]$  (which implies strong existence and pathwise uniqueness for the *SDE* on  $[0, 1-\eta]$ ). Consequently  $(X_t)_{t \in [0, 1-\eta]}$  satisfies a co-monotony principle for every  $\eta \in (0, 1)$ . Then, for every  $\|\cdot\|_{\text{sup}}$ -continuous functional  $F_\eta : \mathcal{C}([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ , the functional  $F_\eta(\alpha) = F((\alpha((1-\eta)t)_{t \in [0, 1]}))$ ,  $\alpha \in \mathcal{C}([0, 1-\eta], \mathbb{R})$  is also continuous. Furthermore  $F_\eta$  converges towards  $F$  as  $\eta \rightarrow 0$  which implies that the Brownian bridge itself satisfies a co-monotony principle.

### 3.3 Continuous Gaussian processes

Let  $X = (X_t)_{t \in [0, T]}$  be a continuous centered Gaussian process. Its (continuous) covariance function  $C_X$  is defined on  $[0, T]^2$  as follows

$$\forall s, t \in [0, T], \quad C_X(s, t) = \mathbb{E} X_s X_t.$$

We establish below the functional counterpart of Pitt's Theorem.

**Theorem 3.1** (Functional Pitt's Theorem). *Let  $X = (X_t)_{t \in [0, T]}$  be a continuous Gaussian process with covariance operator  $C_X$ .*

*$(X_t)_{t \in [0, T]}$  satisfies a co-monotony principle if and only if  $\forall s, t \in [0, T]$ ,  $C_X(s, t) \geq 0$ .*

**Proof.** We follow the lines and the notations of the proof of Proposition 4.2. We replace the monotony preserving argument on the Markov transition by Pitt's Theorem: for every  $n \geq 1$ ,  $(X_{t_k^n}^n)_{0 \leq k \leq n}$  is a centered Gaussian process with a covariance matrix  $\Sigma^n = [C_X(t_k^n, t_\ell^n)]_{0 \leq k, \ell \leq n}$  having nonnegative

entries. Hence, the function  $F_n$  and  $G_n$  being non-decreasing in each of its variables, one has for every  $n \geq 1$ ,

$$\mathbb{E} F_n(X)G_n(X) \geq \mathbb{E} F_n(X)\mathbb{E} G_n(X).$$

One concludes by noting that  $F_n(X) = F(X^{(n)})$  (idem for  $G$ ) and by letting  $n$  go to infinity.  $\square$

We inspect below several classical classes of Gaussian processes.

### 3.3.1 Liouville processes

**Definition 3.3.** Let  $f : [0, T] \rightarrow \mathbb{R}$  be a locally  $\rho$ -Hölder function,  $\rho \in (0, 1]$ , in the following sense: there exists  $\varphi \in \mathcal{L}^2([0, T], dt)$ ,  $\rho \in (0, 1]$ ,  $a \in (0, +\infty)$  such that

$$(L_{\rho, a}) \equiv \begin{cases} (i) & \forall t, t' \in [0, T], \quad |f(t) - f(t')| \leq [f]_{\rho, \varphi} |t - t|^\rho \varphi(t \wedge t') \\ (ii) & \int_0^t f^2(s) ds = O(t^a). \end{cases} \quad (3.3)$$

Then the Gaussian process defined for every  $t \in [0, T]$

$$X_t = \int_0^t f(t-s) dW_s$$

admits a continuous modification called Liouville process (related to  $f$ ) with covariance function

$$C_X(s, t) = \int_0^{s \wedge t} f(t-u)f(s-u) du.$$

JUSTIFICATION: First note that  $f \in \mathcal{L}^2(dt)$  since  $f(t) \leq |f(0)| + t^\rho |\varphi(0)|$ . Then, for every  $t, t' \in [0, T]$ ,  $t \leq t'$ ,

$$X_{t'} - X_t = \int_t^{t'} f(t'-s) dW_s + \int_0^t (f(t'-s) - f(t-s)) dW_s$$

so that

$$\begin{aligned} \mathbb{E}|X_{t'} - X_t|^2 &= \int_t^{t'} f^2(t'-s) ds + \int_0^t (f(t'-s) - f(t-s))^2 ds \\ &\leq \int_0^{t'-t} f^2(s) ds + [f]_{\rho, \varphi}^2 |t' - t|^{2\rho} \int_0^T \varphi^2(s) ds \\ &\leq C_f (|t' - t|^a + |t' - t|^{2\rho}) \\ &\leq C_f |t' - t|^{(2\rho) \wedge a} \end{aligned}$$

so that, using the Gaussian feature of the process  $X$

$$\mathbb{E}|X_{t'} - X_t|^p \leq C_{f,p} |t' - t|^{p(\rho \wedge a/2)}$$

for every  $p \geq 2$  which in turn implies, owing to Kolmogorov's continuity criterion, that  $(X_t)_{t \in [0, T]}$  admits a version  $(\rho \wedge \frac{a}{2} - \eta)$ -Hölder continuous for any small enough  $\eta > 0$ .

**Proposition 3.3.** Let  $X_t = \int_0^t f(t-s) dW_s$ ,  $t \in [0, T]$ , be a continuous Liouville process where  $W$  is a standard B.M. defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  and  $f$  satisfies  $(L_{\rho, a})$  for a couple  $(\rho, a) \in (0, 1] \times (0, +\infty)$ . If furthermore  $f$  is  $\lambda$ -a.e. nonnegative, then  $X$  satisfies a co-monotony principle.

The proof of the proposition is straightforward since  $f \geq 0$   $\lambda_1$ -a.e. implies that the covariance function  $C_X$  is nonnegative.

EXAMPLE: If  $f(u) = u^{H-1/2}$  with  $H \in (0, 1]$ , then  $f$  satisfies  $(L_{a,\rho})$  with  $a = 2H$  and  $\rho \in (0, \frac{1}{2} - H)$  if  $H < \frac{1}{2}$ ,  $\rho = H - \frac{1}{2}$  if  $H > \frac{1}{2}$  (and  $\rho = 1$  if  $H = \frac{1}{2}$ ). This corresponds to the pseudo-fractional Brownian motion with Hurst constant  $H$ .

### 3.3.2 Wiener integrals depending on a parameter

Now we consider a class of processes which is wider than Liouville's class and for which we provide a slightly less refined criterion of existence (as a path wise continuous process).

$$X_t = \int_0^\infty f(t, s) dW_s, \quad t \in [0, T],$$

where  $f : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfies a dominated  $\rho$ -Hölder assumption reading as follows: there exists  $\varphi \in \mathcal{L}^2(\mathbb{R}, dt)$ , non-increasing, and  $\rho \in (0, 1]$  such that

$$(L'_\rho) \equiv \forall t, t' \in [0, T], \forall s \in \mathbb{R}_+, \quad |f(t', s) - f(t, s)| \leq [f]_\rho |t' - t|^\rho \varphi(s).$$

Such a process has a continuous modification since  $t \mapsto X_t$  is  $\rho$ -Hölder from  $[0, T] \rightarrow L^2(\mathbb{P})$  and Gaussian (still owing to Kolmogorov's continuity criterion).

As for Liouville processes, if furthermore, for every  $t \in [0, T]$ ,  $f(t, \cdot)$  is  $\lambda_1$ -a.e. nonnegative, then the process  $X$  satisfies a co-monotony principle.

EXAMPLE. Let  $f_H(t, s) = (t + s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}$ ,  $H \in (0, 1]$ .

- If  $H \geq 1/2$ ,  $f_H$  satisfies  $(L'_{H-\frac{1}{2}})$  with  $\varphi(s) = 1$ .
- If  $H \in (0, \frac{1}{2}]$ ,  $f_H$  satisfies  $(L'_{\frac{1}{2}-H})$  with  $\varphi(s) = s^{2H-1}$ .

### 3.3.3 Fractional Brownian motion with Hurst constant $H \in (0, 1]$

The fractional Brownian motion is a continuous Gaussian process characterized by its covariance function defined by

$$\forall s, t \in [0, T], \quad C^H(s, t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Since  $u \mapsto u^H$  is  $H$ -Hölder and  $|t - s|^2 \leq t^2 + s^2$ , it is clear that  $C(s, t) \geq 0$ . Consequently, the fractional Brownian motion satisfies a co-monotony principle.

**Remark.** An alternative approach could be to rely on the celebrated Mandelbrot-Van Ness representation of the fractional Brownian motion with Hurst constant  $H \in (0, 1]$ , given by

$$B_t^H = \underbrace{\int_0^{+\infty} ((t + s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}) dW_s^1}_{=: B_t^{H,1}} + \underbrace{\int_0^t |t - s|^{H-\frac{1}{2}} dW_s^2}_{=: B_t^{H,2}}$$

where  $W^1$  and  $W^2$  are independent standard Brownian motions. These two Wiener integrals define pathwise continuous independent processes, both satisfying the co-monotony principle for  $\mathbb{P}_{B^{H,i}}$ -a.s.  $\|\cdot\|_{\text{sup}}$ -continuous functionals, consequently their sum satisfies a co-monotony principle for  $\mathbb{P}_{B^H}$ -a.s.  $\|\cdot\|_{\text{sup}}$ -continuous functionals owing to Lemma 3.1(b).

## 4 Comonotony principle for càdlàg processes

### 4.1 Processes with independent increments

We consider a general càdlàg process with independent increments (PII)  $(X_t)_{t \geq 0}$  defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . We rely on its Lévy-Khintchine decomposition as exposed in [14], chap. II, section 3. First one can decompose  $X$  as the sum

$$X = X^{(1)} \dot{+} X^{(2)}$$

where  $X^{(1)}$  and  $X^{(2)}$  are two independent PII:  $X^{(1)}$  is a PII *without fixed discontinuities* and  $X^{(2)}$  is a pure jump PII, possibly jumping only at a deterministic sequence of times, namely

$$X_t^{(2)} = \sum_{n \geq 1} U_n \mathbf{1}_{\{t_n \leq t\}}, \quad t \in [0, T],$$

where  $(t_n)_{n \geq 1}$  is a sequence of  $[0, T]$ -valued real numbers and  $(U_n)_{n \geq 1}$  is a sequence of independent random variables satisfying the usual assumption of the three series theorem

$$\begin{aligned} \sum_n \mathbb{P}(|U_n| \geq 1) &< +\infty, \\ \sum_n \mathbb{E} U_n \mathbf{1}_{\{|U_n| \leq 1\}} &< +\infty, \\ \sum_n \mathbb{E} (U_n^2 \mathbf{1}_{\{|U_n| \leq 1\}} - (\mathbb{E} U_n \mathbf{1}_{\{|U_n| \leq 1\}})^2) &< +\infty. \end{aligned}$$

**Proposition 4.1.** *A PII satisfies a co-monotony principle on  $(\mathcal{ID}([0, T], \mathbb{R}), \|\cdot\|_{\text{sup}})$ .*

**Proof.** Owing to Lemma 3.1(a), we will inspect successively the cases of PII without fixed discontinuities and of pure jumps.

STEP 1. *X is a PII without fixed discontinuities:* This means that  $X^{(2)} \equiv 0$ . The classical (pathwise) Lévy-Khintchine formula for PII without fixed discontinuities says that, a truncation level  $\varepsilon > 0$  being fixed,  $X$  reads as the sum of three mutually independent processes as follows

$$\forall t \in [0, T], \quad X_t = b^\varepsilon(t) + W_{c(t)} \dot{+} \sum_{s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| > \varepsilon\}} \dot{+} M_t^\varepsilon$$

where  $b^\varepsilon$  is a continuous function on  $[0, T]$ ,  $c$  is a nonnegative non-decreasing continuous function on  $[0, T]$  with  $c(0) = 0$  and  $M_t^\varepsilon$  is a pure jump martingale satisfying

$$\mathbb{E} \left( \sup_{s \in [0, t]} |M_s^\varepsilon|^2 \right) \leq 4 \int_{\mathbb{R} \setminus \{0\}} x^2 \mathbf{1}_{\{|x| \leq \varepsilon\}} \nu^X([0, t] \times dx)$$

where the measure  $\nu^X$  is the Lévy measure of  $X$ , *i.e.* the dual predictable projection of the jump measure  $\mu^X(ds, dx) = \sum_{s \in [0, T]} \mathbf{1}_{\{\Delta X_s \neq 0\}} \Delta X_s$ . The Lévy measure is characterized by the fact that, for every bounded Borel function  $g : \mathbb{R} \rightarrow \mathbb{R}$  null in the neighbourhood of 0,

$$\int_{[0, t]} \int_{\mathbb{R} \setminus \{0\}} g(x) (\mu^X(ds, dx) - \nu^X(ds, dx)), \quad t \geq 0, \quad \text{is a local martingale.}$$

In particular for any such function we get the compensation formula

$$\mathbb{E}\left(\sum_{t \leq T} g(\Delta X_t)\right) = \int g(x) \nu^X([0, T] \times dx)$$

which extends to any nonnegative function  $g$  or satisfying  $\int_{\mathbb{R} \setminus \{0\}} |g(x)| \nu^X([0, T] \times dx) < +\infty$ . The Lévy measure  $\nu^X$  satisfies

$$\nu^X(\{0\} \times \mathbb{R}) = \nu^X(\{t\} \times dx) = 0, \quad \int_{\mathbb{R}} (x^2 \wedge 1) \nu^X([0, t] \times dx) < +\infty, \quad t \in \mathbb{R}_+.$$

In what follow, we make the convention that  $\Delta \alpha_\infty = 0$  for any càdlàg function  $\alpha$  defined on  $\mathbb{R}_+$ .

First, owing to Lemma 3.1(a) and the result about the standard Brownian in Section 3, we can assume that  $c \equiv 0$  in what follows, *i.e.* that there is no Brownian component. Then we define two independent marked Poisson processes with positive jumps as follows

$$\tilde{X}_t^{\varepsilon, \pm} = \sum_{s \leq t} (\Delta X_s)_\pm \mathbf{1}_{\{(\Delta X_s)_\pm > \varepsilon\}}$$

and  $\tilde{X}^\varepsilon = \tilde{X}^{\varepsilon, +} - \tilde{X}^{\varepsilon, -}$ . For each process, we define their inter jump sequence  $(\tilde{\Theta}_n^{\varepsilon, \pm})_{n \geq 0}$  *i.e.*, with the convention  $\tilde{\Theta}_0^{\varepsilon, \pm} = 0$ ,

$$\tilde{\Theta}_{n+1}^{\varepsilon, \pm} = \min\{s > \tilde{S}_n^\pm \mid (\Delta X_{\tilde{S}_n^\pm + s})_\pm > \varepsilon\} \in (0, +\infty], \quad n \geq 0,$$

where  $\tilde{S}_n^{\varepsilon, \pm} = \tilde{\Theta}_1^{\varepsilon, \pm} + \dots + \tilde{\Theta}_n^{\varepsilon, \pm}$ .

Both processes  $\tilde{X}^{\varepsilon, +}$  and  $\tilde{X}^{\varepsilon, -}$  are independent since they have no common jumps. Furthermore the four sequences  $(\tilde{\Theta}_n^{\varepsilon, \pm})_{n \geq 1}$  and  $(\Delta \tilde{X}_{\tilde{S}_n^\pm}^{\varepsilon, \pm})_{n \geq 1}$  are mutually independent and made of mutually independent terms.

Let  $F$  be a bounded measurable non-decreasing defined on functional on  $\mathcal{D}([0, T], \mathbb{R})$ . Now, for every  $n \geq 1$ , we define on  $\mathbb{R}^{2n}$  the function  $F_n$  by

$$F_n(\xi_1, \theta_1, \dots, \xi_n, \theta_n) = F\left(\left(\sum_{k=1}^n (\xi_k)_+ \mathbf{1}_{\{\theta_1 + \dots + \theta_n \leq t\}}\right)_{t \in [0, T]}\right), \quad \xi_1, \dots, \xi_n \in \mathbb{R}, \quad \theta_1, \dots, \theta_n \in \overline{\mathbb{R}}.$$

It is straightforward that the functions  $F_n$  are non-decreasing in each variable  $\xi_i$  and non-increasing in each variable  $\theta_i \in \overline{\mathbb{R}}_+$ .

For every  $n \geq 1$ , set  $\tilde{X}_t^{\varepsilon, n, \pm} = \sum_{k=1}^n (\Delta X_{\tilde{S}_k^{\varepsilon, \pm}})_\pm \mathbf{1}_{\{\tilde{S}_k^{\varepsilon, \pm} \leq t\}}$  so that

$$F(\tilde{X}^{\varepsilon, n, \pm}) = F_n\left(\left((\Delta X_{\tilde{S}_k^{\varepsilon, \pm}})_\pm, \tilde{\Theta}_k^{\varepsilon, \pm}\right)_{k=1, \dots, n}\right).$$

Consequently, if  $F$  and  $G$  are co-monotone (measurable) functionals on  $\mathcal{D}([0, T], \mathbb{R})$ , it follows from Proposition 2.1(b) (co-monotony principle for mutually independent random variables) that

$$\mathbb{E} F(\tilde{X}^{\varepsilon, n, \pm}) G(\tilde{X}^{\varepsilon, n, \pm}) \geq \mathbb{E} F(\tilde{X}^{\varepsilon, n, \pm}) \mathbb{E} G(\tilde{X}^{\varepsilon, n, \pm}).$$

Now

$$\sup_{t \in [0, T]} |\tilde{X}_t^{\varepsilon, \pm} - \tilde{X}_t^{\varepsilon, n, \pm}| \leq \sum_{k \geq n+1} (\Delta X_{\tilde{S}_k^{\varepsilon, \pm}})_\pm \mathbf{1}_{\{\tilde{S}_k^{\varepsilon, \pm} \leq T\}}$$

so that

$$\mathbb{P}\left(\sup_{t \in [0, T]} |\widetilde{X}_t^{\varepsilon, \pm} - \widetilde{X}_t^{\varepsilon, n, \pm}| > 0\right) \leq \mathbb{P}(\widetilde{S}_{n+1}^{\pm} \leq T) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

since the process  $X$  has finitely many jumps of size greater than  $\varepsilon$  on any bounded time interval. The continuity of  $F$  and  $G$  transfers the co-monotony inequality to  $\widetilde{X}^{\varepsilon, \pm}$ . In turn, the independence of these two processes, combined with Lemma 3.1(a), propagates co-monotony to the global Poisson process  $\widetilde{X}^\varepsilon$ .

Noting that  $F$  and  $F(b^\varepsilon + \cdot)$  have the same monotony (if any), one derives that  $X - M^\varepsilon$  satisfies the co-monotony principle for every  $\varepsilon > 0$ . One concludes by noting that  $\|M^\varepsilon\|_{\text{sup}} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  in  $L^2$ .  $\square$

## 4.2 Càdlàg Markov processes and $\|\cdot\|_{L_T^p(\mu)}$ -continuous functionals

It is often convenient to consider some path spaces of the form  $L^p([0, T], \mu)$  where  $\mu$  is a  $\sigma$ -finite measure and  $p \in [1, +\infty)$ , especially because of the properties of differentiation on these spaces which allow the natural introduction of gradient fields. Of course, less functionals are continuous for such a topology than with the  $\|\cdot\|_{\text{sup}}$ -norm topology when the process  $X$  has continuous (or even càdlàg) paths.

Then, following the lines of the proof of Proposition 4.2 but with a new canonical approximation procedure of a function  $\alpha$ , this time by a stepwise constant function, one shows the following property.

**Proposition 4.2.** *Let  $(X_t)_{t \in [0, T]}$  be a càdlàg Markov process defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  with transitions operators  $(P_{s,t})_{0 \leq t \leq s \leq T}$  satisfying the monotony property. Let  $\mu$  be a finite measure on  $([0, T], \mathcal{B}or([0, T]))$  and let  $p \in [1, +\infty)$ . Let  $F, G : \mathcal{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be two  $\mu$ -co-monotone functionals,  $\mathbb{P}_X$ -a.s. continuous with respect to the  $L_T^p(\mu)$ -norm on  $\mathcal{D}([0, T], \mathbb{R})$ . If  $F(X)$ ,  $G(X)$  and  $F(X)G(X)$  are integrable or have  $\mathbb{P}_X$ -a.s. a common constant sign, then*

$$\mathbb{E} F(X)G(X) \geq \mathbb{E} F(X)\mathbb{E} G(X).$$

**Proof.** For every  $\alpha \in \mathcal{D}([0, T], \mathbb{R})$ , we define  $\alpha^{(n)} = \sum_{k=0}^{n-1} \alpha(t_{k+1}^n) \mathbf{1}_{[t_k^n, t_{k+1}^n)} + \alpha(T) \mathbf{1}_{\{T\}}$  with, as usual,  $t_k^n = \frac{kT}{n}$ ,  $k = 0, \dots, n$ . It is clear that  $\alpha^{(n)}(t) \rightarrow \alpha(t)$  at every  $t \in [0, T]$  and that the sequence  $(\alpha^{(n)})_{n \geq 1}$  is bounded by  $\|\alpha\|_{\text{sup}}$ . Hence  $\alpha^{(n)}$  converges to  $\alpha$  in every  $L_T^p(\mu)$ ,  $1 \leq p < +\infty$ . The rest of the proof is similar to that of Proposition 4.2.  $\square$

## 5 Applications

### 5.1 Functional antithetic simulation method

Of course, the first natural application is a functional version of the antithetic simulation method which was briefly described in the introduction. In fact, if a process  $X$  taking values in a vector subspace  $E \subset \mathcal{F}([0, T], \mathbb{R})$  (partially ordered by the pointwise order) satisfies a functional co-monotony principle in the sense of Definition 3.2 and is invariant in distribution under a continuous non-increasing mapping  $T : E \rightarrow E$  (by  $T$  non-increasing we mean that  $\alpha \leq \beta \Rightarrow T(\alpha) \geq T(\beta)$ ,  $\alpha, \beta \in E$ ) that for any sup-norm continuous monotone functional  $F : E \rightarrow \mathbb{R}$  (square integrable or with constant sign)

$$\text{Cov}(F(X), F \circ T(X)) = \mathbb{E} F(X)F \circ T(X) - (\mathbb{E} F(X))^2 \leq 0.$$

As a consequence, in order to compute  $\mathbb{E} F(X)$  by a Monte Carlo simulation, it follows that the computation of (independent copies of)  $\frac{F(X) + F \circ T(X)}{2}$  will induce, for a prescribed simulation

budget, a lower variance than than a simulation only computing (independent copies of)  $F(X)$  like in the scalar framework. In practice such simulations rely on discretization schemes of  $X$  for which the co-monotony principle is only true asymptotically (when the discretization step will go to zero). So is the case for the Euler scheme of a Brownian diffusion with non deterministic diffusion coefficient).

It remains that this strongly suggests, in order to compute  $\mathbb{E}F(X)$  where  $X = (X_t)_{t \in [0, T]}$  is a Brownian diffusion and  $F$  is a monotone  $\mathbb{P}_X$ -a.s. sup-norm continuous functional, to simulate systematically two coupled paths of an Euler scheme (with a small enough step): one with a sequence of Brownian increments  $(W_{t_{k+1}^n} - W_{t_k^n})_{k \geq 0}$  and one with its opposite  $-(W_{t_{k+1}^n} - W_{t_k^n})_{k \geq 0}$ .

## 5.2 Sensitivity of a class of path-dependent options

Let  $\mu$  be a finite measure on  $([0, T], \mathcal{B}or([0, T]))$  and, for every  $p \in [1, +\infty)$ , let  $q$  denote its Hölder conjugate. Note that, of course,  $\mathcal{ID}([0, T], \mathbb{R}) \subset L_T^\infty(\mu) \subset \bigcap_{p \geq 1} L_T^p(\mu)$ .

**Definition 5.1.** (a) Let  $p \in [1, +\infty)$ . A measurable functional  $F : L_T^p(\mu) \rightarrow \mathbb{R}$  is regularly differentiable on  $\mathcal{ID}([0, T], \mathbb{R})$  if, for every  $\alpha \in \mathcal{ID}([0, T], \mathbb{R})$ , there exists a measurable “gradient” functional  $\nabla F : ([0, T] \times \mathcal{ID}([0, T], \mathbb{R}), \mathcal{B}or([0, T]) \otimes \mathcal{D}_T) \rightarrow \mathbb{R}$  such that

$$\begin{cases} (i) & \nabla F(\cdot, \alpha) \in L_T^q(\mu) \\ (ii) & \lim_{\|h\|_{L_T^p(\mu)} \rightarrow 0, h \in L_T^p(\mu)} \frac{|F(\alpha+h) - F(\alpha) - \int_0^T \nabla F(s, \alpha) h(s) \mu(ds)|}{\|h\|_{L_T^p(\mu)}} = 0. \end{cases} \quad (5.4)$$

(b) Furthermore, a gradient functional  $\nabla F$  is monotone if, for every  $t \in [0, T]$ ,  $\nabla F(t, \cdot)$  is monotone on  $\mathcal{ID}([0, T], \mathbb{R})$  and if this monotony does not depend on  $t \in [0, T]$ .

**Proposition 5.1.** Let  $X = (X_t)_{t \in [0, T]}$  be a (càdlàg) PII such that, for every  $u \in \mathbb{R}$ ,  $L(u, t) = \mathbb{E}e^{uX_t}$  is bounded and bounded away from 0 over  $[0, T]$  so that, in particular, the function  $\Psi(u, t) = \log \mathbb{E}e^{uX_t}$  can be defined as a real valued function. Let  $F : L_T^p(\mu) \rightarrow \mathbb{R}$  be a measurable functional, regularly differentiable with a monotone gradient  $\nabla F$  on  $\mathcal{ID}([0, T], \mathbb{R})$  satisfying the following Lipschitz continuity assumption

$$\forall \alpha, \beta \in \mathcal{ID}([0, T], \mathbb{R}), \quad |F(\alpha) - F(\beta)| \leq [F]_{\text{Lip}} \|\alpha - \beta\|_{L_T^p(\mu)}.$$

Set, for every  $\sigma > 0$ ,

$$f(\sigma) = \mathbb{E} \left( F \left( e^{\sigma X \cdot - \Psi(\sigma, \cdot)} \right) \right). \quad (5.5)$$

Then, under the above assumptions, the function  $f$  is (differentiable and) non-decreasing.

**Remark.** At least for Lévy processes, the assumption  $\sup_{t \in [0, T]} \mathbb{E}e^{uX_t} < +\infty$ ,  $u \in \mathbb{R}$ , is satisfied as soon as  $\mathbb{E}e^{uX_t} < +\infty$  for every  $u \in \mathbb{R}$  (see [24], Theorem 25.18, p.166).

Before proving the proposition, we need the following technical lemma about the regularity of function  $L$  whose details of proof are left to the reader.

**Lemma 5.1.** Under the assumption made on the function  $L$  in Proposition 5.1, the function  $\Lambda$  defined on  $\mathbb{R}_+^2$  by  $\Lambda(a, t) = \mathbb{E}e^{a|X_t|}$  is finite. Then for every  $a \in (0, +\infty)$ ,  $L$  is Lipschitz continuous in  $u$  on  $[-a, a]$ , uniformly in  $t \in [0, T]$ , with Lipschitz coefficient (upper-bounded by)  $\Lambda(a, T)$ . Furthermore, for every  $u \in \mathbb{R}$ , there exists  $\kappa_{u, T} > 0$  and  $\varepsilon = \varepsilon(u, T) > 0$  such that

$$\forall t \in [0, T], \forall u' \in [u - \varepsilon, u + \varepsilon], L(u, t) \geq \kappa_{u, T}.$$

**Proof of Proposition 5.1.** Formally, the derivative of  $f$  reads

$$\begin{aligned} f'(\sigma) &= \mathbb{E} \left( \int_0^T \nabla F \left( e^{\sigma X_t - \Psi(\sigma, \cdot)}, t \right) e^{\sigma X_t - \Psi(\sigma, t)} (X_t - \Psi'_\sigma(\sigma, t)) \mu(dt) \right) \\ &= \int_0^T \mathbb{E} \left( \nabla F \left( e^{\sigma X_t - \Psi(\sigma, \cdot)}, t \right) e^{\sigma X_t - \Psi(\sigma, t)} (X_t - \Psi'_\sigma(\sigma, t)) \right) \mu(dt). \end{aligned}$$

To justify that this interchange of differentiation and expectation in the first line is valid we need

to prove that the ratio  $\frac{F(e^{\sigma' X_t - \Psi(\sigma', \cdot)}) - F(e^{\sigma X_t - \Psi(\sigma, \cdot)})}{\sigma' - \sigma}$ ,  $\sigma' \neq \sigma$ ,  $\sigma, \sigma' \in [\epsilon_0, 1/\epsilon_0]$ ,  $\epsilon_0 > 0$ , is  $L^{1+\eta}$ -bounded for an  $\eta > 0$ . Without loss of generality, we may assume that  $p = 1 + \eta > 1$  since  $\|\cdot\|_{L_T^p} \leq \mu([0, T])^{\frac{1}{p} - \frac{1}{p'}} \|\cdot\|_{L_T^{p'}}$  if  $1 \leq p \leq p'$ . This follows from the Lipschitz continuity of  $F$  and from the properties of the Laplace transform  $L$  established in Lemma 5.1.

Let  $\mathcal{G}_t := \sigma(X_s - X_t, s \in [t, T])$ . This  $\sigma$ -field is independent of  $\mathcal{F}_t^X$ . Elementary computations show that, for every  $t \in [0, T]$ ,

$$\mathbb{E} \left( \nabla F \left( e^{\sigma X_t - \Psi(\sigma, \cdot)}, t \right) \mid \mathcal{G}_t \right) = \Phi \left( X_t - \frac{\Psi(\sigma, \cdot)}{\sigma}, (X_s - \frac{\Psi(\sigma, \cdot)}{\sigma})_{s \in [0, t]}, t \right)$$

where, for every  $\beta \in \mathcal{D}([0, t], \mathbb{R})$ ,

$$\Phi(\xi, \beta, t) = \mathbb{E} \left( \nabla F \left( e^{\sigma(X_t - X_t) - (\Psi(\sigma, \cdot) - \Psi(\sigma, t)) + \sigma \beta(t)} \mathbf{1}_{(t, T]} + e^{\sigma \beta} \mathbf{1}_{[0, t]}, t \right) \right).$$

Note that, for every  $t \in [0, T]$ , the function  $\Phi(\cdot, \cdot, t)$  is non-decreasing in both remaining arguments.

Now

$$f'(\sigma) = \int_0^T \mathbb{E} \left( \Phi \left( (X_s - \frac{\Psi(\sigma, \cdot)}{\sigma})_{s \in [0, t]}, t \right) e^{\sigma(X_t - \frac{\Psi(\sigma, t)}{\sigma})} (X_t - \Psi'_\sigma(\sigma, t)) \right) \mu(dt). \quad (5.6)$$

Set  $\mathbb{Q}_{(t)} = e^{\sigma X_t - \Psi(\sigma, t)} \mathbb{P}$ . It is classical that  $(X_s)_{s \in [0, t]}$  is still a PII under  $\mathbb{Q}_{(t)}$  with exponential moment at any order and a log-Laplace transform  $\Psi_{(t)}$  given by

$$\Psi_{(t)}(u, s) = \Psi(\sigma + u, s) - \Psi(\sigma, s).$$

Note that  $\Psi_{(t)}$  does not depend on  $t$  but on  $\sigma$ . Consequently, for every  $s \in [0, t]$ ,

$$\mathbb{E}_{\mathbb{Q}_{(t)}}(X_s) = \frac{\partial \Psi_{(t)}}{\partial u}(0, s) = \Psi'_\sigma(\sigma, s)$$

where  $\Psi'_\sigma(\sigma, s)$  denote the partial derivative of  $\psi$  with respect to  $\sigma$ . Putting  $\tilde{X}_s = X_s - \Psi'_\sigma(\sigma, s)$ , we get

$$f'(\sigma) = \int_0^T \mathbb{E}_{\mathbb{Q}_{(t)}} \left( \Phi \left( (\tilde{X}_s + \Psi'_\sigma(\sigma, s) - \frac{\Psi(\sigma, s)}{\sigma})_{s \in [0, t]}, t \right) \tilde{X}_t \right) \mu(dt). \quad (5.7)$$

Applying the co-monotony principle to the PII  $\tilde{X}$  and to the two non-decreasing  $L_T^p(\mu)$ -continuous functionals  $F(\alpha) = \Phi \left( (\alpha(s) + \Psi'_\sigma(\sigma, s) - \frac{\Psi(\sigma, s)}{\sigma})_{s \in [0, t]} \right)$  and  $G(\alpha) = \alpha(t)$  yields that, for every  $t \in [0, T]$ ,

$$\mathbb{E}_{\mathbb{Q}_{(t)}} \left( \Phi \left( (\tilde{X}_s + \Psi'_\sigma(\sigma, s) - \frac{\Psi(\sigma, s)}{\sigma})_{s \in [0, t]}, t \right) \tilde{X}_t \right) \geq 0$$

since  $\mathbb{E}_{\mathbb{Q}_{(t)}} \tilde{X}_t = 0$ . As a consequence,  $f$  is a non-decreasing function.  $\square$

**Corollary 5.1.** *Under the assumptions of Proposition 5.1 on the càdlàg PII  $X$ , the process  $\sigma \mapsto \int_0^T e^{\sigma X_t - \Psi(\sigma, t)} \mu(dt)$ ,  $\sigma \in \mathbb{R}_+$ , is a peacock (with the definition recalled in the introduction).*

**Proof.** Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a convex function and, for every  $A > 0$ , let  $\varphi_A$  be defined by  $\varphi'_A(x) = \varphi(x)$  if  $x \in [-A, A]$  and  $\varphi_A$  affine and differentiable on  $(-\infty, -A] \cup [A, +\infty)$ . It is clear that  $\varphi_A \uparrow \varphi$  since  $\varphi$  takes values in  $(-\infty, +\infty]$ . Then set  $\varphi_{A,\varepsilon} = \mathbb{E} \varphi_A(x + \varepsilon Z)$  where  $Z \stackrel{\mathcal{L}}{\sim} \mathcal{N}(0; 1)$ . The function  $\varphi_{A,\varepsilon}$  is (finite) convex, infinitely differentiable, Lipschitz continuous and converges uniformly to  $\varphi_A$  when  $\varepsilon \rightarrow 0$ . The functional  $F_{A,\varepsilon}(\alpha) = \varphi_{A,\varepsilon}(\int_0^T \alpha(t) \mu(dt))$  satisfies the assumptions of the above Proposition 5.1 so that the function  $f_{A,\varepsilon}$  defined by (5.5) is non-decreasing. Letting  $\varepsilon \rightarrow 0$  and  $A \rightarrow +\infty$  successively implies that the function  $f$  related to the original functional  $F(\alpha) = \varphi(\int_0^T \alpha(t) \mu(dt))$  by (5.5) is non-decreasing which completes the proof.  $\square$

**Remarks.** • In fact this proof remains close in spirit to that proposed in [11]. Roughly speaking we replace the notion of *conditional monotony* used in [11] by a functional co-monotony argument (which also spares a time discretization phase). The notion of conditional monotony and its applications have been extensively investigated in the recent PhD thesis of A. Bogso (see [3]). Conditional monotony has been developed on the basis of finite dimensional distributions of a process but it is clear that a functional version can be derived for (continuous) functionals. Then, when the parameter of interest is time, the connection with functional co-monotony looks clear since it corresponds to a *weak form* of the functional co-monotony principle restricted to couples of functionals of the form  $F(\alpha^t)$  and  $G(\alpha) = g(\alpha(t))$  ( $\alpha^t$  denotes the stopped function  $\alpha$  at  $t$ ).

• As already noticed in [11], specifying  $\mu$  into  $\delta_T$  or  $\frac{1}{T} \lambda_{|[0,T]}$  provides the two main results for peacocks devised from  $e^{\sigma X_t - \Psi(\sigma, t)}$ . When  $\mu = \frac{1}{T} \lambda_{|[0,T]}$  one can combine the above results with some self-similarity property of the PII process  $(X_t)_{t \in [0,T]}$  (if any) to produce other peacocks. So is the case with the seminal example investigated in [4] where the original aim was, for financial purposes, to prove that

$$\left( \frac{1}{t} \int_0^t e^{B_s - \frac{s}{2}} ds \right)_{t \in (0,T]} \text{ is a peacock.}$$

Many other examples of this type are detailed in [11, 3].

APPLICATION TO A CLASS OF ASIAN OPTIONS. As concerns the sensitivity of exotic derivatives, one can derive or retrieve classical results in a Black-Scholes model for the class of Asian options with convex payoff. To be precise, we consider payoffs functionals of the form  $\Phi_T = \varphi(\frac{1}{T} \int_0^T S_s ds)$  where  $\varphi$  is a nonnegative convex function (with linear growth) and  $S_t = s_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t}$ ,  $t \in [0, T]$ , where  $s_0 > 0$ ,  $\sigma > 0$  and  $W$  is a standard Brownian motion ( $r$  is a possibly negative interest rate). The holder of an option contract “written” on this payoff receives in cash at the *maturity*  $T > 0$  the value of the payoff  $\Phi_T$ . Classical arbitrage arguments yield that the premium or price at time 0 of such an option is given by

$$\text{Premium}_0(s_0, \sigma, r, T) = e^{-rT} \mathbb{E} \varphi \left( \frac{1}{T} \int_0^T S_s ds \right).$$

By considering the measure  $\mu(dt) = e^{rt} \frac{1}{T} \lambda_{|[0,T]}(dt)$ , one derives from Corollary 5.1 that  $\sigma \mapsto \text{Premium}_0(s_0, \sigma, r, T)$  is non-decreasing. When  $r = 0$  a change of variable shows that the premium is also non-decreasing as a function of the maturity  $T$ .

### 5.3 Application to bounds on barrier options

Let  $S = (S_t)_{t \in [0, T]}$  be a càdlàg nonnegative stochastic process defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , modeling the price dynamics of a risky asset. We will assume that  $\mathbb{P}$  is a pricing measure in the sense that derivatives products “written” on the asset  $S$  are priced under  $\mathbb{P}$ . In particular we do not ask  $\mathbb{P}$  to be risk-neutral. We assume for convenience that the zero-coupon bond (also known as the riskless asset) is constant equal to 1 (or equivalently that all interest rates are constant equal to 0) but what follows remains true if this bond is deterministic.

For notational convenience, for a càdlàg function  $\alpha : [0, T] \rightarrow \mathbb{R}$ , we will denote by  $_*\alpha_t := \inf_{s \leq t} \alpha_s$ , ( $t \in [0, T]$ ), the running minimum of the function  $\alpha$  and by  $\alpha_t^* := \sup_{s \leq t} \alpha_s$  its running maximum process.

We assume throughout this section that *the asset price dynamics  $(S_t)_{t \in [0, T]}$  satisfies a functional co-monotony principle*. This seems to be a quite natural and general assumption given the various classes of examples detailed above.

We will focus on *Down-and-In Call* and *Down-and-Out Call* with maturity  $T$ . The payoff functional of a *Down-and-In Call* with maturity  $T$  is defined for every strike price  $K > 0$  and every barrier  $L \in (0, S_0)$  by

$$F_{D\&I}(\alpha) = (\alpha(T) - K)_+ \mathbf{1}_{\{*_\alpha T \leq L\}}.$$

This means that the holder of the *Down-and-In Call* written on the risky asset  $S$  contract receives  $F_{D\&I}(S)$  at the maturity  $T$ , namely  $S_T - K$  at the maturity  $T > 0$  provided this flow is positive and the asset attained at least once the (low) barrier  $H$  between 0 and  $T > 0$ .

The premium of such a contract at time 0 is defined by  $\text{Call}_{D\&In}(K, L, T) = \mathbb{E} F_{D\&I}(S)$ . We will denote by  $\text{Call}(K, T) = \mathbb{E} (S_T - K)_+$  the premium of the regular (or vanilla) *Call* option with strike  $K$  (and maturity  $T$ ).

**Proposition 5.2.** *If the nonnegative càdlàg process  $(S_t)_{t \in [0, T]}$  satisfies a functional co-monotony principle, then the following semi-universal bound holds:*

$$\text{Call}_{D\&In}(K, L, T) \leq \text{Call}(K, T) \mathbb{P}(*S_T \leq L).$$

**Proof.** For every  $\alpha \in \mathcal{D}([0, T], \mathbb{R})$  and every  $\varepsilon > 0$ , we have

$$F_{D\&I}(\alpha) \leq (\alpha(T) - K)_+ \left( \left( 1 - \frac{*_\alpha T - L}{\varepsilon} \right)_+ \wedge 1 \right).$$

The two functionals involved in the product of the right hand side of the above equation are clearly anti-monotone, nonnegative and continuous with respect to the sup-norm, consequently

$$\text{Call}_{D\&In}(K, L, T) \leq \text{Call}(K, T) \mathbb{E} \left( \left( 1 - \frac{*_S T - L}{\varepsilon} \right)_+ \wedge 1 \right).$$

The result follows by letting  $\varepsilon \rightarrow 0$  owing to Fatou’s Lemma.  $\square$

As concerns the *Down-and-Out Call* with payoff

$$F_{D\&O}(\alpha) = (\alpha(T) - K)_+ \mathbf{1}_{\{*_\alpha T > L\}}$$

for which the holder of the option receives  $S_T - K$  at the maturity  $T > 0$  if this flow is positive and if the asset did not attain the (low) level  $H$  between 0 and  $T > 0$ , one gets, either by a direct approach or by using the obvious parity equation  $\text{Call}_{D\&In}(K, L, T) + \text{Call}_{D\&Out}(K, L, T) = \text{Call}(K, T)$ ,

$$\text{Call}_{D\&Out}(K, H, T) \geq \text{Call}(K, T) \mathbb{P}(*S_T > L).$$

Similar bounds can be derived for for *Up-and-In* and *Up-and-Out Calls* with barrier  $L > S_0$  (and strike  $K$ ), namely

$$\text{Call}_{U\&Out}(K, L, T) \leq \text{Call}(K, T)\mathbb{P}(*S_T \leq L)$$

and

$$\text{Call}_{U\&In}(K, L, T) \geq \text{Call}(K, T)\mathbb{P}(*S_T > L).$$

#### 5.4 A remark on running extrema

If a càdlàg process  $X = (X_t)_{t \in [0, T]}$  satisfies a co-monotony principle and  $X_T$  and  $\sup_{t \in [0, T]} X_t$  have no atom so that, for every  $x, y \in \mathbb{R}$ ,  $x \leq y$ ,  $\alpha \mapsto (\mathbf{1}_{\{\alpha(T) \geq x\}}, \mathbf{1}_{\{\sup_{t \in [0, T]} \alpha(t) \geq y\}})$  is  $\mathbb{P}_X$ -a.s.  $\|\cdot\|_{\text{sup}}$ -continuous, then

$$\forall y \in \mathbb{R}, \quad \mathbb{P}\left(\sup_{t \in [0, T]} X_t \geq y\right) = \inf_{x \leq y} \mathbb{P}\left(\sup_{t \in [0, T]} X_t \geq y \mid X_T \geq x\right).$$

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