On generalized Cramér-Rao inequalities, generalized Fisher informations and characterizations of generalized q-Gaussian distributions

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Abstract. This paper deals with Cramér-Rao inequalities in the context of nonextensive statistics and in estimation theory. It gives characterizations of generalized q-Gaussian distributions, and introduces generalized versions of Fisher information. The contributions of this paper are (i) the derivation of new extended Cramér-Rao inequalities for the estimation of a parameter, involving general q-moments of the estimation error, (ii) the derivation of Cramér-Rao inequalities saturated by generalized q-Gaussian distributions, (iii) the definition of generalized Fisher informations, (iv) the identification and interpretation of some prior results, and finally, (v) the suggestion of new estimation methods.

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1. Introduction

It is well known that the Gaussian distribution has a central role with respect to classical information measures and inequalities. For instance, the Gaussian distribution maximizes the entropy over all distributions with the same variance; see [1, Lemma 5]. Similarly, the Cramér-Rao inequality, e.g. [1, Theorem 20], shows that the minimum of the Fisher information over all distributions with a given variance is attained for the Gaussian distribution. Generalized q-Gaussian distributions arise as the maximum entropy solution in Tsallis' nonextensive thermostatistics, which is based on the use of the generalized Tsallis entropy. The Generalized q-Gaussian distributions also appear in other fields, namely as the solution of non-linear diffusion equations, or as the distributions that saturate some sharp inequalities in functional analysis. Furthermore, the generalized q-Gaussian distributions form a versatile family that can describe problems with compact support as well as problems with heavy tailed distributions.

Since the standard Gaussian is both a maximum entropy and a minimum Fisher information distribution over all distributions with a given variance, a natural question is to find whether this can be extended to include the case of the generalized q-gaussian distribution, thus improving the information theoretic characterization of these generalized q-Gaussians. This question amounts to look for a definition of a generalized Fisher information, that should include the standard one as a particular case, and whose minimum over all distributions with a given variance is a q-Gaussian. More generally, this should lead to an extension of the Cramér-Rao inequality saturated by q-Gaussian distributions.

Several extensions of the Fisher information and of the Cramér-Rao inequality have been proposed in the literature. In particular, the beautiful work by Lutwak, Yang and Zhang [2] gives an extended Fisher information and a Cramér-Rao inequality saturated by q-Gaussian distributions. In an interesting work, Furuichi [3] defines another generalized Fisher information and a Cramér-Rao inequality saturated by q-Gaussian distributions. In this latter contribution, the statistical expectations are computed with respect to escort distributions. These escort distributions are one parameter deformed versions of the original distributions, and are very useful in different formulations of nonextensive statistics. In the present paper, we will recover these results and show that the two generalized Fisher informations above and the associated Cramér-Rao inequalities are actually linked by a simple transformation.

In section 2, we give the main definitions and describe the ingredients that are used in the paper. In particular, we give the definition and describe the importance of generalized q-Gaussians, we define the notion of escort distributions, and finally give some definitions related to deformed calculus in nonextensive statistics.

The Cramér-Rao inequalities indicated above are inequalities characterizing the probability distribution, and the Fisher information is the Fisher information of the distribution. Actually, the Fisher information is defined in a broader context as the information about a parameter of a parametric family of distributions. In the special case of a location parameter, it reduces to the Fisher information of the distribution. The Cramér-Rao inequality appears in the context of estimation theory, and as it is well known, defines a lower bound on the variance of any estimator of a parameter. In section 3, we describe the problem of estimation, recall the classical Cramér-Rao inequality, and show that the standard Cramér-Rao inequality can be extended in two directions. First we consider moments of any order of the estimation error, and second we use generalized moments computed with respect to an escort distributions. This lead us to two new Cramér-Rao inequalities for a general parameter, together with their equality conditions. In this context, the definitions of the generalized Fisher informations will pop up very naturally. For a very general definition of escort distributions, we also recover a general Cramér-Rao inequality given in a deep paper by Naudts [4].

In section 4, we examine the special case of a translation parameter and show, as an immediate consequence, that the general results enable to easily recover two Cramér-Rao inequalities that characterize the generalized q-Gaussian distributions. So doing, we recover the definitions of generalized Fisher informations previously introduced by Lutwak *et al.* [2] and Furuichi [3]. Furthermore, we show that the related Cramér-Rao inequalities, which are similar to those of [2] and [3], are saturated by the generalized q-Gaussians. Finally, in section 5, we discuss some new estimation rules that emerge from this setting, and in particular point out a connection to the MLq-likelihood method that has been introduced recently, c.f. [5, 6].

2. Definitions and main ingredients

2.1. Generalized q-Gaussian

The generalized Gaussian distribution is a family of distributions which includes the standard Gaussian as a special case. These generalized Gaussians appear in statistical physics, where they are the maximum entropy distributions of the nonextensive thermostatistics [7]. In this context, these distributions have been observed to present a significant agreement with experimental data. They are also analytical solutions of actual physical problems, see [8, 9] [10], [11], and are sometimes known as Barenblatt-Pattle functions, following their identification by [12, 13]. Let us also mention that the generalized Gaussians are the one dimensional versions of explicit extremal functions of Sobolev, log-Sobolev or Gagliardo–Nirenberg inequalities on $\mathbb{R}^n$, as was shown by [14, 15] for $n \geq 2$ and by [16] for $n \geq 1$.

Definition 1. Let x be a random variable on $\mathbb{R}$. For $\alpha \in (0, \infty)$, γ a real positive parameter and $q > 1-\alpha$, the generalized q-Gaussian with scale parameter γ has the symmetric probability density

$$G_\gamma(x) = \begin{cases} \frac{1}{Z(\gamma)}\left(1-(q-1)\,\gamma|x|^\alpha\right)_+^{\frac{1}{q-1}} & \text{for } q \neq 1 \\ \frac{1}{Z(\gamma)}\exp\left(-\gamma|x|^\alpha\right) & \text{if } q = 1 \end{cases} \tag{1}$$

where we use the notation $(x)_+ = \max\{x, 0\}$, and where $Z(\gamma)$ is the partition function such that $G_\gamma(x)$ integrates to one:

$$Z(\gamma) = \frac{2}{\alpha}\,(\gamma)^{-\frac{1}{\alpha}} \times \begin{cases} (1-q)^{-\frac{1}{\alpha}} B\left(\frac{1}{\alpha}, -\frac{1}{q-1}-\frac{1}{\alpha}\right) & \text{for } 1-\alpha < q < 1 \\ (q-1)^{-\frac{1}{\alpha}} B\left(\frac{1}{\alpha}, \frac{1}{q-1}+1\right) & \text{for } q > 1 \\ \Gamma\left(\frac{1}{\alpha}\right) & \text{if } q = 1. \end{cases} \tag{2}$$

where $B(x, y)$ is the Beta function.

For $q > 1$, the density has a compact support, while for $q \leq 1$ it is defined on the whole real line and behaves as a power distribution for $|x| \to \infty$. Notice that the name generalized Gaussian is sometimes restricted to the case $q = 1$ above. In this case, the standard Gaussian is recovered with $\alpha = 2$.

2.2. Escort distributions

The escort distributions are an essential ingredient in the nonextensive statistics context. Actually, the escort distributions have been introduced as an operational tool in the context of multifractals, c.f. [17], [18], with interesting connections with the standard thermodynamics. Discussion of their geometric properties can be found in [19, 20]. Escort distributions also prove useful in source coding, as noticed in [21]. They are defined as follows.

If $f(x)$ is a univariate probability density, then its escort distribution $g(x)$ of order q, $q \geq 0$, is defined by

$$g(x) = \frac{f(x)^q}{\int f(x)^q \mathrm{d}x}, \tag{3}$$

provided that Golomb's "information generating function" [22]

$$M_q[f] = \int f(x)^q \mathrm{d}x \tag{4}$$

is finite.

Given that $g(x)$ is the escort of $f(x)$, we see that $f(x)$ is itself the escort of $g(x)$ of order $\bar{q} = 1/q$.

Accordingly, the (absolute) generalized q-moment of order p is defined by

$$m_{p,q}[f] := E_q\left[|x|^p\right] = \int |x|^p g(x) \mathrm{d}x = \frac{\int |x|^p f(x)^q \mathrm{d}x}{\int f(x)^q \mathrm{d}x}, \tag{5}$$

where $E_q[.]$ denotes the statistical expectation with respect to the escort of order q. Of course, standard moments are recovered in the case $q = 1$.

2.3. Deformed functions and algebra

In Tsallis statistics, it has appeared convenient to use deformed algebra and calculus, c.f. [23, 24]. The q-exponential function is defined by

$$\exp_q(x) := (1 + (1-q)x)_+^{\frac{1}{1-q}}, \tag{6}$$

while its inverse function, the q-logarithm is defined by

$$\ln_q(x) := \frac{x^{1-q} - 1}{1-q}. \tag{7}$$

When q tends to 1, both quantities reduce to the standard functions $\exp(x)$ and $\ln(x)$ respectively. In the following, we will use the notation $\bar{q} = 1/q$ that already appeared above in connection with escort distributions, and the notation $q_* = 2 - q$ that changes the quantity $(1 - q_*)$ into $(q-1)$, e.g. $\exp_{q_*}(x) := (1 + (q-1)x)_+^{\frac{1}{q-1}}$. We note the following expressions for the derivatives of deformed logarithms:

$$\frac{\partial}{\partial\theta} \ln_q\left(f(x;\theta)\right) = \frac{\frac{\partial}{\partial\theta} f(x;\theta)}{f(x;\theta)} f(x;\theta)^{1-q} \quad \text{and} \quad \frac{\partial}{\partial\theta} \ln_{q_*}\left(f(x;\theta)\right) = \frac{\frac{\partial}{\partial\theta} f(x;\theta)}{f(x;\theta)} f(x;\theta)^{q-1}. \tag{8}$$

The q-product is a deformed version of the standard product such that standard properties of exponential and logarithm functions still hold for their deformed versions. The q-product is defined by

$$x \otimes_q y := \left(x^{1-q} + y^{1-q} - 1\right)^{\frac{1}{1-q}} \tag{9}$$

and gives the identities

$$\ln_q(x \otimes_q y) = \ln_q(x) + \ln_q(x) \quad \text{and} \quad \exp_q(x+y) = \exp_q(x) \otimes_q \exp_q(y) \tag{10}$$

2.4. Fisher information

The importance of Fisher information as a measure of the information about a parameter in a distribution is well known, as exemplified in estimation theory by the Cramér-Rao bound which provides a fundamental lower bound on the variance of an estimator. The statement of the standard Cramér-Rao inequality, as well as several extensions, will be given in section 3.

It might be also useful to note that Fisher information is used as a method of inference and understanding in statistical physics and biology, as promoted by Frieden [25, 26]. It is also used as a tool for characterizing complex signals or systems, with applications, e.g. in geophysics, in biology, in reconstruction or in signal processing. Information theoretic inequalities involving Fisher information have attracted lot of attention for characterizing statistical systems through their localization in information planes, e.g. the Fisher-Shannon information plane [27, 28] or the Cramér-Rao information plane [29].

Definition 2. Let $f(x;\theta)$ denote a probability density defined over a subset X of $\mathbb{R}$, and $\theta \in \Theta$ a real parameter. Suppose that $f(x;\theta)$ is differentiable with respect to θ. Then, the Fisher information in the density f about the parameter θ is defined as

$$I_{2,1}[f,\theta] = \int_X \left(\frac{\partial \ln f(x;\theta)}{\partial\theta}\right)^2 f(x;\theta)\mathrm{d}x. \tag{11}$$

When θ is the location parameter, i.e. $f(x;\theta) = f(x-\theta)$, the Fisher information, expressed at $\theta = 0$, becomes a characteristic of the distribution: the Fisher information of the distribution:

$$I_{2,1}[f] = \int_X \left(\frac{\mathrm{d}\ \ln f(x)}{\mathrm{d}x}\right)^2 f(x)\mathrm{d}x. \tag{12}$$

The meaning of the subscripts in the definition will appear in the following.

3. Generalized Cramér-Rao inequalities

In this section, we begin by recalling the context of estimation, the role of Fisher information and the statement of the standard Cramér-Rao theorem. Then, we show how this can be extended to higher moments, and to generalized moments computed with respect to an escort distribution.

3.1. The standard Cramér-Rao inequality

The problem of estimation, in a few words, consists in finding a function $\hat{\theta}(x)$ of the data x, that approaches the unknown value of a characteristic parameter θ (e.g. location, scale or shape parameter) of the probability density of these data.

A standard statement of the Cramér-Rao inequality is recalled now.

Proposition 1. *[Cramér-Rao inequality] Let $f(x;\theta)$ be a univariate probability density function defined over a subset X of $\mathbb{R}$, and $\theta \in \Theta$ a parameter of the density. If $f(x)$ is continuously differentiable with respect to θ, satisfies some regularity conditions that enable to interchange integration with respect to x and differentiation with respect to θ, then for any estimator $\hat{\theta}(x)$ of the parameter θ,*

$$E\left[\left|\hat{\theta}(x)-\theta\right|^2\right] I_{2,1}[f;\theta] \geq \left|1+\frac{\partial}{\partial\theta}E\left[\hat{\theta}(x)-\theta\right]\right|^2. \tag{13}$$

When the estimator is unbiased, that is if $E\left[\hat{\theta}(x)\right]=\theta$, then the inequality reduces to

$$E\left[\left|\hat{\theta}(x)-\theta\right|^2\right] I_{2,1}[f] \geq 1. \tag{14}$$

The estimator is said efficient if it is unbiased and saturates the inequality. This can happen if the probability density and the estimator satisfy $\frac{\partial}{\partial\theta}\ln f(x;\theta)=k(\theta)\left(\hat{\theta}(x)-\theta\right)$.

3.2. Generalized Cramér-Rao inequalities for higher moments and escort distributions

The Fisher information is usually defined as the second order moment of the score function, the derivative of the log-likelihood, but this definition can be extended to other moments, leading to a generalized version of the Cramér-Rao inequality. This extension, which seems not well known, can be traced back to Barakin [30, Corollary 5.1]. This generalized Fisher information, together with the extension of the Cramér-Rao inequality, has also been exhibited by Vajda [31] as a limit of a χ^α-divergence. We will recover this general Cramér-Rao inequality, as well as the standard one, as a particular case of our new q-Cramér-Rao inequalities. The main idea here is to compute the bias, or a moment of the error, with respect to an escort distribution $g(x;\theta)$ of $f(x;\theta)$ instead of the initial distribution. If we first consider the q-bias defined by $B_q(\theta) := E_q\left[\hat{\theta}(x)-\theta\right]$, we have the following general statement

Theorem 1. *[Generalized q-Cramér-Rao inequality] - Let $f(x;\theta)$ be a univariate probability density function defined over a subset X of $\mathbb{R}$, and $\theta \in \Theta$ a parameter of the density. Assume that $f(x;\theta)$ is a jointly measurable function of x and θ, is integrable with respect to x, is absolutely continuous with respect to θ, and that the derivative with respect to θ is locally integrable. Assume also that $q>0$ and that $M_q[f;\theta]$ is finite. For any estimator $\hat{\theta}(x)$ of θ, we have*

$$E\left[\left|\hat{\theta}(x)-\theta\right|^\alpha\right]^{\frac{1}{\alpha}} I_{\beta,q}[f;\theta]^{\frac{1}{\beta}} \geq \left|1+\frac{\partial}{\partial\theta}E_q\left[\hat{\theta}(x)-\theta\right]\right| \tag{15}$$

with α and β Hölder conjugates of each other, i.e. $\alpha^{-1}+\beta^{-1}=1$, $\alpha \geq 1$, and where the quantity

$$I_{\beta,q}[f;\theta] = E\left[\left|\frac{f(x;\theta)^{q-1}}{M_q[f;\theta]}\frac{\partial}{\partial\theta}\ln\left(\frac{f(x;\theta)^q}{M_q[f;\theta]}\right)\right|^\beta\right] = \left(\frac{q}{M_q[f;\theta]^{\frac{1}{q}}}\right)^\beta E\left[\left|\frac{\partial}{\partial\theta}\ln_{q*}\left(\frac{f(x;\theta)}{M_q[f;\theta]^{\frac{1}{q}}}\right)\right|^\beta\right] \tag{16}$$

is the generalized Fisher information of order (β, q) *on the parameter* θ*. The equality case is obtained if*

$$\frac{q}{M_q[f;\theta]^{\frac{1}{q}}} \frac{\partial}{\partial\theta} \ln_{q*}\left(\frac{f(x;\theta)}{M_q[f;\theta]}\right) = c(\theta)\,\mathrm{sign}\left(\hat{\theta}(x) - \theta\right)\left|\hat{\theta}(x) - \theta\right|^{\alpha-1}, \tag{17}$$

with $c(\theta) > 0$.

Observe that in the case $q = 1$, $M_1[f;\theta] = 1$ and the deformed logarithm reduces to the standard one. Immediately, we obtain the extended Barakin-Vajda Cramér-Rao inequality in the $q = 1$ case, as well as the standard Cramér-Rao inequality (13) when $q = 1$ and $\alpha = \beta = 2$.

Corollary 1. *[Barakin-Vajda Cramér-Rao inequality] - Under the same hypotheses as in Theorem 1, we have*

$$E\left[\left|\hat{\theta}(x) - \theta\right|^{\alpha}\right]^{\frac{1}{\alpha}} I_{\beta,1}[f;\theta]^{\frac{1}{\beta}} \geq \left|1 + \frac{\partial}{\partial\theta} E\left[\hat{\theta}(x) - \theta\right]\right| \tag{18}$$

with

$$I_{\beta,1}[f;\theta] = E\left[\left|\frac{\partial}{\partial\theta}\ln\left(f(x;\theta)\right)\right|^{\beta}\right] \tag{19}$$

and equality if $\frac{\partial}{\partial\theta}\ln\left(f(x;\theta)\right) = c(\theta)\mathrm{sign}\left(\hat{\theta}(x) - \theta\right)\left|\hat{\theta}(x) - \theta\right|^{\alpha-1}$.

This inequality generalizes the standard $\alpha = 2$ Cramér-Rao inequality to moments of any order $\alpha > 1$.

Proof. [of Theorem 1] Consider the derivative of the q-bias

$$\frac{\partial}{\partial\theta} B_q(\theta) = \frac{\partial}{\partial\theta}\int_X \left(\hat{\theta}(x) - \theta\right)\frac{f(x;\theta)^q}{M_q[f;\theta]}\mathrm{d}x. \tag{20}$$

The regularity conditions in the statement of the theorem enable to interchange integration with respect to x and differentiation with respect to θ, so that

$$\begin{aligned}\frac{\partial}{\partial\theta}\int_X \left(\hat{\theta}(x) - \theta\right)\frac{f(x;\theta)^q}{M_q[f;\theta]}\mathrm{d}x = &-\int_X \frac{f(x;\theta)^q}{M_q[f;\theta]}\mathrm{d}x \\ &+ \int_X \left(\hat{\theta}(x) - \theta\right)\left[q\frac{\frac{\partial}{\partial\theta}f(x;\theta)}{f(x;\theta)} - \frac{\frac{\partial}{\partial\theta}M_q[f;\theta]}{M_q[f;\theta]}\right]\frac{f(x;\theta)^{q-1}}{M_q[f;\theta)]}f(x;\theta)\,\mathrm{d}x,\end{aligned}$$

or, since the first term on the right is equal to -1 and since the term in bracket can be written as the derivative of the logarithm of the escort distribution of $f(x;\theta)$,

$$1 + \frac{\partial}{\partial\theta}B_q(\theta) = \int_X \left(\hat{\theta}(x) - \theta\right)\frac{\partial}{\partial\theta}\ln\left(\frac{f(x;\theta)^q}{M_q[f;\theta]}\right)\frac{f(x;\theta)^{q-1}}{M_q[f;\theta]}f(x;\theta)\,\mathrm{d}x. \tag{21}$$

Consider the absolute value of the integral above, which is less than the integral of the absolute value of the integrand. By the Hölder inequality, with $\alpha > 1$ and β its Hölder conjugate, we then have

$$\left|1 + \frac{\partial}{\partial\theta}B_q(\theta)\right| \leq \left(\int_X \left|\hat{\theta}(x) - \theta\right|^{\alpha} f(x;\theta)\,\mathrm{d}x\right)^{\frac{1}{\alpha}} \left(\int_X \left|\frac{\partial}{\partial\theta}\ln\left(\frac{f(x;\theta)^q}{M_q[f;\theta]}\right)\frac{f(x;\theta)^{q-1}}{M_q[f;\theta]}\right|^{\beta} f(x;\theta)\,\mathrm{d}x\right)^{\frac{1}{\beta}} \tag{22}$$

which is the generalized Cramér-Rao inequality (15). By elementary calculations, we can identify that the generalized Fisher information above can also be expressed as the derivative of the q_*-logarithm, as indicated in the right side of (16). Finally, the case of equality follows from the condition of equality in the Hölder inequality, and from the requirement that the integrand in (21) is non negative: this gives

$$\left| \frac{q}{M_q[f;\theta]^{\frac{1}{q}}} \frac{\partial}{\partial\theta} \ln_{q*}\left(\frac{f(x;\theta)}{M_q[f;\theta]^{\frac{1}{q}}}\right)\right|^{\beta} = k(\theta)\left|\hat{\theta}(x)-\theta\right|^{\alpha} \quad \text{and} \quad \left(\hat{\theta}(x)-\theta\right)\frac{\partial}{\partial\theta}\ln_{q*}\left(\frac{f(x;\theta)}{M_q[f;\theta]^{\frac{1}{q}}}\right) \geq 0, \tag{23}$$

which can be combined into the single condition (17), with $c(\theta) = k(\theta)^{\frac{1}{\beta}} > 0$. □

By the properties of escort distributions, we can also obtain an inequality that involves the q-moment of the error $\left(\hat{\theta}(x)-\theta\right)$ instead of the standard moment. Indeed, if $g(x;\theta)$ denotes the escort distribution of $f(x,\theta)$ of order q, then, as already mentioned, $f(x;\theta)$ is the escort of order $\bar{q}$ of $g(x;\theta)$, and

$$f(x;\theta) = \frac{g(x;\theta)^{\bar{q}}}{N_q[g;\theta]}, \tag{24}$$

with $N_q[g;\theta] = \int_X g(x;\theta)^{\bar{q}}\mathrm{d}x = M_q[f;\theta]^{-\bar{q}}$.

With these notations, we see that the expectation with respect to f is the $\bar{q}$-expectation with respect to g, and that the q-expectation with respect to f is simply the standard expectation with respect to g. On the other hand, we also have a simple property that links the deformed logarithms of orders q_* and $\bar{q}$:

Proposition 2. *Let $b > 0$, $a = b^q$. With $q_* = 2-q$ and $\bar{q} = 1/q$, the following equality holds:*

$$\ln_{\bar{q}}(a) = q\ln_{q_*}(b). \tag{25}$$

Proof. By direct verification. □

In particular, we note that with $a = g(x;\theta) = b^q = \frac{f(x;\theta)^q}{M_q[f;\theta]}$, we have

$$\ln_{\bar{q}}(g) = q\ln_{q_*}(f(x;\theta)/M_q[f;\theta]^{\frac{1}{q}}). \tag{26}$$

With these elements, the simple expression of the extended Cramér-Rao inequality (15) in terms of the escort $g(x;\theta)$ of $f(x,\theta)$ yields the following corollary.

Corollary 2. *[Generalized escort-q-Cramér-Rao inequality] - Under the same hypotheses as in Theorem 1, we have*

$$E_{\bar{q}}\left[\left|\hat{\theta}(x)-\theta\right|^{\alpha}\right]^{\frac{1}{\alpha}} \bar{I}_{\beta,\bar{q}}[g;\theta]^{\frac{1}{\beta}} \geq \left|1+\frac{\partial}{\partial\theta}E\left[\hat{\theta}(x)-\theta\right]\right| \tag{27}$$

with α and β Hölder conjugates of each other, i.e. $\alpha^{-1}+\beta^{-1}=1$, $\alpha \geq 1$, and where the quantity

$$\bar{I}_{\beta,\bar{q}}[g;\theta] = (N_q[g;\theta])^{\beta}\, E_{\bar{q}}\left[\left|g(x;\theta)^{1-\bar{q}}\frac{\partial}{\partial\theta}\ln(g)\right|^{\beta}\right] = (N_q[g;\theta])^{\beta}\, E_{\bar{q}}\left[\left|\frac{\partial}{\partial\theta}\ln_{\bar{q}}(g(x;\theta)\right|^{\beta}\right] \tag{28}$$

is the generalized Fisher information of order (β, q) on the parameter θ. The equality case is obtained if

$$\frac{\partial}{\partial\theta}\ln_{\bar{q}}\left(g(x;\theta)\right) = c(\theta)\text{sign}\left(\hat{\theta}(x)-\theta\right)\left|\hat{\theta}(x)-\theta\right|^{\alpha-1}. \tag{29}$$

Note that this is simply a rewriting of the initial extended expression of extended Cramér-Rao inequality (15) in terms of the escort $g(x;\theta)$ of $f(x,\theta)$. The generalized Fisher information $\bar{I}_{\beta,\bar{q}}[g]$ is the same as $I_{\beta,q}[f;\theta]$, up to the rewriting in terms of g. The second Cramér-Rao inequality in (27) is nice because it exhibits a fundamental estimation bound for a q-moment on the estimation error, thus making a bridge between concepts in estimation theory and the tools of nonextensive thermostatistics. What we learn from this result is the fact that for all estimators with a given bias, the best estimator that minimizes the q-moment of the error is lower bounded by the inverse of the (generalized) Fisher information.

We shall also discuss in some more details the case of equality in the two Cramér-Rao inequality. It appears that the general solution that saturates the bounds is in the form of a deformed q-exponential.

Consider the conditions of equality (17) and (30) in the two Cramér-Rao inequalities. In the first case, we have that the distribution which attains the bound shall satisfy

$$\frac{\partial}{\partial\theta}\ln_{q*}\left(\frac{f(x;\theta)}{M_q[g;\theta]^{\frac{1}{q}}}\right) = c(\theta)\,\text{sign}\left(\hat{\theta}(x)-\theta\right)\left|\hat{\theta}(x)-\theta\right|^{\alpha-1}, \tag{30}$$

where $c(\theta)$ is a positive function. The general solution of this differential equation has the form

$$f(x;\theta) \propto \exp_{q_*}\left(\int_\Theta c(\theta)\,\text{sign}\left(\hat{\theta}(x)-\theta\right)\left|\hat{\theta}(x)-\theta\right|^{\alpha-1}\mathrm{d}\theta\right). \tag{31}$$

Similarly, in the second Cramér-Rao inequality, we get that

$$g(x;\theta) \propto \exp_{\bar{q}}\left(\int_\Theta c(\theta)\,\text{sign}\left(\hat{\theta}(x)-\theta\right)\left|\hat{\theta}(x)-\theta\right|^{\alpha-1}\mathrm{d}\theta\right), \tag{32}$$

which is the escort of $f(x;\theta)$.

3.3. Yet another pair of Cramér-Rao inequalities

It is quite immediate to extend the Cramér-Rao inequalities above to an even broader context: let us consider a general pair of escort distributions linked by say $g = \phi(f)$, with $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ monotone increasing, and $f = \phi^{-1}(g) = \psi(g)$. Denote E_ϕ and E_ψ the corresponding expectations, e.g. $E_\phi\left[|x|^\alpha\right] = \int_X |x|^\alpha \phi\left(f(x)\right)\mathrm{d}x$. Following the very same steps as in Theorem 1, we readily arrive at

$$E\left[\left|\hat{\theta}(x)-\theta\right|^\alpha\right]^{\frac{1}{\alpha}} I_{\beta,\phi}[f;\theta]^{\frac{1}{\beta}} \geq \left|1+\frac{\partial}{\partial\theta}E_\phi\left[\hat{\theta}(x)-\theta\right]\right| \tag{33}$$

where

$$I_{\beta,\phi}[f;\theta] = \int_X f(x;\theta)\left|\frac{\partial\phi(f)/\partial\theta}{f(x;\theta)}\right|^\beta \mathrm{d}x = E\left[\left|\frac{\partial\phi(f)/\partial\theta}{f(x;\theta)}\right|^\beta\right]. \tag{34}$$

Then, the analog of corollary 2 takes the form

$$E_\psi \left[\left| \hat{\theta}(x) - \theta \right|^\alpha \right]^{\frac{1}{\alpha}} \bar{I}_{\beta,\psi}[g;\theta]^{\frac{1}{\beta}} \geq \left| 1 + \frac{\partial}{\partial\theta} E\left[\hat{\theta}(x) - \theta \right] \right|, \tag{35}$$

with

$$\bar{I}_{\beta,\psi}[g;\theta] = \int_X \psi(g) \left| \frac{\partial g/\partial\theta}{\psi(g)} \right|^\beta \mathrm{d}x = E_\psi \left[\left| \frac{\partial}{\partial\theta} \ln_\psi (g) \right|^\beta \right], \tag{36}$$

where the function $\ln_\psi(u)$ is defined by $\ln_\psi(u) := \int_1^u \frac{1}{\psi(x)} \mathrm{d}x$, and where the equality case in (35) occurs if and only if $g(x;\theta) \propto \exp_\psi \int_\Theta k(\theta) \left| \hat{\theta}(x) - \theta \right|^{\alpha-1} \mathrm{d}\theta$, with $\exp_\psi$ the inverse function of $\ln_\psi$. During the writing of this paper, we realized that a result similar to (35), though obtained using a different approach and with slightly different notations, has been given in a deep paper by Naudts [4]. In this interesting work, the author studied general escort distributions and introduced, in particular, the notion of ψ-exponential families.

4. Inequalities in the case of a translation family

In the particular case of a translation parameter, our q-Cramér-Rao inequalities reduce to two interesting inequalities that characterize the q-Gaussian distributions.

Let θ be a location parameter, and define by $f(x;\theta)$ the family of density $f(x;\theta) = f(x-\theta)$. In this case, we have that $\frac{\partial}{\partial\theta} f(x;\theta) = -\frac{\mathrm{d}}{\mathrm{d}x} f(x-\theta)$, and the Fisher information becomes a characteristic of the information in the distribution.

Let us denote by μ_q the q-mean of $f(x)$, that is of the escort distribution $g(x)$ associated with $f(x)$. We immediately have that the q-mean of $f(x;\theta)$ is $(\mu_q+\theta)$. Thus, the estimator $\hat{\theta}(x) = x - \mu_q$ is a q-unbiased estimator of θ, since $E_q\left[\hat{\theta}(x) - \theta\right] = 0$. Similarly, if we choose $\hat{\theta}(x) = x$, the estimator will be biased, $E_q\left[\hat{\theta}(x) - \theta\right] = \mu_q$, but independent of θ, so that the derivative of the bias with respect to θ is zero. Finally, let us observe that for a translation family, the information generating function $M_q[f;\theta] = M_q[f]$ is independent of the parameter θ.

4.1. Cramér-Rao characterizations of generalized q-Gaussian distributions

These simple observations can be applied directly to our two Cramér-Rao inequalities (15) and (27). This is stated in the two following corollaries.

Corollary 3. *[Generalized q-Cramér-Rao inequality] - Let $f(x)$ be a univariate probability density function defined over a subset X of $\mathbb{R}$. Assume that $f(x)$ is a measurable function of x, is integrable with respect to x. Assume also that $q > 0$ and that $M_q[f]$ is finite. The following generalized Cramér-Rao inequality then holds:*

$$E\left[|x|^\alpha\right]^{\frac{1}{\alpha}} I_{\beta,q}[f]^{\frac{1}{\beta}} \geq 1 \tag{37}$$

with α and β Hölder conjugates of each other, i.e. $\alpha^{-1} + \beta^{-1} = 1$, $\alpha \geq 1$, and where the quantity

$$I_{\beta,q}[f] = E\left[\left| \frac{q}{M_q[f]} f(x)^{q-1} \frac{\mathrm{d}}{\mathrm{d}x} \ln\left(f(x)\right) \right|^\beta \right] = \left(\frac{q}{M_q[f]} \right)^\beta E\left[\left| \frac{\mathrm{d}}{\mathrm{d}x} \ln_{q*}\left(f(x)\right) \right|^\beta \right] \tag{38}$$

is the generalized Fisher information of order (β, q) of the distribution. The equality case is obtained if

$$f(x) \propto \exp_{q_*}\left(-\gamma\,|x|^{\alpha}\right), \text{ with } \gamma > 0. \tag{39}$$

Proof. This is a direct consequence of (15), with $\hat{\theta}(x) = x$ and $\theta = 0$. The case of equality is obtained by integration and simplifications of (30), where the derivative with respect to θ is replaced by the derivative with respect to x; with $\frac{\partial}{\partial\theta} f(x;\theta) = -\frac{\mathrm{d}}{\mathrm{d}x} f(x-\theta)$. □

Corollary 4. *[generalized escort-q-Cramér-Rao inequality] - Under the same hypotheses as in Theorem 1, we have*

$$E_{\bar{q}}\left[|x|^{\alpha}\right]^{\frac{1}{\alpha}}\, \bar{I}_{\beta,\bar{q}}[g]^{\frac{1}{\beta}} \geq 1 \tag{40}$$

with α and β Hölder conjugates of each other, i.e. $\alpha^{-1} + \beta^{-1} = 1$, $\alpha \geq 1$, and where the quantity

$$\bar{I}_{\beta,\bar{q}}[g] = \left(N_q[g]\right)^{\beta}\, E_{\bar{q}}\left[\left|g(x)^{1-\bar{q}}\,\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(g\right)\right|^{\beta}\right] = \left(N_q[g]\right)^{\beta}\, E_{\bar{q}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}\ln_{\bar{q}}\left(g(x)\right)\right|^{\beta}\right] \tag{41}$$

is the generalized Fisher information of order (β, q) of the distribution g. The equality case is obtained if and only if

$$g(x) \propto \exp_{\bar{q}}\left(-\gamma\,|x|^{\alpha}\right), \textit{ with } \gamma > 0. \tag{42}$$

In these two cases, the general extended Cramér-Rao inequalities lead to inequalities for the moments of the distribution, where the equality is achieved for a generalization of the Gaussian distribution.

Let us finally note that by the same reasoning as above, the general inequalities (33) and (35) yield

$$E\left[|x|^{\alpha}\right]^{\frac{1}{\alpha}}\, E\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}\ln_{\psi}\left(\phi(f(x))\right)\right|^{\beta}\right]^{\frac{1}{\beta}} \geq 1 \tag{43}$$

with equality if and only if

$$g(x) = \phi(f(x)) \propto \exp_{\psi}\left(-\gamma\,|x|^{\alpha} + k\right), \text{ with } \gamma > 0 \text{ and } k \text{ a constant.} \tag{44}$$

4.2. Connections with earlier results

In the case $q = 1$, the characterization result in Corollary 3 has first been given by Boekee [32], who studied the generalized Fisher information $I_{\beta,1}[f]$ and gave a Cramér-Rao inequality saturated by the generalized Gaussian $g(x) \propto \exp\left(-\gamma\,|x|^{\alpha}\right)$.

It is also important to link our findings to a result by Lutwak *et al.* [2]. In that remarkable paper, the authors defined a generalized Fisher information, which can be written as

$$\phi_{\beta,q}[f] = E\left[\left|\,f(x)^{q-1}\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(f(x)\right)\right|^{\beta}\right] = E\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}\ln_{q*}\left(f(x)\right)\right|^{\beta}\right] \tag{45}$$

and is similar to our $I_{\beta,q}[f]$ in (38), up to a factor $\left(\frac{q}{M_q[f]}\right)^{\beta}$. Then, they established a general Cramér-Rao inequality in the form

$$E\left[|x|^{\alpha}\right]^{\frac{1}{\alpha}}\, \phi_{\beta,q}[f]^{\frac{1}{\beta q}} \geq E_G\left[|x|^{\alpha}\right]^{\frac{1}{\alpha}}\, \phi_{\beta,q}[G]^{\frac{1}{\beta q}}, \tag{46}$$

where G is any generalized Gaussian as in (1). Actually, their result (obtained in a very different way), can be seen as an improved version of (37). Indeed, rewriting the inequality (37) in terms of $\phi_{\beta,q}[f]$, we have

$$E\left[|x|^{\alpha}\right]^{\frac{1}{\alpha}}\,\phi_{\beta,q}[f]^{\frac{1}{\beta}} \geq q^{-1}M_q[f]. \tag{47}$$

Then, the inequality (46) can be obtained by minimizing the lower bound in the right of (47), as is described in [33].

Similarly, the characterization result in Corollary 4 can be connected to a recent result by Furuichi [3, 34] in the case $\alpha = \beta = 2$. In these very interesting works, the author investigated Cramér-Rao inequalities involving q-expectations. More precisely, he considered unnormalized escort distributions, that is distributions $g(x) = f(x)^q$, and defined expectations $\tilde{E}_q[.]$ as the expectations computed with respect to these unnormalized escort. He defined a generalized Fisher information which is essentially the same as our Fisher information (41) – although it is written in terms of unnormalized q-expectation. Then, he derived a Cramér-Rao inequality [34, Theorem 1], [3, Theorem 4.1], with its case of equality. This inequality can be recovered at once from (40), which is rewritten below in a developed form

$$\left(\int_X \frac{g(x)^{\bar{q}}}{N_q[g]}\,|x|^{\alpha}\,\mathrm{d}x\right)^{\frac{1}{\alpha}} \times N_q[g]\left(\int_X \frac{g(x)^{\bar{q}}}{N_q[g]}\left|\frac{\mathrm{d}}{\mathrm{d}x}\ln_{\bar{q}}\left(g(x)\right)\right|^{\beta}\mathrm{d}x\right)^{\frac{1}{\beta}} \geq 1. \tag{48}$$

It suffices to simplify the normalizations $N_q[g]$, using the fact that $\alpha^{-1}+\beta^{-1} = 1$ to get

$$\tilde{E}_{\bar{q}}\left[|x|^{\alpha}\right]^{\frac{1}{\alpha}}\,\tilde{E}_{\bar{q}}\left[\left|\frac{\mathrm{d}}{\mathrm{d}x}\ln_{\bar{q}}\left(g(x)\right)\right|^{\beta}\right] \geq 1, \tag{49}$$

recovering Furuichi's definition of generalized Fisher information and the associated Cramér-Rao inequality, with equality if and only if $g(x) \propto \exp_{\bar{q}}(-\gamma|x|^{\alpha})$.

5. Further remarks

In this section, we add some further comments on two possible estimation procedures that can be derived by examination of the condition of equality in the q-Cramér-Rao inequalities.

5.1. Maximum escort likelihood

Let us first return to the case of equality in the generalized q-Cramér-Rao inequalities. For the second Cramér-Rao inequality, the condition (29) is

$$\frac{\partial}{\partial\theta}\ln_{\bar{q}}\left(g(x;\theta)\right) = c(\theta)\,\mathrm{sign}\left(\hat{\theta}(x)-\theta\right)\left|\hat{\theta}(x)-\theta\right|^{\alpha-1}. \tag{50}$$

Thus, we see that if the bound is attained (the estimator could then be termed "q-efficient"), then this suggests to look for the parameter that maximizes the escort distribution of the likelihood:

$$\hat{\theta}_{MEL} = \arg\max_{\theta}\, g(x;\theta) = \arg\max_{\theta}\frac{f(x;\theta)^q}{M_q[f(x;\theta)]}, \tag{51}$$

where MEL stands for "maximum escort likelihood". Indeed, in these conditions, we have that the derivative in the left of (50) is zero, and thus that

$$\left.\frac{\partial}{\partial\theta}\ln_{\bar{q}}\left(g(x;\theta)\right)\right|_{\theta=\hat{\theta}_{MEL}} = c(\theta)\,\mathrm{sign}\left(\hat{\theta}(x)-\theta\right)\left.\left|\hat{\theta}(x)-\theta\right|^{\alpha-1}\right|_{\theta=\hat{\theta}_{MEL}} = 0. \tag{52}$$

Therefore, we get from the equality in the right side that $\hat{\theta}(x) = \hat{\theta}_{MEL}$. Hence, we see that if it exists a q-efficient estimator, it is the estimator defined by the maximum of the escort of the likelihood. Of course, we recover the standard maximum likelihood estimator in the $q = 1$ case. The analysis of the properties of this estimator will be the subject of future efforts.

5.2. Maximum Lq-likelihood estimation

We also saw that in case of equality, then the likelihood, or equivalently its escort, must be under the form of a q-exponential:

$$f(x;\theta) \propto \exp_{q_*}\left(\int_{\Theta} c(\theta)\,\mathrm{sign}\left(\hat{\theta}(x)-\theta\right)\left|\hat{\theta}(x)-\theta\right|^{\alpha-1}\mathrm{d}\theta\right). \tag{53}$$

Actually, it seems that there is only some very particular cases where this could occur. For instance, if the measurements consists in a series of independent and identically distributed observations x_i, then $f(x;\theta) = \Pi_i f(x_i;\theta)$ and one would have to find a distribution $f(x_i;\theta)$ such that the product $f(x;\theta)$ writes as a q_*-exponential. A possible amendment to the formulation can be to consider a q_*-product of the densities $f(x_i;\theta)$ instead of the standard product, and define $f^{(q_*)}(x;\theta) = \bigotimes_{q_*,i} f(x_i;\theta)$. Such q-likelihood has already been considered by [35]. Here, the Cramér-Rao inequality still applies for the q_*-likelihood $f^{(q_*)}(x;\theta)$, and the equality is obtained if $f^{(q_*)}(x;\theta)$ is a q_*-exponential. By the properties (10) of the q-product, we see that the individual densities $f(x_i;\theta)$ must be q_*-exponentials. Similarly, we see that the escort-likelihood will have the form of q-exponential if we use the q-product of the escort densities: $g^{(\bar{q})}(x;\theta) = \bigotimes_{\bar{q},i} g(x_i;\theta)$.

The equality condition (29), applied to the $\bar{q}$ escort-likelihood $g^{(\bar{q})}(x;\theta)$ then suggests to define the estimator as the maximizer of the $\bar{q}$ escort-likelihood, or, equivalently, as the maximizer of the $\ln_{\bar{q}}$ escort-likelihood:

$$\hat{\theta}_{MLq} = \arg\max_{\theta}\ln_{\bar{q}}\left(g^{(\bar{q})}(x;\theta)\right) = \arg\max_{\theta}\left(\sum_i \ln_{\bar{q}} g(x_i;\theta)\right). \tag{54}$$

Actually, the rule defined by (54) has been proposed and studied in the literature. It has been introduced by Ferrari [5, 36], and independently by Hasegawa [6]. As a matter of fact, the first authors, defining a problem as in (54) with data distributed according to $f(x;\theta)$, showed that the distribution $g(x;\theta)$ must be the escort of $f(x;\theta)$. These authors have shown that (54) yields a robust estimator with a tuning parameter, q, which balances efficiency and robustness. When the number of data increases, then the estimator appears to be the empirical version of

$$\hat{\theta}_{MLq} = \arg\min_{\theta}\ -E\left[\ln_{\bar{q}} g(x_i;\theta)\right] = \arg\min_{\theta}\frac{1}{N_q[g;\theta]}\,\frac{1}{(1-\bar{q})}\left(\int g(x;\theta)^{\bar{q}}\mathrm{d}x - 1\right), \tag{55}$$

which is nothing but the normalized Tsallis entropy attached to the escort distribution g. Such links between maximum likelihood and the minimization of the entropy with respect to the parameter θ

can be traced back to Akaike in [37]. Here, this gives a direct interpretation of the MLq method as an approximate minimum entropy procedure, and highlights the particular role of escort distributions in this context. Interestingly, it is shown in [5] and [6] that the asymptotical behaviour of the estimator is governed by a generalized Fisher information similar to (28). Our findings add the fact that the MLq estimator satisfies the Cramér-Rao inequality (27), for the product distribution $g^{(\bar{q})}(x;\theta)$.

6. Conclusions

The generalized q-Gaussians form an important and versatile family of probability distributions. These generalized q-Gaussians, which appear in physical problems as well as in functional inequalities, are the maximum entropy distributions associated with Tsallis or Rényi entropy. In this paper, we have shown that the generalized q-Gaussians are also the minimizers of extended versions of the Fisher information, over all distributions with a given moment, just as the standard Gaussian minimizes Fisher information over all distributions with a given variance. Actually, we obtain more precise results in the form of extended versions of the standard Cramér-Rao inequality, which are saturated by the generalized q-Gaussians. These Cramér-Rao inequalities, and the associated generalized Fisher informations, recover, put in perspective and connect earlier results by Lutwak et al. [2] and Furuichi [3, 34].

As a matter of fact, these characterizations of the generalized q-Gaussians appear as simple consequences of more general extended Cramér-Rao inequalities obtained in the context of estimation theory. Indeed, considering moments of any order of the estimation error, and using statistical expectations with respect to an escort distribution, we have derived two general Cramér-Rao inequalities that still include the Barakin-Vajda as well as the standard Cramér-Rao inequality as particular cases. This gives rise to general definitions of generalized Fisher information, which reduce to the standard one as a particular case, and make sense in this context. We have also characterized the case of equality and shown that the lower bounds of the inequalities can be attained if the parametric density belongs to a q-exponential family. Finally, we have indicated that these findings suggest some new estimation procedures, recovering in particular a recent Maximum L_q-likelihood procedure.

These results have been derived and presented in the monodimensional case. An important point will be to extend these results to the multidimensional case. This would be important both for the estimation inequalities as well as for the Cramér-Rao inequalities characterizing the generalized q-Gaussians. As far as the latter point is concerned, some results are already available in [33], and should be connected to estimation problems. As is well-known, the Weyl-Heisenberg uncertainty principle in statistical physics is nothing but the standard Cramér-Rao inequality for the location parameter. Thus it would be of particular interest to investigate on the possible meanings of the uncertainty relationships that could be associated to the extended Cramér-Rao inequalities. Fisher information, Cramér-Rao planes have been identified as useful and versatile tools for characterizing complex systems, see e.g. [29, 38, 39], and it would be therefore interesting to look at the potential benefits of using the extended versions in such problems. An open issue is the possible convexity property of the generalized Fisher information. Indeed, it is known that the standard Fisher information, as well as the generalized versions with $q = 1$, are convex functions of the density. If this were also true for any value of q, then it would be possible to associate to the generalized Fisher information a statistical mechanics with the standard Legendre structure and with the q-Gaussian as canonical distribution. Finally, future work shall also examine the estimation rules suggested by our setting and study their statistical properties.

References

[1] A. Dembo, T.M. Cover, and J.A. Thomas. Information theoretic inequalities. *IEEE Transactions on Information Theory*, 37(6):1501–1518, 1991.

[2] E. Lutwak, D. Yang, and G. Zhang. Cramér-Rao and moment-entropy inequalities for Rényi entropy and generalized Fisher information. *IEEE Transactions on Information Theory*, 51(2):473–478, 2005.

[3] S. Furuichi. On the maximum entropy principle and the minimization of the Fisher information in Tsallis statistics. *Journal of Mathematical Physics*, 50(1):013303–12, January 2009.

[4] J. Naudts. Estimators, escort probabilities, and ϕ-exponential families in statistical physics. *JIPAM. Journal of Inequalities in Pure & Applied Mathematics*, 5(4), 2004.

[5] D. Ferrari. Maximum lq-likelihood estimation. *The Annals of Statistics*, 38(2):753–783, April 2010.

[6] Y. Hasegawa and M .Arita. Properties of the maximum q-likelihood estimator for independent random variables. *Physica A*, 388(17):3399–3412, September 2009.

[7] C. Tsallis. *Introduction to Nonextensive Statistical Mechanics*. Springer, April 2009.

[8] E. Lutz. Anomalous diffusion and Tsallis statistics in an optical lattice. *Physical Review A*, 67(5):051402, 2003.

[9] V. Schwämmle, F. D Nobre, and C. Tsallis. q-Gaussians in the porous-medium equation: stability and time evolution. *The European Physical Journal B-Condensed Matter and Complex Systems*, 66(4):537–546, 2008.

[10] C. Vignat and A. Plastino. Why is the detection of q-Gaussian behavior such a common occurrence? *Physica A*, 388(5):601–608, March 2009.

[11] A. Ohara and T. Wada. Information geometry of q-Gaussian densities and behaviors of solutions to related diffusion equations. *Journal of Physics A: Mathematical and Theoretical*, 43(3):035002, 2010.

[12] G. I. Barenblatt. On some unsteady motions of a liquid and a gas in a porous medium. *Prikladnaja Matematika i Mechanika*, 16:67–78, 1952.

[13] R. E. Pattle. Diffusion from an instantaneous point source with concentration dependent coefficient. *Quart. J. Mech. Appl. Math.*, 12:407–409, 1959.

[14] M. Del Pino and J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *Journal de Mathématiques Pures et Appliquées*, 81(9):847–875, September 2002.

[15] M. Del Pino and J. Dolbeault. The optimal euclidean Lp-Sobolev logarithmic inequality. *Journal of Functional Analysis*, 197(1):151–161, January 2003.

[16] M. Agueh. Sharp Gagliardo-Nirenberg inequalities via p-Laplacian type equations. *Nonlinear Differential Equations and Applications NoDEA*, 15:457–472, December 2008.

[17] A. Chhabra and R. V. Jensen. Direct determination of the $f(\alpha)$ singularity spectrum. *Physical Review Letters*, 62(12):1327, March 1989.

[18] C. Beck and F. Schloegl. *Thermodynamics of Chaotic Systems*. Cambridge University Press, 1993.

[19] S. Abe. Geometry of escort distributions. *Physical Review E*, 68(3):031101, 2003.

[20] A. Ohara, H. Matsuzoe, and S. Amari. A dually flat structure on the space of escort distributions. *Journal of Physics: Conference Series*, 201:012012, 2010.

[21] J.-F. Bercher. Source coding with escort distributions and Rényi entropy bounds. *Physics Letters A*, 373(36):3235–3238, August 2009.

[22] S. Golomb. The information generating function of a probability distribution. *IEEE Transactions on Information Theory*, 12(1):75–77, 1966.

[23] L. Nivanen, A. Le Méhauté, and Q.A. Wang. Generalized algebra within a nonextensive statistics. *Reports on Mathematical Physics*, 52(3):437–444, December 2003.

[24] E. P. Borges. A possible deformed algebra and calculus inspired in nonextensive thermostatistics. *Physica A: Statistical Mechanics and its Applications*, 340(1–3):95–101, September 2004.

[25] B. Roy Frieden and P. M. Binder. Physics from Fisher information: A unification. *American Journal of Physics*, 68(11):1064, 2000.

[26] B. R. Frieden. *Science from Fisher Information: A Unification*. Cambridge University Press, 2004.

[27] C. Vignat and J.-F. Bercher. Analysis of signals in the Fisher-Shannon information plane. *Physics Letters A*, 312(1-2):27–33, June 2003.

[28] E. Romera and J. S. Dehesa. The Fisher-Shannon information plane, an electron correlation tool. *The Journal of Chemical Physics*, 120(19):8906–12, May 2004.

[29] J. S. Dehesa, P. Sánchez-Moreno, and R.J. Yáñez. Cramér-Rao information plane of orthogonal hypergeometric polynomials. *Journal of Computational and Applied Mathematics*, 186(2):523–541, 2006.

[30] E. W. Barankin. Locally best unbiased estimates. *The Annals of Mathematical Statistics*, 20(4):477–501, December 1949.

[31] I. Vajda. χ^α-divergence and generalized Fisher information. In *Transactions of the Sixth Prague Conference on Information Theory, Statistical Decision Functions and Random Processes*, page 223–234, 1973.

[32] D. E. Boekee. An extension of the Fisher information measure. In I Csiszár and P Elias, editors, *Topics in Information Theory*, volume 16, pages 113–123, Keszthely, Hungary, 1977. János Bolyai Mathematical Society and North-Holland.

[33] J.-F. Bercher. On a (β, q)-generalized Fisher information and inequalities involving q-Gaussian distributions. J. Math. Phys., at press. *arXiv:1203.1435*, March 2012.

[34] S. Furuichi. On generalized Fisher informations and Cramér-Rao type inequalities. *Journal of Physics: Conference Series*, 201:012016, 2010.

[35] H. Suyari and M. Tsukada. Law of error in Tsallis statistics. *IEEE Transactions on Information Theory*, 51(2):753–757, 2005.

[36] D. Ferrari and D. La Vecchia. On robust estimation via pseudo-additive information. *Biometrika*, volume 99, issue 1, pp. 238-244, 2012.

[37] H. Akaike. Information theory and the maximum likelihood principle. *2nd International Symposium in Information Theory*, 1973.

[38] J. S. Dehesa, S. López-Rosa, B. Olmos, and R. J Yáñez. Fisher information of d-dimensional hydrogenic systems in position and momentum spaces. *Journal of Mathematical Physics*, 47(5):052104, 13 p, May 2006.

[39] J. S. Dehesa, A.R. Plastino, P. Sánchez-Moreno, and C. Vignat. Generalized Cramér-Rao relations for non-relativistic quantum systems. *Applied Mathematics Letters*, at press.