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# DOWNSIDE RISK AND KAPPA INDEX OF NON-GAUSSIAN PORTFOLIO WITH LPM

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**ABSTRACT.** In this paper, we find analytic expressions of the *lower partial moment* and *kappa index* of linear portfolios when the returns are elliptically distributed. We also introduced the notion of *Target Semi-Kurtosis* of portfolio return and discuss the robust optimization Mean-LPM problem with non-gaussian risk factors. Special attention is given to the particular case of a mixture of multivariate *t*-distributions.

*Key Words:* Lower partial moment; Kappa Index; Linear Portfolio; Elliptical portfolio, Performance Measure.

## 1. Introduction

The concept of risk and its measurement is very important in economics, business and industry problems. Thus the literature about financial or insurance risk and performance measurement has increase continuously in the last three decades. Basel I, II, III and solvency II banking and insurance regulations for the determination of capital allocation have increased the importance or the risk measurement. Variance which is the square of the standard deviation has several mathematical properties that are very useful such as the decomposition rules. Thus it has been the common risk measure used in portfolio optimization since the introduction of mean-variance model in the seminal work of Markowitz (1952). For a long time ago, the standard deviation as a dispersion or risk measure has been used in risk modeling, asset pricing theory and portfolio risk. However, the use of variance as a risk measure have been criticized by several authors. Variance measures upside as well as downside risk. Investors should however be more interested in maximizing upside risk than minimizing it. The use of variance implies that the investor has a quadratic utility function. If an investor disagrees with this view, classical portfolio theory will not maximized his utility. Markowitz(1959) noted that variance is invariant under reflection ; it do not recognize asymmetry with asset return distributions, and it consider favorable and adverse outcomes in the same way. The nobel prize Markowitz suggested that semi-variance which is particular case of the lower partial moment (LPM), may serve as an objective function for portfolio construction. Among the all criticisms against variance as a risk measure, the risk literature has recognized the importance of distinguishing between adverse and favorable results, and this has been a serious area of research. That why, among several alternatives to supplied variance as a risk measure, the LPM plays an important role in the analysis of risks and in other areas such in income/poverty

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studies because it only penalizes the downside deviation. Also, many of the theoretical and experimental research in risk identification and measurement shows that corporate heads are mostly concerned with one sided risk which the so-called "downside risk", that measures the distance between the risky situation and a real fixed threshold. Unlike variance, LPM avoids the minimization of upside risk and its give investor the option to select the order of his utility function. Note that, psychologists studies of Mao (1970), Unser(2000) and Veld and Veld-Merkoulova (2008) suggested LPM over variance as a measure of the investor's perception of risk. In the microeconomics area, the relationship between stochastic dominance and the LPM, under certain nested utility function classes, was developed by several authors such as Bawa (1975) and Bawa (1978). The latter author shows that the ordering of portfolios for an investor with a certain class of utility function is equivalent to the ordering provided by the LPM under certain conditions. Because the ordering involves evaluating the LPM for each threshold accross the entire return domain, its follows that LPM is very attractive in utility theory. In fact, it allows the evaluation of the LPM degree for individual investors with varying preferences.

One of the most popular downside risk measures is the lower partial moment (LPM). The entry of LPM in portfolio theory has mainly been driven by Bawa(1975), Fishburn (1977) and Nawrocki(1991, 1992 and 1999). The LPM recognizes asymmetry within return distributions and does not involve the size of favorable outcomes. The LPM take into account the size of adverse results, that below a define and fix threshold. Because LPM can be viewed as the expected value of a powered put option on the returns of a portfolio, its agrees with intuitive notion of risk. Consider a portfolio with a random return  $X$  and assume individual has a target return  $\tau$ . An outcome larger than  $\tau$  is non-risky and desirable, then individual faces only a one-sided risk called downside risk that occurs when  $X$  falls below  $\tau$ . Therefore LPM provides a measure that a specified minimum return (target return) may not be earned by a financial investment. Is clear that, LPM provides a summary statistics for the downside risk. For a given integer  $n$ , the  $n^{th}$  order LPM of a random variable  $X$  yields a measure known as a Target Semi-Variance (TSV) for  $n = 2$ , the so-called Target Semi-Kurtosis (TSK) for  $n = 4$ ...etc. One reason for the interest in LPM measures of risk such as TSV is that they reflect investor's preferences better than the traditional measure such as variance. Because TSV and TSK penalize any extremely low returns in the same way that the variance and kurtosis penalizes extreme values in either direction, TSV and TSK respectively fit investors risk preferences better than respectively variance and TSK.

In this paper, we find analytic expressions of the lower partial moments and *kappa index* of linear portfolios when the returns are elliptically distributed. Special attention is given to the particular case of a mixture of multivariate  $t$ -distributions. We also introduced the so-called *Target Semi-Kurtosis* that take into account the leptokurticity for the return portfolio that is below the target  $\tau$ .

The paper is organized as follows: section 2 recalls the definitions of the LPM and the *Kappa index* risk-adjusted performance measure. Section 3 describes LPM and *kappa index* for portfolios with mixture of elliptical distributions risk factors. In section 4, we introduce the Mean-LPM portfolio optimization with elliptical distributed risk factors; special attention given to the mixture of  $t$ -Student distributions. Section 5 discuss parameters estimation and section 6 concludes.

## 2. LOWER PARTIAL MOMENTS AND KAPPA INDEX

As it has been mentioned in Unser (2002), investors are often only interested in an evaluation of outcomes values that are below a given real target  $\tau$ . This features

yields the importance of downside risk measures (see, e.g. Ebert 2005) such as LPM. The lower partial moment (LPM) of order  $n > 0$  of the portfolio return  $R_p$ , given a threshold return  $\tau$  is defined as:

$$(1) \quad \text{LPM}_n(\tau) = E[\max(\tau - R_p, 0)^n].$$

Instead of Sharpe Ratio index, the so-called Kappa index can be defined by using the lower partial moment (LPM) as a standard deviation in the denominator of the Sharpe ratio. This yields the Kappa index introduced by Kaplan and Knowles (2004) as follows:

$$(2) \quad K_n(\tau) = \frac{E(R_p) - \tau}{(\text{LPM}_n(\tau))^{\frac{1}{n}}}.$$

When  $\tau = E(R_p) = \mu \cdot w$ ,  $\text{LPM}_2(\tau)$  correspond to the semi-variance. In general, for  $n > 0$  an odd number,  $\text{LPM}_n(E(R_p))$  is the semi-moment of par order  $n$  (i.e. semi-variance (n=1), semi-kurtosis (n=4)). In the Bawa and Linderberg (1977) framework to characterize equilibrium, risk is defined as deviation below the risk-free interest rate (i.e.  $\tau = r_f$ ). In this case the *Kappa index* will produce preference rankings congruent with the use of a piecewise power-linear utility function of the form:

$$(3) \quad u(x) = \begin{cases} x, & x \geq \tau; \\ x - c(x - \tau)^n, & x \leq \tau. \end{cases}$$

Replacing  $n = 1$  in (2), we get the *Omega ratio* performance measure (see Keating and Shadwick (2002)). For  $n = 2$ , we get the *Sortino ratio* (see Kazemi et al.,2004).

**Remark 2.1.** The order  $n$  of the  $\text{LPM}_n$  measure determines the type of the utility functions consistent with that risk measure.  $\text{LPM}_1$  is consistent with all utility functions such that  $(-1)^m u^{(m)} < 0$  for  $m = 1, 2$ .  $\text{LPM}_2$  is valid for all risk averse functions displaying skewness preference ( $(-1)^m u^{(m)} < 0$  for  $m = 1, 2, 3$ ). In general,  $\text{LPM}_{p+1}$  is valid for all risk averse functions displaying  $p^{\text{th}}$ -moment preference ( $(-1)^m u^{(m)} < 0$  for  $m = 1, 2, \dots, p$ ). Thus the familiar HARA class is entirely consistent with  $\text{LPM}_1$ , while all utility function  $u$  displaying DARA are consistent with  $\text{LPM}_2$  since DARA implies  $u^{(3)} > 0$ .

### 3. LPM AND KAPPA INDEX OF PORTFOLIOS WITH LOG-ELLIPTICALLY DISTRIBUTED RETURNS

We will use the following notational conventions for the action of matrices on vectors: single letters  $x, y, \dots$  will denote *row vectors*  $(x_1, \dots, x_n)$ ,  $(y_1, \dots, y_k)$ . The corresponding column vectors will be denoted by  $x^t, y^t$ , the  $^t$  standing more generally for taking the transpose of any matrix. Matrices  $A = (A_{ij})_{i,j}$ ,  $B$ , etc. will be multiplied in the usual way. In particular,  $A$  will act on vectors by left-multiplication on column vectors,  $Ay^t$ , and by right multiplication on row vectors,  $xA$ ;  $x \cdot x = xx^t = x_1^2 + \dots + x_k^2$  will stand for the Euclidean inner product.

A portfolio with time- $t$  value  $\Pi(t)$  is called linear if its profit and loss  $\Delta\Pi(t) = (\Pi(t) - \Pi(0))/\Pi(0)$  over a time window  $[0, t]$ , is a linear function of the returns  $X_1(t), \dots, X_n(t)$  of its constituents over the same time period:

$$(4) \quad R_P = \log \left( \frac{\Pi(t) - \Pi(0)}{\Pi(0)} \right) \simeq \sum_{i=1}^k w_i X_i$$

where  $X_i = \log(S_i(t)/S_i(0))$  is the risk factor of asset  $i$  in the time interval  $[0, t]$ .

This will for instance be the case for ordinary portfolios of common stock, if we use percentage returns, and will also hold to good approximation with log-returns,

provided the time window  $[0, t]$  is small. We will drop the time  $t$  from our notations, since it will be kept fixed, and simply write  $X_j, \Delta\Pi$ , etc. We also put

$$X = (X_1, \dots, X_k),$$

so that  $\Delta\Pi = \delta \cdot X = \delta X^t$ .

We now assume that the  $X_j$  are elliptically distributed with mean  $\mu$  and correlation matrix  $\Sigma = AA^t$ :

$$(X_1, \dots, X_k) \sim N(\mu, \Sigma, \phi).$$

This means that the pdf of  $X$  is of the form

$$f_X(x) = |\Sigma|^{-1} g((x - \mu)\Sigma^{-1}(x - \mu)^t),$$

where  $|\Sigma|$  stands for the determinant of  $\Sigma$ , and where  $g : \mathbb{R}_{\geq 0} \rightarrow 0$  is such that the Fourier transform of  $g(|x|^2)$ , as a generalized function on  $\mathbb{R}^n$ , is equal to  $\phi(|\xi|^2)^1$ . Assuming that  $g$  is continuous, and non-zero everywhere, the LPM is given as follows:

$$\begin{aligned} \text{LPM}_n(\tau, \mathbf{x}) &= E[\max(\tau - R_p, 0)^n] \\ &= |\Sigma|^{-1/2} \int_{\mathbb{R}^k} \max(\tau - R_p, 0)^n g((x - \mu)\Sigma^{-1}(x - \mu)^t) dx \\ &= |\Sigma|^{-1/2} \int_{\{R_p \leq \tau\}} (\tau - R_p)^n g((x - \mu)\Sigma^{-1}(x - \mu)^t) dx \\ (5) \quad &= |\Sigma|^{-1/2} \int_{\{\sum_{i=1}^k w_i x_i \leq \tau\}} \left( \tau - \sum_{i=1}^k w_i x_i \right)^n g((x - \mu)\Sigma^{-1}(x - \mu)^t) dx \end{aligned}$$

where in the precede integral  $x - \mu = (x_1 - \mu_1, \dots, x_k - \mu_k)$  is a row vector and  $E$  denotes the mean probability operator. Changing variables to  $y = (x - \mu)A^{-1}$ ,  $dy = |A| dx$ , where  $\Sigma = A^t A$  is a Cholesky decomposition of  $A$ , this becomes

$$\begin{aligned} \text{LPM}_n(\tau, \mathbf{x}) &= |\Sigma|^{-1/2} \int_{\{\sum_{i=1}^k w_i x_i \leq \tau\}} \left( \tau - \sum_{i=1}^k w_i x_i \right)^n g((x - \mu)\Sigma^{-1}(x - \mu)^t) dx \\ (6) \quad &= \int_{\{wA \cdot y \leq \tau - w \cdot \mu\}} (\tau - w \cdot \mu - wA \cdot y)^n g(|y|^2) dy \end{aligned}$$

Let  $R$  be a rotation which sends  $\delta A$  to  $(|\delta A|, 0, \dots, 0)$ . Changing variables once more to  $y = zR$ , we obtain the equation

$$\begin{aligned} \text{LPM}_n(\tau, \mathbf{x}) &= \int_{\{wA \cdot y \leq \tau - w \cdot \mu\}} (\tau - w \cdot \mu - wA \cdot y)^n g(|y|^2) dy \\ (7) \quad &= \int_{\{|\delta A| z_1 \leq -w \cdot \mu + \tau\}} (\tau - w \cdot \mu - |\delta A| z_1)^n g(|z|^2) dz \end{aligned}$$

If we write that  $|z|^2 = z_1^2 + |z'|^2$  with  $z' \in \mathbb{R}^{n-1}$  then we obtain the following expression:

$$(8) \quad \text{LPM}_n(\tau, \mathbf{x}) = \int_{\mathbf{R}^{k-1}} \left[ \int_{+\infty}^{\frac{\tau - w \cdot \mu}{|\delta A|}} (\tau - w \cdot \mu - |\delta A| z_1)^n g(z_1^2 + |z'|^2) dz_1 \right] dz'$$

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<sup>1</sup>One uses  $\phi$  as a parameter for the class of elliptic distributions, since it is always well-defined as a continuous function:  $\phi(|\xi|^2)$  is simply the characteristic function of an  $X \sim N(0, Id, \phi)$ . Note, however, that in applications we'd rather know  $g$

Next, by using spherical variables  $z' = r\xi$  with  $\xi \in S_{k-2}$ ,  $dz' = r^{k-2}d\sigma(\xi)dr$ , we have the following expression

$$(9) \quad \text{LPM}_n(\tau, \mathbf{x}) = |S_{k-2}| \int_0^{+\infty} r^{k-2} \left[ \int_{-\infty}^{\frac{-w\mu^t + \tau}{|wA|}} (\tau - w\mu^t - |wA|z_1)^n g(z_1^2 + r^2) dz_1 \right] dr,$$

$|S_{k-2}|$  being the surface measure of the unit-sphere in  $\mathbb{R}^{k-1}$ :

$$|S_{k-2}| = \frac{2\pi^{\frac{k-1}{2}}}{\Gamma(\frac{k-1}{2})}.$$

We now introduce the parameter  $s = \frac{w\mu^t - \tau}{|wA|}$  and get the function  $LPM(\tau)$ :

$$(10) \quad \begin{aligned} \text{LPM}_n(\tau, \mathbf{x}) &= \frac{2\pi^{\frac{k-1}{2}}|wA|^n}{\Gamma(\frac{k-1}{2})} \int_{-\infty}^{-s} \left[ \int_0^{+\infty} r^{k-2} (-s - z_1)^n g(z_1^2 + r^2) dr \right] dz_1 \\ &= \frac{\pi^{\frac{k-1}{2}}}{\Gamma(\frac{k-1}{2})} |wA|^n \int_s^{+\infty} \left[ (z_1 - s)^n \int_{z_1^2}^{+\infty} (u - z_1^2)^{\frac{k-3}{2}} g(u) du \right] dz_1, \end{aligned}$$

where for the second line we changed variables  $u = r^2 + z_1^2$  and replaced  $z_1$  by  $-z_1$ . We then have proved the following result:

**Theorem 3.1.** Let's assume that the portfolio's Profit & Loss function over the time window of interest is, to good approximation, given by  $\Delta\Pi = w_1X_1 + w_2X_2 + \dots + w_kX_k$ , with constant portfolio weights  $w_j$ . If the random vector  $X = (X_1, \dots, X_k)$  of risk factors follows a continuous elliptic distribution, with probability density given by  $f_X(x) = |\Sigma|^{-1}g((x - \mu)\Sigma^{-1}(x - \mu)^t)$  where  $\mu$  is the vector mean and  $\Sigma$  is the variance-covariance matrix, and where we suppose that  $g(s^2)$  is integrable over  $\mathbb{R}$ , continuous and nowhere 0, then for a given  $n, \tau \in \mathbb{R}_+^*$ , the portfolio's *Lower Partial Moment LPM* is define as

$$(11) \quad \text{LPM}_n(\tau, \mathbf{w}) = \frac{\pi^{\frac{k-1}{2}}}{\Gamma(\frac{k-1}{2})} |\mathbf{w}A|^n \int_s^{+\infty} \left[ (z_1 - s)^n \int_{z_1^2}^{+\infty} (u - z_1^2)^{\frac{n-3}{2}} g(u) du \right] dz_1,$$

where the parameter  $s = \frac{\mathbf{w}\mu^t - \tau}{|\mathbf{w}A|}$  with  $\mathbf{w} = (w_1, w_2, \dots, w_k)$ .

**Remark 3.2.** Note that  $|\mathbf{w}A|$  has a clear financial interpretation, since

$$(12) \quad |\mathbf{w}A| = \sqrt{\mathbf{w}\Sigma\mathbf{w}^t},$$

which is simply the portfolio's volatility, or the square of its variance. Moreover, if  $\tau = r_f$  is the risk free rate, then

$$(13) \quad s = \frac{\mathbf{w}\mu^t - \tau}{|\mathbf{w}A|} = \frac{\mathbf{w}\mu^t - \tau}{\sqrt{\mathbf{w}\Sigma\mathbf{w}^t}}$$

is the classical Sharpe ratio adjusted-risk measure of portfolio performance. It is clear that  $\text{LPM}_n(\tau, \mathbf{w})$  is a function of the Sharpe ratio.

**Remark 3.3.** One can do the integral over  $z_1$  in (22): by Fubini,

$$\begin{aligned}
\text{LPM}_n(\tau) &= \frac{\pi^{\frac{k-1}{2}}}{\Gamma(\frac{k-1}{2})} |wA|^n \int_s^{+\infty} \left[ \int_{\sqrt{s}}^{\sqrt{u}} (z_1 - s)^n (u - z_1^2)^{\frac{k-3}{2}} dz_1 \right] g(u) du \\
(14) &= \frac{\pi^{\frac{k-1}{2}}}{\Gamma(\frac{k-1}{2})} |wA|^n \sum_{j=0}^n \frac{j!(n-j)!}{n!} s^j \int_s^{+\infty} \left[ \int_{\sqrt{s}}^{\sqrt{u}} (u - z_1^2)^{\frac{k-3}{2}} z_1^{n-j} dz_1 \right] g(u) du \\
&= \frac{\pi^{\frac{k-1}{2}}}{2\Gamma(\frac{k-1}{2})} |wA|^n \sum_{j=0}^n \frac{j!(n-j)!}{n!} s^j \int_s^{+\infty} \left[ \int_0^{u-s} w^{\frac{k-3}{2}} (u-w)^{\frac{n-j-1}{2}} dw \right] g(u) du \\
(15) &= \frac{\pi^{\frac{k-1}{2}}}{2} |wA|^n \sum_{j=0}^n \frac{j!(n-j)!}{n!} \frac{\Gamma(\frac{n-j-1}{2})}{\Gamma(\frac{n+k-j}{2})} s^j \int_s^{+\infty} (u-s)^{\frac{n+k-j}{2}-1} g(u) du
\end{aligned}$$

In the precede remark, we have used the following lemma:

**Lemma 3.4.** (Cf. [12], page 312.) If  $\text{Re}(\nu_1) > 0$  and  $\text{Re}(\mu) > 0$ , then

$$(16) \quad \int_0^z x^{\nu_1-1} (x-w)^{\mu-1} dx = z^{\mu+\nu_1-1} B(\nu_1, \mu),$$

with  $B(\alpha, \beta)$  the Euler Beta function:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

Using the precede lemma with  $\nu_1 = \frac{k-1}{2}$ ,  $\mu = \frac{n-j+1}{2}$  and  $z = u - s$  in (22), we have prove the following theorem:

**Theorem 3.5.** Assume that the risk factors random vector  $X = (X_1, \dots, X_k)$  of the portfolio follows a continuous elliptic distribution, with probability density given by  $f_X(x) = |\Sigma|^{-1} g((x - \mu)\Sigma^{-1}(x - \mu)^t)$  where  $\mu$  is the vector mean and  $\Sigma$  is the variance-covariance matrix, and where we suppose that  $g(s^2)$  is integrable over  $\mathbb{R}$ , continuous and nowhere 0. By applying Fubini to (22), the lower partial moment  $\text{LPM}_n(\tau)$  of the portfolio returns  $R_p$  becomes

$$(17) \quad \text{LPM}_{n,g}(\tau, \mathbf{w}) = \int_s^{+\infty} K_{n,k}(s, u) g(u) du,$$

where  $s = \frac{\mathbf{w}\mu^t - \tau}{|wA|}$ ,  $\mathbf{w} = (w_1, w_2, \dots, w_k)$  and the kernel function  $K_{n,k}$  is given by:

$$(18) \quad K_{n,k}(s, u) = \frac{\pi^{\frac{k-1}{2}}}{2} |wA|^n \sum_{j=0}^n \frac{j!(n-j)!}{n!} \frac{\Gamma(\frac{n-j-1}{2})}{\Gamma(\frac{n+k-j}{2})} s^j (u-s)^{\frac{n+k-j}{2}-1}.$$

The *Kappa index* of order  $n$  is given by

$$(19) \quad \kappa_{n,g}(\tau, \mathbf{w}) := \frac{\mathbf{w}\mu^t - \tau}{(\text{LPM}_{n,g}(\tau, \mathbf{w}))^{\frac{1}{n}}}.$$

**Corollary 3.6.** For  $n = 4$ ,  $\tau = E(R_p) = w \cdot \mu$  then  $s = 0$  and the lower semi-kurtosis of the portfolio return  $R_p$  is given by

$$(20) \quad \text{LPM}_4(w \cdot \mu) = \int_s^{+\infty} K_{4,k}(w \cdot \mu, u) g(u) du$$

**Corollary 3.7.** In short-term Risk Management one can usually assume that  $\mu \simeq 0$ . In that case, we have that

$$(21) \quad \text{LPM}_n(\tau) = \frac{\pi^{\frac{k-1}{2}}}{\Gamma(\frac{k-1}{2})} \int_{\frac{-\tau}{|wA|}}^{+\infty} \left[ (\tau - |wA|z_1)^n \int_{z_1^2}^{+\infty} (u - z_1^2)^{\frac{k-3}{2}} g(u) du \right] dz_1.$$

**3.1. The case of t-student Distributions.** Like the multivariate normal distribution, multivariate  $t$  distributions belongs to the family of elliptically symmetric distributions, but with an additional parameter  $\nu$ , called the degrees of freedom. As  $\nu$  tends to infinity, the  $t$ -distribution approaches the normal distribution. I now consider in detail the case where our elliptic distribution is a multivariate Student- $t$ . We will, unsurprisingly, call the corresponding  $LPM$  the  $\Delta$ -Student  $LPM$ .

In the case of multi-variate t-student distributions, the portfolio probability density function is given by:

$$f_X(x) = \frac{\Gamma(\frac{\nu+k}{2})}{\Gamma(\nu/2) \cdot \sqrt{|\Sigma|} (\nu\pi)^k} \left( 1 + \frac{(x-\mu)^t \Sigma^{-1} (x-\mu)}{\nu} \right)^{\frac{-(\nu+k)}{2}},$$

$x \in \mathbb{R}^k$  and  $\nu > 2$ . Hence  $g$  is given by

$$g(s) = C(\nu, k) (1 + s/\nu)^{-\frac{(k+\nu)}{2}}, \quad s \geq 0,$$

where we have put

$$C(\nu, k) = \frac{\Gamma(\frac{\nu+k}{2})}{\Gamma(\nu/2) \sqrt{(\nu\pi)^n}}.$$

Using this  $g$  in (22), we find that

$$\begin{aligned} LPM_n(\tau) &= C_1(\nu, k) \sum_{j=0}^n \frac{j!(n-j)!}{n!} \frac{\Gamma(\frac{n-j-1}{2})}{\Gamma(\frac{n+k-j}{2})} s^j \int_s^{+\infty} (u-s)^{\frac{n+k-j}{2}-1} (\nu+u)^{-\frac{(k+\nu)}{2}} du \\ (22) \end{aligned}$$

**Lemma 3.8.** (cf. [12], formula 3.197(2)). If  $|\arg(\frac{u}{\beta})| < \pi$  or  $|u| > |\beta|$  and  $0 < \text{Re}(\mu) < \text{Re}(-\nu)$ , then

$$(23) \quad \int_u^{+\infty} (x-u)^{\mu_1-1} (x+\beta)^{\nu_1} dx = u^{\mu_1+\nu_1} B(-\mu_1-\nu_1, \mu_1) {}_2F_1\left(-\nu_1, -\mu_1-\nu_1; -\nu_1; -\frac{\beta}{u}\right).$$

where  ${}_2F_1(\alpha; \beta, \gamma; w)$  is the hypergeometric function.

Applying the precede lemma (33) to the integral part of equation (22) yields the following theorem:

**Theorem 3.9.** Assuming that  $\Delta\Pi \simeq \delta_1 X_1 + \delta_2 X_2 + \dots + \delta_k X_k$  with a multivariate Student- $t$  random vector  $(X_1, X_2, \dots, X_k)$  with vector mean  $\mu$ , and variance-covariance matrix  $\Sigma$ , the  $LPM_n$  of a linear portfolio is given by:

$$(24) \quad LPM_{n,\nu}(\tau) = C_{\nu,n,s} \sum_{j=0}^n \frac{\Gamma(\frac{n-j-1}{2}) \Gamma(\frac{\nu-n+j}{2}) j!(n-j)!}{n!} {}_2F_1\left(\frac{k+\nu}{2}, \frac{\nu+j-n}{2}, \frac{k+\nu}{2}; -\frac{\nu}{s}\right) s^{\frac{j}{2}}$$

where  $C_{\nu,n,s} = \frac{\nu^{\nu/2} |wA|^n}{2\sqrt{\pi} \Gamma(\nu/2)} s^{\frac{n-\nu}{2}}$  and  $s = \frac{w\mu^t - \tau}{|wA|}$ .

The kappa index of order  $n$  is given by

$$(25) \quad \kappa_{n,\nu}(\tau) := \frac{w\mu^t - \tau}{\left( C_{\nu,n,s} \sum_{j=0}^n \frac{\Gamma(\frac{n-j-1}{2}) \Gamma(\frac{\nu-n+j}{2}) j!(n-j)!}{n!} {}_2F_1\left(\frac{k+\nu}{2}, \frac{\nu+j-n}{2}, \frac{k+\nu}{2}; -\frac{\nu}{s}\right) s^{\frac{j}{2}} \right)^{\frac{1}{n}}}.$$

For the proof of the theorem (24), we can also use the following lemma:

**Lemma 3.10.** (cf. [12], formula 3.254(2)). If  $\operatorname{Re}\left(\frac{\beta}{u}\right) > 0$ ,  $|u| > |\beta|$  and  $0 < \operatorname{Re}(\mu) < \operatorname{Re}(1 - 2\nu_1)$ , then

$$(26) \quad \int_u^{+\infty} ((x-u)^{\mu_1-1}(x^2+\beta^2))^{\nu_1} dx = \frac{\Gamma(\mu_1)\Gamma(\lambda_1 - \mu_1)}{\Gamma(\lambda_1 - 2\nu_1)u^{-\mu_1+\lambda_1}} {}_3F_2\left(-\nu_1, \frac{\lambda_1 - \mu_1}{2}, \frac{\lambda_1 + 1 - \mu_1}{2}, \frac{\lambda_1}{2}, \frac{1 + \lambda_1}{2}; -\frac{\beta^2}{u^2}\right),$$

where  $\lambda_1 = 1 - 2\nu_1$  and  ${}_3F_2(\alpha; \beta, \gamma; w)$  is the hypergeometric function.

In fact, using this  $g$  in (22), we find that

$$(27) \quad \operatorname{LPM}_n^{ST}(\tau) = \frac{\pi^{\frac{k-1}{2}} C(\nu, k)}{\Gamma(\frac{k-1}{2})} \int_{\frac{w\mu^t - \tau}{|wA|}}^{\infty} \left[ (\tau - w \cdot \mu + |wA|z_1)^n \int_{z_1^2}^{+\infty} (u - z_1^2)^{\frac{k-3}{2}} \left(1 + \frac{u}{\nu}\right)^{-\frac{(k+\nu)}{2}} du \right] dz_1$$

then

$$(28) \quad \operatorname{LPM}_n^{ST}(\tau) = C_1(k, \nu) |wA|^n \int_s^{\infty} (-s + z_1)^n T(z_1) dz_1$$

where  $C_1(k, \nu) = \frac{\pi^{\frac{k-1}{2}} C(\nu, k)}{\Gamma(\frac{k-1}{2})\nu^{-\frac{(k+\nu)}{2}}} = \frac{\pi^{\frac{k-1}{2}} \Gamma(\frac{\nu+k}{2})}{\Gamma(\frac{k-1}{2})\Gamma(\nu/2)\sqrt{(\nu\pi)^k \nu^{-(k+\nu)}}$  and

$$(29) \quad T(z_1) = \int_{z_1^2}^{+\infty} (u - z_1^2)^{\frac{k-3}{2}} (\nu + u)^{-\frac{(k+\nu)}{2}} du$$

The function  $T(z_1)$  can be evaluated with the help of another one of the integrals in [3]:

**Lemma 3.11.** (Cf. [12], page 314.) If  $|\arg(\frac{w}{\beta})| < \pi$ , and  $\operatorname{Re}(\nu_1) > \operatorname{Re}(\mu) > 0$ , then

$$(30) \quad \int_w^{+\infty} (x-w)^{\mu-1} (\beta+x)^{-\nu_1} dx = (w+\beta)^{\mu-\nu_1} B(\nu_1 - \mu, \mu),$$

with  $B(\alpha, \beta)$  the Euler Beta function:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

Using formula (30) with  $\nu_1 = \frac{(k+\nu)}{2}$ ,  $\mu = \frac{k-1}{2}$ ,  $\beta = \nu$ , and  $w = z_1^2$ , and therefore,  $\mu - \nu_1 = -\frac{1+\nu}{2}$  and  $-\mu + \nu_1 = \frac{1+\nu}{2}$ , we find that

$$(31) \quad T(z_1) = (z_1^2 + \nu)^{-\frac{1+\nu}{2}} B\left(\frac{1+\nu}{2}, \frac{k-1}{2}\right).$$

We have not finished yet, since we still have to integrate over  $z_1$  in (28). We therefore have to evaluate

$$(32) \quad \begin{aligned} J(s, \nu) &= \int_{\frac{w\mu^t - \tau}{|wA|}}^{\infty} (\tau - w \cdot \mu - |wA|z_1)^n (z_1^2 + \nu)^{-\frac{1+\nu}{2}} dz_1 \\ &= |wA|^n \int_s^{\infty} (s - z_1)^n (z_1^2 + \nu)^{-\frac{1+\nu}{2}} dz_1 \end{aligned}$$

with  $s = \frac{w\mu^t - \tau}{|wA|}$ .

For the latter integral, we will use another formula from [3], in the case where  $\lambda = 1$ :

**Lemma 3.12.** (cf. [12], formula 3.254(2)). If  $Re\left(\frac{\beta}{u}\right) > 0$ ,  $|u| > |\beta|$  and  $0 < Re(\mu) < Re(1 - 2\nu_1)$ , then

$$(33) \quad \int_u^{+\infty} ((x-u)^{\mu_1-1}(x^2+\beta^2))^{\nu_1} dx = \frac{\Gamma(\mu_1)\Gamma(\lambda_1 - \mu_1)}{\Gamma(\lambda_1 - 2\nu_1)u^{-\mu_1+\lambda_1}} {}_3F_2\left(-\nu_1, \frac{\lambda_1 - \mu_1}{2}, \frac{\lambda_1 + 1 - \mu_1}{2}, \frac{\lambda_1}{2}, \frac{1 + \lambda_1}{2}; -\frac{\beta^2}{u^2}\right).$$

where  $\lambda_1 = 1 - 2\nu_1$  and  ${}_3F_2(\alpha; \beta, \gamma; w)$  is the hypergeometric function.

In our case,  $\nu_1 = -\frac{1+\nu}{2}$ ,  $(\mu_1 - 1)\nu_1 = n$ . therefore  $\mu_1 = -\frac{2n}{1+\nu} + 1$  and  $\beta^2 = \nu$ . If we replace in (32), we obtain the following expression:

$$(34) \quad \begin{aligned} J(s, \nu) &= |wA|^n \int_u^{\infty} (u - z_1)^n (z_1^2 + \nu)^{-\frac{1+\nu}{2}} dz_1 \\ &= |wA|^n \int_u^{\infty} ((u - z_1)^{\mu_1-1}(z_1^2 + \nu))^{\nu_1} dz_1. \end{aligned}$$

Then

$$\begin{aligned} J(s, \nu) &= |wA|^n \frac{\Gamma(\mu_1)\Gamma(\lambda_1 - \mu_1)}{\Gamma(\lambda_1 - 2\nu_1)u^{-\mu_1+\lambda_1}} {}_3F_2\left(-\nu_1, \frac{\lambda_1 - \mu_1}{2}, \frac{\lambda_1 + 1 - \mu_1}{2}, \frac{\lambda_1}{2}, \frac{1 + \lambda_1}{2}; -\frac{\nu}{u^2}\right) \\ &= \frac{|wA|^{n+\lambda_1-\mu_1}\Gamma(\mu_1)\Gamma(\lambda_1 - \mu_1)}{\Gamma(\lambda_1 - 2\nu_1)(w\mu^t - \tau)^{-\mu_1+\lambda_1}} {}_3F_2\left(-\nu_1, \frac{\lambda_1 - \mu_1}{2}, \frac{\lambda_1 + 1 - \mu_1}{2}, \frac{\lambda_1}{2}, \frac{1 + \lambda_1}{2}; -\frac{\nu|wA|^2}{(w\mu^t - \tau)^2}\right) \end{aligned}$$

with  $u = \frac{w\mu^t - \tau}{|wA|}$ .

Hypergeometric  ${}_3F_2$  and  ${}_2F_1$ 's have been extensively studied, and numerical software for their evaluation is available.

**3.2. LPM with a mixture of elliptic Distributions.** Mixtures of normal distributions have been largely considered in the literature such as McLachlan and Peel [], since the early works of Karl Pearson over 100 years ago. However in the last decade, in presence of data with longer-than-normal tails, mixture of elliptical distributions such as the mixture of  $t$ -Student distributions have proved to be a more useful as they provide more realistic tails for real world data, see e.g Kotz et Nadarajah (2004) and Sadefo Kamdem(2004). In the context of likelihood approach to mixture modeling, many authors follow the seminal works of Dempster et al. (1977) by using the expectation-maximization (EM) algorithm for estimating the parameters of the model.

Mixture distributions can be used to model situations where the data can be viewed as arising from two or more distinct classes of populations; see also [?]. For example, in the context of Risk Management, if we divide trading days into two sets, quiet days and hectic days, a mixture model will be based on the fact that returns are moderate on quiet days, but can be unusually large or small on hectic days. In my knowledge, applications of mixture models to compute LPM have not been published. Here we sketch how to compute LPM when the portfolios risk factors follow a mixture of multivariate elliptic distributions, that is, a convex linear combination of elliptic distributions.

**Definition 3.13.** We say that  $(X_1, \dots, X_n)$  has a joint distribution that is the mixture of  $m$  elliptic distributions  $N(\mu_j, \Sigma_j, \phi_j)^2$ , with weights  $\{\beta_j\}$  ( $j=1, \dots, m$ ;  $\beta_j > 0$ ;  $\sum_{j=1}^m \beta_j = 1$ ), if its cumulative distribution function can be written as

$$F_{X_1, \dots, X_n}(x_1, \dots, x_n) = \sum_{j=1}^m \beta_j F_j(x_1, \dots, x_n)$$

<sup>2</sup>or  $N(\mu_j, \Sigma_j, g_j)$  if we parameterize elliptical distributions using  $g$  instead of  $\phi$

with  $F_j(x_1, \dots, x_n)$  the cdf of  $N(\mu_j, \Sigma_j, \phi_j)$ .

**Remark 3.14.** In practice, one would usually limit oneself to  $m = 2$ , due to estimation and identification problems; see [?].

We will suppose that all our elliptic distributions  $N(\mu_j, \Sigma_j, \phi_j)$  admit a pdf :

$$(35) \quad f_j(x) = |\Sigma_j|^{-1/2} g_j((x - \mu_j)\Sigma_j^{-1}(x - \mu_j)^t).$$

The pdf of the mixture will then simply be  $\sum_{j=1}^m \beta_j f_j(x)$ .

Let

$$\Sigma_j = A_j^t A_j$$

be a Cholesky decomposition of  $\Sigma_j$ .

**Theorem 3.15.** Let  $\Delta\Pi = \delta_1 X_1 + \dots + \delta_n X_n$  with  $(X_1, \dots, X_n)$  is a mixture of elliptic distributions, with density

$$f(x) = \sum_{j=1}^m \beta_j |\Sigma_j|^{-1/2} g_j((x - \mu_j)\Sigma_j^{-1}(x - \mu_j)^t)$$

where  $\mu_j$  is the vector mean, and  $\Sigma_j$  the variance-covariance matrix of the  $j$ -th component of the mixture. We suppose that each  $g_j$  is integrable function over  $\mathbb{R}$ , and that the  $g_j$  never vanish jointly in a point of  $\mathbb{R}^m$ . By a linear combination of (??), the lower partial moment  $LPM_n(\tau)$  of the portfolio returns  $R_p$  is

$$(36) \quad LPM_n(\tau) = \sum_{j=1}^k \beta_j \int_{s_j}^{\infty} K_{n,k}(s_j, u) g_j(u) du,$$

where  $s_j = \frac{w\mu_j^t - \tau}{|wA_j|}$  and the kernel function  $K_{n,k}$  is given by:

$$(37) \quad K_{n,k}(s_j, u) = \frac{\pi^{\frac{k-1}{2}}}{2} |wA|^{n-1} \sum_{h=0}^n \frac{h!(n-h)!}{n!} \frac{\Gamma(\frac{n-h-1}{2})}{\Gamma(\frac{n+k-h}{2})} s_j^h (u - s_j)^{\frac{n+k-h}{2}-1}$$

However, we shall see in the example of the multi-variate  $t$ -distribution which we will treat next, that it can be easier to work directly with the double integral version (3) instead of with (36), (??).

**3.2.1. LPM with mixture of  $m$  Student- $t$  distributions.** Among the finite mixture, the finite  $t$ -Students mixture models (SMM) are tolerant for untypical data outliers. Thus, I now consider in detail the case where our mixture of elliptic distributions is a mixture of multivariate Student- $t$ . We will, unsurprisingly, call the corresponding LPM the *Delta SMM-LPM*.

In the last decade, in presence of data with longer-than-normal tails, mixture of  $t$ -Student distributions have proved to be a more useful as they provide more realistic tails for real world data, see e.g Sadefo Kamdem (2009). Moreover, the degrees of freedom of each  $t$  distribution can act as an adaptive robustness parameter, tuning the heaviness of the tails, see e.g. Lange et al. (1989). Thus, using mixture of  $t$  distributions with different degrees of freedom helps to automatically fit a typical data points.

In the case of the multi-variate  $t$ -Student distributions, the portfolio probability density function is given by:

$$(38) \quad h_X(x) = \sum_{j=1}^m \frac{\beta_j \Gamma(\frac{\nu_j+n}{2})}{\Gamma(\nu_j/2) \cdot \sqrt{|\Sigma_j|} (\nu_j \pi)^n} \left( 1 + \frac{(x - \mu_j)^t \Sigma_j^{-1} (x - \mu_j)}{\nu_j} \right)^{\frac{(-\nu_j-n)}{2}},$$

$x \in \mathbb{R}^n$  and  $\nu_j > 2$ . Hence  $g_j$  is given by

$$g_j(s) = C(\nu_j, n)(1 + s/\nu_j)^{-\frac{(n+\nu_j)}{2}}, \quad s \geq 0,$$

where we have put

$$C(\nu_j, n) = \frac{\Gamma(\frac{\nu_j+n}{2})}{\Gamma(\nu_j/2)\sqrt{(\nu_j\pi)^n}}.$$

**Theorem 3.16.** Assuming that  $\Delta\Pi \simeq \delta_1 X_1 + \delta_2 X_2 + \dots + \delta_k X_k$  with a mixture of  $m$  multivariate Student- $t$  random vector  $(X_1, X_2, \dots, X_k)$  with vector mean  $\mu_j$ , and variance-covariance matrix  $\Sigma_j$ , the the SMM-LPM of a linear portfolio is given by:

$$(39) \quad LPM_{mixt}(\tau) = \sum_{k=1}^m \beta_j C_{\nu_k, n, s} \sum_{j=0}^n \frac{\Gamma(\frac{n-j-1}{2}) \Gamma(\frac{\nu_k-n+j}{2}) j!(n-j)!}{n!} {}_2F_1\left(\frac{k+\nu_k}{2}, \frac{\nu_k+j-n}{2}, \frac{k+\nu_k}{2}; -\frac{\nu_k}{s}\right) s^{\frac{j}{2}}$$

where  $C_{\nu_k, n, s} = \frac{\nu_k^{\nu_k/2} |wA_k|^n}{2\sqrt{\pi}\Gamma(\nu_k/2)} s^{\frac{n-\nu_k}{2}}$  and  $s = \frac{w\mu_k^t - \tau}{|wA_k|}$ .

$$(40) \quad LPM_{mixt}(\tau) = \sum_{l=1}^m \beta_l \sum_{j=0}^n \left[ \frac{\nu_l^{\frac{\nu_l}{2}} \Gamma(\frac{n-j-1}{2}) \Gamma(\frac{\nu_l-n+j}{2}) {}_2F_1\left(\frac{k+\nu_l}{2}, \frac{\nu_l+j-n}{2}, \frac{k+\nu_l}{2}; -\frac{\nu_l\sqrt{w\Sigma_l w^t}}{w\mu_l^t - \tau}\right) j!(n-j)!}{2\sqrt{\pi}\Gamma(\frac{\nu_l}{2}) (w\mu_l^t - \tau)^{\nu_l-n-\frac{j}{2}} (w\Sigma_l w^t)^{\frac{2\nu_l-j}{4}}} \right] \frac{1}{n!}$$

The kappa index of order  $n$  is given by

$$(41) \quad \kappa_{mixt}(\tau) := \frac{w\mu^t - \tau}{(LPM_{mixt}(\tau))^{\frac{1}{n}}}.$$

**Remark 3.17.** One might, in certain situations, try to model with a mixture of  $t$ -Student distributions which all have the same  $\nu_j = \nu$  and the same mean  $\mu_j \approx 0$ , and obtain for example a mixture of different tail behaviors by playing with the  $\Sigma_j$ 's.

#### 4. UTILITY THEORY AND MEAN-LPM PORTFOLIO THEORY

**4.1. Mean-LPM Portfolio theory.** In this section, I modify classical portfolio theory so that the normal distribution can be relaxed by another elliptically distribution. Investors will instead of caring solely about means and variances, care about the so-called Target Semi-Moments such as the *Target Semi-Kurtosis* that will sufficiently consider leptokurticity for returns portfolio that are below the fixed real target  $\tau$ .

**4.2. The portfolio choice problem with LPM.** Consider an investor who is averse to  $LPM_{n,g}(\tau)$  with target rate return  $\tau$ . Let  $w = (w_1, w_2, \dots, w_k)$  and  $R = (R_1, \dots, R_k)$  represent the raw vector of security returns. Assume that  $R$  follows an elliptic distribution with mean vector  $\mu$ , the variance-covariance matrix  $\Sigma$  and the density generator function  $g$ . Following (36), the investor's optimal portfolio choice problem can be represented by

$$(42) \quad \min_{w \in \mathcal{W}} LPM_{n,g}(\tau, w)$$

where  $\mathcal{W} = \{w = (w_1, \dots, w_k) \in [0, 1]^k \mid \sum_{i=1}^k w_i = 1, r_f w_{k+1} + \mu w^t = E\}$  where  $E$  is the expected return of the investor, and  $r_f$  is the risk free interest rate. Equation (42) represents the fact that the (downside risk-averse) investor chooses the optimal portfolio weights such that the relevant risk measure ( $LPM_{n,g}$ ) is minimized for a specified value of the expected portfolio return  $E$ .

In the special case where we consider  $t$ -Student distribution,  $g$  is given by

$$g(s) = C(\nu, k)(1 + s/\nu)^{-\frac{(k+\nu)}{2}}, \quad s \geq 0,$$

with  $C(\nu, k) = \frac{\Gamma(\frac{\nu+k}{2})}{\Gamma(\nu/2)\sqrt{(\nu\pi)^n}}$ . Using this  $g$  in (36) and using (24), we find  $w_0 \in \mathcal{W}$  such that

$$(43) \quad \mathbf{w}_0 := \arg \min_{\mathbf{w} \in \mathcal{W}} \sum_{j=0}^n \left[ \frac{j!(n-j)! \nu^{\nu/2} \Gamma\left(\frac{n-j-1}{2}\right) \Gamma\left(\frac{\nu-n+j}{2}\right) {}_2F_1\left(\frac{k+\nu}{2}, \frac{\nu+j-n}{2}, \frac{k+\nu}{2}; -\frac{\nu}{\frac{\mathbf{w}\mu^t - \tau}{\sqrt{\mathbf{w}\Sigma\mathbf{w}^t}}}\right)}{2n! \sqrt{\pi} \Gamma(\nu/2) \left(\frac{\mathbf{w}\mu^t - \tau}{\sqrt{\mathbf{w}\Sigma\mathbf{w}^t}}\right)^{\nu-n-\frac{j}{2}} (\mathbf{w}\Sigma\mathbf{w}^t)^{-n/2}} \right]$$

where  $\mathcal{W} = \{\mathbf{w} = (w_1, \dots, w_k) \in [0, 1]^k / \sum_{i=1}^k w_i = 1, w_{k+1}r_f + \mu\mathbf{w}^t = E\}$  where  $E$  is the expected return of the investor, and  $r_f$  is the risk free interest rate.

**4.3. Worst case Elliptical LPM.** In this paper, following Duan et al.(2007), we can assume that the random vector  $X$  of portfolio risk factors density generator function is belong to a certain set of density generators functions  $\mathcal{G}$  that characterize elliptic distributions. We can then develop a robust portfolio policies when the density generator function  $g_X \in \mathcal{G}$ .

**Definition 4.1.** *The worst-case  $n$ -degree lower partial moment ( $n \geq 0$ ) of the portfolio return  $R_P$  with respect to  $g$  is defined as*

$$(44) \quad WLPM_n(\tau, \mathbf{w}) := LPM_{n, g_0}(\tau, \mathbf{w}_0) = \sup_{g \in \mathcal{G}} LPM_{n, g}(\tau, \mathbf{w})$$

where  $LPM_{n, g}(\tau, \mathbf{w})$  is given in 22) or in (36).

We can obtain a robust portfolio optimization in the case where only a partial information about the distribution of the portfolio risk factors is known (e.g. elliptic distribution). The robust selection is achieved by minimizing the worst-case downside risk measure  $WLPM_n(\tau, \mathbf{w})$ , which results in the following *min-max* problem:

$$(45) \quad LPM_{n, g_0}(\tau, \mathbf{w}_0) := \min_{\mathbf{w} \in \mathcal{W}} \sup_{g \in \mathcal{G}} LPM_{n, g}(\tau, \mathbf{w})$$

where  $\mathcal{W} = \{\mathbf{w} = (w_1, \dots, w_k) \in [0, 1]^k / \sum_{i=1}^k w_i = 1, \mu\mathbf{w}^t = E\}$  where  $E$  is the expected return of the investor.

**4.3.1. WLPM with Student distribution.** If we consider the family of Student distributions, we obtain the following robust portfolio *min-max* formulation:

$$(46) \quad \min_{\mathbf{w} \in \mathcal{W}} \sup_{\nu \in ]2, +\infty[} \sum_{j=0}^n \frac{j!(n-j)! \nu^{\nu/2} \Gamma\left(\frac{n-j-1}{2}\right) \Gamma\left(\frac{\nu-n+j}{2}\right) {}_2F_1\left(\frac{k+\nu}{2}, \frac{\nu+j-n}{2}, \frac{k+\nu}{2}; -\frac{\nu}{\frac{\mathbf{w}\mu^t - \tau}{\sqrt{\mathbf{w}\Sigma\mathbf{w}^t}}}\right)}{2n! \sqrt{\pi} \Gamma(\nu/2) \left(\frac{\mathbf{w}\mu^t - \tau}{\sqrt{\mathbf{w}\Sigma\mathbf{w}^t}}\right)^{\nu-n-\frac{j}{2}} (\mathbf{w}\Sigma\mathbf{w}^t)^{-n/2}}$$

$$(47) \quad \min_{\mathbf{w} \in \mathcal{W}} \sup_{\nu \in ]2, +\infty[} \sum_{j=0}^n \left[ \frac{\nu^{\frac{\nu}{2}} \Gamma\left(\frac{n-j-1}{2}\right) \Gamma\left(\frac{\nu-n+j}{2}\right) {}_2F_1\left(\frac{k+\nu}{2}, \frac{\nu+j-n}{2}, \frac{k+\nu}{2}; -\frac{\nu\sqrt{\mathbf{w}\Sigma\mathbf{w}^t}}{\mathbf{w}\mu^t - \tau}\right) j!(n-j)!}{2\sqrt{\pi} \Gamma\left(\frac{\nu}{2}\right) (\mathbf{w}\mu^t - \tau)^{\nu-n-\frac{j}{2}} (\mathbf{w}\Sigma\mathbf{w}^t)^{\frac{2\nu-j}{4}} n!} \right]$$

where  $s = \frac{\mathbf{w}\mu^t - \tau}{|\mathbf{w}A|}$ .  $C_{\nu, n, s} = \frac{\nu^{\nu/2}}{2\sqrt{\pi}\Gamma(\nu/2)}$

## 5. PARAMETERS ESTIMATION

**5.1. GO-GARCH covariance matrix.** The GO-GARCH model was proposed by van der Weide [28]. The starting point of the model is that an observed vector of risk factors can be expressed as a non-singular linear transformation of independent latent factors that have GARCH type conditional variance specification. Following Van der Weide [28], I puts forward a three step estimation method that is numerically attractive and easy to implement. The first two steps define a method of moments (MM) estimator for the linear transformation with a good convergence regardless of the dimension. The identification of linear transformation is done by using the fact that latent factors are heteroscedastic. All that is assumed is that the factors exhibit persistence in variance and have finite moments. The third and final step involves estimation of the univariate GARCH-type model for each factors.

**5.1.1. GO-GARCH model with elliptic distribution.** In the definition of elliptic distribution  $N(\mu_t, \Sigma_t, g)$ , we can consider a dynamic covariance matrix  $\Sigma_t$ . The random vector  $\mathbf{X}_t = (X_{1t}, \dots, X_{nt})^\top$  be the  $n$ -dimensional column vector process of risk factors (ie: log-returns) with conditional mean vector  $\mu_t$  that is assumed to be known or correctly fitted, decompose as

$$(48) \quad \mathbf{X}_t | \mathcal{F}_{t-1} = \mu_t + \eta_t \sim N(\mu_t, \Sigma_t, g),$$

where  $\eta_t$  is the zero mean random vector with covariance matrix  $\Sigma_t$ , that is independent dependent on  $\mathcal{F}_{t-1}$  the information set available up to time  $t-1$  (filtration) on which  $\mathbf{X}_t$  is adapted and

$$(49) \quad \Sigma_t = \text{var}(\mathbf{X}_t / \mathcal{F}_{t-1}) = Z_t \mathcal{H}_t Z_t \quad \text{and} \quad \mu_t = E(\mathbf{X}_t / \mathcal{F}_{t-1})$$

are respectively a positive definite  $\mathcal{F}_{t-1}$ -measurable dispersion conditional variance matrix depending on time, where  $H_t$  is the  $n \times n$  diagonal matrix defined by

$$(50) \quad Z_t^2 = \text{diag}(\omega_i) + \text{diag}(\theta_i) \circ \mathbf{X}_{t-1} \mathbf{X}_{t-1}^\top + \text{diag}(\beta_i) \circ Z_{t-1}^2,$$

where the symbol  $\circ$  is the Hadamard product of two matrices, that is element-by-element multiplication and the  $\mathcal{F}_{t-1}$ -measurable conditional mean. The matrix  $\mathcal{H}_t$  is defined via  $Q$  by

$$(51) \quad Q_t = (11' - A_1 - A_2)\Omega + A_1 \circ (\eta_{t-1} \eta_{t-1}^\top) + A_2 \circ Q_{t-1}$$

where  $\eta_t = Z_t^{-1} \mathbf{X}_t \sim N(0, \mathcal{H}_t, g)$ ,  $\Omega = \mathbb{E}[\eta_t \eta_t^\top]$  is the unconditional correlation matrix which is estimated using the sample correlation of the standardized residuals  $\eta_t$ ,  $A_1 = \theta \theta'$  with  $\theta = (\theta_1, \dots, \theta_n)$ ,  $A_2 = \beta \beta'$  with  $\beta = (\beta_1, \dots, \beta_n)$ ,  $\beta'$  is the transpose of  $\beta$  and

$$(52) \quad \mathcal{H}_t = (\text{diag}(Q_t))^{-1/2} Q_t (\text{diag}(Q_t))^{-1/2}.$$

In fact, for each  $i = 1, \dots, n$ , we can write

$$(53) \quad h_{it} = \left( 1 - \sum_{j=1}^n \alpha_{ij} - \beta_j \right) + \sum_{j=1}^n \alpha_{ij} y_{i,t-1}^2 + \beta_i h_{i,t-1}$$

with  $\alpha_{ij}, \beta_i \geq 0$  and  $\sum_{j=1}^n \alpha_{ij} + \beta_i < 0$ .

Paper [8] gives necessary conditions for  $\mathcal{R}_t$  to be positive definite.

step 1 : We find the marginal density function of the risk factor  $X_{it}$  as:

$$f_i(x_i) = |\Sigma|^{-1/2} \int_{\mathbb{R}^{n-1}} g(x \Sigma^{-1} x^t) dx^{-i},$$

where  $dx^{-i} = dx_1 dx_2 \dots dx_{i-1} dx_{i+1} \dots dx_n$ .

step 2 : Since the precede step 0 gives the marginal density function, the parameters  $\omega_i$ ,  $\theta_i$  and  $\beta_i$  of the sequence of the univariate GARCH models of equation (50) may be estimated by maximizing the  $n$  marginals univariate likelihoods

$$l(\theta_i) = \sum_{t=1}^T \log(f_i(X_{it})), \quad i = 1, \dots, n.$$

Then the matrix  $D_t$  and the standardized residuals,  $\eta_t = D_t^{-1} \mathbf{X}_t$  may be estimated.

step 3 : To estimate the matrix  $\Omega$  in equation , we used the sample covariance matrix of the residuals estimated in Step 2.

step 4 : Finally, using the estimated  $D_t$  and  $\Omega$ , the likelihood

$$l(\theta) = \sum_{t=1}^T \left[ -\frac{1}{2} \log|\Sigma_t| + \log(g(X_t \Sigma_t^{-1} X_t')) \right],$$

is maximized with respect to the parameters  $A_1$ ,  $A_2$  and  $g$ , for a moderate number  $k$  of assets in the portfolio.

Consider the polar decomposition of  $Z$ :

$$(54) \quad Z = SU$$

where  $S$  is the positive definite symmetric matrix, and  $U$  is an orthogonal matrix. It is easy to see that  $S$  is a squared root of the unconditional covariance matrix (e.g.  $H$  is an identity matrix) and that it can be written as  $S = PL^{1/2}P^t$ , where  $PLP^t$  is the spectral decomposition of  $\Sigma_t$ . Thus, estimating  $Z$  may be reduced to the problem of identifying the orthogonal matrix  $U$  from the conditional information. It's follows that, the unconditionally standardized returns  $s_t = \Sigma^{-1/2} \eta_t$  follows a GO-GARCH specification  $s_t = U y_t$  with an orthogonal link matrix  $U$ .

**5.2. Parameters estimation for the mixture of elliptic distributions distributions.** Consider the likelihood  $\mathcal{L}(\Theta)$  based on a sample  $\mathcal{X}$  drawn from (38) with parameters

$$(55) \quad \theta = (\beta_1, \dots, \beta_k, \mu_1, \dots, \mu_k, \Sigma_1, \dots, \Sigma_k, \nu_1, \dots, \nu_k)$$

where  $\mu_j$  is the parameter location,  $\Sigma_j$  is a positive definite variance-covariance matrix and  $\nu_j$  is the degree of freedom of the  $j$ -th distribution for  $j = 1, 2, \dots, k$ . The parameter space is

$$(56) \quad \Theta = \left\{ \theta \in \mathbb{R}^{\frac{k(4+3q+q^2)}{2}} / \sum_1^k \beta_k = 1, \beta_k \geq 0, |\Sigma_j| > 0 \text{ for } j = 1, 2, \dots, k \right\}$$

Following Shoham (2002), we recall the main steps of the EM algorithm for the mixture of  $t$  distributions. The EM algorithm generates a sequence of estimates  $\{\theta^{(m)}\}$  where  $\theta^{(0)} \in \Theta$  denotes the initial guess and  $\theta^{(m)} \in \Theta$  for  $m \in \mathbb{N}$  is such that the sequence  $\mathcal{L}(\theta^{(m)})_{m \in \mathbb{N}}$  is not decreasing. The E-step, on the  $(m+1)$ -th iteration of the EM algorithm, requires the calculation of the conditional expectation of the complete-data likelihood function  $Q(\theta, \theta^{(m)}) := \ln(\mathcal{L}_c(\theta))$ , evaluated using the current fit  $\theta^{(m)}$  for  $\theta$ . See [15] for more details on the estimation of the parameters for mixture of elliptical distributions.

## 6. CONCLUSION

This paper provides analytical expressions of the lower partial moment and the risk-adjusted performance (kappa index) of a linear portfolio with mixture of elliptically distributed risk factors. After introducing the the notion of target semi-kurtosis, we also discuss the classical and robust Mean-LPM portfolio optimization problem. After then we provide some discussion concerning estimation of parameters with data.

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