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Paul-Emile Paradan

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ON THE STRUCTURE OF $\mathbf{K}_G^*(\mathbf{T}_G^*M)$

PAUL-EMILE PARADAN

ABSTRACT. In this expository paper, we revisit the results of Atiyah-Singer and de Concini-Procesi-Vergne concerning the structure of the K -theory groups $\mathbf{K}_G^*(\mathbf{T}_G^*M)$.

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1. INTRODUCTION

When a compact Lie group G acts on a compact manifold M , the K -theory group $\mathbf{K}_G^0(\mathbf{T}^*M)$ is the natural receptacle for the principal symbol of any G -invariant elliptic pseudo-differential operators on M . One important point of Atiyah-Singer's Index Theory [4, 3, 5, 6] is that the equivariant index map $\text{Index}_M^G : \mathbf{K}_G^0(\mathbf{T}^*M) \rightarrow R(G)$ can be defined as the composition of a pushforward map $i_! : \mathbf{K}_G^0(\mathbf{T}^*M) \rightarrow \mathbf{K}_G^0(\mathbf{T}^*V)$ associated to an embedding $M \xrightarrow{i} V$ in a G -vector space, with the index map $\text{Index}_V^G : \mathbf{K}_G^0(\mathbf{T}^*V) \rightarrow R(G)$ which is the inverse of the Bott-Thom isomorphism [15].

In his Lecture Notes [1] describing joint work with I.M. Singer, Atiyah extends the index theory to the case of transversally elliptic operators. If we denote by \mathbf{T}_G^*M the closed subset of \mathbf{T}^*M , union of the conormals to the G -orbits, Atiyah explains how the principal symbol of a pseudo-differential transversally elliptic operator on M determines an element of the equivariant K -theory group $\mathbf{K}_G^0(\mathbf{T}_G^*M)$, and how the analytic index induces a map

$$(1) \quad \text{Index}_M^G : \mathbf{K}_G^0(\mathbf{T}_G^*M) \rightarrow R^{-\infty}(G),$$

where $R^{-\infty}(G) := \text{hom}(R(G), \mathbb{Z})$.

Like in the elliptic case the map (1) can be seen as the composition of a push-forward map $i_! : \mathbf{K}_G^0(\mathbf{T}_G^*M) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^*V)$ with the index map $\text{Index}_V^G : \mathbf{K}_G^0(\mathbf{T}_G^*V) \rightarrow R^{-\infty}(G)$. Hence the comprehension of the $R(G)$ -module

$$(2) \quad \mathbf{K}_G^*(\mathbf{T}_G^*V)$$

is fundamental in this context. For example, in [8, 14] the authors gave a cohomological formula for the index and the knowledge of the generators of $\mathbf{K}_{U(1)}^0(\mathbf{T}_{U(1)}^*\mathbb{C})$ was used to establish the formula. In [11], de Concini-Procesi-Vergne proved a formula for the multiplicities of the index by checking it on the generators of (2).

When G is abelian, Atiyah-Singer succeeded to find a set of generators for (2), and recently de Concini-Procesi-Vergne have shown that the index map identifies (2) with a generalized Dahmen-Michelli space [9, 10]. Let us explain their result.

Let \widehat{G} be the set of characters of the abelian compact Lie group G : for any $\chi \in \widehat{G}$ we denote \mathbb{C}_χ the corresponding complex one dimensional representation of G . We associate to any element $\Phi := \sum_{\chi \in \widehat{G}} m_\chi \mathbb{C}_\chi \in R^{-\infty}(G)$ its support $\text{Supp}(\Phi) = \{\chi \mid m_\chi \neq 0\} \subset \widehat{G}$.

For any real G -module V , we denote $\Delta_G(V)$ the set formed by the infinitesimal stabilizer of points in V : we denote \mathfrak{h}_{\min} the minimal stabilizer. For any $\mathfrak{h} \in \Delta_G(V)$, we denote $H := \exp(\mathfrak{h})$ the corresponding torus and we denote $\pi_H : \widehat{G} \rightarrow \widehat{H}$ the restriction map.

We denote $R^{-\infty}(G/H) \subset R^{-\infty}(G)$ the subgroup formed by the elements $\Phi \in R^{-\infty}(G)$ such that $\pi_H(\text{Supp}(\Phi)) \subset \widehat{H}$ is reduced to the trivial representation. Let

$$\langle R^{-\infty}(G/H) \rangle \subset R^{-\infty}(G)$$

be the $R(G)$ -submodule generated by $R^{-\infty}(G/H)$. We have $\Phi \in \langle R^{-\infty}(G/H) \rangle$ if and only if $\pi_H(\text{Supp}(\Phi)) \subset \widehat{H}$ is finite.

For any subspace $\mathfrak{a} \subset \mathfrak{g}$, we denote $V^{\mathfrak{a}} \subset V$ the subspace formed by the vectors fixed by the infinitesimal action of \mathfrak{a} . We fix an invariant complex structure on

$V/V^{\mathfrak{g}}$, hence the vector space $V/V^{\mathfrak{h}} \subset V/V^{\mathfrak{g}}$ is equipped with a complex structure for any $\mathfrak{h} \in \Delta_G(V)$. Following [11], we introduce the following submodule of $R^{-\infty}(G)$: the Dahmen-Michelli submodule

$\mathrm{DM}_G(V) :=$

$$\langle R^{-\infty}(G/H_{\min}) \rangle \cap \left\{ \Phi \in R^{-\infty}(G) \mid \wedge^{\bullet} \overline{V/V^{\mathfrak{h}}} \otimes \Phi = 0, \forall \mathfrak{h} \neq \mathfrak{h}_{\min} \in \Delta_G(V) \right\},$$

and the generalized Dahmen-Michelli submodule

$$\mathcal{F}_G(V) := \left\{ \Phi \in R^{-\infty}(G) \mid \wedge^{\bullet} \overline{V/V^{\mathfrak{h}}} \otimes \Phi \in \langle R^{-\infty}(G/H) \rangle, \forall \mathfrak{h} \in \Delta_G(V) \right\}.$$

Note that the relation $\wedge^{\bullet} \overline{V/V^{\mathfrak{h}}} \otimes \Phi \in \langle R^{-\infty}(G/H) \rangle$ becomes $\Phi \in \langle R^{-\infty}(G/H_{\min}) \rangle$ when $\mathfrak{h} = \mathfrak{h}_{\min}$. Hence $\mathrm{DM}_G(V)$ is contained in $\mathcal{F}_G(V)$. We have the following remarkable result [10].

Theorem 1.1 (de Concini-Procesi-Vergne). *Let G be an abelian compact Lie group, and let V be a real G -module. Let $V^{gen} \subset V$ be its open subset formed by the G -orbits of maximal dimension. The index map defines*

- an isomorphism between $\mathbf{K}_G^0(\mathbf{T}_G^*V)$ and $\mathcal{F}_G(V)$,
- an isomorphism between $\mathbf{K}_G^0(\mathbf{T}_G^*V^{gen})$ and $\mathrm{DM}_G(V)$.

The purpose of this note is to give a comprehensive account on the work of Atiyah-Singer and de Concini-Procesi-Vergne concerning the structure of (2) when G is a compact abelian Lie group. We will explain in details the following facts :

- The decomposition of $\mathbf{K}_G^*(\mathbf{T}_G^*M)$ relatively to the stratification of the manifold M relatively to the type of infinitesimal stabilizers.
- A set of generators of $\mathbf{K}_G^*(\mathbf{T}_G^*V)$.
- A set of generators of $\mathbf{K}_G^*(\mathbf{T}_G^*V^{gen})$.
- The injectivness of the index map $\mathrm{Index}_V^G : \mathbf{K}_G^0(\mathbf{T}_G^*V) \rightarrow R^{-\infty}(G)$.
- The isomorphisms $\mathbf{K}_G^0(\mathbf{T}_G^*V) \simeq \mathcal{F}_G(V)$ and $\mathbf{K}_G^0(\mathbf{T}_G^*V^{gen}) \simeq \mathrm{DM}_G(V)$.

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2. PRELIMINARY ON K -THEORY

In this section, G denotes a compact Lie group. Let $R(G)$ be the representation ring of G and let $R^{-\infty}(G) = \mathrm{hom}(R(G), \mathbb{Z})$.

2.1. Equivariant K -theory. We briefly review the notations for K -theory that we will use, for a systematic treatment see Atiyah [2] and Segal [15].

Let N be a locally compact topological space equipped with a continuous action of G . Let $E^{\pm} \rightarrow N$ be two G -equivariant complex vector bundles. An equivariant morphism σ on N is defined by a vector bundle map $\sigma \in \Gamma(N, \mathrm{hom}(E^+, E^-))$, that we denote also $\sigma : E^+ \rightarrow E^-$: at each point $n \in N$, we have a linear map $\sigma(n) : E_n^+ \rightarrow E_n^-$. The support of the morphism σ is the closed set formed by the point $n \in N$ where $\sigma(n)$ is not an isomorphism. We denote it $\mathrm{Support}(\sigma) \subset N$.

A morphism σ is elliptic when its support is compact, and then it defines a class

$$[\sigma] \in \mathbf{K}_G^0(N)$$

in the equivariant \mathbf{K} -group [15]. The group $\mathbf{K}_G^1(N)$ is by definition the group $\mathbf{K}_G^0(N \times \mathbb{R})$ where G acts trivially on \mathbb{R} .

Let $j : U \hookrightarrow N$ be an invariant open subset, and let us denote by $r : N \setminus U \hookrightarrow N$ the inclusion of the closed complement. We have a push-forward morphism $j_* : \mathbf{K}_G^*(U) \rightarrow \mathbf{K}_G^*(N)$ and a restriction morphism $r^* : \mathbf{K}_G^*(N) \rightarrow \mathbf{K}_G^*(N \setminus U)$ that fit in a six terms exact sequence :

$$(3) \quad \begin{array}{ccccc} \mathbf{K}_G^0(U) & \xrightarrow{j_*} & \mathbf{K}_G^0(N) & \xrightarrow{r^*} & \mathbf{K}_G^0(N \setminus U) \\ \delta \uparrow & & & & \delta \downarrow \\ \mathbf{K}_G^1(N \setminus U) & \xleftarrow{r^*} & \mathbf{K}_G^1(N) & \xleftarrow{j_*} & \mathbf{K}_G^1(U) \end{array}$$

In the next sections we will use the following basic lemma which is a direct consequence of (3).

Lemma 2.1. *Suppose that we have a morphism $S : \mathbf{K}_G^*(N \setminus U) \rightarrow \mathbf{K}_G^*(N)$ of $R(G)$ -module such that $r^* \circ S$ is the identity on $\mathbf{K}_G^*(N \setminus U)$. Then*

$$\mathbf{K}_G^*(N) \simeq \mathbf{K}_G^*(U) \oplus \mathbf{K}_G^*(N \setminus U)$$

as $R(G)$ -module.

We finish this section by considering the case of torus \mathbb{T} belonging to the center of G . Let $i : \mathbb{T} \hookrightarrow G$ be the inclusion map. We still denote $i : \text{Lie}(\mathbb{T}) \rightarrow \mathfrak{g}$ the map of Lie algebra, and $i^* : \mathfrak{g}^* \rightarrow \text{Lie}(\mathbb{T})^*$ the dual map. Note that the restriction to \mathbb{T} of an irreducible representation V_λ^G is isomorphic to $(\mathbb{C}_{i^*(\lambda)})^p$ with $p = \dim(V_\lambda^G)$. The representation ring $R(G)$ contains as a subring $R(G/\mathbb{T})$. At each character μ of \mathbb{T} , we associate the $R(G/\mathbb{T})$ -submodule of $R(G)$ defined by

$$R(G)_\mu = \sum_{i^*(\lambda)=\mu} \mathbb{Z} V_\lambda^G.$$

Note that $R(G)_0 = R(G/\mathbb{T})$.

We have then a grading $R(G) = \bigoplus_{\mu \in \widehat{\mathbb{T}}} R(G)_\mu$ since $R(G)_\mu \cdot R(G)_{\mu'} \subset R(G)_{\mu+\mu'}$. If we work now with the $R(G)$ -module $R^{-\infty}(G)$, we have also a decomposition¹ $R^{-\infty}(G) = \widehat{\bigoplus}_{\mu \in \widehat{\mathbb{T}}} R^{-\infty}(G)_\mu$ such that $R(G)_\mu \cdot R^{-\infty}(G)_{\mu'} \subset R^{-\infty}(G)_{\mu+\mu'}$.

Let us consider now the case of a G -space N , connected, such that the action of the subgroup \mathbb{T} is trivial. Each G -equivariant complex vector bundle $\mathcal{E} \rightarrow N$ decomposes as a finite sum

$$(4) \quad \mathcal{E} = \bigoplus_{\mu \in \mathcal{X}} \mathcal{E}_\mu$$

where $\mathcal{E}_\mu \simeq \text{hom}_{\mathbb{T}}(\mathbb{C}_\mu, \mathcal{E})$ is the G -sub-bundle where \mathbb{T} acts through the character $t \mapsto t^\mu$. Note that a G -equivariant morphism $\sigma : \mathcal{E}^+ \rightarrow \mathcal{E}^-$ is equal to the sum of morphisms $\sigma_\mu : \mathcal{E}_\mu^+ \rightarrow \mathcal{E}_\mu^-$. Hence, at the level of \mathbf{K} -theory we have also a decomposition

$$(5) \quad \mathbf{K}_G^*(N) = \bigoplus_{\mu \in \widehat{\mathbb{T}}} \mathbf{K}_G^*(N)_\mu$$

such that $R(G)_\mu \cdot \mathbf{K}_G^*(N)_{\mu'} \subset \mathbf{K}_G^*(N)_{\mu+\mu'}$.

¹The sign $\widehat{\bigoplus}$ means that one can take infinite sum.

Definition 2.2. We denote $\widehat{\mathbf{K}}_G^*(N)$ or simply $\widehat{\mathbf{K}}_G^*(N)$ the $R(G)$ -module formed by the infinite sum $\sum_{\mu \in \widehat{\mathbb{T}}} \sigma_\mu$ with $\sigma_\mu \in \mathbf{K}_G^*(N)_\mu$. When $N = \{\bullet\}$, $\widehat{\mathbf{K}}_G^*(\bullet) = \widehat{R(G)}$ is a $R(G)$ -submodule of $R^{-\infty}(G)$.

2.2. Index morphism : excision and free action. When M is a compact G -manifold, an equivariant morphism σ on the cotangent bundle \mathbf{T}^*M is called a symbol on M . We denote by \mathbf{T}_G^*M the following subset of \mathbf{T}^*M

$$\mathbf{T}_G^*M := \{(m, \xi) \in \mathbf{T}^*M \mid \langle \xi, X_M(m) \rangle = 0 \text{ for all } X \in \mathfrak{g}\}.$$

where $X_M(m) := \frac{d}{dt}e^{-tX} \cdot m|_{t=0}$ is the vector field generated by the infinitesimal action of $X \in \mathfrak{g}$. More generally, if $D \subset G$ is a distinguished subgroup, we can consider the G -invariant subset

$$(6) \quad \mathbf{T}_D^*M \supset \mathbf{T}_G^*M.$$

An elliptic symbol σ on M defines an element of $\mathbf{K}_G^0(\mathbf{T}^*M)$, and the index of σ is a virtual finite dimensional representation of G that we denote $\text{Index}_M^G(\sigma)$ [4, 3, 5, 6]. An equivariant symbol σ on M is transversally elliptic when $\text{Support}(\sigma) \cap \mathbf{T}_G^*M$ is compact: in this case Atiyah and Singer have shown that its index, still denoted $\text{Index}_M^G(\sigma)$, is well defined in $R^{-\infty}(G)$ and its depends only of the class $[\sigma] \in \mathbf{K}_G^0(\mathbf{T}_G^*M)$ (see [1] for the analytic index and [14] for the cohomological one). It is interesting to look at the index map as a pairing

$$(7) \quad \text{Index}_M^G : \mathbf{K}_G^0(\mathbf{T}_G^*M) \times \mathbf{K}_G^0(M) \rightarrow R^{-\infty}(G).$$

Let σ be a $G_1 \times G_2$ -equivariant symbol σ on a manifold M . If σ is G_1 -transversally elliptic it defines a class

$$[\sigma] \in \mathbf{K}_{G_1 \times G_2}^0(\mathbf{T}_{G_1}^*M),$$

and its index is *smooth* relatively to G_2 . It means that $\text{Index}_M^{G_1 \times G_2}(\sigma) = \sum_{\mu \in \widehat{G_1}} \theta_\mu \otimes V_\mu^{G_1}$ where $\theta_\mu \in R(G_2)$ for any μ . Hence

- the G_1 -index $\text{Index}_M^{G_1}(\sigma) = \sum_{\mu \in \widehat{G_1}} \dim(\theta_\mu) \otimes V_\mu^{G_1}$ is equal to the restriction of $\text{Index}_M^{G_1 \times G_2}(\sigma)$ to $g = 1 \in G_2$.
- the product of $\text{Index}_M^{G_1 \times G_2}(\sigma)$ with any element $\Theta \in R^{-\infty}(G_1)$ is a well defined element $\Theta \cdot \text{Index}_M^{G_1 \times G_2}(\sigma) \in R^{-\infty}(G_1 \times G_2)$.

Remark 2.3. Suppose that a torus \mathbb{T} belonging to the center of G acts trivially on the manifold M . Since the index map Index_M^G is a morphism of $R(G)$ -module, the pairing (7) specializes in a map from $\mathbf{K}_G^0(\mathbf{T}_G^*M)_\mu \times \mathbf{K}_G^0(M)_{\mu'}$ into $R^{-\infty}(G)_{\mu+\mu'}$. Hence one can extend the pairing (7) to

$$(8) \quad \text{Index}_M^G : \mathbf{K}_G^0(\mathbf{T}_G^*M) \times \widehat{\mathbf{K}}_G^0(M) \rightarrow R^{-\infty}(G).$$

See Definition 2.2, for the notation $\widehat{\mathbf{K}}_G^0(M)$.

Let U be a non-compact K -manifold. Lemma 3.6 of [1] tell us that, for any open K -embedding $j : U \hookrightarrow M$ into a compact manifold, we have a push-forward map $j_* : \mathbf{K}_G^*(\mathbf{T}_G^*U) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*M)$.

Let us rephrase Theorem 3.7 of [1].

Theorem 2.4 (Excision property). *The composition*

$$\mathbf{K}_G^0(\mathbf{T}_G^*U) \xrightarrow{j_*} \mathbf{K}_G^0(\mathbf{T}_G^*M) \xrightarrow{\text{index}_M^G} R^{-\infty}(G)$$

is independent of the choice of $j : U \hookrightarrow M$: we denote this map index_U^G .

Note that a *relatively compact* G -invariant open subset U of a G -manifold admits an open G -embedding $j : U \hookrightarrow M$ into a compact G -manifold. So the index map index_U^G is defined in this case. Another important example is when $U \rightarrow N$ is a G -equivariant vector bundle over a compact manifold N : we can imbed U as an open subset of the real projective bundle $\mathbb{P}(U \oplus \mathbb{R})$.

Let K be another compact Lie group. Let P be a compact manifold provided with an action of $K \times G$. We assume that the action of K is free. Then the manifold $M := P/K$ is provided with an action of G and the quotient map $q : P \rightarrow M$ is G -equivariant. Note that we have the natural identification of \mathbf{T}_K^*P with $q^*\mathbf{T}^*M$, hence $(\mathbf{T}_K^*P)/K \simeq \mathbf{T}^*M$ and more generally

$$(\mathbf{T}_{K \times G}^*P)/K \simeq \mathbf{T}_G^*M.$$

This isomorphism induces an isomorphism

$$Q^* : \mathbf{K}_G^0(\mathbf{T}_G^*M) \rightarrow \mathbf{K}_{K \times G}^0(\mathbf{T}_{K \times G}^*P).$$

The following theorem was obtained by Atiyah-Singer in [1]. For any $\Theta \in R^{-\infty}(K \times G)$, we denote $[\Theta]^K \in R^{-\infty}(G)$ its K -invariant part.

Theorem 2.5 (Free action property). *For any $[\sigma] \in \mathbf{K}_G^0(\mathbf{T}_G^*M)$, we have the following equality in $R^{-\infty}(K)$:*

$$\left[\text{index}_P^{K \times G}(Q^*[\sigma]) \right]^K = \text{index}_M^G([\sigma]).$$

2.3. Product. Suppose that we have two G -locally compact topological spaces $N_k, k = 1, 2$. For $j \in \mathbb{Z}/2\mathbb{Z}$, we have a product

$$(9) \quad \odot_{ext} : \mathbf{K}_G^0(N_1) \times \mathbf{K}_G^0(N_2) \longrightarrow \mathbf{K}_G^0(N_1 \times N_2)$$

which is defined as follows [1]. Suppose first that $*$ = 0. For $k = 1, 2$, let $\sigma_k : E_k^+ \rightarrow E_k^-$ be a morphism on N_k . Let E^\pm be the vector bundles on $N_1 \times N_2$ defined as $E^+ = E_1^+ \otimes E_2^+ \oplus E_1^- \otimes E_2^-$ and $E^- = E_1^- \otimes E_2^+ \oplus E_1^+ \otimes E_2^-$. On $N_1 \times N_2$, the morphism $\sigma_1 \odot_{ext} \sigma_2 : E^+ \rightarrow E^-$, is defined by the matrix

$$\sigma_1 \odot_{ext} \sigma_2(a, b) = \begin{pmatrix} \sigma_1(a) \otimes Id & -Id \otimes \sigma_2(b)^* \\ Id \otimes \sigma_2(b) & \sigma_1(a)^* \otimes Id \end{pmatrix}.$$

for $(a, b) \in N_1 \times N_2$. Note that $\text{Support}(\sigma_1 \odot_{ext} \sigma_2) = \text{Support}(\sigma_1) \times \text{Support}(\sigma_2)$. Hence the product $\sigma_1 \odot_{ext} \sigma_2$ is elliptic when each σ_k is elliptic, and the product $[\sigma_1] \odot_{ext} [\sigma_2]$ is defined as the class $[\sigma_1 \odot_{ext} \sigma_2]$. When $*$ = 1, we make the same construction with the spaces N_1 and $N_2 \times \mathbb{R}$.

Two particular cases of this product are noteworthy:

- When $N_1 = N_2 = N$, the inner product on $\mathbf{K}_G^0(N)$ is defined as $a \odot b = \Delta^*(a \odot_{ext} b)$, where $\Delta^* : \mathbf{K}_G^0(N \times N) \rightarrow \mathbf{K}_G^0(N)$ is the restriction morphism associated to the diagonal mapping $\Delta : N \rightarrow N \times N$.
- The structure of $R(G)$ -module of $\mathbf{K}_G^*(N_2)$ can be understood as a particular case of the exterior product, when N_1 is reduced to a point.

Let us recall the multiplicative property of the index for the product of manifolds. Consider a compact Lie group G_2 acting on two manifolds M_1 and M_2 , and assume that another compact Lie group G_1 acts on M_1 commuting with the action of G_2 . The external product of complexes on \mathbf{T}^*M_1 and \mathbf{T}^*M_2 induces a multiplication (see (9)):

$$\odot_{ext} : \mathbf{K}_{G_1 \times G_2}^0(\mathbf{T}_{G_1}^*M_1) \times \mathbf{K}_{G_2}^*(\mathbf{T}_{G_2}^*M_2) \longrightarrow \mathbf{K}_{G_1 \times G_2}^*(\mathbf{T}_{G_1 \times G_2}^*(M_1 \times M_2)).$$

Since $\mathbf{T}_{G_1 \times G_2}^*(M_1 \times M_2) \neq \mathbf{T}_{G_1}^*M_1 \times \mathbf{T}_{G_2}^*M_2$ in general, the product $[\sigma_1] \odot_{ext} [\sigma_2]$ of transversally elliptic symbols need some care: we have to take representative σ_2 that are almost homogeneous (see Lemma 4.9 in [13]).

Theorem 2.6 (Multiplicative property). *For any $[\sigma_1] \in \mathbf{K}_{G_1 \times G_2}^0(\mathbf{T}_{G_1}^*M_1)$ and any $[\sigma_2] \in \mathbf{K}_{G_2}^0(\mathbf{T}_{G_2}^*M_2)$ we have*

$$\text{index}_{M_1 \times M_2}^{G_1 \times G_2}([\sigma_1] \odot_{ext} [\sigma_2]) = \text{index}_{M_1}^{G_1 \times G_2}([\sigma_1]) \text{index}_{M_2}^{G_2}([\sigma_2]).$$

In the last theorem, the product of $\text{index}_{M_1}^{G_1 \times G_2}([\sigma_1]) \in R^{-\infty}(G_1 \times G_2)$ and $\text{index}_{M_2}^{G_2}([\sigma_2]) \in R^{-\infty}(G_2)$ is well defined since $\text{index}_{M_1}^{G_1 \times G_2}([\sigma_1])$ is *smooth* relatively to G_2 (see Section 2.2).

Suppose now that G is **abelian**. For a generalized character $\Phi \in R^{-\infty}(G)$, we consider its support $\text{Supp}(\Phi) \subset \widehat{G}$ and the corresponding subset $\overline{\text{Supp}(\Phi)} \subset \mathfrak{g}^*$ formed by the differentials.

Let $\mathfrak{a} \subset \mathfrak{g}$ a rational² subspace, and let $\pi_{\mathfrak{a}} : \mathfrak{g}^* \rightarrow \mathfrak{a}^*$ be the projection. We will be interested to the K -groups $\mathbf{K}_G^*(\mathbf{T}_{\mathfrak{a}}^*M)$ associated to the G -spaces $\mathbf{T}_{\mathfrak{a}}^*M := \{(m, \xi) \in \mathbf{T}^*M \mid \langle \xi, X_M(m) \rangle = 0 \text{ for all } X \in \mathfrak{a}\}$. We can prove that if $\sigma \in \mathbf{K}_G^0(\mathbf{T}_{\mathfrak{a}}^*M)$, then its index $\Phi := \text{Index}_G^M(\sigma) \in R^{-\infty}(G)$ has the following property : the projection $\pi_{\mathfrak{a}}$, when restricted to $\overline{\text{Supp}(\Phi)}$, is *proper* (see [8]).

We have another version of Theorem 2.6.

Theorem 2.7 (Multiplicative property - Abelian case). *Let M_1 and M_2 be two G -manifolds (with G abelian), and let $\mathfrak{a}_1, \mathfrak{a}_2$ be two rationnal subspaces of \mathfrak{g} such that $\mathfrak{a}_1 \cap \mathfrak{a}_2 = \{0\}$. If the infinitesimal action of \mathfrak{a}_1 is **trivial** on M_2 , we have an external product*

$$\odot_{ext} : \mathbf{K}_G^0(\mathbf{T}_{\mathfrak{a}_1}^*M_1) \times \mathbf{K}_G^*(\mathbf{T}_{\mathfrak{a}_2}^*M_2) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_{\mathfrak{a}_1 \oplus \mathfrak{a}_2}^*(M_1 \times M_2)),$$

and for any $[\sigma_k] \in \mathbf{K}_G^0(\mathbf{T}_{\mathfrak{a}_k}^*M_k)$ we have

$$\text{index}_{M_1 \times M_2}^G([\sigma_1] \odot_{ext} [\sigma_2]) = \text{index}_{M_1}^G([\sigma_1]) \text{index}_{M_2}^G([\sigma_2]).$$

Let us briefly explain why the product of the generalized characters $\Phi_k := \text{index}_{M_k}^G([\sigma_k]) \in R^{-\infty}(G)$ is well-defined. We know that the projection $\pi_k : \mathfrak{g}^* \rightarrow \mathfrak{a}_k^*$ is proper when restricted to the infinitesimal support $\overline{\text{Supp}(\Phi_k)} \subset \mathfrak{g}^*$. Since the infinitesimal action of \mathfrak{a}_1 is trivial on M_2 , we know also that the image of $\overline{\text{Supp}(\Phi_2)}$ by π_1 is finite (see Remark 2.8). These three facts insure that for any $\chi \in \widehat{G}$ the set $\{(\chi_1, \chi_2) \in \text{Supp}(\Phi_1) \times \text{Supp}(\Phi_2) \mid \chi_1 + \chi_2 = \chi\}$ is finite. Hence we can define the product $\Phi_1 \otimes \Phi_2$ as the restriction of $(\Phi_1, \Phi_2) \in R^{-\infty}(G \times G)$ to the diagonal.

²A subspace $\mathfrak{a} \subset \mathfrak{g}$ is rational when it is the Lie algebra of a closed subgroup.

Remark 2.8. Consider an action of a compact **abelian** Lie group G on a manifold M . Suppose that a torus subgroup $H \subset G$ acts **trivially** on M . Let H' be a closed subgroup of G such that $G \simeq H \times H'$. In this case we have an isomorphism $\mathbf{K}_G^*(\mathbf{T}_G^*M) \simeq R(H) \otimes \mathbf{K}_{H'}^*(\mathbf{T}_{H'}^*M)$ and we see that the index map sends $\mathbf{K}_G^0(\mathbf{T}_G^*M)$ into $R(H) \otimes R^{-\infty}(H') \simeq \langle R^{-\infty}(G/H) \rangle$. See the introduction where the submodule $\langle R^{-\infty}(G/H) \rangle$ is defined without using a decomposition $G \simeq H \times H'$.

2.4. Direct images and Bott symbols. Let $\pi : \mathcal{E} \rightarrow N$ be a G -equivariant complex vector bundle. We define the Bott morphism on \mathcal{E}

$$\text{Bott}(\mathcal{E}) : \wedge^+ \pi^* \mathcal{E} \rightarrow \wedge^- \pi^* \mathcal{E},$$

by the relation $\text{Bott}(\mathcal{E})(n, v) = \text{Cl}(v) : \wedge^+ \mathcal{E}_n \rightarrow \wedge^- \mathcal{E}_n$. Here the Clifford map is defined after the choice of a G -invariant Hermitian product on \mathcal{E} .

Let $s : N \rightarrow \mathcal{E}$ be the 0-section map. Since the support of $\text{Bott}(\mathcal{E})$ is the zero section, we have a push-forward morphism

$$(10) \quad \begin{aligned} s_! : \mathbf{K}_G^*(N) &\longrightarrow \mathbf{K}_G^*(\mathcal{E}) \\ \sigma &\longmapsto \text{Bott}(\mathcal{E}) \odot_{\text{ext}} \pi^*(\sigma) \end{aligned}$$

which is bijective: it is the Bott-Thom isomorphism [15].

Consider now an Euclidean vector space V . Then its complexification $V_{\mathbb{C}}$ is an Hermitian vector space. The cotangent bundle \mathbf{T}^*V is identified with $V_{\mathbb{C}}$: we associate to the covector $\xi \in \mathbf{T}_v^*V$ the element $v + i\hat{\xi} \in V_{\mathbb{C}}$, where $\xi \in V^* \rightarrow \hat{\xi} \in V$ is the identification given by the Euclidean structure.

Then $\text{Bott}(V_{\mathbb{C}})$ defines an elliptic symbol on V which is equivariant relative to the action of the orthogonal group $O(V)$. Its analytic index is computed in [1]. We have the equality

$$(11) \quad \text{index}_V^{O(V)}(\text{Bott}(V_{\mathbb{C}})) = 1$$

in $R(O(V))$.

Let $\pi : \mathcal{V} \rightarrow M$ be a G -equivariant *real* vector bundle over a compact manifold. We have the fundamental fact

Proposition 2.9. *We have a push-forward morphism*

$$(12) \quad s_! : \mathbf{K}_G^*(\mathbf{T}_G^*M) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{V})$$

such that $\text{index}_{\mathcal{V}}^G \circ s_! = \text{index}_M^G$ on $\mathbf{K}_G^0(\mathbf{T}_G^*M)$.

Proof. We fix a G -invariant euclidean structure on \mathcal{V} . Let $n = \text{rank } \mathcal{V}$. Let P be the associated orthogonal frame bundle. We have $M = P/O$ and $\mathcal{V} = P \times_O V$ where $V = \mathbb{R}^n$ and O is the orthogonal group of V . For the cotangent bundle we have canonical isomorphisms

$$\mathbf{T}_G^*M \simeq \mathbf{T}_{G \times O}^*(P/O) \quad \text{and} \quad \mathbf{T}_G^*\mathcal{V} \simeq \mathbf{T}_{G \times O}^*(P \times V)/O$$

which induces isomorphisms between \mathbf{K} -groups

$$\begin{aligned} Q_1^* : \mathbf{K}_G^*(\mathbf{T}_G^*M) &\longrightarrow \mathbf{K}_{G \times O}^*(\mathbf{T}_{G \times O}^*P), \\ Q_2^* : \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{V}) &\longrightarrow \mathbf{K}_{G \times O}^*(\mathbf{T}_{G \times O}^*(P \times V)). \end{aligned}$$

Let us use the multiplicative property (see Section 2.3) with the groups $G_2 = G \times O, G_1 = \{1\}$ and the manifolds $M_1 = V, M_2 = P$. We have a map

$$(13) \quad \begin{aligned} s'_! : \mathbf{K}_{G \times O}^*(\mathbf{T}_{G \times O}^*P) &\longrightarrow \mathbf{K}_{G \times O}^*(\mathbf{T}_{G \times O}^*(P \times V)) \\ \sigma &\longmapsto \text{Bott}(V_{\mathbb{C}}) \odot_{\text{ext}} \sigma \end{aligned}$$

The map $s_! : \mathbf{K}_G^*(\mathbf{T}_G^*M) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{V})$ is defined by the relation $s_! = Q_1^* \circ s'_! \circ (Q_2^*)^{-1}$.

Thanks to Theorem 2.6, the relation (11) implies that $\text{index}_{P \times V}^{G \times O} \circ s'_! = \text{index}_P^{G \times O}$ on $\mathbf{K}_{G \times O}^0(\mathbf{T}_{G \times O}^*P)$. Thanks to Theorem 2.5 we have

$$\begin{aligned} \text{index}_{\mathcal{V}}^G(s_!(\sigma)) &= \left[\text{index}_{P \times V}^{G \times O}(s'_! \circ (Q_2^*)^{-1}(\sigma)) \right]^O \\ &= \left[\text{index}_P^{G \times O}((Q_2^*)^{-1}(\sigma)) \right]^O \\ &= \text{index}_M^G(\sigma). \end{aligned}$$

for any $\sigma \in \mathbf{K}_{G \times O}^0(\mathbf{T}_{G \times O}^*P)$. \square

We finish this section by considering the case of a G -equivariant embedding $i : Z \hookrightarrow M$ between G -manifolds.

Proposition 2.10. *We have a push-forward morphism*

$$(14) \quad i_! : \mathbf{K}_G^*(\mathbf{T}_G^*Z) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*M)$$

such that $\text{index}_M^G \circ i_! = \text{index}_Z^G$ on $\mathbf{K}_G^0(\mathbf{T}_G^*Z)$.

Proof. Let $\mathcal{N} = \mathbf{T}M|_Z / \mathbf{T}Z$ be the normal bundle. We know that an open G -invariant tubular neighborhood U of Z is equivariantly diffeomorphic with \mathcal{N} : let us denote by $\varphi : U \rightarrow \mathcal{N}$ this equivariant diffeomorphism. Let $j : U \hookrightarrow M$ be the inclusion. We consider the morphism $s_! : \mathbf{K}_G^*(\mathbf{T}_G^*Z) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{N})$ defined in Proposition 2.9, the isomorphism $\varphi^* : \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{N}) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*U)$ and the push-forward morphism $j_* : \mathbf{K}_G^*(\mathbf{T}_G^*U) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*M)$. Thanks to Proposition 2.9, one sees that the composition $i_! = j_* \circ \varphi^* \circ s_!$ satisfies $\text{index}_M^G \circ i_! = \text{index}_Z^G$ on $\mathbf{K}_G^0(\mathbf{T}_G^*Z)$. \square

2.5. Restriction : the vector bundle case. Let $\mathcal{E} \rightarrow M$ be a G -equivariant complex vector bundle. Let us introduce the invariant open subset $\mathbf{T}_G^*(\mathcal{E} \setminus \{0\})$ of $\mathbf{T}_G^*\mathcal{E}$ and its complement $\mathbf{T}_G^*\mathcal{E}|_{0\text{-section}} = \mathbf{T}_G^*M \times \mathcal{E}^*$. We denote

$$(15) \quad \mathbf{R} : \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E}) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*M)$$

the composition of the restriction morphism $\mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E}) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*M \times \mathcal{E}^*)$ with the Bott-Thom isomorphism $\mathbf{K}_G^*(\mathbf{T}_G^*M \times \mathcal{E}^*) \simeq \mathbf{K}_G^*(\mathbf{T}_G^*M)$. Note that the morphism

$$(16) \quad \mathbf{R} : \mathbf{K}_G^*(\mathbf{T}_D^*\mathcal{E}) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_D^*M)$$

is also defined when $D \subset G$ is a **distinguished** subgroup.

If $\mathcal{S} = \{v \in \mathcal{E} \mid \|v\|^2 = 1\}$ is the sphere bundle, we have $\mathcal{E} \setminus \{0\} \simeq \mathcal{S} \times \mathbb{R}$ and then $\mathbf{T}_G^*(\mathcal{E} \setminus \{0\}) \simeq \mathbf{T}_G^*\mathcal{S} \times \mathbf{T}^*\mathbb{R}$. Let $i : \mathcal{S} \hookrightarrow \mathcal{E}$ be the canonical immersion. The composition of the Bott-Thom isomorphism $\mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{S}) \simeq \mathbf{K}_G^*(\mathbf{T}_G^*(\mathcal{E} \setminus \{0\}))$ with

the morphism $j_* : \mathbf{K}_G^*(\mathbf{T}_G^*(\mathcal{E} \setminus \{0\})) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E})$ correspond to the push-forward map $i_!$ defined in Proposition 2.10. The six term exact sequence (3) becomes

$$(17) \quad \begin{array}{ccccc} \mathbf{K}_G^0(\mathbf{T}_G^*\mathcal{S}) & \xrightarrow{i_!} & \mathbf{K}_G^0(\mathbf{T}_G^*\mathcal{E}) & \xrightarrow{\mathbf{R}} & \mathbf{K}_G^0(\mathbf{T}_G^*M) \\ \delta \uparrow & & & & \delta \downarrow \\ \mathbf{K}_G^1(\mathbf{T}_G^*M) & \xleftarrow{\mathbf{R}} & \mathbf{K}_G^1(\mathbf{T}_G^*\mathcal{E}) & \xleftarrow{i_!} & \mathbf{K}_G^1(\mathbf{T}_G^*\mathcal{S}). \end{array}$$

Let $s_! : \mathbf{K}_G^*(\mathbf{T}_G^*M) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E})$ be the push-forward morphism associated to the zero section $s : M \hookrightarrow \mathcal{E}$ (see Proposition 2.10). We have the fundamental

Proposition 2.11.

- The composition $\mathbf{R} \circ s_! : \mathbf{K}_G^*(\mathbf{T}_G^*M) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E})$ is the map $\sigma \rightarrow \sigma \otimes \wedge^\bullet \overline{\mathcal{E}}$.
- The composition $s_! \circ \mathbf{R} : \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E}) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E})$ is defined by $\sigma \rightarrow \sigma \otimes \wedge^\bullet \pi^* \overline{\mathcal{E}}$.
- We have $\text{index}_M^G(\mathbf{R}(\sigma)) = \text{index}_\mathcal{E}^G(\sigma \otimes \wedge^\bullet \pi^* \overline{\mathcal{E}})$ for any $\sigma \in \mathbf{K}_G^0(\mathbf{T}_G^*\mathcal{E})$.

Proof. The third point is a consequence of second point. Let us check the first two points.

We use the notations of the proof of proposition 2.9: we have a principal bundle $P \rightarrow M = P/O$ and \mathcal{E} coincides as a real vector bundle with $P \times_O E$. Since \mathcal{E} has an invariant complex structure, we can consider the frame bundle $Q \subset P$ formed by the unitary basis of \mathcal{E} . Here $E = \mathbb{R}^{2n} = \mathbb{C}^n$. Let $U \subset O$ be the unitary group of E . Here the map $s_!$ and \mathbf{R} can be defined with the reduced data (Q, U) through the maps

$$\begin{aligned} s'_! : \mathbf{K}_{G \times U}^*(\mathbf{T}_{G \times U}^*Q) &\rightarrow \mathbf{K}_{G \times U}^*(\mathbf{T}_{G \times U}^*(Q \times E)) \\ \sigma &\mapsto \text{Bott}(E_\mathbb{C}) \odot_{\text{ext}} \sigma \end{aligned}$$

and $\mathbf{R}' : \mathbf{K}_{G \times U}^*(\mathbf{T}_{G \times U}^*(Q \times E)) \rightarrow \mathbf{K}_{G \times U}^*(\mathbf{T}_{G \times U}^*Q)$. Since E admits a complex structure J , the map $w \oplus iv \mapsto (w + Jv, w - Jv)$ is an isomorphism between $E_\mathbb{C}$ and the orthogonal sum $E \oplus \overline{E}$. Hence on $E_\mathbb{C}$ the Bott morphism $\text{Cl}(w \oplus iv) : \wedge^+ E_\mathbb{C} \rightarrow \wedge^- E_\mathbb{C}$ is equal to the product of the morphisms $\text{Cl}(w + Jv) : \wedge^+ E \rightarrow \wedge^- E$ and $\text{Cl}(w - Jv) : \wedge^+ \overline{E} \rightarrow \wedge^- \overline{E}$. When we restrict the Bott symbol $\text{Bott}(E_\mathbb{C}) \in \mathbf{K}_U^0(\mathbf{T}^*E)$ to the 0-section, we get

$$\left(\wedge^+ E \xrightarrow{\text{Cl}(w)} \wedge^- E \right) \odot \left(\wedge^+ \overline{E} \xrightarrow{\text{Cl}(w)} \wedge^- \overline{E} \right)$$

which is equal to the class $\text{Bott}(E) \otimes \wedge^\bullet \overline{E}$ in $\mathbf{K}_U^0(E)$. Finally the composition $\mathbf{R}' \circ s'_! : \mathbf{K}_{G \times U}^*(\mathbf{T}_{G \times U}^*Q) \rightarrow \mathbf{K}_{G \times U}^*(\mathbf{T}_{G \times U}^*Q)$ is equal to the map $\sigma \rightarrow \sigma \otimes \wedge^\bullet \overline{E}$. We get the first point through the isomorphism $\mathbf{K}_{G \times U}^*(\mathbf{T}_{G \times U}^*Q) \simeq \mathbf{K}_G^*(\mathbf{T}_G^*M)$.

Let $\sigma \in \mathbf{K}_{G \times U}^*(\mathbf{T}_{G \times U}^*(Q \times E))$. For $(x, \xi; v, w) \in \mathbf{T}^*Q \times \mathbf{T}^*E$, the transversally elliptic symbols

$$\begin{aligned} \sigma(x, \xi; v, w) &\otimes \wedge^\bullet \overline{E} \\ \sigma(x, \xi; v, w) &\odot \text{Cl}(v) \\ \sigma(x, \xi; 0, w) &\odot \text{Cl}(v) \\ \mathbf{R}'(\sigma)(x, \xi) &\odot \text{Cl}(w) \odot \text{Cl}(v) \\ \mathbf{R}'(\sigma)(x, \xi) &\odot \text{Cl}(w + Jv) \odot (w - Jv) \\ s'_! &\circ \mathbf{R}'(\sigma)(x, \xi; v, w) \end{aligned}$$

define the same class in $\mathbf{K}_{G \times U}^*(\mathbf{T}_{G \times U}^*(Q \times E))$. We have proved that $s'_! \circ \mathbf{R}'(\sigma) = \sigma \otimes \wedge^\bullet \overline{E}$, and we get the second point through the isomorphism $\mathbf{K}_{G \times U}^*(\mathbf{T}_{G \times U}^*Q) \simeq \mathbf{K}_G^*(\mathbf{T}_G^*M)$. \square

2.6. Restriction to a sub-manifold. Let M be a G -manifold and let Z be a closed G -invariant sub-manifold of M . Let us consider the open subset $\mathbf{T}_G^*(M \setminus Z)$ of \mathbf{T}_G^*M . Its complement is the closed subset $\mathbf{T}_G^*M|_Z$. Let \mathcal{N} be the normal bundle of Z in M . We have $\mathbf{T}^*M|_Z = \mathbf{T}^*Z \times \mathcal{N}^*$ and then $\mathbf{T}_G^*M|_Z = \mathbf{T}_G^*Z \times \mathcal{N}^*$.

We make the following **hypothesis** : the real vector bundle $\mathcal{N}^* \rightarrow Z$ has a G -equivariant complex structure. Then we can define the map

$$(18) \quad \mathbf{R}_Z : \mathbf{K}_G^*(\mathbf{T}_G^*M) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*Z)$$

as the composition of the restriction $\mathbf{K}_G^*(\mathbf{T}_G^*M) \rightarrow \mathbf{K}^*(\mathbf{T}_G^*M|_Z) = \mathbf{K}_G^*(\mathbf{T}_G^*Z \times \mathcal{N}^*)$ with the Bott-Thom isomorphism $\mathbf{K}_G^*(\mathbf{T}_G^*Z \times \mathcal{N}^*) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*Z)$.

3. LOCALIZATION

In this section, $\beta \in \mathfrak{g}$ denotes a non-zero G -invariant element, and $\pi : \mathcal{E} \rightarrow M$ is a G -equivariant *hermitian* vector bundle such that

$$(19) \quad \mathcal{E}^\beta = M.$$

Remark 3.1. Note that (19) imposes the existence of a G -invariant complex structure on the fibers of \mathcal{E} . We can take³ $J_\beta := \mathcal{L}(\beta)(-\mathcal{L}(\beta)^2)^{\frac{-1}{2}}$, where $\mathcal{L}(\beta)$ denotes the linear action on the fibers of \mathcal{E} .

The aim of this section is the following

Theorem 3.2. *There exists a morphism $\mathbf{S}_\beta : \mathbf{K}_G^*(\mathbf{T}_G^*M) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E})$ satisfying the following properties:*

- (1) *The composition $\mathbf{R} \circ \mathbf{S}_\beta$ is the identity on $\mathbf{K}_G^*(\mathbf{T}_G^*M)$.*
- (2) *For any $a \in \mathbf{K}_G^*(\mathbf{T}_G^*M)$, we have $\mathbf{S}_\beta(a) \otimes \wedge^\bullet \pi^* \overline{\mathcal{E}} = s_!(a)$.*
- (3) *For any $\sigma \in \mathbf{K}_G^0(\mathbf{T}_G^*M)$, we have the following equality*

$$\text{Index}_{\mathcal{E}}^G(\mathbf{S}_\beta(\sigma)) = \text{Index}_M^G(\sigma \otimes [\wedge^\bullet \overline{\mathcal{E}}]_\beta^{-1})$$

in $R^{-\infty}(G)$, where $[\wedge^\bullet \overline{\mathcal{E}}]_\beta^{-1}$ is a polarized inverse of $\wedge^\bullet \overline{\mathcal{E}}$.

Remark 3.3. *The maps \mathbf{R} and \mathbf{S}_β depend on the choice of the G -invariant complex structure on \mathcal{E} .*

Theorem 3.2 tells us that (17) breaks in an exact sequence

$$0 \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{S}) \xrightarrow{i_!} \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E}) \xrightarrow{\mathbf{R}} \mathbf{K}_G^*(\mathbf{T}_G^*M) \rightarrow 0.$$

Since $\mathbf{R} \circ \mathbf{S}_\beta = \mathbf{R} \circ \mathbf{S}_{-\beta}$ the image of the map $\mathbf{S}_\beta - \mathbf{S}_{-\beta} : \mathbf{K}_G^*(\mathbf{T}_G^*M) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E})$ belongs to the image of the push-forward map $i_! : \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{S}) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{E})$.

Let us work now with the complex structure J_β on \mathcal{E} . We denote $\mathbf{S}_{\pm\beta}^o$ the corresponding morphism. In Section 3.5.3 we will prove the following

Theorem 3.4. *There exists a morphism $\theta_\beta : \mathbf{K}_G^*(\mathbf{T}_G^*M) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{S})$ such that*

$$\mathbf{S}_{-\beta}^o - \mathbf{S}_\beta^o = i_! \circ \theta_\beta.$$

³Relatively to a G -invariant Euclidean metric on \mathcal{E} , the linear map $-\mathcal{L}(\beta)^2$ is positive definite, hence one can take its square root.

3.1. Atiyah-Singer pushed symbols. Let M be a G -manifold with an invariant almost complex structure J . Then the cotangent bundle \mathbf{T}^*M is canonically equipped with a complex structure, still denoted J . The Bott morphism on \mathbf{T}^*M associated to the complex vector bundle $(\mathbf{T}^*M, J) \rightarrow M$, is called the Thom symbol of M , and is denoted⁴ $\text{Thom}(M, J)$. Note that the product by the Thom symbol induces an isomorphism $\mathbf{K}_G^*(M) \simeq \mathbf{K}_G^*(\mathbf{T}^*M)$.

For any $X \in \mathfrak{g}$, we denote $X_M(m) := \frac{d}{dt}|_0 e^{-tX} \cdot m$ the corresponding vector field on M . Thanks to an invariant Riemannian metric on M , we define the 1-form

$$\tilde{X}_M(m) = (X_M(m), -).$$

From now on, we take $X = \beta$ a non-zero G -invariant element. Then the corresponding 1-form $\tilde{\beta}_M$ is G -invariant, and we define following Atiyah-Singer the equivariant morphism

$$\text{Thom}_\beta(M, J)(m, \xi) := \text{Thom}(M, J)(m, \xi - \tilde{\beta}_M(m)), \quad (\xi, m) \in \mathbf{T}^*M.$$

We check easily that

$$\text{Support}(\text{Thom}_\beta(M, J)) \cap \mathbf{T}_{\mathbb{T}_\beta}^*M = \{(m, 0); m \in M^\beta\}.$$

where $\mathbb{T}_\beta = \overline{\exp(\mathbb{R}\beta)}$ is the torus generated by β . In particular, we get a class

$$(20) \quad \text{Thom}_\beta(M, J) \in \mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^*M)$$

when M^β is compact.

3.2. Atiyah-Singer pushed symbols : the linear case. Let us consider the case of a G -Hermitian vector space E such that $E^\beta = \{0\}$.

Let $i_! : \mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^*S) \rightarrow \mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^*E)$ be the push-forward morphism associated to the inclusion $i : S \hookrightarrow E$ of the sphere of radius one. Let $\mathbf{R} : \mathbf{K}_G^*(\mathbf{T}_{\mathbb{T}_\beta}^*E) \rightarrow \mathbf{K}_G^*(\{\bullet\})$ be the restriction morphism. Since $\mathbf{K}_G^1(\{\bullet\}) = 0$, the six term exact sequence (17) becomes

$$(21) \quad 0 \longrightarrow \mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^*S_E) \xrightarrow{i_!} \mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^*E) \xrightarrow{\mathbf{R}} R(G).$$

The pushed Thom symbol on E defines a class $\text{Thom}_\beta(E) \in \mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^*E)$.

Proposition 3.5. • We have $\mathbf{R}(\text{Thom}_\beta(E)) = 1$ in $R(G)$.

• The sequence (21) breaks down: we have a decomposition

$$\mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^*E) = \mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^*S_E) \oplus \langle \text{Thom}_\beta(E) \rangle,$$

where $\langle \text{Thom}_\beta(E) \rangle$ denotes the free $R(G)$ -module generated by $\text{Thom}_\beta(E)$.

Proof. At $(x, \xi) \in \mathbf{T}^*E$ the map $\text{Thom}_\beta(E)(x, \xi) : \wedge^+ E \rightarrow \wedge^- E$ is equal to $\text{Cl}(\hat{\xi} - \beta_E(x))$, where $\xi \in E^* \mapsto \hat{\xi} \in E$ is the identification given by the Euclidean structure. We see that the restriction of $\text{Thom}_\beta(E)$ to $\mathbf{T}_{\mathbb{T}_\beta}^*E|_0 = E^*$ is equal to $\text{Bott}(E^*)$ and then the first point follows. The second point is a direct consequence of the first one. \square

⁴When the almost complex structure is understood, we will use the notation $\text{Thom}(M)$.

Let $\widehat{\mathbb{T}}_\beta$ the group of characters of the torus \mathbb{T}_β . The complex G -module E decomposes into weight spaces $E = \sum_{\alpha \in \widehat{\mathbb{T}}_\beta} E_\alpha$ where each $E_\alpha = \{v \in E \mid t \cdot v = t^\alpha v\}$ are G -submodules. We define the β -positive and negative part of E ,

$$E^{+, \beta} = \sum_{\substack{\alpha \in \widehat{\mathbb{T}}_\beta \\ \langle \alpha, \beta \rangle > 0}} E_\alpha, \quad E^{-, \beta} = \sum_{\substack{\alpha \in \widehat{\mathbb{T}}_\beta \\ \langle \alpha, \beta \rangle < 0}} E_\alpha$$

and the β -polarized module $|E|^\beta = E^{+, \beta} \oplus \overline{E^{-, \beta}}$. It is important to note that the complex G -module $|E|^\beta$ is isomorphic to⁵ (E, J_β) , and so it does not depend on the initial complex structure of E .

Let $\widehat{R(G)}$ be the $R(G)$ -submodule of $R^{-\infty}(G)$ defined by the torus \mathbb{T}_β (see Definition 2.2). Since all the $\widehat{\mathbb{T}}_\beta$ -weights in $|E|^\beta$ satisfy the condition $\langle \alpha, \beta \rangle > 0$, the symmetric space $S^\bullet(|E|^\beta)$ decomposes as a sum $\sum_{\mu \in \widehat{\mathbb{T}}_\beta} S^\bullet(|E|^\beta)_\mu$ with $S^\bullet(|E|^\beta)_\mu \in R(G)_\mu$. Hence $S^\bullet(|E|^\beta)$ defines an element of $\widehat{R(G)}$.

The following computation is done in [1][Lecture 5] (see also [12][Section 5.1]).

Proposition 3.6. *We have the following equality in $R^{-\infty}(G)$:*

$$(22) \quad \text{Index}_E^G(\text{Thom}_\beta(E)) = (-1)^{\dim_{\mathbb{C}} E^{+, \beta}} \det(E^{+, \beta}) \otimes S^\bullet(|E|^\beta),$$

where $\det(E^{+, \beta})$ is a character of G .

Example 3.7. *Let $V = \mathbb{C}$ with the canonical action of $G = S^1$. Let $\beta = \pm 1$ in $\text{Lie}(S^1) = \mathbb{R}$. The class $\text{Thom}_{\pm 1}(\mathbb{C}) \in \mathbf{K}_{S^1}^0(\mathbf{T}_{S^1}^*\mathbb{C})$ are represented by the symbols*

$$\text{Cl}(\xi \pm ix) : \mathbb{C} \longrightarrow \mathbb{C}, \quad (x, \xi) \in \mathbf{T}^*\mathbb{C} \simeq \mathbb{C}^2.$$

We have $\text{Index}_{\mathbb{C}}^{S^1}(\text{Cl}(\xi + ix)) = -\sum_{k \geq 1} t^k$, and $\text{Index}_{\mathbb{C}}^{S^1}(\text{Cl}(\xi - ix)) = \sum_{k \leq 0} t^k$ in $R^{-\infty}(S^1)$.

Remark 3.8. *Let $J_k, k = 0, 1$ be two invariant complex structures on E , and let $\text{Thom}_\beta(E, J_k)$ be the corresponding pushed symbols. There exists an invertible element $\Phi \in R(G)$ such that*

$$\text{Index}_E^G(\text{Thom}_\beta(E, J_0)) = \Phi \cdot \text{Index}_E^G(\text{Thom}_\beta(E, J_1)).$$

3.3. Pushed symbols : functoriality. Suppose now that we have a decomposition $V = W \oplus E$ of G -complex vector spaces such that $V^\beta = \{0\}$.

Proposition 3.9. *In $\mathbf{K}_G^0(\mathbf{T}_G^*V)$, we have⁶ the equalities*

$$\begin{aligned} \text{Thom}_\beta(V) \otimes \wedge_{\mathbb{C}}^\bullet \overline{V} &= \text{Bott}(V_{\mathbb{C}}), \\ \text{Thom}_\beta(V) \otimes \wedge_{\mathbb{C}}^\bullet \overline{E} &= \text{Thom}_\beta(W) \odot \text{Bott}(E_{\mathbb{C}}). \end{aligned}$$

Proof. Note that the first relation is a particular case of the second one when $W = 0$.

A covector $(x, \xi) \in \mathbf{T}^*V$ decomposes in $x = x_W \oplus x_E$, and $\xi = \xi_W \oplus \xi_E$. The morphism $\sigma := \text{Thom}_\beta(W) \odot \text{Bott}(E_{\mathbb{C}})$ defines at (x, ξ) the map

$$\text{Cl}(\hat{\xi}_W - \gamma_W(x_W)) \odot \text{Cl}(x_E + i\hat{\xi}_E)$$

from $(\wedge W \otimes \wedge E_{\mathbb{C}})^+$ to $(\wedge W \otimes \wedge E_{\mathbb{C}})^-$.

⁵With $J_\beta = \mathcal{L}(\beta)(-\mathcal{L}(\beta)^2)^{-1/2}$.

⁶These equalities holds also in $\mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^*V)$.

We have an isomorphism of complex G -modules : $E_{\mathbb{C}} \simeq E \times \overline{E}$. We have two classes $\text{Bott}(E)$ and $\text{Bott}(\overline{E})$ in $\mathbf{K}_G^0(E)$ and $\text{Bott}(E_{\mathbb{C}}) = \text{Bott}(E) \odot \text{Bott}(\overline{E})$. At the level of endomorphism on $\wedge E_{\mathbb{C}} \simeq \wedge E \otimes \wedge \overline{E}$, one has

$$(23) \quad \text{Cl}(x_E + i\xi_E) = \text{Cl}(\xi_E - J_E x_E) \odot \text{Cl}(\xi_E + J_E x_E)$$

where J_E is the complex structure on E . We consider the family of maps $\sigma_s(x, \xi) : (\wedge W \otimes \wedge E \otimes \wedge \overline{E})^+ \longrightarrow (\wedge W \otimes \wedge E \otimes \wedge \overline{E})^-$ defined by $\text{Cl}(\xi_W - \beta_W(x_W)) \odot \text{Cl}(\xi_E - \theta_s(x_E)) \odot \text{Cl}(\xi_E + J_E x_E)$ where $\theta_s = (1-s)J_E + s\beta_E$. One checks easily that $\text{Support}(\sigma_s) \cap \mathbf{T}_G V = \{x_W = x_E = \xi_W = \xi_E = 0\}$ for any $s \in [0, 1]$. Hence $\sigma = \sigma_0$ is equal to σ_1 in $\mathbf{K}^0(\mathbf{T}_G^* E)$. Finally we check that $\sigma_1(x, \xi) = \text{Cl}(\xi - \beta_V(x)) \odot \text{Cl}(\xi_E + J_E x_E)$ can be deformed in

$$\text{Cl}(\xi - \beta_V(x)) \odot \text{Cl}(0) = \text{Thom}_{\beta}(V) \otimes \wedge^{\bullet} \overline{E},$$

without changing its class in $\mathbf{K}_G^0(\mathbf{T}_G^* V)$. \square

Since $\text{Index}_V^G(\text{Bott}(V_{\mathbb{C}})) = 1$, the first relation of Proposition 3.9 gives that

$$(24) \quad \text{Index}_V^G(\text{Thom}_{\beta}(V)) \cdot \wedge^{\bullet} \overline{V} = 1$$

in $R^{-\infty}(G)$.

Definition 3.10. Let V be a complex G -vector space such that $V^{\beta} = \{0\}$. We denote $[\wedge^{\bullet} V]_{\beta}^{-1} \in R^{-\infty}(G)$ the element $(-1)^{\dim_{\mathbb{C}} V^{-, \beta}} \det(V^{-, \beta}) \otimes S^{\bullet}(|V|^{\beta})$.

We come back to the morphism

$$(25) \quad \mathbf{R} : \mathbf{K}_G^0(\mathbf{T}_G^* V) \longrightarrow \mathbf{K}_G^0(\mathbf{T}_G^* W)$$

which is the composition of the restriction morphism $\mathbf{K}_G^0(\mathbf{T}_G^* V) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^* W \times E^*)$ with the Thom isomorphism $\mathbf{K}_G^0(\mathbf{T}_G^* W \times E^*) \simeq \mathbf{K}_G^0(\mathbf{T}_G^* W)$. We are interested by the image of the transversally elliptic symbols $\text{Thom}_{\beta}(V) \in \mathbf{K}_G^0(\mathbf{T}_G^* V)$ by the morphism \mathbf{R} .

Proposition 3.11. We have the following equality in $\mathbf{K}_G^0(\mathbf{T}_G^* W)$

$$\mathbf{R}(\text{Thom}_{\beta}(V)) = \text{Thom}_{\beta}(W).$$

Proof. The class $\text{Thom}_{\beta}(V)$ are defined by the symbols $\text{Cl}(\xi - \tilde{\beta}(x)) : \wedge^+ V \rightarrow \wedge^- V$, for $(x, \xi) \in \mathbf{T}V$. Relatively to the decomposition $V = W \oplus E$, we write $x = x_W \oplus x_E$ and $\xi = \xi_W \oplus \xi_E$. If we restrict $\text{Cl}(\xi - \tilde{\beta}(x))$ to $\mathbf{T}^* V|_W = \mathbf{T}^* W \times E^*$ we get $\text{Cl}(\xi_W - \tilde{\beta}(x_W)) \odot \text{Cl}(\xi_E)$ acting from $(\wedge W \otimes \wedge E)^+$ to $(\wedge W \otimes \wedge E)^-$. By definition of the map \mathbf{R} we find that $\mathbf{R}(\text{Thom}_{\beta}(V)) = \text{Thom}_{\beta}(W)$. \square

We consider now the case of a product of pushed symbols. Suppose that we have an invariant decomposition $E = E_1 \oplus E_2$ and invariant elements $\beta_1, \beta_2 \in \mathfrak{g}$ such that

- $E_1^{\beta_1} = E_2^{\beta_2} = \{0\}$,
- β_2 acts trivially on E_1 .

We consider then $\beta^t = t\beta_1 + \beta_2$ with $t > 0$. We have $V_1^{\beta^t} = \{0\}$ for any $t > 0$ and $V_2^{\beta^t} = \{0\}$ if $t > 0$ is small enough.

Lemma 3.12. *Let $J = J_1 \oplus J_2$ be an invariant complex structure on $V = V_1 \oplus V_2$. Then if $t > 0$ is small enough, we have the following equality in $\mathbf{K}_G^0(\mathbf{T}_G^*V)$:*

$$\text{Thom}_{\beta^t}(V, J) = \text{Thom}_{\beta_1}(V_1, J_1) \odot \text{Thom}_{\beta_2}(V_2, J_2).$$

Proof. Both symbols are maps from $(\wedge V_1 \otimes \wedge V_2)^+$ into $(\wedge V_1 \otimes \wedge V_2)^-$. We write a tangent vector $(\xi, x) \in \mathbf{T}V$ as $\xi = \xi_1 \oplus \xi_2$ and $x = x_1 \oplus x_2$. The symbol $\text{Thom}_{\beta^t}(V, J)$ is equal to

$$\text{Cl}(\xi_1 + \tilde{\beta}^t(x_1)) \odot \text{Cl}(\xi_2 + \tilde{\beta}^t(x_2)) = \text{Cl}(\xi_1 + t\tilde{\beta}_1(x_1)) \odot \text{Cl}(\xi_2 + (t\tilde{\beta}_1 + \tilde{\beta}_2)(x_2))$$

Note that $\tilde{\beta}_2 : V_2 \rightarrow V_2$ is invertible, so there exist $c > 0$ such that $t\tilde{\beta}_1 + \tilde{\beta}_2$ is invertible for any $t \in [0, c]$. Hence $\text{Thom}_{\beta^t}(V, J)$ is transversally elliptic for $0 < t \leq c$. We consider the deformation

$$\sigma_s = \text{Cl}(\xi_1 + (st + (1-s))\tilde{\beta}_1(x_1)) \odot \text{Cl}(\xi_2 + (st\tilde{\beta}_1 + \tilde{\beta}_2)(x_2))$$

for $s \in [0, 1]$. We check easily that $\text{Support}(\sigma_s) \cap \mathbf{T}_G V = \{(0, 0)\}$ for any $s \in [0, 1]$. Hence $\sigma_1 = \text{Thom}_{\beta^t}(V, J)$ and $\sigma_0 = \text{Thom}_{\beta_1}(V_1, J_1) \odot \text{Thom}_{\beta_2}(V_2, J_2)$ defines the same class in $\mathbf{K}_G^0(\mathbf{T}_G^*V)$. \square

3.4. The map \mathbf{S}_β . We come back to the situation of a G -equivariant complex vector bundle $\pi : \mathcal{E} \rightarrow M$ such that $\mathcal{E}^\beta = M$. Since the torus \mathbb{T}_β acts trivially on M , we have a decomposition $\mathcal{E} = \bigoplus_{\alpha \in \mathcal{X}} \mathcal{E}_\alpha$ where \mathcal{X} is a finite set of character of \mathbb{T}_β , and \mathcal{E}_α is the complex sub-bundle of \mathcal{E} where \mathbb{T}_β acts through the character $t \mapsto t^\alpha$. Definition 3.10 can be extended as follows. We denote

$$(26) \quad [\wedge^\bullet \mathcal{E}]_\beta^{-1} = (-1)^{\dim_{\mathbb{C}} \mathcal{E}^{-, \beta}} \det(\mathcal{E}^{-, \beta}) \otimes S^\bullet(|\mathcal{E}|^\beta).$$

where $\mathcal{E}^{\pm, \beta} = \sum_{\pm \langle \alpha, \beta \rangle > 0} \mathcal{E}_\alpha$ and $|\mathcal{E}|^\beta = \mathcal{E}^{+, \beta} \oplus \overline{\mathcal{E}^{-, \beta}}$. Note that $[\wedge^\bullet \mathcal{E}]_\beta^{-1}$ belongs to $\widehat{\mathbf{K}}_G^0(M)$ (see Definition 2.2).

Let n_α be the complex rank of \mathcal{E}_α , and let E be the following \mathbb{T}_β -complex vector space

$$E = \bigoplus_{\alpha \in \mathcal{X}} (\mathbb{C}_\alpha)^{n_\alpha},$$

which is equipped with the standard Hermitian structure.

Let U be the unitary group of E , and let U' be the subgroup of elements that commute with the action of \mathbb{T}_β : we have $U' \simeq \prod_{\alpha \in \mathcal{X}} U(\mathbb{C}^{n_\alpha})$. Let $P' \rightarrow M$ be the U' -principal bundle defined as follows: for $m \in M$, the fiber P'_m is defined as the set of maps $f : E \rightarrow \mathcal{E}_m$ preserving the Hermitian structures and which are \mathbb{T}_β -equivariant. By definition, the bundle $P' \rightarrow M$ is G -equivariant. We consider the following groups action:

- $G \times U'$ acts on P' ,
- $U' \times \mathbb{T}_\beta$ acts on E ,
- \mathbb{T}_β and G acts trivially respectively on P' and on E .

Let us use the multiplicative property (see Section 2.3) with the groups $G_2 = G \times U'$, $G_1 = \mathbb{T}_\beta$ and the manifolds $M_1 = E$, $M_2 = P'$. We have a product

$$\mathbf{K}_{\mathbb{T}_\beta \times G \times U'}^0(\mathbf{T}_{\mathbb{T}_\beta}^* E) \times \mathbf{K}_{G \times U'}^*(\mathbf{T}_{G \times U'}^* P') \longrightarrow \mathbf{K}_{\mathbb{T}_\beta \times G \times U'}^*(\mathbf{T}_{\mathbb{T}_\beta \times G \times U'}^*(P' \times E)),$$

and the Thom class $\text{Thom}_\beta(E) \in \mathbf{K}_{\mathbb{T}_\beta \times U'}^0(\mathbf{T}_{\mathbb{T}_\beta}^* E)$ induces the map

$$(27) \quad \begin{aligned} \mathbf{S}_\beta'' : \mathbf{K}_{G \times U'}^*(\mathbf{T}_{G \times U'}^* P) &\longrightarrow \mathbf{K}_{\mathbb{T}_\beta \times G \times U'}^*(\mathbf{T}_{\mathbb{T}_\beta \times G \times U'}^*(P' \times E)) \\ \sigma &\longmapsto \text{Thom}_\beta(E) \odot_{\text{ext}} \sigma \end{aligned}$$

After taking the quotient by U' , we get a map

$$\mathbf{S}_\beta' : \mathbf{K}_G^*(\mathbf{T}_G^* M) \longrightarrow \mathbf{K}_{\mathbb{T}_\beta \times G}^*(\mathbf{T}_{\mathbb{T}_\beta \times G}^* \mathcal{E})$$

Finally, since $\mathbf{T}_{\mathbb{T}_\beta \times G}^* \mathcal{E} = \mathbf{T}_G^* \mathcal{E}$, we can compose \mathbf{S}_β' with the forgetful map $\mathbf{K}_{\mathbb{T}_\beta \times G}^*(\mathbf{T}_G^* \mathcal{E}) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^* \mathcal{E})$ to get

$$\mathbf{S}_\beta : \mathbf{K}_G^*(\mathbf{T}_G^* M) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^* \mathcal{E}).$$

Now we see that in Theorem 3.2 :

- The relation $\mathbf{R} \circ \mathbf{S}_\beta = \text{Id}$ is induced by the relation $\mathbf{R}(\text{Thom}_\beta(E)) = 1$, where $\mathbf{R} : \mathbf{K}_{\mathbb{T}_\beta \times U'}^0(\mathbf{T}_{\mathbb{T}_\beta}^* E) \rightarrow R(\mathbb{T}_\beta \times U')$ (see Proposition 3.5).
- The relation $\mathbf{S}_\beta(a) \otimes \wedge^\bullet \pi^* \bar{\mathcal{E}} = s_!(a)$ is induced by the relation $\text{Thom}_\beta(E) \otimes \wedge^\bullet \bar{E} = \text{Bott}(E_C)$ proved in Proposition 3.9.

Let us prove the last point of Theorem 3.2. Let $\sigma \in \mathbf{K}_G^0(\mathbf{T}_G^* M)$ and let $\tilde{\sigma}$ be the corresponding element in $\mathbf{K}_{G \times U'}^0(\mathbf{T}_{G \times U'}^* P)$. The index $\text{Index}_\mathcal{E}^G(\mathbf{S}_\beta(\sigma)) \in R^{-\infty}(G)$ is equal to the restriction of $\text{Index}_\mathcal{E}^{G \times \mathbb{T}_\beta}(\mathbf{S}_\beta'(\sigma)) \in R^{-\infty}(G \times \mathbb{T})$ at $t = 1 \in \mathbb{T}_\beta$ (see Section 2.2). By definition we have the following equalities in $R^{-\infty}(G \times \mathbb{T}_\beta)$

$$\begin{aligned} \text{Index}_\mathcal{E}^{G \times \mathbb{T}_\beta}(\mathbf{S}_\beta'(\sigma)) &= \left[\text{Index}_{P' \times E}^{U' \times G \times \mathbb{T}_\beta}(\mathbf{S}_\beta''(\tilde{\sigma})) \right]^{U'} \\ &= \left[\text{Index}_{P' \times E}^{U' \times G}(\tilde{\sigma}) \cdot \text{Index}_E^{U' \times \mathbb{T}_\beta}(\text{Thom}_\beta(E)) \right]^{U'} \\ &= \sum_{\mu \in \widehat{\mathbb{T}}_\beta} \text{Index}_M^G(\sigma \otimes \mathcal{W}_\mu) \otimes \mathbb{C}_\mu \end{aligned}$$

where $\text{Index}_E^{U' \times \mathbb{T}_\beta}(\text{Thom}_\beta(E)) = [\wedge^\bullet \bar{E}]_\beta^{-1} = \sum_{\mu \in \widehat{\mathbb{T}}} W_\mu \otimes \mathbb{C}_\mu$ with $W_\mu \in R(U')$. We denote $\mathcal{W}_\mu = P' \times_{U'} W_\mu$ the corresponding element in $\mathbf{K}_G^0(M)_\mu$. Finally we get

$$\begin{aligned} \text{Index}_\mathcal{E}^G(\mathbf{S}_\beta(\sigma)) &= \sum_{\mu \in \widehat{\mathbb{T}}_\beta} \text{Index}_M^G(\sigma \otimes \mathcal{W}_\mu) \\ &= \text{Index}_M^G \left(\sigma \otimes [\wedge^\bullet \bar{\mathcal{E}}]_\beta^{-1} \right), \end{aligned}$$

where $[\wedge^\bullet \bar{\mathcal{E}}]_\beta^{-1} = \sum_{\mu \in \widehat{\mathbb{T}}} \mathcal{W}_\mu \in \widehat{\mathbf{K}}_G^0(M)$.

3.5. The map θ_β . We keep the same notation than the previous section: $\pi : \mathcal{E} \rightarrow M$ is a G -equivariant complex vector bundle such that $\mathcal{E}^\beta = M$, but here we work with the complex structure J_β on \mathcal{E} . Since the map $\mathbf{S}_{\pm\beta}^o$ are defined through the pushed Thom classes $\text{Thom}_\beta(E) \in \mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^* E)$ (see (27)), we have to study the class $\text{Thom}_{-\beta}(E) - \text{Thom}_\beta(E)$ in order to understand how the map $\mathbf{S}_{-\beta}^o - \mathbf{S}_\beta^o : \mathbf{K}_G^*(\mathbf{T}_G^* M) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^* \mathcal{E})$ factorizes through the push-forward morphism $i_! : \mathbf{K}_G^*(\mathbf{T}_G^* \mathcal{S}) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^* \mathcal{E})$.

3.5.1. *The tangential Cauchy Riemann operator.* Let E be a Euclidean G -module such that $E^\beta = \{0\}$. We equipped E with the invariant complex structure J_β (see Remark 3.1). Let $S \subset E$ be the sphere of radius one. Let us defined the tangential Cauchy Riemann operator on S . For $y \in S$, we have

$$\begin{aligned} \mathbf{T}_y S &= \{ \xi \mid (\xi, y) = 0 \} \\ &= \mathcal{H}_y \oplus \mathbb{R}J_\beta y, \end{aligned}$$

where $\mathcal{H}_y = (\mathbb{C}y)^\perp$ is a complex invariant subspace of (E, J_β) . Let $\mathcal{H} \rightarrow S$ be the corresponding Hermitian vector bundle. For $\xi \in \mathbf{T}_y S$, we denote ξ' its component in \mathcal{H}_y . Since $(\beta_E(y), J_\beta y) \neq 0$ for $y \neq 0$, we see that for $\xi \in \mathbf{T}_G S|_y$, we have $\xi' = 0 \Leftrightarrow \xi = 0$.

Definition 3.13. *The Cauchy Riemann symbol⁷ $\sigma_\partial^E : \wedge^+ \mathcal{H} \rightarrow \wedge^- \mathcal{H}$ is defined by $\sigma_\partial^E(y, \xi) = \text{Cl}(\xi') : \wedge^+ \mathcal{H}_y \rightarrow \wedge^- \mathcal{H}_y$. It defines⁸ a class $\sigma_\partial^E \in \mathbf{K}_G^0(\mathbf{T}_G^* S)$.*

The Thom isomorphism tells us that $\mathbf{K}_G^0(\mathbf{T}_G^* S) \simeq \mathbf{K}_G^0(\mathbf{T}_G^*(E \setminus \{0\}))$ and we know that $i_! : \mathbf{K}_G^0(\mathbf{T}_G^* S) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^* E)$ is injective. Hence, it will be convenient to use the same notations for σ_∂^E and $i_!(\sigma_\partial^E)$ and to consider them as a class in $\mathbf{K}_G^0(\mathbf{T}_G^*(E \setminus \{0\}))$ or in $\mathbf{K}_G^0(\mathbf{T}_G^* E)$.

Example 3.14. *Consider the Cauchy Riemann symbol $\sigma_\partial^{\mathbb{C}_\chi} \in \mathbf{K}_G^0(\mathbf{T}_G^* \mathbb{C}_\chi)$ associated to the one dimensional representation \mathbb{C}_χ of G . We check that $\sigma_\partial^{\mathbb{C}_\chi}$ is represented by the map $\rho : \mathbf{T}^* \mathbb{C}_\chi \rightarrow \mathbb{C}$ defined by: $\rho(w, z) = \Re(w\bar{z}) + \imath(\|z\| - 1)$.*

We come back to the setting of Section 3.1. We have an exact sequence $0 \rightarrow \mathbf{K}_G^*(\mathbf{T}_{\mathbb{T}_\beta}^* S) \xrightarrow{i_!} \mathbf{K}_G^*(\mathbf{T}_{\mathbb{T}_\beta}^* E) \xrightarrow{\mathbf{R}} R(G) \rightarrow 0$, and we know that $\mathbf{R}(\text{Thom}_{\pm\beta}(E)) = 1$. Then $\text{Thom}_\beta(E) - \text{Thom}_{-\beta}(E)$ belongs to $\ker(\mathbf{R}) = \text{Im}(i_!)$.

The following result is due to Atiyah-Singer when G is the circle group (see [1][Lemma 6.3]). The proof in the general case is given in Appendix B.

Proposition 3.15. *Let E be a G -module equipped with the invariant complex structure J_β . We have the following equality*

$$\text{Thom}_{-\beta}(E) - \text{Thom}_\beta(E) = i_!(\sigma_\partial^E).$$

in $\mathbf{K}_G^0(\mathbf{T}_G^* E)$.

3.5.2. *Functoriality.* Suppose that $V = W \oplus E$ with $W^\beta = E^\beta = \{0\}$. We equipped V, W, E by the invariant complex structures defined by β . Let $\sigma_\partial^V \in \mathbf{K}_G^0(\mathbf{T}_G^*(V \setminus \{0\}))$, $\sigma_\partial^W \in \mathbf{K}_G^0(\mathbf{T}_G^*(W \setminus \{0\}))$ be the corresponding Cauchy Riemann classes. We have a natural product

$$\mathbf{K}_G^0(\mathbf{T}_G^*(W \setminus \{0\})) \times \mathbf{K}_G^0(\mathbf{T}_G^* E) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^*(V \setminus \{0\})).$$

and a restriction morphism $\mathbf{R} : \mathbf{K}_G^0(\mathbf{T}_G^* V) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^* W)$ (see (25)).

Proposition 3.16. *We have*

- $\sigma_\partial^{W, \beta} \otimes \text{Bott}(E_\mathbb{C}) = \sigma_\partial^{V, \beta} \otimes \wedge^\bullet \overline{E}$ in $\mathbf{K}_G^0(\mathbf{T}_G^*(V \setminus \{0\}))$,
- $\mathbf{R}(\sigma_\partial^{V, \beta}) = \sigma_\partial^{W, \beta}$ in $\mathbf{K}_G^0(\mathbf{T}_G^* W)$.

⁷Here we use an identification $\mathbf{T}^* S \simeq \mathbf{T} S$ given by the Euclidean structure.

⁸Note that σ_∂^E defines also a class in $\mathbf{K}_G^0(\mathbf{T}_{\mathbb{T}_\beta}^* S)$.

Proof. These are direct consequences of Proposition 3.15. For the first point, we use it together with Proposition 3.9, and for the second one we use it together with Proposition 3.11. \square

The element $\sigma_{\bar{\partial}}^{V,\beta}$ belongs to the subspace $\mathbf{K}_G^0(\mathbf{T}_G^*(V \setminus \{0\})) \hookrightarrow \mathbf{K}_G^0(\mathbf{T}_G^*V)$, and the restriction map \mathbf{R} sends $\mathbf{K}_G^0(\mathbf{T}_G^*(V \setminus \{0\}))$ into $\mathbf{K}_G^0(\mathbf{T}_G^*(W \setminus \{0\}))$ (see remark 5.5). We can precise the last statement of Proposition 3.16, by saying that the equality $\mathbf{R}(\sigma_{\bar{\partial}}^{V,\beta}) = \sigma_{\bar{\partial}}^{W,\beta}$ holds in $\mathbf{K}_G^0(\mathbf{T}_G^*(W \setminus \{0\}))$.

3.5.3. Definition of the map θ_β . We come back to the setting of Section 3.4. The complex vector bundle $\mathcal{E} \rightarrow M$ corresponds to $P' \times_{U'} E \rightarrow P'/U'$, and the sphere bundle is $\mathcal{S} = P' \times_{U'} S_E$.

Let us use the multiplicative property (see Section 2.3) with the groups $G_2 = G \times U'$, $G_1 = \mathbb{T}_\beta$ and the manifolds $M_1 = S_E$, $M_2 = P'$. Thanks to the product

$$\mathbf{K}_{\mathbb{T}_\beta \times G \times U'}^0(\mathbf{T}_{\mathbb{T}_\beta}^* S_E) \times \mathbf{K}_{G \times U'}^*(\mathbf{T}_{G \times U'}^* P') \longrightarrow \mathbf{K}_{\mathbb{T}_\beta \times G \times U'}^*(\mathbf{T}_{\mathbb{T}_\beta \times G \times U'}^*(P' \times S_E))$$

we can define

$$(28) \quad \begin{aligned} \theta''_\beta : \mathbf{K}_{G \times U'}^*(\mathbf{T}_{G \times U'}^* P') &\longrightarrow \mathbf{K}_{\mathbb{T}_\beta \times G \times U'}^*(\mathbf{T}_{\mathbb{T}_\beta \times G \times U'}^*(P' \times S_E)) \\ \sigma &\longmapsto \sigma_{\bar{\partial}}^{E,\beta} \odot_{\text{ext}} \sigma. \end{aligned}$$

After taking the quotient by U' , we get a map

$$\theta'_\beta : \mathbf{K}_G^*(\mathbf{T}_G^* M) \longrightarrow \mathbf{K}_{\mathbb{T}_\beta \times G}^*(\mathbf{T}_{\mathbb{T}_\beta \times G}^* \mathcal{S})$$

Finally, since $\mathbf{T}_{\mathbb{T}_\beta \times G}^* \mathcal{S} = \mathbf{T}_G^* \mathcal{S}$, we can compose θ'_β with the forgetful map $\mathbf{K}_{\mathbb{T}_\beta \times G}^*(\mathbf{T}_G^* \mathcal{S}) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^* \mathcal{S})$ to get $\theta_\beta : \mathbf{K}_G^*(\mathbf{T}_G^* M) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^* \mathcal{S})$.

The identity $\text{Thom}_{-\beta}(E) - \text{Thom}_\beta(E) = i_!(\sigma_{\bar{\partial}}^{E,\beta})$ shows that we have a commutative diagram

$$\begin{array}{ccc} \mathbf{K}_{G \times U'}^*(\mathbf{T}_{G \times U'}^* P') & \xrightarrow{\theta''_\beta} & \mathbf{K}_{\mathbb{T}_\beta \times G \times U'}^*(\mathbf{T}_{\mathbb{T}_\beta \times G \times U'}^*(P' \times S_E)) \\ & \searrow \mathbf{s}''_{-\beta} - \mathbf{s}''_\beta & \downarrow i_! \\ & & \mathbf{K}_{\mathbb{T}_\beta \times G \times U'}^*(\mathbf{T}_{\mathbb{T}_\beta \times G \times U'}^*(P' \times E)) . \end{array}$$

After taking the quotient by U' , we get the commutative diagram

$$\begin{array}{ccc} \mathbf{K}_G^*(\mathbf{T}_G^* M) & \xrightarrow{\theta_\beta} & \mathbf{K}_G^*(\mathbf{T}_G^* \mathcal{S}) \\ & \searrow \mathbf{s}^o_{-\beta} - \mathbf{s}^o_\beta & \downarrow i_! \\ & & \mathbf{K}_G^*(\mathbf{T}_G^* \mathcal{E}) . \end{array}$$

which is the content of Theorem 3.4.

3.6. Restriction to a fixed point sub-manifold. Let M be a G -manifold and let $\beta \in \mathfrak{g}$ be a G -invariant element. Let Z be a connected component of the fixed point set M^β . Note that β defines a complex structure J_β on the normal bundle of

Z in M . Following Section 2.6 we have a restriction morphism \mathbf{R}_Z that fits in the six term exact sequence

$$\begin{array}{ccccc} \mathbf{K}_G^0(\mathbf{T}_G^*(M \setminus Z)) & \xrightarrow{j_*} & \mathbf{K}_G^0(\mathbf{T}_G^*M) & \xrightarrow{\mathbf{R}_Z} & \mathbf{K}_G^0(\mathbf{T}_G^*Z) \\ \delta \uparrow & & & & \delta \downarrow \\ \mathbf{K}_G^1(\mathbf{T}_G^*Z) & \xleftarrow{\mathbf{R}_Z} & \mathbf{K}_G^1(\mathbf{T}_G^*M) & \xleftarrow{j_*} & \mathbf{K}_G^1(\mathbf{T}_G^*(M \setminus Z)). \end{array}$$

Proposition 3.17.

- There exists a morphism $\mathbf{S}_{\beta,Z} : \mathbf{K}_G^*(\mathbf{T}_G^*Z) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*M)$ such that $\mathbf{R}_Z \circ \mathbf{S}_{\beta,Z}$ is the identity on $\mathbf{K}_G^*(\mathbf{T}_G^*Z)$.

• We have an isomorphism of $R(G)$ -modules :

$$\mathbf{K}_G^*(\mathbf{T}_G^*M) \simeq \mathbf{K}_G^*(\mathbf{T}_G^*Z) \oplus \mathbf{K}_G^*(\mathbf{T}_G^*(M \setminus Z)).$$

Proof. Let \mathcal{N} be the normal bundle of Z in M . Let \mathcal{U} be an invariant tubular neighborhood of Z , which is small enough so that we have an equivariant diffeomorphism $\phi : \mathcal{U} \rightarrow \mathcal{N}$ which is the identity on Z . Let $\mathbf{S}_{\beta,\mathcal{N}} : \mathbf{K}_G^*(\mathbf{T}_G^*Z) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{N})$ the map that we have constructed in Section 3.4. Let $j_* : \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{U}) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*M)$ be the push-forward map associated to the inclusion $j : \mathcal{U} \hookrightarrow M$. Let $\phi^* : \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{N}) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*\mathcal{U})$ be the isomorphism associated to ϕ . We can consider the composition

$$\mathbf{S}_{\beta,Z} := j_* \circ \phi^* \circ \mathbf{S}_{\beta,\mathcal{N}},$$

and we leave to the reader the verification that $\mathbf{R}_Z \circ \mathbf{S}_{\beta,Z} = \text{Id}$. The last point is a direct consequence of the first one. \square

4. DECOMPOSITION OF $\mathbf{K}_G^*(\mathbf{T}_G^*M)$ WHEN G IS ABELIAN

In this section G denotes a compact **abelian** Lie group, with Lie algebra \mathfrak{g} . Let M be a (connected) manifold equipped with an action of G . For any $m \in M$, we denote $\mathfrak{g}_m \subset \mathfrak{g}$ its infinitesimal stabilizer.

Let $\Delta_G(M)$ be the set formed by the infinitesimal stabilizer of points in M . During this section, we suppose that $\Delta_G(M)$ is **finite**: it is the case if M is compact or when M is embedded equivariantly in a G -module. We have a partition

$$M = \bigsqcup_{\mathfrak{h} \in \Delta_G(M)} M_{\mathfrak{h}}$$

where $M_{\mathfrak{h}} := \{m \in M \mid \mathfrak{h} = \mathfrak{g}_m\}$ is an invariant open subset of the smooth submanifold $M^{\mathfrak{h}} := \{m \in M \mid \mathfrak{h} \subset \mathfrak{g}_m\}$.

On the other hand, we consider for $0 \leq k \leq s = \dim G$ the *closed* subset

$$M^{\leq k} \subset M$$

formed by the points $m \in M$ such that $\dim(G \cdot m) = \text{codim}(\mathfrak{g}_m) \leq k$. We have

$$M^{\leq k} = \bigsqcup_{\text{codim } \mathfrak{h} \leq k} M_{\mathfrak{h}} = \bigcup_{\text{codim } \mathfrak{h} \leq k} M^{\mathfrak{h}}$$

Let $M^{=k} = M^{\leq k} \setminus M^{\leq k-1}$ and $M^{>k} = M \setminus M^{\leq k-1}$. We note that

$$M^{=k} = \bigsqcup_{\text{codim } \mathfrak{h} = k} M_{\mathfrak{h}}$$

Let s_o be the maximal dimension of the G -orbit in M . We will use the increasing sequence of invariant open subsets

$$M^{>s_o-1} \subset \dots \subset M^{>1} \subset M^{>0} \subset M.$$

Here $M^{>0} = M \setminus M^{\mathfrak{g}}$, and $M^{>s_o-1} = M^{gen}$ is the dense open subset formed by the G -orbits of maximal dimension. Note also that M^{gen} corresponds to $M_{\mathfrak{h}_{min}}$ where \mathfrak{h}_{min} is the minimal stabilizer.

Let us consider the related sequences of open subspaces

$$\mathbf{T}_G^* M^{>s_o-1} \subset \dots \subset \mathbf{T}_G^* M^{>1} \subset \mathbf{T}_G^* M^{>0} \subset \mathbf{T}_G^* M.$$

At level of K -theory the inclusion $j_k : M^{>k} \hookrightarrow M^{>k-1}$ gives rise to the map

$$(j_k)_* : \mathbf{K}_G^*(\mathbf{T}_G^* M^{>k}) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^* M^{>k-1}).$$

Let $0 \leq k \leq s_o - 1$. We have the decomposition

$$\begin{aligned} \mathbf{T}_G^* M^{>k-1} &= \mathbf{T}_G^* M^{>k} \bigsqcup \mathbf{T}_G^* M^{>k-1}|_{M^k} \\ &= \mathbf{T}_G^* M^{>k} \bigsqcup \bigsqcup_{\text{codim } \mathfrak{h}=k} \mathbf{T}_G^* M^{>k-1}|_{M_{\mathfrak{h}}} \\ &= \mathbf{T}_G^* M^{>k} \bigsqcup \bigsqcup_{\text{codim } \mathfrak{h}=k} \mathbf{T}_G^* M_{\mathfrak{h}} \times \mathcal{N}_{\mathfrak{h}}^* \end{aligned}$$

where $\mathcal{N}_{\mathfrak{h}}$ is the normal bundle of $M_{\mathfrak{h}}$ in M . Note that $M_{\mathfrak{h}}$ is a closed sub-manifold of the open subset $M^{>k-1}$, when $\text{codim } \mathfrak{h} = k$.

Lemma 4.1. *Let $\mathfrak{h} \in \Delta_G(M)$ with $\text{codim } \mathfrak{h} = k$. There exists $\gamma_{\mathfrak{h}} \in \mathfrak{h}$ so that $M_{\mathfrak{h}}$ is equal to the fixed point set $(M^{>k-1})^{\gamma_{\mathfrak{h}}} := \{m \in M^{>k-1} \mid \gamma_{\mathfrak{h}} \in \mathfrak{g}_m\}$. The element $\gamma_{\mathfrak{h}}$ defines then a complex structure $J_{\gamma_{\mathfrak{h}}}$ on the normal bundle $\mathcal{N}_{\mathfrak{h}}$.*

Proof. Let H be the closed connected subgroup of G with Lie algebra \mathfrak{h} . Let $\gamma_{\mathfrak{h}} \in \mathfrak{h}$ generic so that the closure of $\{\exp(t\gamma_{\mathfrak{h}}), t \in \mathbb{R}\}$ is equal to H . Then for any $m \in M$, $\gamma_{\mathfrak{h}} \in \mathfrak{g}_m \Leftrightarrow \mathfrak{h} \subset \mathfrak{g}_m$. Then

$$\{m \in M^{>k-1} \mid \gamma_{\mathfrak{h}} \in \mathfrak{g}_m\} = \{m \in M^{>k-1} \mid \mathfrak{h} \subset \mathfrak{g}_m\} = \{m \in M \mid \mathfrak{h} = \mathfrak{g}_m\} = M_{\mathfrak{h}}.$$

□

Thanks to Lemma 4.1, we can exploit Section 3.6. For any $\mathfrak{h} \in \Delta_G(M)$ of codimension k , we have a restriction morphism

$$(29) \quad \mathbf{R}_{\mathfrak{h}} : \mathbf{K}_G^*(\mathbf{T}_G^* M^{>k-1}) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^* M_{\mathfrak{h}})$$

and a section

$$\mathbf{S}_{\mathfrak{h}} := \mathbf{S}_{\gamma_{\mathfrak{h}}, M_{\mathfrak{h}}} : \mathbf{K}_G^*(\mathbf{T}_G^* M_{\mathfrak{h}}) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^* M^{>k-1})$$

such that $\mathbf{R}_{\mathfrak{h}} \circ \mathbf{S}_{\mathfrak{h}}$ is the identity on $\mathbf{K}_G^*(\mathbf{T}_G^* M_{\mathfrak{h}})$.

We have also a long exact sequence

$$\begin{array}{ccccccc} \mathbf{K}_G^0(\mathbf{T}_G^* M^{>k}) & \xrightarrow{(j_k)_*} & \mathbf{K}^0(\mathbf{T}_G^* M^{>k-1}) & \xrightarrow{\mathbf{R}_k} & \bigoplus_{\text{codim } \mathfrak{h}=k} \mathbf{K}_G^0(\mathbf{T}_G^* M_{\mathfrak{h}}) \\ \delta \uparrow & & & & \downarrow \delta \\ \bigoplus_{\text{codim } \mathfrak{h}=k} \mathbf{K}_G^1(\mathbf{T}_G^* M_{\mathfrak{h}}) & \xleftarrow{\mathbf{R}_k} & \mathbf{K}_G^1(\mathbf{T}_G^* M^{>k-1}) & \xleftarrow{(j_k)_*} & \mathbf{K}_G^1(\mathbf{T}_G^* M^{>k}), \end{array}$$

where $\mathbf{R}_k = \oplus_{\text{codim}\mathfrak{h}=k} \mathbf{R}_{\mathfrak{h}}$. We define $\mathbf{S}_k : \oplus_{\text{codim}\mathfrak{h}=k} \mathbf{K}_G^*(\mathbf{T}_G^*M_{\mathfrak{h}}) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*M^{>k-1})$ by

$$\mathbf{S}_k(\oplus_{\text{codim}\mathfrak{h}=k} \sigma_{\mathfrak{h}}) = \sum_{\text{codim}\mathfrak{h}=k} \mathbf{S}_{\mathfrak{h}}(\sigma_{\mathfrak{h}}).$$

Lemma 4.2. *Let $\mathfrak{a}, \mathfrak{b} \in \Delta_G(M)$.*

- *We have $\mathbf{R}_{\mathfrak{a}} \circ \mathbf{S}_{\mathfrak{a}} = \text{Id}$ in $\mathbf{K}_G^*(\mathbf{T}_G^*M_{\mathfrak{a}})$.*
- *We have $\mathbf{R}_{\mathfrak{a}} \circ \mathbf{S}_{\mathfrak{b}} = 0$ if $\mathfrak{a} \neq \mathfrak{b}$.*
- *The map $\mathbf{R}_k \circ \mathbf{S}_k$ is the identity on $\oplus_{\text{codim}\mathfrak{h}=k} \mathbf{K}_G^*(\mathbf{T}_G^*M_{\mathfrak{h}})$.*

Proof. The last point is a direct consequence of the firsts one. The first point is known, and the second assertion is due to the fact that $M_{\mathfrak{a}} \cap M_{\mathfrak{b}} = \emptyset$ when $\mathfrak{a} \neq \mathfrak{b}$. \square

The previous lemma shows that the map

$$(30) \quad ((j_k)_*, \mathbf{S}_k) : \mathbf{K}^*(\mathbf{T}_G^*M^{>k}) \times \oplus_{\text{codim}\mathfrak{h}=k} \mathbf{K}_G^*(\mathbf{T}_G^*M_{\mathfrak{h}}) \longrightarrow \mathbf{K}^*(\mathbf{T}_G^*M^{>k-1})$$

is an isomorphism of $R(G)$ -module. In particular the maps $(j_k)_*$ are injective.

Remark 4.3. *If we consider the open subset $j : M^{\text{gen}} \hookrightarrow M$ formed by the G -orbits of maximal dimension, we know then that*

$$j_* : \mathbf{K}_G^*(\mathbf{T}_G^*M^{\text{gen}}) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*M)$$

is injective, since j is the composition of all the j_k .

The isomorphisms (30) all together give the following Theorem (which was given in a less precise version in [1][Theorem 8.4]).

Theorem 4.4 (Atiyah-Singer). *Let $\gamma := \{\gamma_{\mathfrak{h}}, \mathfrak{h} \in \Delta_G(M)\}$ such that $M_{\mathfrak{h}} = \{m \in M^{>\text{codim}\mathfrak{h}-1} \mid \gamma_{\mathfrak{h}} \in \mathfrak{g}_m\}$. We have an isomorphism*

$$(31) \quad \Phi_{\gamma} : \bigoplus_{\mathfrak{h} \in \Delta_G(M)} \mathbf{K}_G^*(\mathbf{T}_G^*M_{\mathfrak{h}}) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*M)$$

of $R(G)$ -module such that

$$\text{Index}_M^G(\Phi_{\gamma}(\oplus_{\mathfrak{h} \in \Delta} \sigma_{\mathfrak{h}})) = \sum_{\mathfrak{h} \in \Delta} \text{Index}_{M_{\mathfrak{h}}}^G(\sigma_{\mathfrak{h}} \otimes S^{\bullet}(\mathcal{N}_{\mathfrak{h}}))$$

*for any $\oplus_{\mathfrak{h} \in \Delta} \sigma_{\mathfrak{h}} \in \bigoplus_{\mathfrak{h} \in \Delta} \mathbf{K}_G^0(\mathbf{T}_G^*M_{\mathfrak{h}})$. Here $\mathcal{N}_{\mathfrak{h}}$ is the normal bundle of $M_{\mathfrak{h}}$ in M which is equipped with the complex structure defined by $-\gamma_{\mathfrak{h}}$*

For any $\mathfrak{h} \in \Delta_G(M)$ we denote $H \subset G$ the closed connected subgroup with Lie algebra \mathfrak{h} . Let us denote $H' \subset G$ be a Lie subgroup such that $G \simeq H \times H'$. Then the $R(G)$ -module $\mathbf{K}_G^*(\mathbf{T}_G^*M_{\mathfrak{h}})$ is equal to

$$\mathbf{K}_{H'}^*(\mathbf{T}_{H'}^*M_{\mathfrak{h}}) \otimes R(H).$$

Thus Theorem 4.4 says that $\mathbf{K}_G^*(\mathbf{T}_G^*M)$ is isomorphic to

$$\bigoplus_{\mathfrak{h} \in \Delta_G(M)} \mathbf{K}_{H'}^*(\mathbf{T}_{H'}^*M_{\mathfrak{h}}) \otimes R(H).$$

Note that the action of H' on $M_{\mathfrak{h}}$ has finite stabilizers, hence the group $\mathbf{K}_{H'}^*(\mathbf{T}_{H'}^*M_{\mathfrak{h}})$ is equal to $\mathbf{K}_{orb}^*(\mathbf{T}^*\mathcal{M}_{\mathfrak{h}})$, where $\mathcal{M}_{\mathfrak{h}} = M_{\mathfrak{h}}/H'$ is an orbifold.

5. THE LINEAR CASE

In this section, the group G is a compact **abelian** Lie group. Let V be a real G -module. Let V^{gen} be the open subset formed by the G -orbits of maximal dimension. We equip $V/V^{\mathfrak{g}}$ with an invariant complex structure. For any $\gamma \in \mathfrak{g}$ such that $V^\gamma = V^{\mathfrak{g}}$ we associate the class

$$\text{Thom}_\gamma(V/V^{\mathfrak{g}}) \odot \text{Bott}(V^{\mathfrak{g}}) \in \mathbf{K}_G^0(\mathbf{T}_G^*V).$$

Let $H_{min} \subset G$ be the minimal stabilizer for the G -action on V . Let $s := \dim G - \dim H_{min}$.

Definition 5.1. A (G, V) -flag φ corresponds to a decomposition $V/V^{\mathfrak{g}} = V_1^\varphi \oplus \dots \oplus V_s^\varphi$ in complex G -subspaces, and a decomposition $\mathfrak{g} = \mathfrak{h}_{min} \oplus \mathbb{R}\beta_1^\varphi \oplus \dots \oplus \mathbb{R}\beta_s^\varphi$ such that for any $1 \leq k \leq s$

- c1** β_k^φ acts trivially on V_j^φ when $j < k$,
- c2** β_k^φ acts bijectively⁹ on V_k^φ .

We can associate to the data φ above, the flags $V^{\mathfrak{g}} = V^{[0],\varphi} \subset V^{[1],\varphi} \subset \dots \subset V^{[s],\varphi} = V$ and $\mathfrak{h}_{min} = \mathfrak{g}^{[0],\varphi} \subset \mathfrak{g}^{[1],\varphi} \subset \dots \subset \mathfrak{g}^{[s],\varphi} = \mathfrak{g}$ where

$$V^{[j],\varphi} = V^{\mathfrak{g}} \oplus \sum_{1 \leq k \leq j} V_k^\varphi, \quad \text{and} \quad \mathfrak{g}^{[j],\varphi} = \mathfrak{h}_{min} \oplus \mathbb{R}\beta_{j+1}^\varphi \oplus \dots \oplus \mathbb{R}\beta_s^\varphi.$$

We see that conditions **c1** and **c2** are equivalent to saying that the generic infinitesimal stabilizer of the G -action on the vector space $V^{[j],\varphi}$ is equal to $\mathfrak{g}^{[j],\varphi}$.

Thanks to **c2**, the Cauchy-Riemann symbol

$$\sigma_{\overline{\partial}}^{\varphi,k} \in \mathbf{K}_G^0(\mathbf{T}_{\mathbb{R}\beta_k}^*(V_k^\varphi \setminus \{0\})),$$

is well defined. Conditions **c1** and **c2** tell us also that $(V_1^\varphi \setminus \{0\}) \times \dots \times (V_s^\varphi \setminus \{0\})$ is an open subset of $(V/V^{\mathfrak{g}})^{gen}$, and thanks to Theorem 2.7 we know that the following product

$$\sigma_{\overline{\partial}}^{V/V^{\mathfrak{g}},\varphi} := \sigma_{\overline{\partial}}^{\varphi,1} \odot \dots \odot \sigma_{\overline{\partial}}^{\varphi,s}$$

is a well defined class in $\mathbf{K}_G^0(\mathbf{T}_G^*(V/V^{\mathfrak{g}})^{gen})$.

We need the following submodule of $R^{-\infty}(G)$ defined by the relations

$$\Phi \in \mathcal{F}_G(V) \iff \wedge^{\bullet} \overline{V/V^{\mathfrak{h}}} \otimes \Phi \in \langle R^{-\infty}(G/H) \rangle, \forall \mathfrak{h} \in \Delta_G(V),$$

$$\Phi \in \text{DM}_G(V) \iff \wedge^{\bullet} \overline{V/V^{\mathfrak{h}}} \otimes \Phi = 0, \forall \mathfrak{h} \neq \mathfrak{h}_{min} \quad \text{and} \quad \Phi \in \langle R^{-\infty}(G/H_{min}) \rangle.$$

The purpose of this section is to give a detailed proof of the following theorem.

Theorem 5.2. Let G a compact abelian Lie group and let V be a real G -module. We have

- a.** $\mathbf{K}_G^1(\mathbf{T}_G^*V) = \mathbf{K}_G^1(\mathbf{T}_G^*V^{gen}) = 0$
- b.** The index map $\text{Index}_V^G : \mathbf{K}_G^0(\mathbf{T}_G^*V) \longrightarrow R^{-\infty}(G)$ is one to one.
- c.** The elements $\text{Bott}(V_{\mathbb{C}}^{\mathfrak{g}}) \odot \text{Thom}_\gamma(V/V^{\mathfrak{g}})$ generate $\mathbf{K}_G^0(\mathbf{T}_G^*V)$, when γ runs over the elements such that $V^\gamma = V^{\mathfrak{g}}$.
- d.** The elements $\text{Bott}(V_{\mathbb{C}}^{\mathfrak{g}}) \odot \sigma_{\overline{\partial}}^{V/V^{\mathfrak{g}},\varphi}$ generate $\mathbf{K}_G^0(\mathbf{T}_G^*V^{gen})$, when φ runs over the (G, V) -flag.
- e.** The image $\mathbf{K}_G^*(\mathbf{T}_G^*V)$ by Index_V^G is equal to $\mathcal{F}_G(V)$.
- f.** The image $\mathbf{K}_G^*(\mathbf{T}_G^*V^{gen})$ by Index_V^G is equal to $\text{DM}_G(V)$.

⁹ β acts bijectively on a vector space V if $V^\beta = \{0\}$.

Hence **b.**, **e.** and **f.** say that the $R(G)$ -modules $\mathbf{K}_G^*(\mathbf{T}_G^*V)$ and $\mathbf{K}_G^*(\mathbf{T}_G^*V^{gen})$ are respectively isomorphic to $\mathcal{F}_G(V)$ and $\mathrm{DM}_G(V)$.

Note that, when $\dim V/V^\mathfrak{g} = 0$, we have $\mathbf{T}_G^*V = \mathbf{T}_G^*V^{gen} = \mathbf{T}^*V$ and all the points are direct consequences of the Bott isomorphism. Point **d.** is proved in [1], and points **a.**, **e.** and **f.** are due to de Concini-Procesi-Vergne [9, 10]. Point **b.** is proved in [1] for the circle group, and in [9, 10] for the general case. In [9, 10], **c.** is obtained as a consequence of **d.** together with the decomposition formula (31).

We will give a proof by induction on $\dim V/V^\mathfrak{g}$ that is based on the work of [9, 10]. But here our treatment differs from those of [1, 9, 10], since the proof of all points of Theorem 5.2 follows directly by a careful analysis of the exact sequence

$$0 \longrightarrow \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^*W) \xrightarrow{\mathbf{J}} \mathbf{K}_G^0(\mathbf{T}_G^*V) \xrightarrow{\mathbf{R}} \mathbf{K}_G^0(\mathbf{T}_G^*W) \longrightarrow 0.$$

associated to an invariant decomposition $V = \mathbb{C}_\chi \oplus W$.

5.1. Restriction to a subspace. Suppose that $V \neq V^\mathfrak{g}$. Then V contains a complex representation \mathbb{C}_χ attached to a **surjective** character $\chi : G \rightarrow S^1$. Let $G_\chi = \ker(\chi)$ with Lie algebra \mathfrak{g}_χ . The differential of χ is $i\bar{\chi}$ with $\bar{\chi} \in \mathfrak{g}^*$. Here $\mathbb{C}_\chi \cap V^\mathfrak{g} = \{0\}$ since $\bar{\chi} \neq 0$.

Let us consider an invariant decomposition $V = W \oplus \mathbb{C}_\chi$.

Remark 5.3. We check that $\dim W/W^\mathfrak{g} = \dim V/V^\mathfrak{g} - 1$, and $\dim W/W^{\mathfrak{g}_\chi} \leq \dim V/V^\mathfrak{g} - 1$.

We look at the open subset $j : \mathbf{T}_G^*(W \times \mathbb{C}_\chi \setminus \{0\}) \hookrightarrow \mathbf{T}_G^*V$. Its complement is the closed subset $\mathbf{T}_G^*V|_{W \times \{0\}} \simeq \mathbf{T}_G^*W \times \mathbb{C}_\chi$. We have the six term exact sequence (32)

$$\begin{array}{ccccc} \mathbf{K}_G^0(\mathbf{T}_G^*(W \times \mathbb{C}_\chi \setminus \{0\})) & \xrightarrow{j_*} & \mathbf{K}_G^0(\mathbf{T}_G^*V) & \xrightarrow{r} & \mathbf{K}_G^0(\mathbf{T}_G^*W \times \mathbb{C}_\chi) \\ \delta \uparrow & & & & \delta \downarrow \\ \mathbf{K}_G^1(\mathbf{T}_G^*W \times \mathbb{C}_\chi) & \xleftarrow{r} & \mathbf{K}_G^1(\mathbf{T}_G^*V) & \xleftarrow{j_*} & \mathbf{K}_G^1(\mathbf{T}_G^*(W \times \mathbb{C}_\chi \setminus \{0\})). \end{array}$$

Let $\mathbf{R} : \mathbf{K}_G^*(\mathbf{T}_G^*V) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*W)$ be the composition of the map r with the Bott isomorphism $\mathbf{K}_G^*(\mathbf{T}_G^*W \times \mathbb{C}_\chi) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*W)$. Note that \mathbf{R} depends of the choice of the canonical complex structure on \mathbb{C}_χ .

The open subset $\mathbb{C}_\chi \setminus \{0\}$ with the G -action is isomorphic to $G/G_\chi \times \mathbb{R}$. Hence $\mathbf{T}_G^*(W \times \mathbb{C}_\chi \setminus \{0\}) \simeq \mathbf{T}_G^*(W \times G/G_\chi) \times \mathbf{T}\mathbb{R}$. Since the G -manifold $W \times G/G_\chi$ is isomorphic to $G \times_{G_\chi} W$, we get finally

$$\begin{aligned} \mathbf{K}_G^*(\mathbf{T}_G^*(W \times \mathbb{C}_\chi \setminus \{0\})) &= \mathbf{K}_G^*(\mathbf{T}_G^*(W \times G/G_\chi) \times \mathbf{T}\mathbb{R}) \\ &\simeq \mathbf{K}_G^*(\mathbf{T}_G^*(W \times G/G_\chi)) \\ &\simeq \mathbf{K}_G^*(\mathbf{T}_G^*(G \times_{G_\chi} W)) \\ &\simeq \mathbf{K}_{G_\chi}^*(\mathbf{T}_{G_\chi}^*W). \end{aligned}$$

Let $\mathbf{J} : \mathbf{K}_{G_\chi}^*(\mathbf{T}_{G_\chi}^*W) \rightarrow \mathbf{K}_G^*(\mathbf{T}_G^*V)$ be the composition of the map j_* with the previous isomorphism $\mathbf{K}_{G_\chi}^*(\mathbf{T}_{G_\chi}^*W) \simeq \mathbf{K}_G^*(\mathbf{T}_G^*(W \times \mathbb{C}_\chi \setminus \{0\}))$. The sequence (32)

becomes

$$(33) \quad \begin{array}{ccccc} \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^* W) & \xrightarrow{\mathbf{J}} & \mathbf{K}_G^0(\mathbf{T}_G^* V) & \xrightarrow{\mathbf{R}} & \mathbf{K}_G^0(\mathbf{T}_G^* W) \\ \delta \uparrow & & & & \delta \downarrow \\ \mathbf{K}_G^1(\mathbf{T}_G^* W) & \xleftarrow{\mathbf{R}} & \mathbf{K}_G^1(\mathbf{T}_G^* V) & \xleftarrow{\mathbf{J}} & \mathbf{K}_{G_\chi}^1(\mathbf{T}_{G_\chi}^* W). \end{array}$$

The following description of the morphism \mathbf{J} will be used in the next sections. Let $\beta \in \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_\chi \oplus \mathbb{R}\beta$. Since the action of G_χ is trivial on \mathbb{C}_χ , the product

$$\mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^* W) \times \mathbf{K}_G^0(\mathbf{T}_{\mathbb{R}\beta}^* \mathbb{C}_\chi) \xrightarrow{\odot} \mathbf{K}_G^0(\mathbf{T}_G^* V)$$

is well defined. Let $\sigma_{\frac{\mathbb{C}_\chi}{\partial}} \in \mathbf{K}_G^0(\mathbf{T}_{\mathbb{R}\beta}^* \mathbb{C}_\chi)$ be the Cauchy-Riemann class.

Lemma 5.4. *Let $[\sigma] \in \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^* W)$ be a class that is represented by a G -equivariant, G_χ -transversally elliptic morphism σ . Then the product $\sigma \odot \sigma_{\frac{\mathbb{C}_\chi}{\partial}}$ is G -transversally elliptic and $\mathbf{J}([\sigma]) = [\sigma \odot \sigma_{\frac{\mathbb{C}_\chi}{\partial}}]$ in $\mathbf{K}_G^0(\mathbf{T}_G^* V)$.*

Proof. The character χ defines the inclusion $i : G/G_\chi \rightarrow \mathbb{C}_\chi, g \mapsto \chi(g)$. Let $i_! : \mathbf{K}_G^0(\mathbf{T}_G^*(G/G_\chi \times W)) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^* V)$ be the push-forward morphism.

The manifold $G \times W$ is equipped with two $G \times G_\chi$ -actions: $(g, h) \cdot_1 (x, w) := (gxh^{-1}, h \cdot w)$ and $(g, h) \cdot_2 (x, w) := (gxh^{-1}, g \cdot w)$. The map $\theta(x, w) = (x, x^{-1} \cdot w)$ is an isomorphism between $G \times_2 W$ and $G \times_1 W$. The quotients by G and G_χ give us the maps $\pi_G : G \times_1 W \rightarrow W$, and $\pi_{G_\chi} : G \times_2 W \rightarrow G/G_\chi \times W$.

We have

$$(34) \quad \mathbf{J} = i_! \circ (\pi_{G_\chi}^*)^{-1} \circ \theta^* \circ \pi_G^*$$

where $(\pi_{G_\chi}^*)^{-1} \circ \theta^* \circ \pi_G^* : \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^* W) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^*(G/G_\chi \times W))$ is an isomorphism.

It is an easy matter to check that if the class $[\sigma] \in \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^* W)$ is represented by a **G-equivariant**, G_χ -transversally elliptic morphism σ , then $(\pi_{G_\chi}^*)^{-1} \circ \theta^* \circ \pi_G^*(\sigma) = \sigma \odot [0]$ where $[0] : \mathbb{C} \rightarrow \{0\}$ is the zero symbol on G/G_χ . Finally $\mathbf{J}(\sigma) = i_!(\sigma \odot [0]) = \sigma \odot \sigma_{\frac{\mathbb{C}_\chi}{\partial}}$. \square

Remark 5.5. *In the next sections, we will use the exact sequence (33), when V is replaced by an invariant open subset \mathcal{U}_V . Suppose that there exist invariant open subsets $\mathcal{U}_W^1, \mathcal{U}_W^2 \subset W$ such that $\mathcal{U}_V = \mathcal{U}_W^1 \sqcup \mathcal{U}_W^2 \times \mathbb{C}_\chi \setminus \{0\}$. Then (33) becomes*

$$(35) \quad \begin{array}{ccccc} \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^* \mathcal{U}_W^2) & \xrightarrow{\mathbf{J}} & \mathbf{K}_G^0(\mathbf{T}_G^* \mathcal{U}_V) & \xrightarrow{\mathbf{R}} & \mathbf{K}_G^0(\mathbf{T}_G^* \mathcal{U}_W^1) \\ \delta \uparrow & & & & \delta \downarrow \\ \mathbf{K}_G^1(\mathbf{T}_G^* \mathcal{U}_W^1) & \xleftarrow{\mathbf{R}} & \mathbf{K}_G^1(\mathbf{T}_G^* \mathcal{U}_V) & \xleftarrow{\mathbf{J}} & \mathbf{K}_{G_\chi}^1(\mathbf{T}_{G_\chi}^* \mathcal{U}_W^2). \end{array}$$

For example, if $\mathcal{U}_V = V^{gen}$, we take $\mathcal{U}_W^1 = W \cap V^{gen}$ and $\mathcal{U}_W^2 = W^{gen, G_\chi}$.

5.2. The index map is injective. Let us prove by induction on $n \geq 0$ the following fact

$$(H_n) \quad \text{Index}_V^G : \mathbf{K}_G^0(\mathbf{T}_G^*V) \longrightarrow R^{-\infty}(G) \text{ is one to one if } \dim V/V^\mathfrak{g} \leq n.$$

If $\dim V/V^\mathfrak{g} = 0$, we have $\mathbf{T}_G^*V = \mathbf{T}^*V$ and the index map $\mathbf{K}_G^0(\mathbf{T}^*V) \rightarrow R(G)$ is the inverse of the Bott isomorphism.

Suppose now that (H_n) is true, and consider $G \circ V$ such that $\dim V/V^\mathfrak{g} = n+1$. We start with a decomposition $V = W \oplus \mathbb{C}_\chi$ and the exact sequence (33). The induction map $\text{Ind}_{G_\chi}^G : R^{-\infty}(G_\chi) \rightarrow R^{-\infty}(G)$ is defined by the relation $\text{Ind}_{G_\chi}^G(E) = [L^2(G) \otimes E]^{G_\chi}$. We denote $\wedge^\bullet \overline{\mathbb{C}_\chi} : R^{-\infty}(G) \rightarrow R^{-\infty}(G)$ the product by $1 - \overline{\mathbb{C}_\chi}$.

Proposition 5.6. *The following diagram is commutative*

$$(36) \quad \begin{array}{ccccc} \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^*W) & \xrightarrow{\mathbf{J}} & \mathbf{K}_G^0(\mathbf{T}_G^*V) & \xrightarrow{\mathbf{R}} & \mathbf{K}_G^0(\mathbf{T}_G^*W) \\ \downarrow \text{Index}_W^{G_\chi} & & \downarrow \text{Index}_V^G & & \downarrow \text{Index}_W^G \\ R^{-\infty}(G_\chi) & \xrightarrow{\text{Ind}_{G_\chi}^G} & R^{-\infty}(G) & \xrightarrow{\wedge^\bullet \overline{\mathbb{C}_\chi}} & R^{-\infty}(G). \end{array}$$

Proof. Let $\sigma \in \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^*W)$. We have $\pi_G^*(\sigma) = \sigma \odot [0]$ where $[0] : \mathbb{C} \rightarrow \{0\}$ is the zero symbol on G . Then the product formula says that $\text{Index}_{G \times W^\chi}^{G \times G_\chi}(\pi_G^*(\sigma)) = \text{Index}_W^{G_\chi}(\sigma) \otimes L^2(G)$ and thanks to (34), we see that

$$\begin{aligned} \text{Index}_V^G(\mathbf{J}(\sigma)) &= \text{Index}_{G/G_\chi \times W}^G((\pi_{G_\chi}^*)^{-1} \circ \theta^* \circ \pi_G^*(\sigma)) \\ &= \left[\text{Index}_{G \times W^\chi}^{G \times G_\chi}(\pi_G^*(\sigma)) \right]^{G_\chi} \\ &= \text{Ind}_{G_\chi}^G \left(\text{Index}_W^{G_\chi}(\sigma) \right). \end{aligned}$$

This proved the commutativity of the left part of the diagram, and the commutativity of the right part of the diagram is a particular case of Proposition 2.11. \square

We need now the following result that will be proved in Appendix A

Lemma 5.7. *The sequence*

$$(37) \quad 0 \longrightarrow R^{-\infty}(G_\chi) \xrightarrow{\text{Ind}_{G_\chi}^G} R^{-\infty}(G) \xrightarrow{\wedge^\bullet \overline{\mathbb{C}_\chi}} R^{-\infty}(G)$$

is exact.

Lemma 5.7 tells us in particular that $\text{Ind}_{G_\chi}^G$ is one to one. We can now finish the proof of the induction. In the commutative diagram (36), the maps $\text{Index}_W^G, \text{Index}_W^{G_\chi}$ and $\text{Ind}_{G_\chi}^G$ are one to one. It is an easy matter to deduces that Index_V^G is one to one.

We end up this section with the following statement which is the direct consequence of the injectivity of Index_V^G (see Remark 3.8).

Remark 5.8. *Let $J_k, k = 0, 1$ be two invariants complex structures on V , and let $\text{Thom}_\beta(V, J_k)$ be the corresponding pushed symbols attached to an element β satisfying $V^\beta = \{0\}$. There exists an invertible element $\Phi \in R(G)$ such that*

$$\text{Thom}_\beta(V, J_0) = \Phi \cdot \text{Thom}_\beta(V, J_1)$$

*in $\mathbf{K}_G^0(\mathbf{T}_G^*V)$.*

5.3. Generators of $\mathbf{K}_G^*(\mathbf{T}_G^*V)$. Let V be a real G -module : we equip $V/V^\mathfrak{g}$ with an invariant complex structure. Let $A_G(V) \subset \mathbf{K}_G^0(\mathbf{T}_G^*V)$ be the submodule generated by the family $\text{Bott}(V_\mathbb{C}^\mathfrak{g}) \odot \text{Thom}_\gamma(V/V^\mathfrak{g})$, where γ runs over the element of \mathfrak{g} satisfying $V^\gamma = V^\mathfrak{g}$. Remark 5.8 tells us that $A_G(V)$ is independent of the choice of the complex structure on $V/V^\mathfrak{g}$.

In this section we will prove by induction on $n \geq 0$ the following fact

$$(H_n) \quad \mathbf{K}_G^1(\mathbf{T}_G^*V) = 0 \quad \text{and} \quad \mathbf{K}_G^0(\mathbf{T}_G^*V) = A_G(V) \quad \text{if} \quad \dim V/V^\mathfrak{g} \leq n.$$

If $\dim V/V^\mathfrak{g} = 0$, we have $\mathbf{T}_G^*V = \mathbf{T}^*V$ and assertion (H_0) is a direct consequence of the Bott isomorphism.

Suppose now that (H_n) and is true, and consider $G \odot V$ such that $\dim V/V^\mathfrak{g} = n + 1$. We have a decomposition $V = W \oplus \mathbb{C}_\chi$ with $\bar{\chi} \neq 0$. If we apply¹⁰ (H_n) to $G \odot W$ and $G_\chi \odot W$, we get first that $\mathbf{K}_G^1(\mathbf{T}_G^*W) = 0$ and $\mathbf{K}_{G_\chi}^1(\mathbf{T}_{G_\chi}^*W) = 0$.

The long exact sequence (33) implies then that $\mathbf{K}_G^1(\mathbf{T}_G^*V) = 0$, and induces the short exact sequence

$$(38) \quad 0 \longrightarrow \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^*W) \xrightarrow{\mathbf{J}} \mathbf{K}_G^0(\mathbf{T}_G^*V) \xrightarrow{\mathbf{R}} \mathbf{K}_G^0(\mathbf{T}_G^*W) \longrightarrow 0.$$

The assertion (H_n) gives also $\mathbf{K}_G^0(\mathbf{T}_G^*W) = A_G(W)$ and $\mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^*W) = A_{G_\chi}(W)$. With the help of (38), the equality $\mathbf{K}_G^0(\mathbf{T}_G^*V) = A_G(V)$ will follows from following Lemma.

Lemma 5.9. *We have*

- $\mathbf{J}(A_{G_\chi}(W)) \subset A_G(V)$,
- $A_G(W) \subset \mathbf{R}(A_G(V))$.

Proof. We equip $V/V^\mathfrak{g} = W/W^\mathfrak{g} \oplus \mathbb{C}_\chi$ with the complex structure $J := J_\beta$ where $\langle \bar{\chi}, \beta \rangle > 0$. We will use the decomposition of complex G -vector spaces $W/W^\mathfrak{g} \simeq W/W^{\mathfrak{g}_\times} \oplus W^{\mathfrak{g}_\times}/W^\mathfrak{g}$, and the fact that $V^\mathfrak{g} = W^\mathfrak{g}$.

Let $\alpha := \text{Bott}(W_\mathbb{C}^{\mathfrak{g}_\times}) \odot \text{Thom}_\gamma(W/W^{\mathfrak{g}_\times})$ be a generator of $A_{G_\chi}(W)$. It is a G -equivariant symbol, hence Lemma 5.4 applies: its image by \mathbf{J} is equal to $\mathbf{J}(\alpha) = \text{Bott}(W_\mathbb{C}^{\mathfrak{g}_\times}) \odot \text{Thom}_\gamma(W/W^{\mathfrak{g}_\times}) \odot \sigma_{\bar{\beta}}^{\mathbb{C}_\chi}$. If we use the fact that $\sigma_{\bar{\beta}}^{\mathbb{C}_\chi} = \text{Thom}_{-\beta}(\mathbb{C}_\chi) - \text{Thom}_\beta(\mathbb{C}_\chi)$, we see that $\mathbf{J}(\alpha) = U_- - U_+$ where

$$\begin{aligned} U_\pm &= \text{Bott}(W_\mathbb{C}^{\mathfrak{g}_\times}) \odot \text{Thom}_\gamma(W/W^{\mathfrak{g}_\times}) \odot \text{Thom}_{\pm\beta}(\mathbb{C}_\chi) \\ &= \text{Bott}(W_\mathbb{C}^{\mathfrak{g}_\times}) \odot \text{Thom}_{\gamma_\pm}(W/W^{\mathfrak{g}_\times} \oplus \mathbb{C}_\chi) \quad [1] \\ &= \text{Bott}(V_\mathbb{C}^\mathfrak{g}) \odot \text{Bott}((W^{\mathfrak{g}_\times}/W^\mathfrak{g})_\mathbb{C}) \odot \text{Thom}_{\gamma_\pm}(W/W^{\mathfrak{g}_\times} \oplus \mathbb{C}_\chi) \quad [2] \\ &= \wedge^\bullet \overline{W^{\mathfrak{g}_\times}/W^\mathfrak{g}} \otimes \text{Bott}(V_\mathbb{C}^\mathfrak{g}) \odot \text{Thom}_{\gamma_\pm}(V/V^\mathfrak{g}) \quad [3]. \end{aligned}$$

In [1], the term γ_\pm is equal to $\gamma \pm t\beta$ with $0 < t < 1$ (see Lemma 3.12). In [2], we use that $W^{\mathfrak{g}_\times} \simeq V^\mathfrak{g} \oplus W^{\mathfrak{g}_\times}/W^\mathfrak{g}$. In [3] we use that $V/V^\mathfrak{g} = W/W^{\mathfrak{g}_\times} \oplus W^{\mathfrak{g}_\times}/W^\mathfrak{g} \oplus \mathbb{C}_\chi$ (see Proposition 3.9).

We have proved that $\mathbf{J}(\alpha)$ belongs to $A_G(V)$ for any generator α of $A_{G_\chi}(W)$. We get the first point, since the restriction $R(G) \rightarrow R(G_\chi)$ is surjective.

Let $\alpha' := \text{Bott}(W_\mathbb{C}^\mathfrak{g}) \odot \text{Thom}_\gamma(W/W^\mathfrak{g})$ be a generator of $A_G(W)$. Thanks to Proposition 3.11, we see that $\alpha' = \mathbf{R}(\alpha'')$ with $\alpha'' = \text{Bott}(V_\mathbb{C}^\mathfrak{g}) \odot \text{Thom}_\gamma(V/V^\mathfrak{g})$. The second point is then proved. \square

¹⁰See Remark 5.3.

5.4. Generators of $\mathbf{K}_G^*(\mathbf{T}_G^*V^{gen})$. Let $B_G(V)$ be the submodule of $\mathbf{K}_G^0(\mathbf{T}_G^*V^{gen})$ generated by the family $\text{Bott}(V_{\mathbb{C}}^{\mathfrak{g}}) \odot \sigma_{\frac{V/V^{\mathfrak{g}}}{\partial}}^{\varphi}$ where φ runs over the (G, V) -flag.

In this section we will prove by induction on $n \geq 0$ the following fact

$$(H'_n) \quad \mathbf{K}_G^1(\mathbf{T}_G^*V^{gen}) = 0 \quad \text{and} \quad \mathbf{K}_G^0(\mathbf{T}_G^*V^{gen}) = B_G(V) \quad \text{if} \quad \dim V/V^{\mathfrak{g}} \leq n.$$

If $\dim V/V^{\mathfrak{g}} = 0$, we have $\mathbf{T}_G^*V^{gen} = \mathbf{T}^*V$ and (H'_0) is a direct consequence of the Bott isomorphism. Suppose now that (H'_n) is true, and consider $G \curvearrowright V$ such that $\dim V/V^{\mathfrak{g}} = n + 1$. We have an invariant decomposition $V = W \oplus \mathbb{C}_{\chi}$, with $\bar{\chi} \neq 0$, and

$$V^{gen} = V^{gen} \cap W \sqcup W^{gen, G_{\chi}} \times \mathbb{C}_{\chi} \setminus \{0\}.$$

Note that $V^{gen} \cap W$ is either equal to W^{gen} (if the G -orbits in V and W have the same maximal dimension) or is empty. Following Remark 5.5, we have the exact sequence

$$(39) \quad \begin{array}{ccccc} \mathbf{K}_{G_{\chi}}^0(\mathbf{T}_{G_{\chi}}^*W^{gen, G_{\chi}}) & \xrightarrow{\mathbf{J}} & \mathbf{K}_G^0(\mathbf{T}_G^*V^{gen}) & \xrightarrow{\mathbf{R}} & \mathbf{K}_G^0(\mathbf{T}_G^*W^{gen}) \\ \delta \uparrow & & & & \delta \downarrow \\ \mathbf{K}_G^1(\mathbf{T}_G^*W^{gen}) & \xleftarrow{\mathbf{R}} & \mathbf{K}_G^1(\mathbf{T}_G^*V^{gen}) & \xleftarrow{\mathbf{J}} & \mathbf{K}_{G_{\chi}}^1(\mathbf{T}_{G_{\chi}}^*W^{gen, G_{\chi}}) \end{array}$$

when $V^{gen} \cap W \neq \emptyset$. On the other hand, when $V^{gen} \cap W = \emptyset$, we have an isomorphism

$$(40) \quad \mathbf{J} : \mathbf{K}_{G_{\chi}}^*(\mathbf{T}_{G_{\chi}}^*W^{gen, G_{\chi}}) \longrightarrow \mathbf{K}_G^*(\mathbf{T}_G^*V^{gen}).$$

If we apply (H'_n) to $G \curvearrowright W$ and $G_{\chi} \curvearrowright W$, we get first $\mathbf{K}_G^1(\mathbf{T}_G^*W^{gen}) = 0$ and $\mathbf{K}_{G_{\chi}}^1(\mathbf{T}_{G_{\chi}}^*W^{gen, G_{\chi}}) = 0$. Using the bottom of the diagram (39) and the isomorphism (40), we get $\mathbf{K}_G^1(\mathbf{T}_G^*V) = 0$. Moreover, the long exact sequence (39) induces the short exact sequence

$$0 \longrightarrow \mathbf{K}_{G_{\chi}}^0(\mathbf{T}_{G_{\chi}}^*W^{gen, G_{\chi}}) \xrightarrow{\mathbf{J}} \mathbf{K}_G^0(\mathbf{T}_G^*V^{gen}) \xrightarrow{\mathbf{R}} \mathbf{K}_G^0(\mathbf{T}_G^*(V^{gen} \cap W)) \longrightarrow 0.$$

Since the assertion (H'_n) gives also

$$\mathbf{K}_G^0(\mathbf{T}_G^*W^{gen}) = B_G(W) \quad \text{and} \quad \mathbf{K}_{G_{\chi}}^0(\mathbf{T}_{G_{\chi}}^*W^{gen, G_{\chi}}) = B_{G_{\chi}}(W),$$

the equality $\mathbf{K}_G^0(\mathbf{T}_G^*V^{gen}) = A_G(V)$ will follow from following Lemma.

Lemma 5.10. *We have*

- $\mathbf{J}(B_{G_{\chi}}(W)) \subset B_G(V)$,
- $B_G(W) \subset \mathbf{R}(B_G(V))$, when $V^{gen} \cap W \neq \emptyset$.

Proof. Let $\beta \in \mathfrak{g}$ such that $\langle \bar{\chi}, \beta \rangle > 0$: we have $\mathfrak{g} = \mathfrak{g}_{\chi} \oplus \mathbb{R}\beta$. For any (W, G_{χ}) -flag φ , we consider the element

$$\alpha := \text{Bott}(W_{\mathbb{C}}^{\mathfrak{g}_{\chi}}) \odot \sigma_{\frac{W/W^{\mathfrak{g}_{\chi}}}{\partial}}^{\varphi} \in \mathbf{K}_{G_{\chi}}^0(\mathbf{T}_{G_{\chi}}^*W^{gen})$$

and we want to compute its image by \mathbf{J} .

We note that the minimal stabilizer $H_{min} \subset G$ for the G -action on V is equal to the minimal stabilizer for the G_{χ} -action on W . Let $s := \dim G_{\chi} - \dim H_{min}$. A (G_{χ}, W) -flag φ corresponds to

- a decomposition $W/W^{\mathfrak{g}_{\chi}} = W_1^{\varphi} \oplus \cdots \oplus W_s^{\varphi}$ in complex G -subspaces

• a decomposition $\mathfrak{g}_\chi = \mathfrak{h}_{min} \oplus \mathbb{R}\beta_1^\varphi \oplus \cdots \oplus \mathbb{R}\beta_s^\varphi$
such that for any $1 \leq k \leq s$, β_k^φ acts trivially on $W_1^\varphi \oplus \cdots \oplus W_{k-1}^\varphi$ and β_k^φ acts bijectively on W_k^φ . The term $\sigma_{\bar{\partial}}^{W/W^{\mathfrak{g}_\chi}, \varphi} \in \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^*(W/W^{\mathfrak{g}_\chi})^{gen})$ is equal to the product of $\sigma_{\bar{\partial}}^{\varphi, k} \in \mathbf{K}_{G_\chi}^0(\mathbf{T}_{\mathbb{R}\beta_k^\varphi}^* W_k^\varphi)$, for $1 \leq k \leq s$.

Since $V/V^{\mathfrak{g}} \simeq W^{\mathfrak{g}_\chi}/W^{\mathfrak{g}} \oplus \mathbb{C}_\chi \oplus W/W^{\mathfrak{g}_\chi}$, we can define a (V, G) -flag ψ as follows:

- $V_1^\psi := W^{\mathfrak{g}_\chi}/W^{\mathfrak{g}} \oplus \mathbb{C}_\chi$ and $\beta_1^\psi = \beta$,
- $V_k^\psi := W_{k-1}^\varphi$ and $\beta_k^\psi := \beta_{k-1}^\varphi$ for $2 \leq k \leq s+1$.

We note that the G_χ -transversally symbols $\sigma_{\bar{\partial}}^{\varphi, k} \in \mathbf{K}_{G_\chi}^0(\mathbf{T}_{\mathbb{R}\beta_k^\varphi}^* W_k^\varphi)$ correspond to the restriction of the G -transversally symbols $\sigma_{\bar{\partial}}^{\psi, k+1} \in \mathbf{K}_G^0(\mathbf{T}_{\mathbb{R}\beta_{k+1}^\psi}^* V_{k+1}^\psi)$.

Finally, thanks to Lemma 5.4 we have

$$\begin{aligned} \mathbf{J}(\alpha) &= \sigma_{\bar{\partial}}^{\mathbb{C}_\chi} \odot \text{Bott}(V_{\mathbb{C}}^{\mathfrak{g}}) \odot \text{Bott}((W^{\mathfrak{g}_\chi}/W^{\mathfrak{g}})_{\mathbb{C}}) \odot \sigma_{\bar{\partial}}^{\psi, 2} \odot \cdots \odot \sigma_{\bar{\partial}}^{\psi, s} \\ &= \wedge^\bullet \overline{W^{\mathfrak{g}_\chi}/W^{\mathfrak{g}}} \otimes \text{Bott}(V_{\mathbb{C}}^{\mathfrak{g}}) \odot \sigma_{\bar{\partial}}^{\psi, 1} \odot \sigma_{\bar{\partial}}^{\psi, 2} \odot \cdots \odot \sigma_{\bar{\partial}}^{\psi, s} \\ &= \wedge^\bullet \overline{W^{\mathfrak{g}_\chi}/W^{\mathfrak{g}}} \otimes \text{Bott}(V_{\mathbb{C}}^{\mathfrak{g}}) \odot \sigma_{\bar{\partial}}^{V/V^{\mathfrak{g}}, \psi}. \end{aligned}$$

Here we use the identity $\sigma_{\bar{\partial}}^{\mathbb{C}_\chi} \odot \text{Bott}(W^{\mathfrak{g}_\chi}/W_{\mathbb{C}}^{\mathfrak{g}}) = \wedge^\bullet \overline{W^{\mathfrak{g}_\chi}/W^{\mathfrak{g}}} \otimes \sigma_{\bar{\partial}}^{\psi, 1}$, valid in $\mathbf{K}_G^0(\mathbf{T}_{\mathbb{R}\beta}^*(\mathbb{C}_\chi \oplus W^{\mathfrak{g}_\chi}/W^{\mathfrak{g}}))$, which is proved in Proposition 3.16. Since $R(G) \rightarrow R(G_\chi)$ is onto, we have proved that $\mathbf{J}(A_{G_\chi}(W)) \subset A_G(V)$.

Suppose now that $V^{gen} \cap W \neq \emptyset$, and let us prove now that $A_G(W) \subset \mathbf{R}(A_G(V))$. Let φ be a (G, W) -flag : let $W/W^{\mathfrak{g}} = W_1^\varphi \oplus \cdots \oplus W_s^\varphi$ and $\mathfrak{g} = \mathfrak{h}_{min} \oplus \mathbb{R}\beta_1^\varphi \oplus \cdots \oplus \mathbb{R}\beta_s^\varphi$ be the corresponding decompositions. The hypothesis $V^{gen} \cap W \neq \emptyset$ means that the minimal stabilizer \mathfrak{h}_{min} for the \mathfrak{g} -action in W is contained in \mathfrak{g}_χ . Hence $\bar{\chi}$ does not belongs to $(\mathbb{R}\beta_1^\varphi \oplus \cdots \oplus \mathbb{R}\beta_s^\varphi)^\perp$. Let

$$k = \max\{i \mid \langle \bar{\chi}, \beta_i^\varphi \rangle \neq 0\}.$$

Let ψ be the (G, V) -flag defined as follows:

- $V_i^\psi = W_i^\varphi$ is $i \neq k$, and $V_k^\psi = W_k^\varphi \oplus \mathbb{C}_\chi$,
- $\beta_s^\varphi = \beta_s^\varphi$ for $1 \leq k \leq s$.

Then we have

$$\begin{aligned} \mathbf{R}(\text{Bott}(V_{\mathbb{C}}^{\mathfrak{g}}) \odot \sigma_{\bar{\partial}}^{V/V^{\mathfrak{g}}, \psi}) &= \text{Bott}(W_{\mathbb{C}}^{\mathfrak{g}}) \odot \sigma_{\bar{\partial}}^{\varphi, 1} \odot \cdots \odot \mathbf{R}(\sigma_{\bar{\partial}}^{\psi, k}) \cdots \odot \sigma_{\bar{\partial}}^{\varphi, s} \\ &= \text{Bott}(W_{\mathbb{C}}^{\mathfrak{g}}) \odot \sigma_{\bar{\partial}}^{\varphi, 1} \odot \cdots \odot \sigma_{\bar{\partial}}^{\varphi, k} \cdots \odot \sigma_{\bar{\partial}}^{\varphi, s} \\ &= \text{Bott}(W_{\mathbb{C}}^{\mathfrak{g}}) \odot \sigma_{\bar{\partial}}^{W/W^{\mathfrak{g}}, \varphi} \end{aligned}$$

We use here the relation $\mathbf{R}(\sigma_{\bar{\partial}}^{\psi, k}) = \sigma_{\bar{\partial}}^{\varphi, k}$ (see Proposition 3.11). It proves that $A_G(W) \subset \mathbf{R}(A_G(V))$. \square

5.5. $\mathbf{K}_G^0(\mathbf{T}_G^*V)$ is isomorphic to $\mathcal{F}_G(V)$. For any G -module V , we denote $\mathcal{F}_G(V)'$ the image of $\mathbf{K}_G^0(\mathbf{T}_G^*V)$ by Index_V^G . We know from Section 5.2 that the index map Index_V^G is injective, hence $\mathcal{F}_G(V)' \simeq \mathbf{K}_G^0(\mathbf{T}_G^*V)$. Let $\mathcal{F}_G(V)$ be the generalized Dahmen-Michelli submodule defined in the introduction. We start with the following

Lemma 5.11. *We have $\mathcal{F}_G(V)' \subset \mathcal{F}_G(V)$.*

Proof. Let $\sigma \in \mathbf{K}_G^0(\mathbf{T}_G^*V)$ and let $\mathfrak{h} \in \Delta_G(V)$. Since the vector space $V/V^\mathfrak{h}$ carries an invariant complex structure we have a restriction morphism $\mathbf{R}_\mathfrak{h} : \mathbf{K}_G^0(\mathbf{T}_G^*V) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^*V^\mathfrak{h})$. Let $i_! : \mathbf{K}_G^0(\mathbf{T}_G^*V^\mathfrak{h}) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^*V)$ be the push-forward morphism associated to the inclusion $V^\mathfrak{h} \hookrightarrow V$. Thanks to Proposition 2.11 we know that $i_! \circ \mathbf{R}_\mathfrak{h}(\sigma) = \sigma \otimes \wedge^\bullet \overline{V/V^\mathfrak{h}}$, and then

$$(41) \quad \wedge^\bullet \overline{V/V^\mathfrak{h}} \otimes \text{Index}_V^G(\sigma) = \text{Index}_{V^\mathfrak{h}}^G(\mathbf{R}_\mathfrak{h}(\sigma)).$$

But since the action of H is trivial on $V^\mathfrak{h}$, we know that $\text{Index}_{V^\mathfrak{h}}^G(\mathbf{R}_\mathfrak{h}(\sigma)) \in \langle R^{-\infty}(G/H) \rangle$ (see Remark 2.8). The inclusion $\mathcal{F}_G(V)' \subset \mathcal{F}_G(V)$ is proved. \square

We will now prove by induction on $n \geq 0$ the following fact

$$(H_n'') \quad \mathcal{F}_G(V)' = \mathcal{F}_G(V) \quad \text{if} \quad \dim V/V^\mathfrak{g} \leq n.$$

If $\dim V/V^\mathfrak{g} = 0$, we have $\mathbf{T}_G^*V = \mathbf{T}^*V$ and $\Delta_G(V) = \{\mathfrak{g}\}$. In this situation, $\mathfrak{h}_{\min} = \mathfrak{g}$ and $\langle R^{-\infty}(G/H_{\min}) \rangle = R(G)$. We have then $\mathcal{F}_G(V) = R(G)$, and (H_0'') is a direct consequence of the Bott isomorphism.

Suppose now that (H_n'') and is true, and consider $G \circ V$ such that $\dim V/V^\mathfrak{g} = n + 1$. We have a decomposition $V = W \oplus \mathbb{C}_\chi$ with $\bar{\chi} \neq 0$. If we apply (H_n'') to $G \circ W$ and $G_\chi \circ W$, we get $\mathcal{F}_G(W)' = \mathcal{F}_G(W)$ and $\mathcal{F}_{G_\chi}(W)' = \mathcal{F}_{G_\chi}(W)$. The following Lemma will be the key point of our induction.

Lemma 5.12. • *Let $H \subset G_\chi$ be a closed subgroup (G is abelian). For any $\Phi \in R^{-\infty}(G_\chi)$, we have the equivalences*

$$(42) \quad \Phi \in \langle R^{-\infty}(G_\chi/H) \rangle \iff \text{Ind}_{G_\chi}^G(\Phi) \in \langle R^{-\infty}(G/H) \rangle,$$

$$(43) \quad \Phi \in \mathcal{F}_{G_\chi}(W) \iff \text{Ind}_{G_\chi}^G(\Phi) \in \mathcal{F}_G(V).$$

• *The exact sequence (37) specializes in the exact sequence*

$$(44) \quad 0 \longrightarrow \mathcal{F}_{G_\chi}(W) \xrightarrow{\text{Ind}_{G_\chi}^G} \mathcal{F}_G(V) \xrightarrow{\wedge^\bullet \overline{\mathbb{C}_\chi}} \mathcal{F}_G(W).$$

Proof. Let us consider the first point. For $\Phi := \sum_{\mu \in \widehat{G_\chi}} m(\mu) \mathbb{C}_\mu \in R^{-\infty}(G_\chi)$, we have $\text{Ind}_{G_\chi}^G(\Phi) = \sum_{\varphi \in \widehat{G}} m(\pi_{G_\chi}(\varphi)) \mathbb{C}_\varphi$, where $\pi_{G_\chi} : \widehat{G} \rightarrow \widehat{G_\chi}$. We see then that $\text{Supp}(\text{Ind}_{G_\chi}^G(\Phi)) = \pi_{G_\chi}^{-1}(\text{Supp}(\Phi))$. If $\pi_H : \widehat{G} \rightarrow \widehat{H}$ and $\pi'_H : \widehat{G_\chi} \rightarrow \widehat{H}$ denote the projections, we have then the following relation

$$\pi_H \left(\text{Supp} \left(\text{Ind}_{G_\chi}^G(\Phi) \right) \right) = \pi'_H(\text{Supp}(\Phi))$$

that induces (42).

For any $\Phi \in R^{-\infty}(G_\chi)$ and any subspace $\mathfrak{h} \in \Delta_G(V)$, we consider the expression $\Omega := \wedge^\bullet \overline{V/V^\mathfrak{h}} \otimes \text{Ind}_{G_\chi}^G(\Phi)$. We have two cases:

- Either $\mathfrak{h} \not\subset \mathfrak{g}_\chi$: here $\mathbb{C}_\chi \subset V/V^\mathfrak{h}$ and $\wedge^\bullet \overline{V/V^\mathfrak{h}} = \wedge^\bullet \overline{\mathbb{C}_\chi} \otimes \delta$. In this case, $\Omega = 0$ because $\wedge^\bullet \overline{\mathbb{C}_\chi} \circ \text{Ind}_{G_\chi}^G = 0$.
- Or $\mathfrak{h} \subset \mathfrak{g}_\chi$: here $\mathfrak{h} \in \Delta_{G_\chi}(W)$ and $V/V^\mathfrak{h} = W/W^\mathfrak{h}$. In this case $\Omega = \text{Ind}_{G_\chi}^G(\wedge^\bullet \overline{W/W^\mathfrak{h}} \otimes \Phi)$.

It is then immediate that the equivalence (43) follows from (42).

Thanks to (43) it is an easy matter to check that the sequence (44) is exact at $\mathcal{F}_G(V)$. We leave to the reader the checking that $\wedge^\bullet \overline{\mathbb{C}_\chi} \cdot \mathcal{F}_G(V) \subset \mathcal{F}_G(W)$. So the second point is proved. \square

Let $I_{\mathcal{F}} : \mathcal{F}_G(V)' \hookrightarrow \mathcal{F}_G(V)$ be the inclusion. Finally, we have the following commutative diagram, where all the horizontal sequences are exact :

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{K}_{G_\chi}^0(\mathbf{T}_{G_\chi}^* W) & \xrightarrow{\mathbf{J}} & \mathbf{K}_G^0(\mathbf{T}_G^* V) & \xrightarrow{\mathbf{R}} & \mathbf{K}_G^0(\mathbf{T}_G^* W) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}_{G_\chi}(W)' & \xrightarrow{\text{Ind}_{G_\chi}^G} & \mathcal{F}_G(V)' & \xrightarrow{\wedge^\bullet \overline{\mathbb{C}_\chi}} & \mathcal{F}_G(W)' \longrightarrow 0 \\
& & \downarrow & & \downarrow I_{\mathcal{F}} & & \downarrow \\
& & \mathcal{F}_{G_\chi}(W) & \xrightarrow{\text{Ind}_{G_\chi}^G} & \mathcal{F}_G(V) & \xrightarrow{\wedge^\bullet \overline{\mathbb{C}_\chi}} & \mathcal{F}_G(W).
\end{array}$$

Except for $I_{\mathcal{F}}$, we know that all the vertical arrows are isomorphism. It is an easy exercise to check that $I_{\mathcal{F}}$ must be an isomorphism.

5.6. $\mathbf{K}_G^0(\mathbf{T}_G^* V^{gen})$ is isomorphic to $\text{DM}_G(V)$. For any G -module V , we denote $\text{DM}_G(V)'$ the image of $\mathbf{K}_G^0(\mathbf{T}_G^* V^{gen})$ by Index_V^G . Since the maps $j_* : \mathbf{K}_G^0(\mathbf{T}_G^* V^{gen}) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^* V)$ and Index_V^G are injective (see Remark 4.3 and Section 5.2), we have

$$\text{DM}_G(V)' \simeq \mathbf{K}_G^0(\mathbf{T}_G^* V^{gen}).$$

for any G -module. Let $\text{DM}_G(V)$ be the generalized Dahmen-Michelli submodules defined in the introduction. We start with the following

Lemma 5.13. *We have $\text{DM}_G(V)' \subset \text{DM}_G(V)$.*

Proof. Let $\tau \in \mathbf{K}_G^0(\mathbf{T}_G^* V^{gen})$ and $j_*(\tau) \in \mathbf{K}_G^0(\mathbf{T}_G^* V)$. First we remark that $\text{Index}_V^G(\tau) \in \langle R^{-\infty}(G/H_{min}) \rangle$ since H_{min} acts trivially on V (see Remark 2.8).

Let $\mathfrak{h} \neq \mathfrak{h}_{min}$ be a stabilizer in $\Delta_G(V)$. Since $V^{\mathfrak{h}} \cap V^{gen} = \emptyset$ the composition $\mathbf{R}_{\mathfrak{h}} \circ j_*$ is the zero map, and (41) gives in this case that $\wedge^\bullet \overline{V/V^{\mathfrak{h}}} \otimes \text{Index}_V^G(j_*(\tau)) = 0$. Since by definition $\text{Index}_V^G(\tau) = \text{Index}_V^G(j_*(\tau))$, the inclusion $\text{DM}_G(V)' \subset \text{DM}_G(V)$ is proved. \square

We will now prove by induction on $n \geq 0$ the following fact

$$(H_n''') \quad \text{DM}_G(V)' = \text{DM}_G(V) \quad \text{if} \quad \dim V/V^{\mathfrak{g}} \leq n.$$

If $\dim V/V^{\mathfrak{g}} = 0$, we have $\mathbf{T}_G^* V = \mathbf{T}^* V$, $V^{gen} = V$ and $\Delta_G(V) = \{\mathfrak{g}\}$. In this situation, $\mathfrak{h}_{min} = \mathfrak{g}$ and $\langle R^{-\infty}(G/H_{min}) \rangle = R(G)$. We have then $\text{DM}_G(V) = R(G)$, and assertion (H_0''') is a direct consequence of the Bott isomorphism.

Suppose now that (H_n''') is true, and consider $G \circ V$ such that $\dim V/V^{\mathfrak{g}} = n + 1$. We have a decomposition $V = W \oplus \mathbb{C}_\chi$ with $\bar{\chi} \neq 0$. If we apply (H_n''') to $G \circ W$ and $G_\chi \circ W$, we get $\text{DM}_G(W)' = \text{DM}_G(W)$ and $\text{DM}_{G_\chi}(W)' = \text{DM}_{G_\chi}(W)$.

It works like in the previous section, apart for the dichotomy concerning $V^{gen} \cap W$. We have the following

Lemma 5.14. • *Let $H \subset G_\chi$ be a closed subgroup (G is abelian). For any $\Phi \in R^{-\infty}(G_\chi)$, we have the equivalences*

$$(45) \quad \Phi \in \mathrm{DM}_{G_\chi}(W) \iff \mathrm{Ind}_{G_\chi}^G(\Phi) \in \mathrm{DM}_G(V).$$

• *If $V^{gen} \cap W \neq \emptyset$, the exact sequence (37) specializes in the exact sequence*

$$(46) \quad 0 \longrightarrow \mathrm{DM}_{G_\chi}(W) \xrightarrow{\mathrm{Ind}_{G_\chi}^G} \mathrm{DM}_G(V) \xrightarrow{\wedge^\bullet \overline{\mathbb{C}_\chi}} \mathrm{DM}_G(W).$$

• *If $V^{gen} \cap W = \emptyset$, the exact sequence (37) induces the isomorphism*

$$(47) \quad \mathrm{Ind}_{G_\chi}^G : \mathrm{DM}_{G_\chi}(W) \xrightarrow{\sim} \mathrm{DM}_G(V).$$

Proof. Let $\Phi \in R^{-\infty}(G_\chi)$ and $\mathfrak{h} \in \Delta_G(V)$. We consider the term $\Omega := \wedge^\bullet \overline{V/V^\mathfrak{h}} \otimes \mathrm{Ind}_{G_\chi}^G(\Phi)$. Like in the proof of lemma 5.12, we have two cases:

- Either $\mathfrak{h} \not\subseteq \mathfrak{g}_\chi$: in this case $\Omega = 0$.
- Or $\mathfrak{h} \subset \mathfrak{g}_\chi$: here $\mathfrak{h} \in \Delta_{G_\chi}(W)$ and $V/V^\mathfrak{h} = W/W^\mathfrak{h}$. In this case $\Omega = \mathrm{Ind}_{G_\chi}^G(\eta)$ with $\eta = \wedge^\bullet \overline{W/W^\mathfrak{h}} \otimes \Phi$.

Since the minimal stabilizer¹¹ for the G_χ action on W coincides with the minimal stabilizer for the G action on V , the relation (42) induces the equivalence $\Phi \in \langle R^{-\infty}(G_\chi/H_{min}) \rangle \iff \mathrm{Ind}_{G_\chi}^G(\Phi) \in \langle R^{-\infty}(G/H_{min}) \rangle$. For the stabilizers $\mathfrak{h}_{min} \subsetneq \mathfrak{h} \subset \mathfrak{g}_\chi$, using the fact that $\mathrm{Ind}_{G_\chi}^G$ is injective, we see that $\wedge^\bullet \overline{V/V^\mathfrak{h}} \otimes \mathrm{Ind}_{G_\chi}^G(\Phi) = 0$ if and only if $\wedge^\bullet \overline{W/W^\mathfrak{h}} \otimes \Phi = 0$. The first point follows.

Thanks to (45) it is an easy matter to check that the sequence (37) specializes in the exact sequence $0 \rightarrow \mathrm{DM}_{G_\chi}(W) \xrightarrow{\alpha} \mathrm{DM}_G(V) \xrightarrow{\beta} R^{-\infty}(G)$, where $\alpha = \mathrm{Ind}_{G_\chi}^G$ and $\beta = \wedge^\bullet \overline{\mathbb{C}_\chi}$. We can precise this sequence as follows.

Let $\mathfrak{h}_{min}(W), \mathfrak{h}_{min}(V)$ be respectively the minimal infinitesimal stabilizer for the G -action on W and V . We note that $V^{gen} \cap W \neq \emptyset \iff \mathfrak{h}_{min}(W) \subset \mathfrak{g}_\chi \iff \mathfrak{h}_{min}(W) = \mathfrak{h}_{min}(V)$.

Suppose that $V^{gen} \cap W \neq \emptyset$, and let us check that the image of β is contained in $\mathrm{DM}_G(W)$. Take $\Phi \in \mathrm{DM}_G(V)$ and $\mathfrak{h} \in \Delta_G(W)$. Let $\beta(\Phi) = \wedge^\bullet \overline{\mathbb{C}_\chi} \otimes \Phi$. We have to consider three cases :

- (1) If $\mathfrak{h} = \mathfrak{h}_{min}(W)$, then Φ and $\beta(\Phi)$ belong to $\langle R^{-\infty}(G/H_{min}(W)) \rangle = \langle R^{-\infty}(G/H_{min}(V)) \rangle$.
- (2) If $\mathfrak{h}_{min}(W) \subsetneq \mathfrak{h} \subset \mathfrak{g}_\chi$, then $\wedge^\bullet \overline{V/V^\mathfrak{h}} = \wedge^\bullet \overline{W/W^\mathfrak{h}}$ and $\wedge^\bullet \overline{W/W^\mathfrak{h}} \otimes \beta(\Phi) = \beta(\wedge^\bullet \overline{V/V^\mathfrak{h}} \otimes \Phi) = 0$.
- (3) If $\mathfrak{h} \not\subseteq \mathfrak{g}_\chi$, then $V/V^\mathfrak{h} = W/W^\mathfrak{h} \oplus \mathbb{C}_\chi$. We get then $\wedge^\bullet \overline{W/W^\mathfrak{h}} \otimes \beta(\Phi) = \wedge^\bullet \overline{V/V^\mathfrak{h}} \otimes \Phi = 0$.

We have proved that $\beta(\Phi) \in \mathrm{DM}_G(W)$.

Suppose now that $V^{gen} \cap W = \emptyset$, and let us check that β is the zero map. Let $\mathfrak{h} := \mathfrak{h}_{min}(W) \in \Delta_G(V)$. We have $V/V^\mathfrak{h} = \mathbb{C}_\chi$ since $\mathfrak{h} \not\subseteq \mathfrak{g}_\chi$, and by definition we have $\beta(\Phi) = \wedge^\bullet \overline{\mathbb{C}_\chi} \otimes \Phi = \wedge^\bullet \overline{V/V^\mathfrak{h}} \otimes \Phi = 0$ for any $\Phi \in \mathrm{DM}_G(V)$. \square

¹¹Says \mathfrak{h}_{min} with corresponding group $H_{min} = \exp(\mathfrak{h}_{min})$.

Let $I_{\text{DM}} : \text{DM}_G(V)' \hookrightarrow \text{DM}_G(V)$ be the inclusion. If $V^{\text{gen}} \cap W \neq \emptyset$, we have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{K}_{G_x}^0(\mathbf{T}_{G_x}^* W^{\text{gen}, G_x}) & \xrightarrow{\mathbf{J}} & \mathbf{K}_G^0(\mathbf{T}_G^* V^{\text{gen}}) & \xrightarrow{\mathbf{R}} & \mathbf{K}_G^0(\mathbf{T}_G^* W^{\text{gen}}) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{DM}_{G_x}(W)' & \xrightarrow{\text{Ind}_{G_x}^G} & \text{DM}_G(V)' & \xrightarrow{\wedge \bullet \overline{\mathbb{C}_x}} & \text{DM}_G(W)' \longrightarrow 0 \\
& & \downarrow & & \downarrow I_{\text{DM}} & & \downarrow \\
0 & \longrightarrow & \text{DM}_{G_x}(W) & \xrightarrow{\text{Ind}_{G_x}^G} & \text{DM}_G(V) & \xrightarrow{\wedge \bullet \overline{\mathbb{C}_x}} & \text{DM}_G(W),
\end{array}$$

and if $V^{\text{gen}} \cap W = \emptyset$, we have the other commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{K}_{G_x}^0(\mathbf{T}_{G_x}^* W^{\text{gen}, G_x}) & \xrightarrow{\mathbf{J}} & \mathbf{K}_G^0(\mathbf{T}_G^* V^{\text{gen}}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{DM}_{G_x}(W)' & \xrightarrow{\text{Ind}_{G_x}^G} & \text{DM}_G(V)' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow I_{\text{DM}} & & \\
0 & \longrightarrow & \text{DM}_{G_x}(W) & \xrightarrow{\text{Ind}_{G_x}^G} & \text{DM}_G(V) & \longrightarrow & 0.
\end{array}$$

In both diagrams, all the horizontal sequences are exact, and except for I_{DM} , we know that all the vertical arrows are isomorphisms. It is an easy exercise to check that in both cases I_{DM} must be an isomorphism.

5.7. Decomposition of $\mathbf{K}_G^0(\mathbf{T}_G^* V) \simeq \text{DM}_G(V)$. Let V be a real G -module such that $V^{\mathfrak{g}} = \{0\}$. Let J be an invariant complex structure on V . Let $\mathcal{W} \subset \widehat{G}$ be the set of weights: $\chi \in \mathcal{W}$ if $V_\chi := \{v \in V \mid g \cdot v = \chi(g)v\} \neq \{0\}$. The differential of η is denoted $i\bar{\eta}$ with $\bar{\eta} \in \mathfrak{g}^*$. Let $\overline{\mathcal{W}} := \{\bar{\eta} \mid \eta \in \mathcal{W}\}$: it is the set of infinitesimal weights for the action of \mathfrak{g} on V .

Let $\Delta_G(V)$ be the finite set formed by the infinitesimal stabilizer of points in V . For a subspace $\mathfrak{h} \subset \mathfrak{g}$, we see that $\mathfrak{h} \in \Delta_G(V)$ if and only if $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is generated by $\mathfrak{h}^\perp \cap \overline{\mathcal{W}}$.

Any vector $v \in V$ decomposes as $v = \sum_\eta v_\eta$ with $v_\eta \in V_\eta$. The subalgebra \mathfrak{g}_v that stabilizes v is equal to $\cap_{v_\eta \neq 0} \ker(\bar{\eta}) = (\sum_{v_\eta \neq 0} \mathbb{R}\bar{\eta})^\perp$. For a subspace $\mathfrak{h} \subset \Delta_G(V)$, we see that the subspace $V^\mathfrak{h} := \{v \mid \mathfrak{h} \subset \mathfrak{g}_v\}$ is equal to $\oplus_{\bar{\eta} \in \mathfrak{h}^\perp} V_\eta$ and $V_\mathfrak{h} := \{v \mid \mathfrak{h} = \mathfrak{g}_v\}$ is the subspace $(V^\mathfrak{h})^{\text{gen}}$ formed by the vectors $v := \sum_{\bar{\eta} \in \mathfrak{h}^\perp} v_\eta$ such that $\sum_{v_\eta \neq 0} \mathbb{R}\bar{\eta} = \mathfrak{h}^\perp$.

We have $V/V^\mathfrak{h} \simeq \sum_{\bar{\eta} \notin \mathfrak{h}^\perp} V_\eta$. Following Section 4, we consider a collection $\gamma := \{\gamma_\mathfrak{h} \in \mathfrak{h}, \mathfrak{h} \in \Delta_G(V)\}$ such that $(V/V^\mathfrak{h})^{\gamma_\mathfrak{h}} = \{0\}$. We look at the H -transversally elliptic symbol $\text{Thom}_{\gamma_\mathfrak{h}}(V/V^\mathfrak{h})$ on $V/V^\mathfrak{h}$. Since the action of H is trivial on $V^\mathfrak{h}$, the following map

$$\begin{array}{ccc}
\mathbf{K}_G^0(\mathbf{T}_G^*(V^\mathfrak{h})^{\text{gen}}) & \longrightarrow & \mathbf{K}_G^0(\mathbf{T}_G^*((V^\mathfrak{h})^{\text{gen}} \times V/V^\mathfrak{h})) \\
\sigma & \longrightarrow & \sigma \odot \text{Thom}_{\gamma_\mathfrak{h}}(V/V^\mathfrak{h})
\end{array}$$

is well defined. We can compose the previous map with the push-forward morphism $\mathbf{K}_G^0(\mathbf{T}_G^*((V^\mathfrak{h})^{\text{gen}} \times V/V^\mathfrak{h})) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^* V)$: let us denote $\mathbf{S}_\gamma^\mathfrak{h}$ the resulting map.

We can now state Theorem 4.4 in our linear setting.

Theorem 5.15. *The map*

$$\mathbf{S}_\gamma := \bigoplus_{\mathfrak{h} \in \Delta_G(V)} \mathbf{S}_\gamma^{\mathfrak{h}} : \bigoplus_{\mathfrak{h} \in \Delta_G(V)} \mathbf{K}_G^0(\mathbf{T}_G^*(V^{\mathfrak{h}})^{gen}) \longrightarrow \mathbf{K}_G^0(\mathbf{T}_G^*V)$$

is an isomorphism of $R(G)$ -modules.

Now we can translate the previous decomposition through the index map. For $\mathfrak{h} \in \Delta_G(V)$, we consider the element $[\wedge^\bullet \overline{V/V^{\mathfrak{h}}}]_{\gamma_{\mathfrak{h}}}^{-1} \in R^{-\infty}(G)$ which is equal to the G -index of $\text{Thom}_{\gamma_{\mathfrak{h}}}(V/V^{\mathfrak{h}})$ (see Definition 3.10 and Proposition 3.6).

We need first the following

Lemma 5.16. *The product by $[\wedge^\bullet \overline{V/V^{\mathfrak{h}}}]_{\gamma_{\mathfrak{h}}}^{-1}$ defines a map from $\text{DM}_G(V^{\mathfrak{h}})$ into $\mathcal{F}_G(V)$.*

Proof. Since the symbol $\text{Thom}_{\gamma_{\mathfrak{h}}}(V/V^{\mathfrak{h}})$ is H -transversally elliptic, the projection $\pi_{\mathfrak{h}} : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ is proper when restricted to the infinitesimal support $\overline{\text{Supp}(\Omega)}$ of $\Omega := [\wedge^\bullet \overline{V/V^{\mathfrak{h}}}]_{\gamma_{\mathfrak{h}}}^{-1}$. Let $\Phi \in \langle R^{-\infty}(G/H) \rangle$: the image of $\overline{\text{Supp}(\Phi)}$ by $\pi_{\mathfrak{h}}$ is finite. It is now easy to check that for any $\chi \in \widehat{G}$ the set $\{(\chi_1, \chi_2) \in \text{Supp}(\Omega) \times \text{Supp}(\Phi) \mid \chi_1 + \chi_2 = \chi\}$ is finite: the product $[\wedge^\bullet \overline{V/V^{\mathfrak{h}}}]_{\gamma_{\mathfrak{h}}}^{-1} \otimes \Phi$ is well-defined.

Let $\Phi \in \langle R^{-\infty}(G/H) \rangle$. For any $\mathfrak{a} \in \Delta_G(V)$ we have the ‘mother’ formula¹²

$$(48) \quad \wedge^\bullet \overline{V/V^{\mathfrak{a}}} \otimes [\wedge^\bullet \overline{V/V^{\mathfrak{h}}}]_{\gamma_{\mathfrak{h}}}^{-1} \otimes \Phi = \wedge^\bullet \overline{V^{\mathfrak{h}}/V^{\mathfrak{h}+\mathfrak{a}}} \otimes [\wedge^\bullet \overline{V^{\mathfrak{a}}/V^{\mathfrak{h}+\mathfrak{a}}}]_{\gamma_{\mathfrak{h}}}^{-1} \otimes \Phi$$

which is due to the isomorphisms $V/V^{\mathfrak{a}} \simeq V/(V^{\mathfrak{h}} + V^{\mathfrak{a}}) \oplus V^{\mathfrak{h}}/V^{\mathfrak{h}+\mathfrak{a}}$, $V/V^{\mathfrak{h}} \simeq V/(V^{\mathfrak{h}} + V^{\mathfrak{a}}) \oplus V^{\mathfrak{a}}/V^{\mathfrak{h}+\mathfrak{a}}$, and the relation

$$\wedge^\bullet W \otimes [\wedge^\bullet W]_{\gamma}^{-1} = 1$$

that holds for any G -module such that $W^\gamma = \{0\}$.

Note that for any $\mathfrak{a}, \mathfrak{h} \in \Delta_G(V)$ we have the equivalence $V^{\mathfrak{h}+\mathfrak{a}} = V^{\mathfrak{h}} \iff \mathfrak{a} \subset \mathfrak{h}$. Suppose now that $\Phi \in \text{DM}_G(V^{\mathfrak{h}})$ and consider the product $\Omega := [\wedge^\bullet \overline{V/V^{\mathfrak{h}}}]_{\gamma_{\mathfrak{h}}}^{-1} \otimes \Phi \in R^{-\infty}(G)$. If $\mathfrak{a} \subset \mathfrak{h}$, we have

$$\wedge^\bullet \overline{V/V^{\mathfrak{a}}} \otimes \Omega = [\wedge^\bullet \overline{V^{\mathfrak{a}}/V^{\mathfrak{h}}}]_{\gamma_{\mathfrak{h}}}^{-1} \otimes \Phi \in \langle R^{-\infty}(G/A) \rangle$$

since $[\wedge^\bullet \overline{V^{\mathfrak{a}}/V^{\mathfrak{h}}}]_{\gamma_{\mathfrak{h}}}^{-1} \in \langle R^{-\infty}(G/A) \rangle$ and $\Phi \in \langle R^{-\infty}(G/H) \rangle \subset \langle R^{-\infty}(G/A) \rangle$. In the other hand, if $\mathfrak{a} \not\subset \mathfrak{h}$, we have $\wedge^\bullet \overline{V/V^{\mathfrak{a}}} \otimes \Omega = 0$ since $\wedge^\bullet \overline{V^{\mathfrak{h}}/V^{\mathfrak{h}+\mathfrak{a}}} \otimes \Phi = 0$.

We have proved that $\Omega = [\wedge^\bullet \overline{V/V^{\mathfrak{h}}}]_{\gamma_{\mathfrak{h}}}^{-1} \otimes \Phi$ belongs to $\mathcal{F}_G(V)$. \square

The map

$$\mathcal{S}_\gamma : \bigoplus_{\mathfrak{h} \in \Delta_G(V)} \text{DM}_G(V^{\mathfrak{h}}) \longrightarrow \mathcal{F}_G(V)$$

¹²See formula (2) in [9].

defined by $\mathcal{S}_\gamma(\oplus_{\mathfrak{h}} \Phi_{\mathfrak{h}}) := \sum_{\mathfrak{h} \in \Delta_G(V)} [\wedge^{\bullet} \overline{V/V^{\mathfrak{h}}}]_{\gamma_{\mathfrak{h}}}^{-1} \otimes \Phi_{\mathfrak{h}}$ satisfies the following commutative diagram

$$\begin{array}{ccc} \oplus_{\mathfrak{h}} \mathbf{K}_G^0(\mathbf{T}_G^*(V^{\mathfrak{h}})^{gen}) & \xrightarrow{\mathbf{S}_\gamma} & \mathbf{K}_G^0(\mathbf{T}_G^*V) \\ \downarrow \oplus_{\mathfrak{h}} \text{Index}_{V^{\mathfrak{h}}}^G & & \downarrow \text{Index}_V^G \\ \oplus_{\mathfrak{h}} \text{DM}_G(V^{\mathfrak{h}}) & \xrightarrow{\mathcal{S}_\gamma} & \mathcal{F}_G(V). \end{array}$$

Since \mathbf{S}_γ and the index maps $\text{Index}_{V^{\mathfrak{h}}}^G, \text{Index}_V^G$ are isomorphisms we recover the following theorem of de Concini-Procesi-Vergne [9].

Theorem 5.17. *The map \mathcal{S}_γ is an isomorphism of $R(G)$ -modules.*

6. APPENDIX

6.1. Appendix A. Let G be a compact abelian Lie group, and let $\chi : G \rightarrow U(1)$ be a surjective morphism. We want to prove that the sequence

$$(49) \quad 0 \longrightarrow R^{-\infty}(G_\chi) \xrightarrow{\text{Ind}_{G_\chi}^G} R^{-\infty}(G) \xrightarrow{\wedge^{\bullet} \overline{\mathbb{C}_\chi}} R^{-\infty}(G)$$

is exact. Note that the induction map $\text{Ind}_{G_\chi}^G : R^{-\infty}(G_\chi) \rightarrow R^{-\infty}(G)$ is the dual of the restriction morphism $R(G) \rightarrow R(G_\chi)$. Hence the injectivity of $\text{Ind}_{G_\chi}^G$ will follow from the classical

Lemma 6.1. *Let H be a closed subgroup of a compact abelian Lie group G . The restriction $R(G) \rightarrow R(H)$ is onto.*

Proof. Let θ be a character of H . For any L^1 -function $\phi : G \rightarrow \mathbb{C}$, we consider the average $\tilde{\phi}(g) = \int_H \phi(gh)\theta(h)^{-1}dh$: we have then

$$(50) \quad \tilde{\phi}(gh) = \tilde{\phi}(g)\theta(h) \quad \text{for any } (g, h) \in G \times H.$$

Let us choose ϕ such that $\tilde{\phi} \neq 0$. For any character $\chi : G \rightarrow \mathbb{C}$, we consider the function

$$\tilde{\phi}_\chi(t) := \int_G \tilde{\phi}(tg)\chi(g)^{-1}dg.$$

We have $\tilde{\phi}_\chi = (\tilde{\phi}, \chi)\chi$ where $(\tilde{\phi}, \chi) = \int_G \tilde{\phi}(g)\chi(g)^{-1}dg \in \mathbb{C}$. It is immediate that (50) gives that $\tilde{\phi}_\chi(h) = (\tilde{\phi}, \chi)\theta(h)$ for $h \in H$. Hence the restriction of χ to H is equal to θ when $(\tilde{\phi}, \chi) \neq 0$. By a density argument, we know that such χ exists. \square

Now we want to prove that $\text{Image}(\text{Ind}_{G_\chi}^G) = \ker(\wedge^{\bullet} \overline{\mathbb{C}_\chi})$. The inclusion $\text{Image}(\text{Ind}_{G_\chi}^G) \subset \ker(\wedge^{\bullet} \overline{\mathbb{C}_\chi})$ comes from the fact that $\wedge^{\bullet} \overline{\mathbb{C}_\chi} = 0$ in $R(G_\chi)$.

For the other inclusion, we consider $\Phi := \sum_{\mu \in \widehat{G}} m(\mu)\mathbb{C}_\mu \in \ker(\wedge^{\bullet} \overline{\mathbb{C}_\chi})$. We have the relation $\Phi \otimes \mathbb{C}_\chi = \Phi$, which means that $m(\mu + \chi) = m(\mu)$ for all $\mu \in \widehat{G}$. Let $\pi : \widehat{G} \rightarrow \widehat{G}_\chi$ be the restriction morphism. Thanks to Lemma 6.1, we know that π is surjective, and we see that for $\theta \in \widehat{G}_\chi$, $\pi^{-1}(\theta)$ is of the form $\{k\chi + \theta', k \in \mathbb{Z}\}$.

For $\theta \in \widehat{G}_\chi$, we denote $n(\theta) \in \mathbb{Z}$ the integer $m(\mu)$ for $\mu \in \pi^{-1}(\theta)$. We have then

$$\begin{aligned} \Phi &= \sum_{\mu \in \widehat{G}} m(\mu) \mathbb{C}_\mu = \sum_{\theta \in \widehat{G}_\chi} \sum_{\mu \in \pi^{-1}(\theta)} m(\mu) \mathbb{C}_\mu = \sum_{\theta \in \widehat{G}_\chi} n_\theta \sum_{k \in \mathbb{Z}} \mathbb{C}_{k\chi + \theta'} \\ &= \text{Ind}_{G_\chi}^G \left(\sum_{\theta \in \widehat{G}_\chi} n_\theta \mathbb{C}_\theta \right). \end{aligned}$$

6.2. Appendix B. This section is devoted to the proof of Proposition 3.15. Let V be equipped with the complex structure $J := J_\beta$. The class $\text{Thom}_{\pm\beta}(V) \in \mathbf{K}_G^0(\mathbf{T}_G^*V)$ are represented by the symbols $\text{Cl}(\xi \pm \beta(x)) : \wedge^+ V \rightarrow \wedge^- V$. Since $-\text{Thom}_{\pm\beta}(V)$ is represented by $-\text{Cl}(\xi \pm \beta(x)) : \wedge^- V \rightarrow \wedge^+ V$, the class $\text{Thom}_{-\beta}(V) - \text{Thom}_\beta(V)$ is represented by the symbol

$$\tau(x, \xi) : \wedge^\bullet V \rightarrow \wedge^\bullet V$$

defined by $\tau(x, \xi) = \text{Cl}(\xi) \circ \epsilon - \text{Cl}(\beta(x))$, where $\epsilon(w) = (-1)^{|w|}w$. We consider the family $\tau_s(x, \xi) = (s\text{Id} + \text{Cl}(\xi)) \circ \epsilon - \text{Cl}(\beta^s(x))$, $s \in [0, 1]$, where $\beta^s = sJ + (1-s)\beta$. Note that β^s is invertible for any $s \in [0, 1]$.

Lemma 6.2. *The family $\tau_s, s \in [0, 1]$ is an homotopy of transversally elliptic symbols.*

Thanks to the last lemma, we know that $\tau = \tau_1$ in $\mathbf{K}_G^0(\mathbf{T}_G^*V)$. Since $\text{Support}(\tau_1) \cap \mathbf{T}_G^*V \subset \mathbf{T}_G^*(V \setminus \{0\})$, the restriction $\tau' := \tau_1|_{V \setminus \{0\}}$ is a G -transversally elliptic symbol on $V \setminus \{0\}$, and the excision property tells us that $j_!(\tau') = \tau_1 = \tau$ in $\mathbf{K}_G^0(\mathbf{T}_G^*V)$.

For $(x, \xi) \in \mathbf{T}^*(V \setminus \{0\})$, the map $\tau'(x, \xi) : \wedge^\bullet V \rightarrow \wedge^\bullet V$ is given by

$$\tau'(x, \xi) = (\text{Id} + \text{Cl}(\xi)) \circ \epsilon - \text{Cl}(Jx).$$

Let S be the sphere of radius one of V . We work with the isomorphism $S \times \mathbb{R} \simeq V \setminus \{0\}$, $(y, t) \mapsto e^t y$. Let $\underline{\mathbb{C}} = S \times \mathbb{R} \times \mathbb{C}$ be the trivial complex vector bundle. Let $\mathcal{H} \rightarrow S \times \mathbb{R}$ be the vector bundle defined by $\mathcal{H}_{(y,t)} := (\mathbb{C}y)^\perp \subset \mathbf{T}_y S$. We use the isomorphism of vector bundle

$$\phi : \mathcal{H} \oplus \underline{\mathbb{C}} \longrightarrow \mathbf{T}(S \times \mathbb{R})$$

defined by $\phi_{(y,t)}(\xi' \oplus a + ib) = (\xi' + bJ(y), a) \in \mathbf{T}_y S \times \mathbf{T}_t \mathbb{R}$. Through ϕ the bundle map $\text{Cl}(\xi) : \wedge^+ V \rightarrow \wedge^- V$ for $\xi \in \mathbf{T}_x V$ becomes

$$\text{Cl}_{(y,t)}(\xi' \oplus z) : (\wedge \mathcal{H}_y \otimes \wedge \mathbb{C})^+ \rightarrow (\wedge \mathcal{H}_y \otimes \wedge \mathbb{C})^-$$

Through ϕ , the vector field $x \mapsto J_\epsilon x$ becomes the section of $\underline{\mathbb{C}}$ given by $(y, t) \mapsto e^t i$, and the morphism τ' is defined as follows: for $(y, t) \in S \times \mathbb{R}$, and $\xi' \oplus z \in \mathcal{H}_y \oplus \mathbb{C}$, the map $\tau'_{(y,t)}(\xi' \oplus z) : \wedge \mathcal{H}_y \otimes \wedge \mathbb{C} \rightarrow \wedge \mathcal{H}_y \otimes \wedge \mathbb{C}$ is defined by

$$\tau'_{(y,t)}(\xi' \oplus z) = (\text{Id} + \text{Cl}(\xi' \oplus z)) \circ \epsilon - e^t \text{Cl}(i).$$

Let $A_{z, \xi'} = \text{Cl}(\xi' \oplus z)$ and $B = \text{Cl}(i)$ be the maps from $(\wedge \mathcal{H}_y \otimes \wedge \underline{\mathbb{C}})^+$ into $(\wedge \mathcal{H}_y \otimes \wedge \underline{\mathbb{C}})^-$. The matrix of $\tau'_{(y,t)}(\xi' \oplus z)$ relatively to the grading of $\wedge \mathcal{H}_y \otimes \wedge \mathbb{C}$ is

$$\begin{pmatrix} \text{Id} & A_{z, \xi'}^* + e^t B^* \\ A_{z, \xi'} - e^t B & -\text{Id} \end{pmatrix}.$$

Let us consider the deformation of τ' in a family

$$\sigma_s := \begin{pmatrix} \text{Id} & sA_{z,\xi'}^* + e^t B^* \\ A_{z,\xi'} - e^t B & -\text{Id} \end{pmatrix}, \quad s \in [0, 1].$$

Lemma 6.3. *The family $\sigma_s, s \in [0, 1]$ is an homotopy of transversally elliptic symbols.*

The symbol $\sigma_0 := \begin{pmatrix} \text{Id} & e^t B^* \\ A_{z,\xi'} - e^t B & -\text{Id} \end{pmatrix}$ is clearly homotopic to

$$\begin{aligned} \sigma_2 &:= \begin{pmatrix} \text{Id} & 0 \\ e^{-t} B & \text{Id} \end{pmatrix} \sigma_0 \begin{pmatrix} \text{Id} & 0 \\ -e^{-t} B & \text{Id} \end{pmatrix} \\ &= \begin{pmatrix} 0 & e^t B^* \\ A_{z,\xi'} - (e^t - e^{-t})B & 0 \end{pmatrix} \end{aligned}$$

Since the morphism $e^t B : (\wedge \mathcal{H} \otimes \wedge \underline{\mathbb{C}})^- \rightarrow (\wedge \mathcal{H} \otimes \wedge \underline{\mathbb{C}})^+$ is always invertible, its class vanishes. Hence we have

$$[\tau'] = [\sigma_0] = [\sigma_2] = [A_{z,\xi'} - (e^t - e^{-t})B] \quad \text{in} \quad \mathbf{K}_G^0(\mathbf{T}_G^*(S \times \mathbb{R})).$$

We are now working with the morphism $\sigma_3 : (\wedge \mathcal{H} \otimes \wedge \underline{\mathbb{C}})^+ \rightarrow (\wedge \mathcal{H} \otimes \wedge \underline{\mathbb{C}})^-$ defined by

$$\sigma_3(\xi' \oplus z) = \text{Cl}(\xi') + \text{Cl}(z - (e^t - e^{-t})i).$$

Since $\frac{e^t - e^{-t}}{t} > 0$ on \mathbb{R} , we can deform the term $z - (e^t - e^{-t})i$ in $t + i\text{Re}(z)$ without changing the intersection of the support with $\mathbf{T}_G^*(S \times \mathbb{R})$.

Finally we have proved that $\text{Thom}_{-\beta}(V) - \text{Thom}_{\beta}(V)$ is represented on $S \times \mathbb{R}$ by the morphism $\text{Cl}(\xi') + \text{Cl}(t + i\text{Re}(z)) : (\wedge \mathcal{H} \otimes \wedge \underline{\mathbb{C}})^+ \rightarrow (\wedge \mathcal{H} \otimes \wedge \underline{\mathbb{C}})^-$ which is by definition equal to $\sigma_{\frac{V}{\partial}}^V \odot \text{Bott}(\mathbf{T}\mathbb{R}) = i_!(\sigma_{\frac{V}{\partial}}^V)$.

We finish this section with the proofs of the deformation Lemmas. For the family $\tau_s(x, \xi) = (s\text{Id} + \text{Cl}(\xi)) \circ \epsilon - \text{Cl}(\beta^s(x))$, we have

$$(\tau_s(x, \xi))^* \tau_s(x, \xi) = \begin{pmatrix} s^2 + \|\xi - \beta^s(x)\|^2 & -2s\text{Cl}(\beta^s(x)) \\ 2s\text{Cl}(\beta^s(x)) & s^2 + \|\xi + \beta^s(x)\|^2 \end{pmatrix}$$

Then $\det(\tau_s(x, \xi)) = 0$ if and only if

$$(s^2 + \|\xi - \beta^s(x)\|^2)(s^2 + \|\xi + \beta^s(x)\|^2) = 4s^2\|\beta^s(x)\|^2$$

which is equivalent to the equality $(s^2 + \|\xi\|^2 + \|\beta^s(x)\|^2)^2 = 4s^2\|\beta^s(x)\|^2 + 4(\xi, \beta^s(x))^2$. If $\xi \notin \mathbb{R}\beta^s(x)$, we have $(\xi, \beta^s(x))^2 < \|\xi\|^2\|\beta^s(x)\|^2$, and then

$$(s^2 + \|\xi\|^2 + \|\beta^s(x)\|^2)^2 < 4(s^2 + \|\xi\|^2)\|\beta^s(x)\|^2$$

which gives $(s^2 + \|\xi\|^2 - \|\beta^s(x)\|^2)^2 < 0$ which is contradictory. Then $\det(\tau_s(x, \xi)) = 0$ if and only if $\xi \in \mathbb{R}\beta^s(x)$ and $s^2 + \|\xi\|^2 - \|\beta^s(x)\|^2 = 0$. If furthermore $\xi \in \mathbf{T}_G^*V|_x$, then¹³ $\xi = 0$. We have proved that $\text{Support}(\tau_s) \cap \mathbf{T}_G^*V$ is equal to the compact set $\{(x, \xi) \mid \xi = 0 \text{ and } s^2 - \|\beta^s(x)\|^2 = 0\}$. So $s \in [0, 1] \rightarrow \tau_s$ is an homotopy of transversally elliptic symbols.

¹³It is due to the fact that $(\beta^s(x), \beta(x)) > 0$ when $x \neq 0$.

For the family¹⁴ $\sigma_s := \begin{pmatrix} \text{Id} & sA^* + e^t B^* \\ A - e^t B & -\text{Id} \end{pmatrix}$, we have

$$(\sigma_s)^* \sigma_s = \begin{pmatrix} \rho^+ \text{Id} & (1-s)A^* + 2e^t B^* \\ (1-s)A + 2e^t B & \rho^-(s) \text{Id} \end{pmatrix}$$

with $\rho^+ = 1 + \|\xi'\|^2 + \|z - e^t i\|^2$ and $\rho^-(s) = 1 + \|s\xi'\|^2 + \|sz + e^t i\|^2$. We check easily that $((1-s)A^* + 2e^t B^*)((1-s)A + 2e^t B) = \rho(s)\text{Id}$ with

$$\rho(s) = \|(s-1)\xi'\|^2 + \|(s-1)z + 2e^t i\|^2.$$

Finally $\det(\sigma_s) = 0$ if and only if $\rho(s) = \rho^-(s)\rho^+$. In other words, $(y, t; \xi' \oplus z)$ belongs to the support of σ_s if and only if

$$\|(s-1)\xi'\|^2 + \|(s-1)z + 2e^t i\|^2 = (1 + \|s\xi'\|^2 + \|sz + e^t i\|^2)(1 + \|\xi'\|^2 + \|z - e^t i\|^2).$$

Let us suppose now that $\xi' \oplus z \in \mathbf{T}_G^*(S \times \mathbb{R})$. It imposes $\text{Im}(z) = 0$, and the last relation becomes $(s-1)^2\Theta + 4e^{2t} = (1 + e^{2t} + s^2\Theta)(1 + e^{2t} + \Theta)$ with $\Theta = \|\xi'\|^2 + \|z\|^2$. It is easy to see that the last relation holds if and only if $t = \Theta = 0$. Finally we have proved that

$$\text{Support}(\sigma_s) \cap \mathbf{T}_G^*(S \times \mathbb{R}) = \{(y, t; \xi' \oplus z) \mid t = 0, \xi' = 0, z = 0\},$$

and then $\sigma_s, s \in [0, 1]$ defines an homotopy of transversally elliptic symbols.

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¹⁴We write A for $A_{z, \xi'}$.

INSTITUT DE MATHÉMATIQUES ET DE MODÉLISATION DE MONTPELLIER (I3M), UNIVERSITÉ
MONTPELLIER 2

E-mail address: `Paul-Emile.Paradan@math.univ-montp2.fr`