



**HAL**  
open science

# On the conforming contact problem in a reinforced pin-loaded structure with a non-zero second Dundurs' constant

Quy-Dong To, Qi-Chang He

► **To cite this version:**

Quy-Dong To, Qi-Chang He. On the conforming contact problem in a reinforced pin-loaded structure with a non-zero second Dundurs' constant. *International Journal of Solids and Structures*, 2008, 45 (14-15), pp.3935-3950. 10.1016/j.ijsolstr.2008.02.010 . hal-00733145

**HAL Id: hal-00733145**

**<https://hal.science/hal-00733145>**

Submitted on 2 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the conforming contact problem in a reinforced pin-loaded structure with a non-zero second Dundurs' constant

Q.D. To, Q.-C. He\*

*Université Paris-Est, Laboratoire de Modélisation et Simulation Multi Echelle, FRE 3160 CNRS, 5 Boulevard Descartes, 77454 Marne-la-Vallée Cedex 2, France*

Within the framework of two-dimensional linear elasticity, the unilateral frictionless contact between two conformal cylindrical surfaces is governed by an integral equation in which the first and second Dundurs' constants are involved. In the case of elastic similarity characterized by the zero second Dundurs' constant, the integral equation is considerably simplified so as to lend itself to a closed-form solution. However, in the case of elastic dissimilarity defined by the non-zero second Dundurs' constant, the question of obtaining a closed-form solution to the integral equation is a much tougher one. Starting from the integral equation established by To et al. [To, Q.D., He Q.-C., Cossavella, M., Morcant, K., Panait, A., 2007. Closed-form solution for the contact problem of reinforced pin-loaded joints used in glass structures. *Int. J. Solids Struct.* 44, 3887–3903] for the conformal contact problem originating from a reinforced pin-loaded joint used in tempered glass structures, the present work proposes a new approximate analytical method to solve it in the case of elastic dissimilarity by minimizing an error function. The derived closed-form solution, valid not only for the conformal contact between a pin and an infinite holed plate but also for the one between a pin and a finite holed plate, is shown to be in very good agreement with available numerical results.

*Keywords:* Conformal contact; Pin-loaded joints; Integral equation; Dundurs constants; Least-square method

## 1. Introduction

The contact between the border of a circular hole in a plate and the surface of a circular pin inserted into the hole and subjected to a force is the prototype of a great number of unilateral conformal contact problems encountered in civil, mechanical and aerospace engineering. The study of this prototype conformal problem has a long history. A rather comprehensive list of relevant references can be found in a recent interesting paper of Ciavarella et al. (2006).

---

\* Corresponding author. Tel.: +33 0 160 957 786; fax: +33 0 160 957 799.  
*E-mail address:* qi-chang.he@univ-paris-est.fr (Q.-C. He).

When the friction between the pin surface and the hole border is neglected, the aforementioned conformal contact problem formulated with the framework of two-dimensional (2D) elasticity is governed by an integral equation involving both the first and second Dundurs' constants (see, e.g., Barber, 2002) of the materials constituting the pin and plate. The contacting materials are said to be elastically similar or dissimilar according as the second Dundurs' constant is equal or not equal to zero. In the case of elastic similarity, the integral equation governing the conformal contact problem is considerably simplified so as to lend itself to a closed-form solution. However, in the case of elastic dissimilarity, the problem of obtaining a closed-form solution to the integral equation is a very tough one.

In his Ph.D. thesis, Persson (1964) was the first to give a closed-form solution when the contacting materials are similar (see Johnson, 1985, section 5.3). Using a different method and limiting themselves also to the case of elastic similarity and zero-clearance, Noble and Hussain (1969) derived another closed-form solution. In a paper consisting of two parts, Ciavarella and Decuzzi (2001a,b) improved the results of Persson (1964) and Noble and Hussain (1969) in the case of elastic similarity and proposed a method to deal with the case of elastic dissimilarity.

The present work is a continuation of a previous investigation by the authors (To et al., 2007). In the latter, the problem of conformal contact between the border of a circular ring reinforcing a holed glass plate of finite breadth and the surface of a circular pin in the ring and subjected to a force (Fig. 1) was formulated as an integral equation and a closed-form solution was given for the case where the materials forming the ring and pin are elastically similar. This paper presents a new approximate analytical method to derive closed-form solutions in the unsolved case of elastic dissimilarity. Our method is rather different from the one proposed by Ciavarella and Decuzzi (2001b) to treat the case of elastic dissimilarity. Indeed, on the basis of the numerical observation that the second Dundurs' constant has little influence on the contact pressure but significantly affects the relation between the contact area and the normalized dimensionless loading parameter, Ciavarella and Decuzzi (2001b) adopted as an approximate one the contact pressure distribution corresponding to the case of elastic similarity and deduced a formula for the contact area in the case of elastic dissimilarity. By contrast, in our method, the effects of the second Dundurs' constant on the contact pressure and area are directly taken into account through approximating the integral term involving the second Dundurs' constant in the integral equation by an analytically tractable function and by minimizing the resulting error. Furthermore, our results hold not only for a pin in a plate of infinite breadth but also for a pin in a plate of finite breadth.

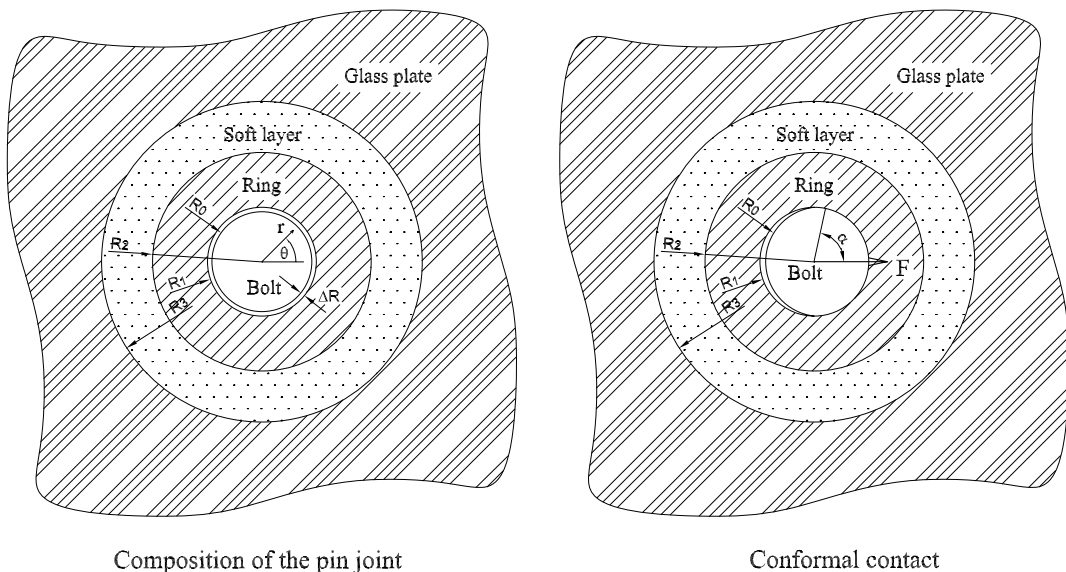


Fig. 1. Composition of a reinforced pin-loaded joint and conformal contact between the pin and ring.

The paper is organized as follows. In the next section, we recall the setting of the problem and rewrite the governing integral equation in a more tractable form. In Section 3, after approximating the integral related to the second Dundurs' constant by a finite series and minimizing the error in the sense of the least-square method, a general closed-form solution to the governing integral equation is obtained, allowing the achievement of any desired degree of accuracy. In Section 4, the general solution derived in Section 4 is applied by considering an infinite ring and a finite ring. The resulting approximate analytical results are compared with the numerical and approximate analytical results provided by Ciavarella and Decuzzi (2001b) in the case of an infinite plate and with the numerical results obtained by us with the aid of the finite element method in the case of a plate of a finite breadth. These comparisons show that our approximate analytical results are very accurate. We conclude this paper by giving a few closing remarks in Section 5.

## 2. Setting of the problem and governing integral equation

Consider the problem of conformal contact between a bolt and a ring which is, by a resin layer, glued to the border of a hole in a glass plate as in Fig. 1. Relative to a system of polar coordinates  $(r, \theta)$  with the origin coinciding with the hole center, the components of this reinforced pin-loaded joint are specified as follows

- the bolt:  $r < R_0$ ;
- the reinforcement ring:  $R_1 < r < R_2$ ;
- the resin layer:  $R_2 < r < R_3$ ;
- the glass plate:  $r > R_3$ .

The materials constituting the components are all taken to be linearly elastic, homogeneous and isotropic. The conforming contact problem will be studied within the framework of plane elasticity. Thus, it is convenient to characterize material  $i$  ( $= 0, 1, 2, 3$ ) by the Kolosov constants  $\mu_i$  and  $\kappa_i$  which are expressed in terms of the Young modulus  $E_i$  and the Poisson ratios  $\nu_i$  by

$$\mu_i = \frac{E_i}{2(1 + \nu_i)}, \quad \kappa_i = \begin{cases} \frac{3 - \nu_i}{1 + \nu_i} & (\text{plane stress}), \\ 3 - 4\nu_i & (\text{plane strain}). \end{cases} \quad (1)$$

The integer subscript  $i$  taking the values 0, 1, 2, 3 denotes the bolt, the ring, the resin and the glass plate, respectively. The frictionless unilateral contact between the bolt and the ring must verify the following conditions of Signorini type:

$$\begin{aligned} u_{r0}(\theta) - u_{r1}(\theta) &= \Delta R, & p(\theta) &\geq 0, & \text{if } \theta \in [-\alpha, \alpha]; \\ u_{r0}(\theta) - u_{r1}(\theta) &< \Delta R, & p(\theta) &= 0, & \text{if } \theta \notin [-\alpha, \alpha]. \end{aligned} \quad (2)$$

Here,  $u_{r0}(\theta)$  is the radial displacement of a point  $(r, R_0)$  on the surface of the bolt,  $u_{r1}(\theta)$  is the radial displacement of a point  $(r, R_1)$  on the interior surface of the ring, the radius difference  $\Delta R = R_1 - R_0$  stands for the clearance of the joint,  $p(\theta)$  represents the contact pressure which is taken to be positive, and  $\alpha$  designates half of the contact angle.

Under the condition that the shear modulus  $\mu_2$  of the resin layer is much smaller than the shear moduli  $\mu_1$  and  $\mu_3$  of the bolt and glass plate, i.e.  $\mu_2 \ll \min(\mu_1, \mu_3)$ , To et al. (2007) have shown that the integral equation governing the foregoing frictionless contact problem takes the form

$$\int_0^\alpha \frac{p(\xi) d\xi}{\cos \theta - \cos \xi} = (\beta_0 G_0 + \beta - \gamma_2 G_0) \frac{\theta}{\sin \theta} + (\beta_1 - 2\gamma_2) G_1 + \sum_{n=2}^{\infty} \beta_n G_n \frac{\sin n\theta}{\sin \theta} + \frac{\gamma_2 \pi}{\sin \theta} \int_0^\theta p(\xi) d\xi, \quad (3)$$

which corresponds to Eq. (43) in To et al. (2007). Above, the coefficients  $\beta_n$ ,  $G_n$ ,  $\beta$  and  $\gamma_2$  are defined by

$$\begin{aligned}
G_n &= \int_0^\alpha p(\xi) \cos n\xi d\xi \quad \text{for } n = 0, 1, 2, \dots, \\
\beta_0 &= \frac{1}{\left(\frac{\kappa_1+1}{\mu_1} + \frac{\kappa_0+1}{\mu_0}\right)} \left( \frac{1-\kappa_0}{\mu_0} - \frac{2\rho + \kappa_1 - 1}{(\rho-1)\mu_1} \right), \quad \beta_1 = -2, \\
\beta_n &= \frac{2[(n^2+n)\rho^{n+1} - 2(n^2-1)\rho^n + (n^2-n)\rho^{n-1} - 2]}{\left(1 + \frac{\kappa_0+1}{\kappa_1+1} \frac{\mu_1}{\mu_0}\right) [\rho^{2n} - n^2\rho^{n+1} + 2(n^2-1)\rho^n - n^2\rho^{n-1} + 1]} \quad \text{for } n \geq 2, \\
\beta &= -\frac{4\pi\Delta R}{R\left(\frac{\kappa_1+1}{\mu_1} + \frac{\kappa_0+1}{\mu_0}\right)}, \quad \gamma_2 = \frac{\frac{\kappa_1-1}{\mu_1} - \frac{\kappa_0-1}{\mu_0}}{\frac{\kappa_1+1}{\mu_1} + \frac{\kappa_0+1}{\mu_0}}. \tag{4}
\end{aligned}$$

In these expressions,  $\gamma_2$  is the well-known second Dundurs' constant (Barber, 2002) and  $\rho$  is defined by

$$\rho = (R_2/R_1)^2, \tag{5}$$

which characterizes the size of the ring relative to that of the pin and whose value range is  $]1, +\infty]$ . Moreover, we have an explicit relation for  $G_1$  due to the force equilibrium condition

$$G_1 = \int_0^\alpha p(\xi) \cos \xi d\xi = F/2R. \tag{6}$$

Normalizing the contact pressure  $p(\xi)$  as

$$q(\xi) = \frac{Rp(\xi)}{F} \tag{7}$$

with  $R = (R_0 + R_1)/2$ , and introducing the first Dundurs' constant

$$\gamma_1 = \frac{\frac{\kappa_1+1}{\mu_1} - \frac{\kappa_0+1}{\mu_0}}{\frac{\kappa_1+1}{\mu_1} + \frac{\kappa_0+1}{\mu_0}} \tag{8}$$

in the integral equation (3), the latter can be written in the following more compact and more tractable form:

$$\int_0^\alpha \frac{q(\xi)d\xi}{\cos \theta - \cos \xi} = (\beta_0^* H_0 - \lambda) \frac{\theta}{\sin \theta} - 2(1 + \gamma_2)H_1 + \sum_{n=2}^{\infty} \beta_n H_n \frac{\sin n\theta}{\sin \theta} + \frac{\gamma_2 \pi}{\sin \theta} \int_0^\theta q(\xi)d\xi \tag{9}$$

with

$$\begin{aligned}
H_n &= \int_0^\alpha q(\xi) \cos n\xi d\xi \quad \text{for } n = 0, 2, 3, \dots, \quad H_1 = 1/2, \\
\lambda &= -\frac{\beta R}{F} = \frac{4\pi\Delta R}{F\left(\frac{\kappa_1+1}{\mu_1} + \frac{\kappa_0+1}{\mu_0}\right)}, \quad \beta_0^* = -\frac{\rho(1 + \gamma_1)}{2(\rho - 1)}, \\
\beta_n &= \frac{(1 + \gamma_1)[(n^2 + n)\rho^{n+1} - 2(n^2 - 1)\rho^n + (n^2 - n)\rho^{n-1} - 2]}{\rho^{2n} - n^2\rho^{n+1} + 2(n^2 - 1)\rho^n - n^2\rho^{n-1} + 1} \quad \text{for } n \geq 2. \tag{10}
\end{aligned}$$

Owing to the previous reformulation, we clearly see that the normalized contact pressure  $q(\xi)$  depends only on the four parameters, i.e.  $\gamma_1$ ,  $\gamma_2$ ,  $\lambda$  and  $\rho$ . The first two ones  $\gamma_1$  and  $\gamma_2$  are the Dundurs' constants; the third one  $\lambda$  represents a combination of the clearance  $\Delta R$ , applied external force  $F$  and material elastic properties; the fourth parameter  $\rho$  describes the relative size of the ring.

In Appendix A, it is shown that, by a variable change, Eq. (9) is equivalent to a generalized Prandtl integral equation.

In the special case where the second Dundurs' constant  $\gamma_2 = 0$ , Eq. (9) reduces to

$$\int_0^\alpha \frac{q(\xi)d\xi}{\cos \theta - \cos \xi} = (\beta_0^* H_0 - \lambda) \frac{\theta}{\sin \theta} - 1 + \sum_{n=2}^{\infty} \beta_n H_n \frac{\sin n\theta}{\sin \theta}, \tag{11}$$

which involves three parameters, i.e.  $\gamma_1$ ,  $\lambda$  and  $\rho$ . This simpler problem has already been solved in our previous work (To et al., 2007). In what follows, we are concerned with treating the much more complicated situation where  $\gamma_2 \neq 0$ .

### 3. General approximate analytical solution

As the right-hand side terms of Eq. (9) contains an infinite series, it is analytically very difficult or even impossible to find an exact analytical solution to it. However, approximate solutions can be obtained by making some physically based simplifications. It is known from To et al. (2007) that, if  $\rho$  is sufficiently large, the infinite series  $\sum_{n=2}^{\infty} \beta_n H_n \sin n\theta / \sin \theta$  can be approximated by a finite one  $\sum_{n=2}^k \beta_n H_n \sin n\theta / \sin \theta$  so as to derive a simple closed-form solution to Eq. (11).

When the material dissimilarity defined by  $\gamma_2 \neq 0$  takes place and when the ring becomes infinite, i.e.  $\rho = \infty$ , Ciavarella and Decuzzi (2001b) proposed an approximate solution by retaining the solution for  $\gamma_2 = 0$  while making the contact angle  $\alpha$  dependent on  $\gamma_2$ . At the same time, Iyer (2001), using the finite element method, found that the material dissimilarity has little effect on the contact pressure distribution in both infinite and finite pin-loaded plates in the particular case where the pins and plates are all made of metallic materials. Below, we propose a different and more efficient approach for obtaining approximate analytical solutions in the general case when  $\gamma_2 \neq 0$  and  $\rho < \infty$ . Remark that other approximation methods for solving integral equations, such as the collocation method and the Bubnov–Galerkin method, can be found in the handbook by Polyanin and Manzhirov (1998).

First, assuming that the value of the finiteness parameter  $\rho$  is large enough, the infinite series  $\sum_{n=2}^{\infty} \beta_n H_n \sin n\theta / \sin \theta$  in Eq. (9) can be replaced by a finite series as follows:

$$\sum_{n=2}^{\infty} \beta_n H_n \frac{\sin n\theta}{\sin \theta} \simeq \sum_{n=2}^k \beta_n H_n \frac{\sin n\theta}{\sin \theta}, \quad (12)$$

where the number  $k$  of initial terms involved in the finite series depends on the desired degree of accuracy. Next, approximating the normalized pressure distribution  $q(\xi)$  by a finite series

$$q(\xi) \simeq a_0 + \sum_{n=1}^l a_n \cos n\xi, \quad (13)$$

the last term in the right-hand side of Eq. (9) involving the second Dundurs' constant  $\gamma_2$  has then the following approximate expression:

$$\frac{\gamma_2 \pi}{\sin \theta} \int_0^\theta q(\xi) d\xi \simeq \gamma_2 \pi \left[ \sum_{n=1}^l \frac{\sin n\theta}{n \sin \theta} a_n + \frac{\theta}{\sin \theta} a_0 \right]. \quad (14)$$

The main advantage of the previous two approximations is that they lead to expressing the right-hand side of Eq. (9) in terms of the base functions  $\theta / \sin \theta$  and  $\sin n\theta / \sin \theta$  with  $n \geq 1$ . Consequently, the technique elaborated in To et al. (2007) can be directly used to solve the resulting integral equation.

To determine the values of the coefficients  $a_n$  ( $n = 0, 1, 2, \dots, l$ ), one simple and direct possibility is to consider the approximation (13) as a finite Fourier series with the coefficients determined by the usual formulae for an even function:

$$a_0 = \frac{H_0}{\pi}, \quad a_n = \frac{2H_n}{\pi} \quad \text{for } n = 1, 2, \dots, l, \quad (15)$$

where  $H_n$  ( $n = 0, 1, 2, \dots, l$ ) are given by the first formula in (10). However, this way of specifying the values of  $a_n$  does not provide the best approximation for the function  $q(\xi)$  with the finite series (13). To gain the best one, we use the least-square method to calculate  $a_n$ . More precisely, the values of  $a_n$  ( $n = 0, 1, 2, \dots, l$ ) are determined by minimizing the error function

$$E(a_0, a_1, \dots, a_k) = \int_0^\alpha \left[ q(\xi) - a_0 - \sum_{n=1}^l a_n \cos n\xi \right]^2 d\xi. \quad (16)$$

The necessary condition to achieving the minimum of this function leads to a system of linear equations

$$\int_0^\alpha \left[ a_0 + \sum_{n=1}^l a_n \cos n\theta \right] \cos m\theta d\theta = H_m \quad \text{with } m = 0, 1, \dots, l, \quad (17)$$

which allows the determination of  $a_n$ .

After choosing the values of  $k$  and  $l$  and after calculating all the coefficients  $a_n$ , the final equation approximating Eq. (9) can be written in the general compact form

$$\int_0^\alpha \frac{q(\xi) d\xi}{\cos \theta - \cos \xi} = (\beta_0^* H_0 - \lambda + \gamma_2 \pi a_0) \frac{\theta}{\sin \theta} - (1 + \gamma_2 - \gamma_2 \pi a_1) + \sum_{n=2}^{\max(k,l)} [\beta_n H_n \text{sign}(k-n) + \gamma_2 \pi a_n \text{sign}(l-n)] \frac{\sin n\theta}{n \sin \theta}, \quad (18)$$

where  $\text{sign}(x)$  is the step function defined by  $\text{sign}(x) = 0$  for  $x < 0$  and  $\text{sign}(x) = 1$  for  $x \geq 0$ .

As in To et al. (2007), by a suitable change of variables, the integral equation (18) governing the contact pressure distribution can be transformed into a Cauchy singular integral equation whose solution is given for example by Peters (1963). To avoid repetition, no details are given here and only the solution for the contact pressure is provided:

$$q(\xi) = \frac{H_0 \sin \xi}{2m\pi\sqrt{t(1-t)}} - \frac{\sin \xi}{\pi^2\sqrt{t}} \frac{d}{dt} \int_t^1 \frac{d\sigma}{\sqrt{\sigma-t}} \int_0^\sigma \frac{\varphi(x)\sqrt{x}dx}{\sqrt{\sigma-x}}, \quad (19)$$

where

$$t = \sin^2(\xi/2)/m, \quad m = \sin^2(\alpha/2), \quad x = \sin^2(\theta/2)/m, \\ \varphi[x(\theta)] = (\beta_0^* H_0 - \lambda + \gamma_2 \pi a_0) \frac{\theta}{\sin \theta} - (1 + \gamma_2 - \gamma_2 \pi a_1) + \sum_{n=2}^{\max(k,l)} (\beta_n H_n \text{sign}(k-n) + \gamma_2 \pi a_n \text{sign}(l-n)) \frac{\sin n\theta}{n \sin \theta}. \quad (20)$$

After carrying out the integrations in the expression (19) of  $q(\xi)$ , we can write (see To et al., 2007 for more details)

$$q(\xi) = \frac{Q(m) \cos(\xi/2)}{\sqrt{m - \sin^2(\xi/2)}} + C \ln \left| \frac{\cos(\xi) + \sqrt{m - \sin^2(\xi/2)}}{\sqrt{1-m}} \right| + \cos(\xi/2) R[\sin^2(\xi/2)] \sqrt{m - \sin^2(\xi/2)}, \quad (21)$$

where  $Q(m)$  is a function of  $m$ ,  $C$  is a constant and  $R[\sin^2(\xi/2)]$  is a polynomial function of  $\sin^2(\xi/2)$  whose coefficients depend on  $H_n$  ( $n = 0, 1, \dots, \max(k, l)$ ) and  $m$ . The coefficients  $H_n$  and  $m$  are determined by the system of equations:

$$\begin{cases} H_1 = 1/2, \\ \int_0^\alpha q(\xi) \cos n\xi d\xi = H_n \quad \text{with } n = 0, 1, \dots, \max(k, l), \\ Q(m) = 0. \end{cases} \quad (22)$$

The last condition  $Q(m) = 0$  in (22) is derived from the condition ensuring non singularity of  $q(\xi)$  at  $\xi = \alpha$  determined by Eq. (21).

To illustrate the above general results, we now consider two important particular cases and present the corresponding results.

*Case I:  $k = 2$  and  $l = 0$*

From the system (17), we deduce  $a_0 = H_0/\alpha$ . The normalized contact pressure admits the closed-form solution

$$q(\xi) = -\frac{2(\beta_0^*H_0 - \lambda + a_0\gamma_2\pi)}{\pi} \ln \left| \frac{\cos \xi/2 + \sqrt{m - \sin^2 \xi/2}}{\sqrt{1-m}} \right| + \frac{8\beta_2H_2}{\pi} \cos \xi/2 \sin^2 \xi/2 \\ \times \sqrt{m - \sin^2 \xi/2} - \frac{2 \cos \xi/2}{\pi} \sqrt{m - \sin^2 \xi/2} [2(1-m)\beta_2H_2 - 1 - \gamma_2], \quad (23)$$

where the contact angle  $\alpha$  expressed in terms of  $m$  and the coefficients  $H_0$  and  $H_2$  are specified upon solving the three nonlinear equations

$$\begin{cases} -m(\beta_0^*H_0 - \lambda + \gamma_2\pi a_0) + \frac{1}{2}m(2-m)(1 + \gamma_2) \\ \quad -2m(1-m)^2\beta_2H_2 = \frac{1}{2}, \\ -\frac{1}{2}m(2-3m)(\beta_0^*H_0 - \lambda + \gamma_2\pi a_0) + m(1-m)^2(1 + \gamma_2) \\ \quad -\frac{1}{2}m(4-14m+20m^2-9m^3)\beta_2H_2 = H_2, \\ H_0 - (\beta_0^*H_0 - \lambda + a_0\gamma_2\pi) \ln(1-m) - (1 + \gamma_2)m \\ \quad -\beta_2H_2m(3m-2) = 0. \end{cases} \quad (24)$$

Case 2:  $k = 2$  and  $l = 1$

In this case, from (17) we obtain

$$a_0 = \frac{(\alpha + \sin \alpha \cos \alpha)H_0 - \sin \alpha}{\alpha \sin \alpha \cos \alpha + \alpha^2 - 2 \sin^2 \alpha}, \quad a_1 = \frac{2 \sin \alpha H_0 - \alpha}{\alpha \sin \alpha \cos \alpha + \alpha^2 - 2 \sin^2 \alpha}. \quad (25)$$

Concerning the contact pressure, we have

$$q(\xi) = -\frac{2(\beta_0^*H_0 - \lambda + a_0\gamma_2\pi)}{\pi} \ln \left| \frac{\cos \xi/2 + \sqrt{m - \sin^2 \xi/2}}{\sqrt{1-m}} \right| + \frac{8\beta_2H_2}{\pi} \cos \xi/2 \sin^2 \xi/2 \\ \times \sqrt{m - \sin^2 \xi/2} - \frac{2 \cos \xi/2}{\pi} \sqrt{m - \sin^2 \xi/2} [2(1-m)\beta_2H_2 - 1 - \gamma_2 + \gamma_2\pi a_1]. \quad (26)$$

The system of equations allowing us to compute  $\alpha, H_0, H_2$  is provided by

$$\begin{cases} -m(\beta_0^*H_0 - \lambda + \gamma_2\pi a_0) + \frac{1}{2}m(2-m)(1 + \gamma_2 - \gamma_2\pi a_1) \\ \quad -2m(1-m)^2\beta_2H_2 = \frac{1}{2}, \\ -\frac{1}{2}m(2-3m)(\beta_0^*H_0 - \lambda + \gamma_2\pi a_0) + m(1-m)^2(1 + \gamma_2 - \gamma_2\pi a_1) \\ \quad -\frac{1}{2}m(4-14m+20m^2-9m^3)\beta_2H_2 = H_2, \\ H_0 - (\beta_0^*H_0 - \lambda + a_0\gamma_2\pi) \ln(1-m) - (1 + \gamma_2 - \gamma_2\pi a_1)m \\ \quad -\beta_2H_2m(3m-2) = 0. \end{cases} \quad (27)$$

Up to now, we have elaborated a general approach to obtain an approximate analytical solution to the integral equation governing the conformal contact problem in a reinforced pin-loaded joint with a non-zero second Dundurs' constant  $\gamma_2 \neq 0$ . In particular, by setting  $\gamma_2 = 0$ , we recover the solution provided by our previous paper (To et al., 2007). In the next section, we apply our foregoing general solution to the important special case of an infinite two-dimensional body by posing  $\rho = \infty$  and compare the corresponding results with the relevant ones existing in the literature; to validate the general solution for a two-dimensional body of finite breadth, it is confronted with the finite element solution obtained for a real reinforced pin-loaded joint.



## 4. Application and validation of the approximate analytical solution

### 4.1. Infinite ring $\rho = \infty$

The results derived in the previous section hold for any value  $\rho \in ]1, +\infty]$ . The special case  $\rho = \infty$  associated to an infinite ring is of important interest, since all the analytical results reported in the literature about the conformal contact problem in a pin-loaded joint with  $\gamma_2 \neq 0$  are limited to this case and can be used for comparison.

Setting  $\rho = \infty$ , the governing integral equation (9) is simplified enormously, because

$$\beta_0^* = -\frac{1 + \gamma_1}{2}, \quad \beta_n = 0 \quad \text{for } n \geq 2. \quad (28)$$

In particular, accounting for these conditions in (23)–(27), we obtain the explicit results detailed below.

*Case 1:  $k = 2$  and  $l = 0$*

Recalling  $a_0 = H_0/\alpha$ , using (28) in (23) and (24) and noting that the second equation in (24) is redundant for the case under consideration, we have

$$q(\xi) = -\frac{2(\beta_0^* H_0 - \lambda + H_0 \gamma_2 \pi / \alpha)}{\pi} \ln \left| \frac{\cos \xi/2 + \sqrt{m - \sin^2 \xi/2}}{\sqrt{1 - m}} \right| - \frac{2(1 + \gamma_2)}{\pi} \cos \xi/2 \sqrt{m - \sin^2 \xi/2}, \quad (29)$$

$$\begin{cases} -m(\beta_0^* H_0 - \lambda + \gamma_2 \pi H_0 / \alpha) + \frac{m(2-m)}{2}(1 + \gamma_2) = \frac{1}{2}, \\ H_0 - (\beta_0^* H_0 - \lambda + H_0 \gamma_2 \pi / \alpha) \ln(1 - m) - m(1 + \gamma_2) = 0. \end{cases} \quad (30)$$

Combining the equations in (30) leads to the nonlinear equation characterizing  $\alpha$ :

$$\begin{aligned} & -\lambda + \frac{(1 - m)^2}{2m} + \frac{1 + \gamma_1}{2} \left[ \frac{(1 - m)^2}{2m} \ln(1 - m) - m \right] \\ & + \gamma_2 \ln(1 - m) \left[ \frac{(2 - m)\pi\gamma_2}{2\alpha} - \frac{(1 - m)^2\pi}{2m\alpha} - \frac{(1 + \gamma_1)(2 - m)}{4} \right] + \gamma_2 m \left[ \frac{\pi\gamma_2}{\alpha} + \frac{\pi}{\alpha} - \frac{1 + \gamma_1}{2} - \frac{2 - m}{2m} \right] = 0 \end{aligned} \quad (31)$$

with  $m = \sin^2 \alpha/2$ . Note that the first three terms in the left-hand member of this equation do not involve the second Dundurs' constant  $\gamma_2$ .

*Case 2:  $k = 2$  and  $l = 1$*

In this case, it is convenient to rewrite (25) in the form

$$a_0 = b_0 H_0 + c_0, \quad a_1 = b_1 H_0 + c_1 \quad (32)$$

with

$$\begin{aligned} b_0 &= \frac{\alpha + \sin \alpha \cos \alpha}{\alpha \sin \alpha \cos \alpha + \alpha^2 - 2 \sin^2 \alpha}, & c_0 &= -\frac{\sin \alpha}{\alpha \sin \alpha \cos \alpha + \alpha^2 - 2 \sin^2 \alpha}, \\ b_1 &= \frac{2 \sin \alpha}{\alpha \sin \alpha \cos \alpha + \alpha^2 - 2 \sin^2 \alpha}, & c_1 &= -\frac{\alpha}{\alpha \sin \alpha \cos \alpha + \alpha^2 - 2 \sin^2 \alpha}. \end{aligned} \quad (33)$$

Then, introducing (28) into (26) and (27) and observing that the second equation in (27) is not useful for the case under consideration, it follows that

$$\begin{aligned} q(\xi) &= -\frac{2(\beta_0^* H_0 - \lambda + a_0 \gamma_2 \pi)}{\pi} \ln \left| \frac{\cos \xi/2 + \sqrt{m - \sin^2 \xi/2}}{\sqrt{1 - m}} \right| - \frac{2(1 + \gamma_2 - a_1 \gamma_2 \pi)}{\pi} \cos \xi/2 \\ &\quad \times \sqrt{m - \sin^2 \xi/2}, \end{aligned} \quad (34)$$

$$\begin{cases} -m(\beta_0^* H_0 - \lambda + \gamma_2 \pi a_0) + \frac{m(2-m)}{2}(1 + \gamma_2 - \gamma_2 \pi a_1) = \frac{1}{2}, \\ H_0 - (\beta_0^* H_0 - \lambda + a_0 \gamma_2 \pi) \ln(1-m) - m(1 + \gamma_2 - \gamma_2 \pi a_1) = 0. \end{cases} \quad (35)$$

Combining (32) and (35) yields the following nonlinear equation for  $\alpha$ :

$$\begin{aligned} & -\lambda + \frac{(1-m)^2}{2m} + \frac{1+\gamma_1}{2} \left[ \frac{(1-m)^2}{2m} \ln(1-m) - m \right] \\ & + \gamma_2 [D_1 D_5 + D_2 - D_3 D_7 - D_4 D_6 + \gamma_2 (D_2 D_5 - D_4 D_7)] = 0 \end{aligned} \quad (36)$$

with

$$\begin{aligned} D_1 &= \frac{(m-1)^2}{2m} - \lambda, & D_2 &= \pi c_0 - \frac{(2-m)}{2}(1 - \pi c_1), \\ D_3 &= \beta_0^* = -\frac{1+\gamma_1}{2}, & D_4 &= \pi b_0 + \frac{(2-m)}{2} \pi b_1, \\ D_5 &= \left[ \frac{(2-m)}{2} \ln(1-m) + m \right] \pi b_1, \\ D_6 &= \frac{(1-m)^2}{2m} \ln(1-m) - m, \\ D_7 &= \left[ \frac{(2-m)}{2} \ln(1-m) + m \right] (\pi c_1 - 1). \end{aligned} \quad (37)$$

Observe that the first three terms in the left-hand side of (36) are identical to those in the left-hand side of (31) and do not include  $\gamma_2$ .

In Ciavarella and Decuzzi (2001b), the equation characterizing the contact angle  $\alpha$  takes the form

$$\frac{E_0^* \Delta R}{F} = \frac{2 - (\gamma_1 + 1)[\ln(b^2 + 1) + 2b^4]}{\pi(1 - \gamma_1)(b^2 + 1)b^2} + \frac{4\gamma_2}{\pi(1 - \gamma_1)}. \quad (38)$$

Above,  $b = \tan(\alpha/2)$ ,  $E_0^* = E_0/(1 - \nu_0^2)$  for plane strain and  $E_0^* = E_0$  for plane stress. It is convenient to express  $E_0^*$  in terms of  $\mu_0$  and  $\kappa_0$  as

$$E_0^* = \frac{8\mu_0}{\kappa_0 + 1}, \quad (39)$$

which is valid both for the plane strain and stress cases. In addition, the left-hand member of (38) is related to the quantity  $\lambda$  in Eq. (10) by

$$\frac{\Delta R E_0^*}{F} = \frac{4\lambda}{(1 - \gamma_1)\pi}. \quad (40)$$

Remark that  $b^2 = \tan^2(\alpha/2)$  can be also expressed in terms of  $m = \sin^2(\alpha/2)$  by

$$b^2 = \frac{m}{1-m}. \quad (41)$$

Finally, the formula (38) of Ciavarella and Decuzzi (2001b) can be recast into

$$-\lambda + \frac{(1-m)^2}{2m} + \frac{1+\gamma_1}{2} \left[ \frac{(1-m)^2}{2m} \ln(1-m) - m \right] + \gamma_2 = 0. \quad (42)$$

It is interesting to note that this equation is different from (31) or (36) only in the terms involving  $\gamma_2$ . In other words, when  $\gamma_2 = 0$ , (31), (36) and (42) reduce to the exact nonlinear equation for  $\alpha$  with  $\lambda$  and  $\gamma_1$  as the two parameters, i.e.

$$-\lambda + \frac{(1-m)^2}{2m} + \frac{1+\gamma_1}{2} \left[ \frac{(1-m)^2}{2m} \ln(1-m) - m \right] = 0; \quad (43)$$

when  $\gamma_2 \neq 0$ , they are different and have  $\lambda$ ,  $\gamma_1$  and  $\gamma_2$  as the three parameters.

To demonstrate the robustness of our approximate analytical solution, we compare it with the numerical solution and approximate analytical solution given by Ciavarella and Decuzzi (2001b). In Table 1, the contact angle  $\alpha = 22.62^\circ$  is fixed in advance but  $\lambda$  and  $q_0 = q(0)$  are computed. From Table 1, it is seen that our solution is very accurate in comparison with the numerical results of Table 2 in Ciavarella and Decuzzi (2001b), in particular when  $l = 1$ . In Table 2,  $\lambda = 0$  (neat fit contact) is considered and the contact angle  $\alpha$  is calculated. It is remarked that the results obtained by the present method with  $l = 1$  are closer to the numerical results than those provided by the approximate method of Ciavarella and Decuzzi (2001b). In this sense, our approximate analytical method improves the one of Ciavarella and Decuzzi (2001b).

The variation of the contact angle  $\alpha$  with the normalized loading parameter  $\lambda$  is illustrated in Fig. 3 with one extreme value  $\gamma_1 = 1$  of  $\gamma_1$  and in Fig. 4 with another extreme value  $\gamma_1 = -1$  of  $\gamma_1$ . In each of these two figures, the dependence of  $\alpha$  on  $\gamma_2$  is shown by taking  $\gamma_2 = 0$  and  $\gamma_2 = 0.5$ . In addition to the curves corresponding to the solutions of our Eqs. (31) and (36), to the solution of Eq. (42) of Ciavarella and Decuzzi (2001b) and to the solution of (43), the Hertzian contact curve is also plotted in Figs. 3 and 4 according to the formula

$$\alpha = \sqrt{\frac{2}{\lambda}} = \sqrt{\frac{F}{2\pi\Delta R} \left( \frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_0 + 1}{\mu_0} \right)}. \quad (44)$$

Table 1

Comparison between the present solution and the numerical solution given in Table 2 of Ciavarella and Decuzzi (2001b)

$\gamma_1$	$\gamma_2$	$\alpha$	$\lambda_{\text{num}}$	$q_{0\text{num}}$	$l = 0$		$l = 1$	
					$\tilde{\lambda}_{(l=0)}$	$q_{0(l=0)}$	$\tilde{\lambda}_{(l=1)}$	$q_{0(l=1)}$
0.5	0.175	22.62	12.32	1.639	12.18	1.623	12.23	1.638
1/3	0.117	22.62	12.14	1.641	12.04	1.631	12.08	1.641
0	0	22.62	11.77	1.645	11.76	1.645	11.76	1.645
-1/3	-0.117	22.62	11.44	1.649	11.49	1.660	11.45	1.650
-0.5	-0.175	22.62	11.26	1.651	11.35	1.667	11.29	1.652

Table 2

Comparison between the present solution and the numerical solution and approximative solution given in Table 3 of Ciavarella and Decuzzi (2001b) (C-D)

$\gamma_1$	$\gamma_2$	$\lambda$	$\alpha_{\text{num}}$	C-D		Present	
				$\alpha_{\text{CD}}$	$\alpha_{(l=0)}$	$\alpha_{(l=1)}$	
1	0.50	0	87.76	94.85	83.94	87.69	
0	0.25	0	98.02	103.9	94.34	97.98	
0	-0.25	0	75.73	74.14	77.25	75.75	
-1	-0.50	0	80.96	76.32	85.67	81.01	

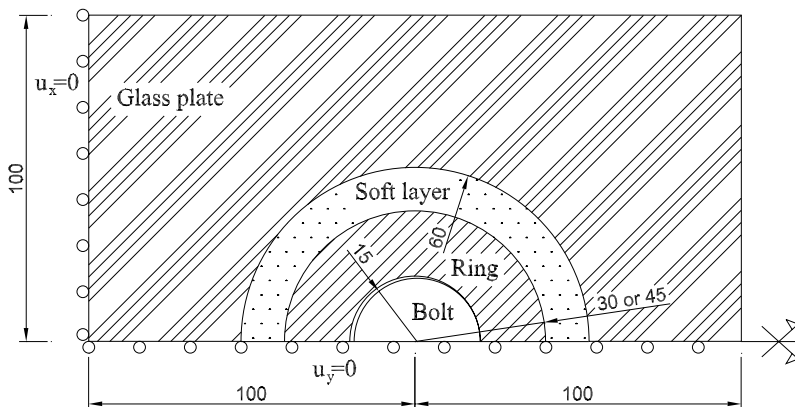


Fig. 2. A quarter of the pin-loaded joint used in the numerical analysis.

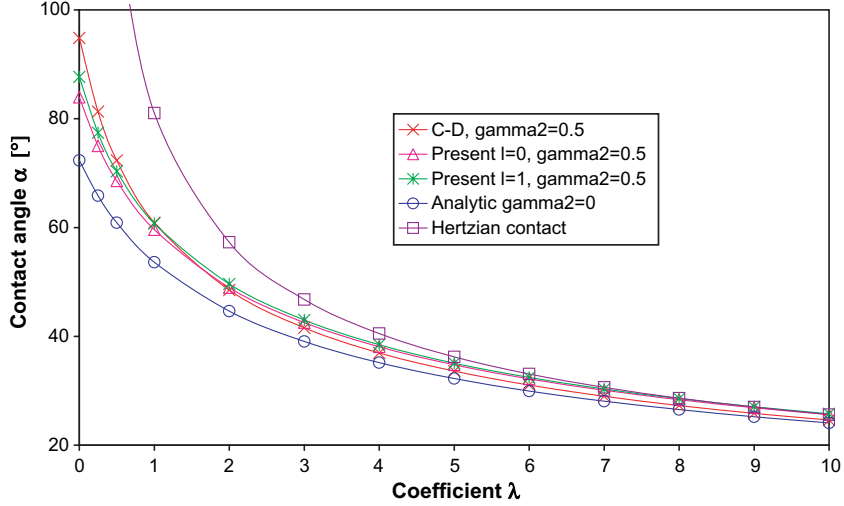


Fig. 3. Relation between coefficient  $\lambda$  and contact angle  $\alpha$  (case  $\gamma_1 = 1$ ;  $\gamma_2 = 0$  and  $0.5$ ).

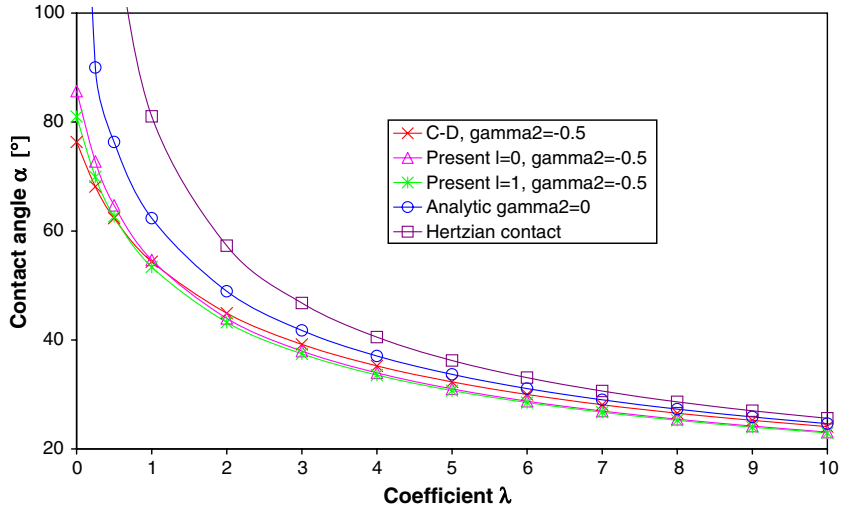


Fig. 4. Relation between coefficient  $\lambda$  and contact angle  $\alpha$  (case  $\gamma_1 = -1$ ;  $\gamma_2 = 0$  and  $-0.5$ ).

The derivation of this formula can be found in Johnson (1985), Hills et al. (1993) or Ciavarella and Decuzzi (2001a).

It is seen from Figs. 3 and 4 that, when  $\lambda$  is relatively large, the contact angle is small and all the curves are very close to each other, so that the Hertzian contact regime prevails as expected. Physically, this situation can be achieved by a small force  $F$  or/and a large clearance  $\Delta R$ . However, when  $\lambda$  has a small value which occurs for a large force  $F$  or/and a small clearance  $\Delta R$ , the curve associated to the Hertzian contact is quite far from the other curves. According to the Hertz formula (44), the contact angle tends to infinity as  $\lambda$  goes to 0, which is physically inadmissible.

From Figs. 3 and 4, we also see that: (i) the effect of the second Dundurs' constant  $\gamma_2$  on the contact angle  $\alpha$  is negligible when  $\lambda$  is large or equivalently when  $\alpha$  is small; (ii) the effect becomes very important when  $\lambda$  is small or equivalently when  $\alpha$  is large. This conclusion refines the relevant one made by Ciavarella and Decuzzi (2001b). The three curves corresponding to our Eqs. (31) and (36) and to Eq. (42) of Ciavarella and Decuzzi (2001b) are close to each other, though our approach is different from theirs. Our approach, based on a direct

approximate solution of the governing integral equation, appears to be more accurate (see Table 1). However, our expressions for the contact pressure and angle are more complicated than those given by Ciavarella and Decuzzi (2001b). At the same time, our approach has the definitive advantage of being applicable to the case of a finite joint with  $\gamma_2 \neq 0$ , which has not been treated up to now.

#### 4.2. Finite ring $\rho < \infty$

In this case,  $\beta_n \neq 0$  for  $n \geq 2$ . The accuracy of the solution depends on the choice of both  $k$  and  $l$ . To evaluate the performance of the approximate formula, we consider a realistic case where the bolt and the ring are made of stainless steel (Steel) and aluminium (Al) respectively, and vice versa (see Table 3). In regard to finiteness parameter  $\rho$ , two values are considered:  $\rho = 4$  when  $R_2 = 2R_1$  and  $\rho = 9$  when  $R_2 = 3R_1$ . The analytic solution is first obtained by the general method presented in the previous section and then compared with the result by the finite element method. The parameters used by the analytical solution are presented in Table 4 (case  $l = 0$ ) and Table 5 (case  $l = 1$ ).

Because the contact pressure does not depend on the geometry of the glass plate and the soft layer as long as the rigidity of the latter is sufficiently soft, we consider a case of a square glass plate of dimension  $200 \times 200 \times 19$  mm as an example. The following geometrical and mechanical parameters are used:

- Geometric parameters:  $R_0 = 15$  mm,  $R_1 = 15$  mm,  $R_2 = 30$  mm or 45 mm,  $R_3 = 60$  mm,  $L = 200$  mm (width and length of the glass plate),  $e = 19$  mm (thickness of the glass plate);
- Soft layer ( $R_2 \leq r \leq R_3$ ):  $E_2 = 0.5$  GPa,  $\nu_2 = 0.2$ ;
- Glass plate ( $R_3 \leq r$  and  $|x| \leq L/2$  and  $|y| \leq L/2$ ):  $R_3 = 60$  mm,  $E_3 = 70$  GPa,  $\nu_3 = 0.2$ ;
- Total force applied at the center of the bolt:  $P = P_x = 19$  kN;

Table 3  
Recapitulation of the mechanical properties involved in the governing equation

$E_0$	$\nu_0$	$E_1$	$\nu_1$	$\gamma_1$	$\gamma_2$
<i>Al-steel</i>					
70	0.33	200	0.3	-0.481	-0.157
<i>Steel-Al</i>					
200	0.3	70	0.33	0.481	0.157

Table 4  
Coefficients calculated for the case  $l = 0$

$\gamma_1$	$\gamma_2$	$\rho$	$\beta_0^*$	$\beta_2$	$\alpha$	$H_0$	$H_2$	$a_0$
<i>Al-steel</i>								
-0.481	-0.157	9	-0.292	0.494	89.31	0.680	0.144	0.436
		4	-0.345	1.882	90.76	0.701	0.113	0.443
<i>Steel-Al</i>								
0.481	0.157	9	-0.833	1.412	84.99	0.668	0.157	0.450
		4	-0.987	5.377	87.16	0.704	0.100	0.423

Table 5  
Coefficients calculated for the case  $l = 1$

$\gamma_1$	$\gamma_2$	$\rho$	$\beta_0^*$	$\beta_2$	$\alpha$	$H_0$	$H_2$	$a_0$	$a_1$
<i>Al-steel</i>									
-0.481	-0.157	9	-0.292	0.494	88.09	0.671	0.157	0.131	0.470
		4	-0.345	1.882	90.08	0.695	0.121	0.195	0.388
<i>Steel-Al</i>									
0.481	0.157	9	-0.833	1.412	85.56	0.672	0.150	0.166	0.426
		4	-0.987	5.377	87.34	0.706	0.097	0.275	0.286

- Force per thickness is equal  $F = P/e = 1$  (kN/mm);
- Boundary conditions:  $u_x(x = -L/2, y) = 0$ ,  $u_y(x, y = 0) = 0$ .

The glass structure is analyzed under the plane stress assumption and using its geometric symmetry. The model and its boundary conditions are shown in Fig. 2 and is identical to the one depicted by Fig. 6 in To et al. (2007).

The comparison between the analytic solution and the numerical solution given by FEM is presented in Figs 5–8. This comparison leads to the following two comments:

- The approximate analytical solution is in good agreement with the numerical one for  $\rho = 4$  and  $\rho = 9$ .
- As  $\rho$  decreases and  $\gamma_1$  increases, the difference between the analytical and numerical solutions becomes non-negligible (see Fig. 8). This can be explained by the fact the coefficient  $\beta_n$  in the series  $\sum_{n=2}^{\infty} \beta_n H_n \sin n\theta / \sin \theta$  is function of both  $(1 + \gamma_1)$  and  $\rho$ . To improve the accuracy in such a case, the number  $k$  should be increased, i.e. more initial terms in  $\sum_{n=2}^{\infty} \beta_n H_n \sin n\theta / \sin \theta$  have to be used.

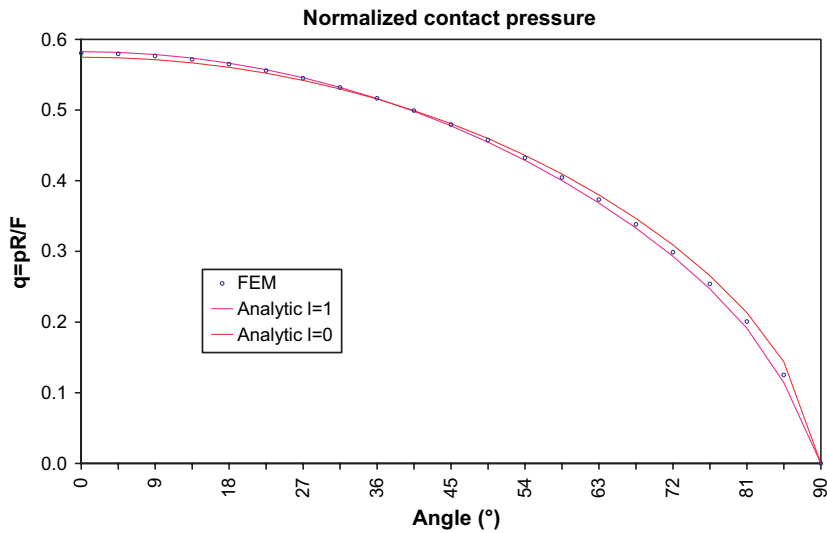


Fig. 5. Contact pressure distribution for the case of aluminium pin, steel ring and  $R_2 = 45$  mm.

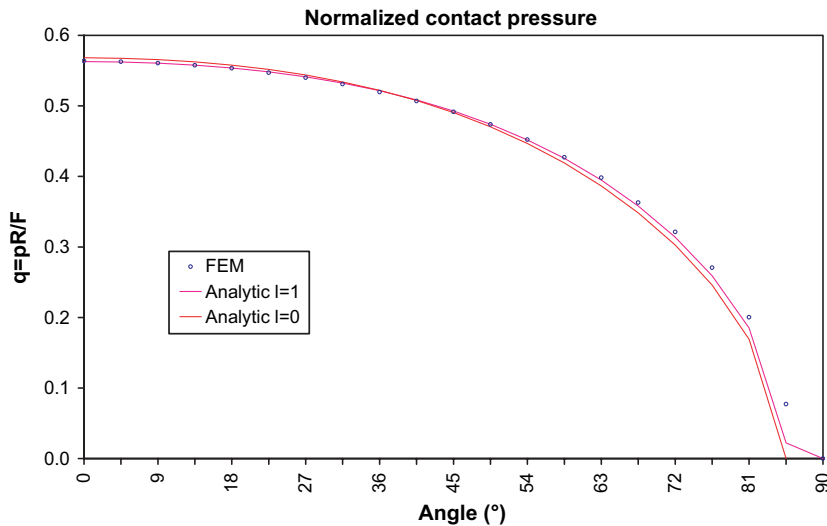


Fig. 6. Contact pressure distribution for the case of steel pin, aluminium ring and  $R_2 = 45$  mm.

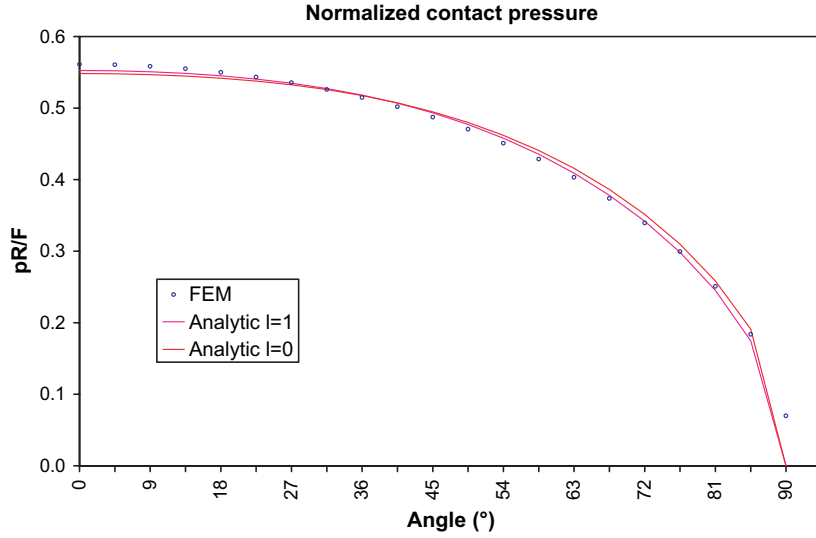


Fig. 7. Contact pressure distribution for the case of aluminium pin, steel ring and  $R_2 = 30$  mm.

## 5. Closing remarks

In this paper, an approximate analytical method has been proposed to derive closed-form solutions to the conformal contact problem in a reinforced pin-loaded joint in the case of elastic dissimilarity. This method, quite different from that used by Ciavarella and Decuzzi (2001b) to treat the latter, is based on the approximation of the terms involving the second Dundurs' material constant  $\gamma_2$  by a finite series and on the minimization of the error function and has led to very accurate analytical results in comparison with the available numerical ones. The work presented above has satisfactorily completed the previous work (To et al., 2007). However, as pointed out in the latter, we believe that taking into account the friction between the pin and ring could be only done numerically (see e.g Renaud and Feng, 2003; Iyer, 2001; Lin and Lin, 1999; Hyer and Klang, 1985).

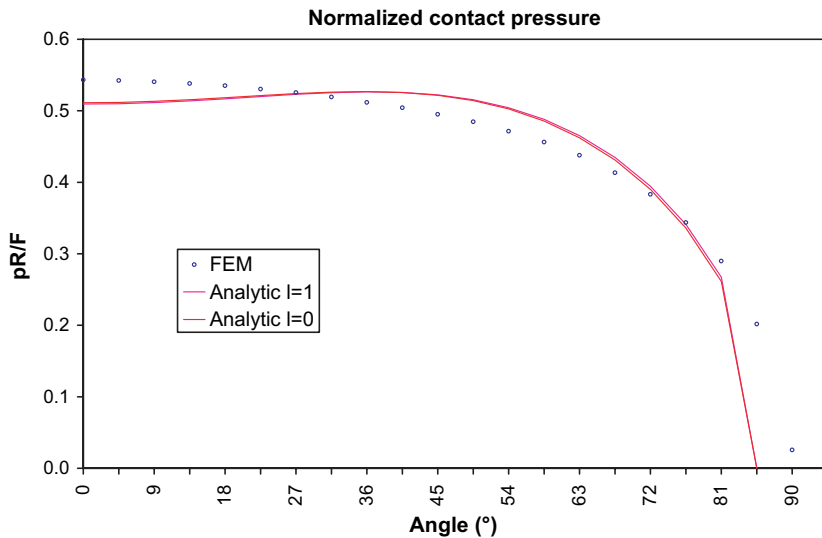


Fig. 8. Contact pressure distribution for the case of steel pin, aluminium ring and  $R_2 = 30$  mm.

The elastic conforming contact problem in a ball-loaded structure is the three-dimensional counterpart of the one in a pin-loaded structure. This problem is of both theoretical and practical importance (see, e.g., Ciavarella et al., 2007). Solving this three-dimensional problem in a closed-form way would constitute a challenge.

## Appendix A

Let us show that Eq. (9) can be recast into a generalized Prandtl equation (see Gori et al., 1998). First, we make the following variable change:

$$x = \sin^2(\theta/2)/m, \quad t = \sin^2(\xi/2)/m,$$

which implies that

$$\begin{aligned} \theta &= \cos^{-1}(1 - 2xm), \quad \xi = \cos^{-1}(1 - 2tm), \\ \cos \theta - \cos \xi &= 2m(t - x), \quad d\xi = \frac{2m dt}{\sqrt{1 - (1 - 2tm)^2}}. \end{aligned}$$

With the help of the above variable change, Eq. (9) becomes

$$\int_0^1 \frac{q_1(t) dt}{t - x} + g(x) \int_0^x q_1(t) dt = f(x),$$

where

$$\begin{aligned} q_1(t) &= \frac{q(\cos^{-1}(1 - 2tm))}{\sqrt{1 - (1 - 2tm)^2}}, \quad g(x) = -\frac{2m\gamma_2\pi}{\sin \theta(x)}, \\ f(x) &= (\beta_0^* H_0 - \lambda) \frac{\theta(x)}{\sin \theta(x)} + 2(1 + \gamma_2) H_1 + \sum_{n=2}^{\infty} \beta_n H_n \frac{\sin n\theta(x)}{\sin \theta(x)}. \end{aligned}$$

Then, posing

$$\int_0^x q_1(t) dt = y(x),$$

Eq. (9) takes the final form

$$\int_0^1 \frac{y'(t) dt}{t - x} + g(x)y(x) = f(x),$$

which is a generalized Prandtl equation.

## References

- Barber, J.R., 2002. *Elasticity*. Kluwer Academic Publisher.
- Ciavarella, M., Strozzi, A., Baldini, A., Giacomini, M., 2007. Normalization of load and clearance effects in ball-in-socket-like replacements. *Proc. I. Mech. Eng. H: J. Eng. Med.* 221, 601–611.
- Ciavarella, M., Baldini, A., Barber, J.R., Strozzi, A., 2006. Reduced dependence on loading parameters in almost conforming contacts. *Int. J. Mech. Sci.* 48, 917–925.
- Ciavarella, M., Decuzzi, P., 2001a. The state of stress induced by the plane frictionless cylindrical contact. I. The case of elastic similarity. *Int. J. Solids Struct.* 38, 4507–4523.
- Ciavarella, M., Decuzzi, P., 2001b. The state of stress induced by the plane frictionless cylindrical contact. II. The case of elastic dissimilarity. *Int. J. Solids Struct.* 38, 4525–4533.
- Gori, L., Santi, E., Cimoroni, M.G., 1998. Projector-splines in the numerical solution of integro-differential equations. *Comput. Math. Appl.* 35, 107–116.
- Hills, D.A., Nowell, D., Sackfield, A., 1993. *Mechanics of Elastic Contacts*. Butterworth-Heinemann, Oxford.
- Hyer, M.W., Klang, E.C., 1985. Contact stresses in pin-loaded orthotropic plates. *Int. J. Solids Struct.* 21, 957–975.
- Iyer, K., 2001. Solutions for contact in pinned connections. *Int. J. Solids Struct.* 38, 9133–9148.
- Johnson, K.L., 1985. *Contact Mechanics*. Cambridge University Press.



- Lin, C.C., Lin, C.H., 1999. Stresses around pin-loaded hole in composite laminates using direct boundary element method. *Int. J. Solids Struct.* 36, 763–783.
- Noble, B., Hussain, M.A., 1969. Exact solution of certain dual series for indentation and inclusion problem. *Int. J. Eng. Sci.* 7, 1149–1161.
- Polyanin, A.D., Manzhirov, A.V., 1998. *Handbook of Integral Equations*. CRC Press.
- Peters, A.S., 1963. A note on the integral equation of the first kind with a Cauchy kernel. *Commun. Pure Appl. Math.* 16, 57–61.
- Renaud, C., Feng, Z.-Q., 2003. BEM and FEM analysis of Signorini contact problems with friction. *Comput. Mech.* 31, 390–399.
- To, Q.D., He, Q.-C., Cossavella, M., Morcant, K., Panait, A., 2007. Closed-form solution for the contact problem of reinforced pin-loaded joints used in glass structures. *Int. J. Solids Struct.* 44, 3887–3903.