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# Locally equilibrated superconvergent patch recovery for efficient error estimation in quantities of interest

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## Abstract

During the last decade there has been an increase on the use of goal-oriented error estimates which help to quantify and control the local error on a quantity of interest (QoI) that might result relevant for design purposes (e.g. the mean stress or mean displacement in a particular area, the stress intensity factor for fracture problems,...). Residual-based error estimators have been used to estimate the error in quantities of interest for finite element approximations. This work presents a recovery-based error estimation technique for QoI whose main characteristic is the use of an enhanced version of the Superconvergent Patch Recovery (SPR) technique developed by Zienkiewicz and Zhu. This enhanced version of the SPR technique, used to recover the primal and dual solutions, provides a nearly statically admissible stress field that results in accurate estimations of the error in the QoI.

KEY WORDS: goal oriented; error estimation; recovery; quantities of interest; error control; mesh adaptivity

## 1 INTRODUCTION

The verification and quality assessment of numerical simulations is a critical area of research. Techniques to control the error in the numerical models and validate

the approximate solutions have been subject of concern for many years and their further development is expected to have a profound impact on the reliability and utility of simulation methods in the future. Effective methods for assessing the global discretization error in the the energy norm in finite element analysis have been extensively developed since the late 70s and can be classified into different families [1, 2]: residual based error estimators, recovery based error estimators, dual techniques, *etc.* Residual based error estimators originally introduced by Babuška and Rheinboldt [3] are characterised by a strong mathematical basis. To estimate the error, they consider local residuals of the numerical solution. By investigating the residuals occurring in a patch of elements or even in a single element it is possible to estimate the errors which arise locally. These methods were improved with the introduction of the *residual equilibration* by Ainsworth and Oden [1]. Recovery based error estimates were first introduced by Zienkiewicz and Zhu [4] and are often preferred by practitioners because they are robust, simple to use and provide an enhanced solution. Further improvements were made with the introduction of new recovery procedures such as the superconvergent patch recovery (SPR) proposed by the same authors [5, 6] and many papers following [7, 8, 9, 10, 11, 12, 13]. Dual techniques are based on the evaluation of two different fields, one compatible in displacements and another equilibrated in stresses [14].

Most of the techniques used to obtain error estimates prior to the mid 90s were aimed to evaluate the global error in the energy norm. Nevertheless, the goal of many numerical simulations is the determination of a particular quantity of interest (QoI) which is needed for taking decisions during the design process. Therefore, it is important to guarantee the quality of such analyses by controlling the error of the approximation in terms of the QoI rather than the global energy norm. Significant advances in the late 90s introduced a new approach focused on the evaluation of error estimates of local quantities [1, 15, 16]. In order to obtain an error estimate for the QoI two different problems are solved: the primal problem which is the problem under consideration, and the dual or adjoint problem which is related to the QoI. Goal oriented error estimators have been usually developed from the basis of residual formulations and the widely used strategy of solving the dual problem [1, 17]. Although limited, the use of recovery techniques to evaluate error in quantities of interest can be found in [18, 19], and considering dual analysis in [20].

In this paper we propose a technique to evaluate accurate error estimates for linear QoIs adapting the technique presented in [12, 21, 13, 22] which requires the analytical expressions of the body loads and boundary tractions also for the dual problem. In Section 2 we define the problem and the finite element approximation used. Section 3 focus on the representation of the error in the QoIs and the solution of the dual problem. Section 4 deals with the definitions of the dual problem and describes the expressions of body loads and boundary tractions required for the stress recovery of the dual problem for different linear QoI. In Section 5 we introduce the nearly equilibrated recovery technique used to obtain enhanced

stress fields for both the primal and dual solutions. In Section 6 we present some numerical results and, finally, conclusions are drawn in Section 7.

## 2 PROBLEM STATEMENT AND SOLUTION

In this section, we briefly present the model for the 2D linear elastic problem. The unknown displacement field  $\mathbf{u}$ , taking values in  $\Omega \subset \mathbb{R}^2$ , is the solution of the boundary value problem given by

$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{b} \quad \text{in } \Omega \quad (1)$$

$$\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{t} \quad \text{on } \Gamma_N \quad (2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad (3)$$

where  $\Gamma_N$  and  $\Gamma_D$  denote the Neumann and Dirichlet boundaries with  $\partial\Omega = \Gamma_N \cup \Gamma_D$  and  $\Gamma_N \cap \Gamma_D = \emptyset$ ,  $\mathbf{b}$  are body loads and  $\mathbf{t}$  are the tractions imposed along  $\Gamma_N$ . The Dirichlet boundary condition in (3) is taken homogeneous for the sake of simplicity.

The variational form of the problem reads

$$\text{Find } \mathbf{u} \in V : \forall \mathbf{v} \in V \quad a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \quad (4)$$

where  $V$  is the standard test space for the elasticity problem such that  $V = \{\mathbf{v} \mid \mathbf{v} \in H^1(\Omega), \mathbf{v}|_{\Gamma_D}(\mathbf{x}) = \mathbf{0}\}$ . The symmetric and bilinear form  $a : V \times V \rightarrow \mathbb{R}$  and the continuous linear form  $l : V \rightarrow \mathbb{R}$  are defined in vectorial form by

$$a(\mathbf{u}, \mathbf{v}) := \int_{\Omega} \boldsymbol{\sigma}^T(\mathbf{u}) \boldsymbol{\varepsilon}(\mathbf{v}) d\Omega = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u})^T \mathbf{D}^{-1} \boldsymbol{\sigma}(\mathbf{v}) d\Omega \quad (5)$$

$$l(\mathbf{v}) := \int_{\Omega} \mathbf{b}^T \mathbf{v} d\Omega + \int_{\Gamma_N} \mathbf{t}^T \mathbf{v} d\Gamma, \quad (6)$$

where  $\boldsymbol{\sigma}$  and  $\boldsymbol{\varepsilon}$  denote the stresses and strains, and  $\mathbf{D}$  is the elasticity matrix of the constitutive relation  $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ .

Let  $\mathbf{u}^h$  be a finite element approximation of  $\mathbf{u}$ . The solution for the discrete counterpart of the variational problem in (4) lies in a subspace  $V^h \subset V$  associated with a mesh of isoparametric finite elements of characteristic size  $h$ , and it is such that

$$\forall \mathbf{v} \in V^h \quad a(\mathbf{u}^h, \mathbf{v}) = l(\mathbf{v}) \quad (7)$$

## 3 ERROR CONTROL

### 3.1 Error representation

In this section we define a general framework for goal oriented error analysis. First, we assume that the finite element discretization error is given by  $\mathbf{e} = \mathbf{u} - \mathbf{u}^h$ . To

quantify the quality of  $\mathbf{u}^h$  in terms of  $\mathbf{e}$  different methods have been proposed, generally based on the energy norm induced by  $\sqrt{a(\cdot, \cdot)}$  and denoted by  $\|\cdot\|$ . That is, the quantity to be assessed is  $\|\mathbf{e}\| = \sqrt{a(\mathbf{e}, \mathbf{e})}$ . Following Zienkiewicz-Zhu [4], an estimate of the error  $\|\mathbf{e}_{es}\|$  can be formulated in the context of FE elasticity problems by introducing the approximation

$$\|\mathbf{e}\|^2 \approx \|\mathbf{e}_{es}\|^2 = \int_{\Omega} (\boldsymbol{\sigma}^* - \boldsymbol{\sigma}^h)^T \mathbf{D}^{-1} (\boldsymbol{\sigma}^* - \boldsymbol{\sigma}^h) d\Omega, \quad (8)$$

where  $\Omega$  represents the domain of the problem,  $\boldsymbol{\sigma}^h$  is the stress field provided by the FE solution and  $\boldsymbol{\sigma}^*$  is the recovered stress field, which is a better approximation to the exact solution than  $\boldsymbol{\sigma}^h$ . Similarly, local element contributions can be evaluated from (8) considering the domain of the element  $\Omega_e$ .

The error estimate measured in the energy norm as given in (8) can overestimate or underestimate the exact error. Thus, it is common to use error bounding techniques that provide upper or lower bounds of the error. Recent works by Díez *et al.* [21] and Ródenas *et al.* [22] have presented a methodology to obtain practical upper bounds of the error in the energy norm  $\|\mathbf{e}\|$  using a SPR-based approach where equilibrium was locally imposed on each patch. The recovered stresses obtained with this technique were proven to provide very accurate estimations of the error in the energy norm.

### 3.2 Error in quantities of interest

In this section we show how error estimators measured in the energy norm may be utilised to estimate the error in a particular quantity of interest [1]. The strategy consists in solving a primal problem, which is the problem at hand, and a dual problem useful to extract information on the quantity of interest (identical to the primal problem except for the applied boundary conditions). From the FE solutions of both problems it is possible to estimate the contribution of each of the elements to the error in the QoI. This error measure allows to adapt the mesh using procedures similar to traditional techniques based on the estimated error in the energy norm.

Consider the linear elasticity problem given in (4) and its approximate FE solution  $\mathbf{u}^h \in V^h \subset V$ . This problem is related to the original problem to be solved, that henceforth will be called the *primal problem*.

Let us define  $Q : V \rightarrow \mathbb{R}$  as a bounded linear functional representing some quantity of interest, acting on the space  $V$  of admissible functions for the problem at hand. The goal is to estimate the error in  $Q(\mathbf{u})$  when calculated using the value of the approximate solution  $\mathbf{u}^h$ :

$$Q(\mathbf{u}) - Q(\mathbf{u}^h) = Q(\mathbf{u} - \mathbf{u}^h) = Q(\mathbf{e}) \quad (9)$$

As will be shown later,  $Q(\mathbf{v})$  may be interpreted as the work associated with a displacement field  $\mathbf{v}$  and a distribution of forces specific to each type of quantity

of interest. If we particularize  $Q(\mathbf{v})$  for  $\mathbf{v} = \mathbf{u}$ , this force distribution will allow to extract information concerning the quantity of interest associated with the solution of the problem in (4).

A standard procedure to evaluate  $Q(\mathbf{e})$  consists in solving the auxiliary *dual* problem (also called *adjoint* or *extraction* problem) defined as:

$$\text{Find } \mathbf{w}_Q \in V : \forall \mathbf{v} \in V \quad a(\mathbf{v}, \mathbf{w}_Q) = Q(\mathbf{v}). \quad (10)$$

An exact representation for the error  $Q(\mathbf{e})$  in terms of the solution of the dual problem can be simply obtained by substituting  $\mathbf{v} = \mathbf{e}$  in (10) and remarking that for all  $\mathbf{w}_Q^h \in V^h$ , due to the Galerkin orthogonality,  $a(\mathbf{e}, \mathbf{w}_Q^h) = 0$  such that

$$Q(\mathbf{e}) = a(\mathbf{e}, \mathbf{w}_Q) = a(\mathbf{e}, \mathbf{w}_Q) - a(\mathbf{e}, \mathbf{w}_Q^h) = a(\mathbf{e}, \mathbf{w}_Q - \mathbf{w}_Q^h) = a(\mathbf{e}, \mathbf{e}_Q) \quad (11)$$

Therefore, the error in evaluating  $Q(\mathbf{u})$  using  $\mathbf{u}^h$  is given by

$$Q(\mathbf{u}) - Q(\mathbf{u}^h) = Q(\mathbf{e}) = a(\mathbf{e}, \mathbf{e}_Q) = \int_{\Omega} (\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_p^h) \mathbf{D}^{-1} (\boldsymbol{\sigma}_d - \boldsymbol{\sigma}_d^h) d\Omega \quad (12)$$

where  $\boldsymbol{\sigma}_p$  is the stress field associated with the primal solution and  $\boldsymbol{\sigma}_d$  is the one associated with the dual solution.

Following [19] we can introduce a first upper error bound for the QoI considering the Cauchy–Schwarz inequality that results in:

$$Q(\mathbf{e}) = a(\mathbf{e}, \mathbf{e}_Q) \leq \|\mathbf{e}\| \|\mathbf{e}_Q\|$$

Then, the evaluation of the error in the QoI is now expressed in terms of the error in the energy norm for the dual and primal solutions, for which several techniques are already available. However, it is clear that this upper bound is rather conservative due to the unknown angle between the two discretization errors  $\mathbf{e}$  and  $\mathbf{e}_Q$ . Moreover, it does not provide a local indicator that can be used to guide the adaptivity process.

Let us consider  $n_e$  as the number of elements in the discretization and  $a_e : V_e \times V_e \rightarrow \mathbb{R}$  as the associated bilinear form  $a$  to an element  $\Omega_e : \Omega = \bigcup_{n_e} \Omega_e$  such that

$$\forall \mathbf{u}, \mathbf{v} \in V \quad a(\mathbf{u}, \mathbf{v}) = \sum_{n_e} a_e(\mathbf{u}, \mathbf{v}) \quad (13)$$

To obtain sharper error measures and local indicators, we can decouple the element contributions as shown in [19] such that:

$$Q(\mathbf{e}) = \sum_{n_e} a_e(\mathbf{e}, \mathbf{e}_Q) \quad (14a)$$

$$\leq \sum_{n_e} |a_e(\mathbf{e}, \mathbf{e}_Q)| \quad (14b)$$

$$\leq \sum_{n_e} \|\mathbf{e}\|_{\Omega_e} \|\mathbf{e}_Q\|_{\Omega_e} \quad (14c)$$

$$\leq \|\mathbf{e}\| \|\mathbf{e}_Q\| \quad (14d)$$

Note that in (12) and the indicators derived afterwards, the error in the QoI is related to the errors in the FE approximations  $\mathbf{u}^h$  and  $\mathbf{w}_Q^h$ . On that account, any of the available procedures to estimate the error in the energy norm may be considered to obtain estimates of the error in the quantity of interest. Using the expression for the Zienkiewicz-Zhu error estimator in (8) with (12) we can derive an estimate for the error in the QoI which reads

$$Q(\mathbf{e}) \approx Q(\mathbf{e}_{es}) = \int_{\Omega} \left( \boldsymbol{\sigma}_p^* - \boldsymbol{\sigma}_p^h \right) \mathbf{D}^{-1} \left( \boldsymbol{\sigma}_d^* - \boldsymbol{\sigma}_d^h \right) d\Omega \quad (15)$$

where  $\boldsymbol{\sigma}_p^*$  and  $\boldsymbol{\sigma}_d^*$  represent the recovered stress fields for the primal and dual problems. Here, we expect to have a sharp estimate of the error in the QoI if the recovered stress fields are accurate approximations to their exact counterparts.

In order to obtain accurate representations of the exact stress fields both for the primal and dual solutions, we propose the use of the locally equilibrated recovery techniques introduced in [12, 13, 22]. This technique, which is an enhancement of the superconvergent patch recovery (SPR) proposed in [5], enforces the fulfilment of the internal and boundary equilibrium equations locally on patches. For problems with singularities the stress field is also decomposed into two parts: smooth and singular, which are separately recovered (see Section 5).

Continuing with the ideas in [19], and using the recovery technique to obtain accurate estimates of the error with the decoupled approach, we can formulate using (14) the following error indicators:

$$Q(\mathbf{e}) \approx Q(\mathbf{e}_{es}) = \mathcal{E}_1 = \sum_{n_e} a_e(\mathbf{e}_{es}, \mathbf{e}_{Q,es}) \quad (16a)$$

$$\leq \mathcal{E}_2 = \sum_{n_e} |a_e(\mathbf{e}_{es}, \mathbf{e}_{Q,es})| \quad (16b)$$

$$\leq \mathcal{E}_3 = \sum_{n_e} \|\mathbf{e}_{es}\|_{\Omega_e} \|\mathbf{e}_{Q,es}\|_{\Omega_e} \quad (16c)$$

$$\leq \mathcal{E}_4 = \|\mathbf{e}_{es}\| \|\mathbf{e}_{Q,es}\| \quad (16d)$$

The properties of these error estimates are discussed in [19] in more detail. In a general sense, the idea is that the more accurate the estimate, the less guaranteed is the upper bound property. Strictly speaking, the only guaranteed upper error bound is given by (16d), under the assumption that the recovery technique correctly enforces equilibrium conditions and provides upper error bounds in the energy norm for both the primal and dual solutions [21].

## 4 ANALYTICAL DEFINITIONS FOR THE DUAL PROBLEM

The recovery procedure based on the SPR technique denoted as SPR-CX, and described in [12, 13, 21, 22], relies on the *a priori* known values of  $\mathbf{b}$  and  $\mathbf{t}$  to impose

the internal and boundary equilibrium equations. These values are already available for the primal problem ( $\mathbf{b}_p$  and  $\mathbf{t}_p$ ). Although the body forces and boundary tractions  $\mathbf{b}_d$  and  $\mathbf{t}_d$  are not known for the dual problem, we can easily derive expressions associated to certain linear QoIs, like the mean values of displacements and stresses in a sub-domain of interest  $\Omega_i$ , that can be interpreted in terms of  $\mathbf{b}_d$  and  $\mathbf{t}_d$ . This approach was first introduced in [23] and presented later in [24]. Similarly, in [25] the authors defined the relation between the natural quantities of interest and dual loading data.

#### 4.1 Mean displacement in $\Omega_i$

Let us assume that the objective is to evaluate the mean value of the displacements along the direction  $\alpha$  in a sub-domain of interest  $\Omega_i \subset \Omega$ . The functional for the quantity of interest can be written as:

$$Q(\mathbf{u}) = \bar{u}_\alpha|_{\Omega_i} = \frac{1}{|\Omega_i|} \int_{\Omega_i} \mathbf{u}^T \mathbf{c}_{u_\alpha} d\Omega \quad (17)$$

where  $|\Omega_i|$  is the volume of  $\Omega_i$  and  $\mathbf{c}_{u_\alpha}$  is a vector used to select the appropriate combination of components of  $\mathbf{u}$ . For example,  $\mathbf{c}_{u_\alpha} = \{1, 0\}^T$  if  $\alpha$  is parallel to the  $x$ -axis. Now, considering  $\mathbf{v} \in V$  in (17) results in:

$$Q(\mathbf{v}) = \int_{\Omega_i} \mathbf{v}^T \left( \frac{\mathbf{c}_{u_\alpha}}{|\Omega_i|} \right) d\Omega = \int_{\Omega_i} \mathbf{v}^T \mathbf{b}_d d\Omega \quad (18)$$

Note that the term  $\mathbf{c}_{u_\alpha}/|\Omega_i|$  formally corresponds to a vector of body forces in the problem defined in (4). Therefore, we can say that  $\mathbf{b}_d = \mathbf{c}_{u_\alpha}/|\Omega_i|$  is a constant vector of body loads that applied in the sub-domain of interest  $\Omega_i$  can be used in the dual problem to extract the mean displacements.

#### 4.2 Mean displacement along $\Gamma_i$

For the case where the quantity of interest is the functional that evaluates the mean value of the displacements along a given boundary  $\Gamma_i$  the expression reads:

$$Q(\mathbf{u}) = \bar{u}_\alpha|_{\Gamma_i} = \frac{1}{|\Gamma_i|} \int_{\Gamma_i} \mathbf{u}^T \mathbf{c}_{u_\alpha} d\Gamma \quad (19)$$

$|\Gamma_i|$  being the length of  $\Gamma_i$  and  $\mathbf{c}_{u_\alpha}$  a vector used to select the appropriate component of  $\mathbf{u}$ . Again, considering  $\mathbf{v} \in V$  in (19) we have:

$$Q(\mathbf{v}) = \int_{\Gamma_i} \mathbf{v}^T \left( \frac{\mathbf{c}_{u_\alpha}}{|\Gamma_i|} \right) d\Gamma = \int_{\Gamma_i} \mathbf{v}^T \mathbf{t}_d d\Gamma \quad (20)$$

Note that the term  $\mathbf{c}_{u_\alpha}/|\Gamma_i|$  can be interpreted as a vector of tractions applied along the boundary in the problem defined in (4). Thus,  $\mathbf{t}_d = \mathbf{c}_{u_\alpha}/|\Gamma_i|$  is a vector of tractions applied on  $\Gamma_i$  and that can be used in the dual problem to extract the mean displacements along  $\Gamma_i$ .



### 4.3 Mean strain in $\Omega_i$

In this case we are dealing with first derivatives of displacements. Consider for instance the component  $\epsilon_{xx}$ . This term can be expressed as:

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \left\{ \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right\} \begin{Bmatrix} u \\ 0 \end{Bmatrix} = \nabla \cdot \begin{Bmatrix} u \\ 0 \end{Bmatrix} = \nabla \cdot \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \right) = \nabla \cdot (\mathbf{C}_{xx} \mathbf{u}) \quad (21)$$

Operating similarly for the other components of the strain vector we would have:

$$\epsilon_{xx} = \nabla \cdot (\mathbf{C}_{xx} \mathbf{u}) \quad \epsilon_{yy} = \nabla \cdot (\mathbf{C}_{yy} \mathbf{u}) \quad \epsilon_{xy} = \nabla \cdot (\mathbf{C}_{xy} \mathbf{u}) \quad (22)$$

Taking (22), the mean value of the strain  $\epsilon_{xx}$  in  $\Omega_i$  can be evaluated as:

$$Q(\mathbf{u}) = \bar{\epsilon}_{xx} = \frac{1}{|\Omega_i|} \int_{\Omega_i} \epsilon_{xx} d\Omega = \frac{1}{|\Omega_i|} \int_{\Omega_i} \nabla \cdot (\mathbf{C}_{xx} \mathbf{u}) d\Omega \quad (23)$$

Considering the divergence theorem the previous equation can be written as:

$$Q(\mathbf{u}) = \bar{\epsilon}_{xx} = \frac{1}{|\Omega_i|} \int_{\Gamma_i} (\mathbf{C}_{xx} \mathbf{u})^T \mathbf{n} d\Gamma \quad (24)$$

where  $\mathbf{n}$  is the unit outward normal vector. The domain integral is then transformed into a contour integral. Finally, considering  $\mathbf{v} \in V$  in the previous expression and reordering we obtain:

$$Q(\mathbf{v}) = \int_{\Gamma_i} \mathbf{v}^T \left( \frac{\mathbf{C}_{xx}^T \mathbf{n}}{|\Omega_i|} \right) d\Gamma = \int_{\Gamma_i} \mathbf{v}^T \mathbf{t}_d d\Gamma \quad (25)$$

Note that the term  $(\mathbf{C}_{xx}^T \mathbf{n}) / |\Omega_i|$  formally corresponds to a vector of boundary tractions along the boundary of  $\Omega_i$ . Therefore, we can interpret  $\mathbf{t}_d = (\mathbf{C}_{xx}^T \mathbf{n}) / |\Omega_i|$  as a vector of boundary tractions that applied along the boundary of the subdomain of interest  $\Gamma_i$  can be used to extract the mean strain in  $\Omega_i$ .

For any given linear combination of strains  $\epsilon_\eta$ , denoted as

$$\epsilon_\eta = c_{\epsilon_{xx}} \epsilon_{xx} + c_{\epsilon_{yy}} \epsilon_{yy} + c_{\epsilon_{xy}} \epsilon_{xy} = \mathbf{c}_{\epsilon_\eta}^T \boldsymbol{\epsilon} = \nabla \cdot (\mathbf{R} \mathbf{u}) \quad (26)$$

where

$$\mathbf{R} = c_{\epsilon_{xx}} \mathbf{C}_{xx} + c_{\epsilon_{yy}} \mathbf{C}_{yy} + c_{\epsilon_{xy}} \mathbf{C}_{xy} \quad (27)$$

we could evaluate an expression similar to (25) such that

$$Q(\mathbf{v}) = \int_{\Gamma_i} \mathbf{v}^T \left( \frac{\mathbf{R}^T \mathbf{n}}{|\Omega_i|} \right) d\Gamma = \int_{\Gamma_i} \mathbf{v}^T \mathbf{t}_d d\Gamma. \quad (28)$$

#### 4.4 Mean stress value in $\Omega_i$

Let us consider now as QoI the mean stress value  $\bar{\sigma}_\alpha$  given by a combination of the stress components in a domain of interest which reads:

$$Q(\mathbf{u}) = \bar{\sigma}_\alpha|\Omega_i = \frac{1}{|\Omega_i|} \int_{\Omega_i} \sigma_i d\Omega_i = \frac{1}{|\Omega_i|} \int_{\Omega_i} \mathbf{c}_{\sigma_\alpha}^T \boldsymbol{\sigma} d\Omega_i \quad (29)$$

The stress components are linear combinations of the strain components. Thus, any combination of stress components  $\sigma_\alpha$  can be expressed as:

$$\sigma_\alpha = \mathbf{c}_{\sigma_\alpha}^T \boldsymbol{\sigma} = \mathbf{c}_{\sigma_\alpha}^T \mathbf{D} \boldsymbol{\epsilon} = \mathbf{c}_{\epsilon_\eta}^T \boldsymbol{\epsilon} \quad (30)$$

where  $\mathbf{c}_{\sigma_\alpha}$  is the vector used to define the combination of stress components and  $\mathbf{c}_{\epsilon_\eta}$  defines its corresponding combination of strain components. Therefore, one could evaluate the mean value of any combination  $\mathbf{c}_{\sigma_\alpha}$  of stress components in  $\Omega_i$  using (28), where  $\mathbf{R}$  is evaluated using the coefficients of  $\mathbf{c}_{\epsilon_\eta}^T = \mathbf{c}_{\sigma_\alpha}^T \mathbf{D}$  in (27).

## 5 RECOVERY TECHNIQUE

As noted in previous sections, a widely used technique to control the error in the energy norm in the finite element discretization is the Zienkiewicz-Zhu error estimator shown in (8), which is based on the recovery of an enhanced stress field  $\boldsymbol{\sigma}^*$ . The accuracy of such estimation depends on the quality of the recovered field. In this work we consider the SPR-CX recovery technique, which is an enhancement of the error estimator introduced in [21]. The technique incorporates the ideas in [12] to guarantee locally on patches the exact satisfaction of the equilibrium equations, and the extension in [13] to singular problems.

In the SPR-CX technique, as in the original SPR technique, a patch is defined as the set of elements connected to a vertex node. On each patch, a polynomial expansion for each one of the components of the recovered stress field is expressed in the form:

$$\sigma_k^*(\mathbf{x}) = \mathbf{p}(\mathbf{x}) \mathbf{a}_k \quad k = xx, yy, xy \quad (31)$$

where  $\mathbf{p}$  represents a polynomial basis and  $\mathbf{a}$  are unknown coefficients. Usually, the polynomial basis is chosen equal to the finite element basis for the displacements.

For the 2D case, the linear system of equations to evaluate the recovered stress field coupling the three stress components reads:

$$\boldsymbol{\sigma}^*(\mathbf{x}) = \begin{Bmatrix} \sigma_{xx}^*(\mathbf{x}) \\ \sigma_{yy}^*(\mathbf{x}) \\ \sigma_{xy}^*(\mathbf{x}) \end{Bmatrix} = \mathbf{P}(\mathbf{x}) \mathbf{A} = \begin{bmatrix} \mathbf{p}(\mathbf{x}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{p}(\mathbf{x}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{p}(\mathbf{x}) \end{bmatrix} \begin{Bmatrix} \mathbf{a}_{xx} \\ \mathbf{a}_{yy} \\ \mathbf{a}_{xy} \end{Bmatrix} \quad (32)$$

Constraint equations are introduced via Lagrange multipliers into the linear system used to solve for the coefficients  $\mathbf{A}$  on each patch in order to enforce the satisfaction of the:

- Internal equilibrium equation.
- Boundary equilibrium equation: A point collocation approach is used to impose the satisfaction of a polynomial approximation to the tractions along the Neumann boundary intersecting the patch.
- Compatibility equation: This additional constraint is also imposed to further increase the accuracy of the recovered stress field.

For singular problems the exact stress field  $\boldsymbol{\sigma}$  is decomposed into two stress fields, a smooth field  $\boldsymbol{\sigma}_{smo}$  and a singular field  $\boldsymbol{\sigma}_{sing}$ :

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{smo} + \boldsymbol{\sigma}_{sing}. \quad (33)$$

Then, the recovered field  $\boldsymbol{\sigma}^*$  required to compute the error estimate given in (8) can be expressed as the contribution of two recovered stress fields, one smooth  $\boldsymbol{\sigma}_{smo}^*$  and one singular  $\boldsymbol{\sigma}_{sing}^*$ :

$$\boldsymbol{\sigma}^* = \boldsymbol{\sigma}_{smo}^* + \boldsymbol{\sigma}_{sing}^*. \quad (34)$$

For the recovery of the singular part, the expressions which describe the asymptotic fields near the crack tip are used. To evaluate  $\boldsymbol{\sigma}_{sing}^*$  we first obtain estimated values of the stress intensity factors  $K_I$  and  $K_{II}$  using a domain integral method based on extraction functions [26, 27]. Notice that the recovered part  $\boldsymbol{\sigma}_{sing}^*$  is an equilibrated field as it satisfies the equilibrium equations.

Once the field  $\boldsymbol{\sigma}_{sing}^*$  has been evaluated, an FE approximation to the smooth part  $\boldsymbol{\sigma}_{smo}^h$  can be obtained subtracting  $\boldsymbol{\sigma}_{sing}^*$  from the raw FE field:

$$\boldsymbol{\sigma}_{smo}^h = \boldsymbol{\sigma}^h - \boldsymbol{\sigma}_{sing}^*. \quad (35)$$

Then, the field  $\boldsymbol{\sigma}_{smo}^*$  is evaluated applying the enhancements of the SPR technique previously described, i.e. satisfaction of equilibrium and compatibility equations at each patch. Note that as both  $\boldsymbol{\sigma}_{smo}^*$  and  $\boldsymbol{\sigma}_{sing}^*$  satisfy the equilibrium equations,  $\boldsymbol{\sigma}^*$  also satisfy equilibrium at each patch.

To obtain a continuous field, the recovered stresses  $\boldsymbol{\sigma}^*$  are directly evaluated at an integration point  $\mathbf{x}$  through the use of a partition of unity procedure [28], properly weighting the stress interpolation polynomials obtained from the different patches formed at the vertex nodes of the element containing  $\mathbf{x}$ :

$$\boldsymbol{\sigma}^*(\mathbf{x}) = \sum_{j=1}^{n_v} N_j(\mathbf{x}) \boldsymbol{\sigma}_j^*(\mathbf{x}_j), \quad (36)$$

where  $N_j$  are the shape functions associated with the vertex nodes  $n_v$ . However, this process introduces a small lack of equilibrium as explained in [21, 22].

This technique, used to obtain the recovered field for the primal problem  $\boldsymbol{\sigma}_p^*$ , is also used to evaluate  $\boldsymbol{\sigma}_d^*$  considering different loading conditions. Two considerations are worth making. First, the analytical expressions that define the tractions

$\mathbf{t}$  and body forces  $\mathbf{b}$  for the dual problem are obtained from the interpretation of the functional  $Q(\mathbf{u})$  in terms of  $\mathbf{t}_d$  and  $\mathbf{b}_d$ , as seen in Section 4. Second, to enforce equilibrium conditions along the boundary of the DoI, we consider a different polynomial expansions on each side of the boundary and enforce statical admissibility of the normal and tangential stresses. Suppose that we have a patch intersected by  $\Gamma_i$  such that  $\Omega_e = \Omega_{1,e} \cup \Omega_{2,e}$  for intersected elements, as indicated in Figure 1. To enforce equilibrium conditions along  $\Gamma_i$  we define the stresses  $\sigma_{\Omega_1}^*|_{\Gamma_i}$ ,  $\sigma_{\Omega_2}^*|_{\Gamma_i}$  at each side of the internal boundary. Then, the boundary equilibrium along  $\Gamma_i$  is:

$$\mathbf{Q}(\sigma_{\Omega_1}^*|_{\Gamma_i} - \sigma_{\Omega_2}^*|_{\Gamma_i}) = \mathbf{t}_{\Gamma_i} \quad (37)$$

where  $\mathbf{t}_{\Gamma_i} = [t_x \ t_y]^T$  are the prescribed tractions and  $\mathbf{Q}$  is the matrix form of Cauchy's law considering the unit normal  $\mathbf{n}$  to  $\Gamma_i$  such that

$$\mathbf{Q} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} \quad (38)$$

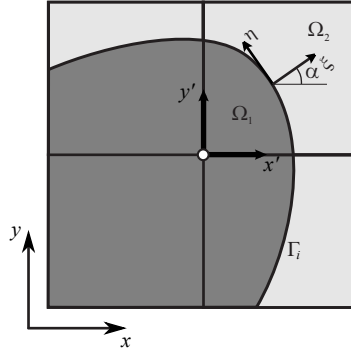


Fig. 1: Equilibrium conditions along internal boundaries.

## 6 NUMERICAL RESULTS

In this section numerical tests using 2D benchmark problems with exact solutions are used to investigate the quality of the proposed technique. The first problem has a smooth solution whilst the second is a singular problem. For both models we assume plane strain conditions. The  $h$ -adaptive FE code used in the numerical examples is based on Cartesian meshes independent of the problem geometry [29], with linear quadrilateral (Q4) elements.

To assess the performance of the proposed technique we consider the effectivity index of the error estimator  $\theta$  defined as:

$$\theta = \frac{Q(\mathbf{e}_{es})}{Q(\mathbf{e})} \quad (39)$$

where  $Q(\mathbf{e})$  denotes the exact error in the quantity of interest, and  $Q(\mathbf{e}_{es})$  represents the evaluated error estimate. We can also represent the effectivity in the QoI defined as

$$\theta_{QoI} = \frac{Q(\mathbf{u}^h) + Q(\mathbf{e}_{es})}{Q(\mathbf{u})} \quad (40)$$

and the relative error in the QoI for the exact and estimated error

$$\eta^Q(\mathbf{e}) = \frac{|Q(\mathbf{e})|}{|Q(\mathbf{u})|}, \quad \eta^Q(\mathbf{e}_{es}) = \frac{|Q(\mathbf{e}_{es})|}{|Q(\mathbf{u}^h) + Q(\mathbf{e}_{es})|} \quad (41)$$

The distribution of the local effectivity index  $D$  is analysed at element level, as described in [12]:

$$\begin{aligned} D &= \theta^e - 1 & \text{if } \theta^e &\geq 1 \\ D &= 1 - \frac{1}{\theta^e} & \text{if } \theta^e < 1 \end{aligned} \quad \text{with} \quad \theta^e = \frac{\|\mathbf{e}_{es}^e\|}{\|\mathbf{e}^e\|} \quad (42)$$

where superscript  $e$  denotes evaluation at element level. To evaluate  $\|\mathbf{e}^e\|$  we use (12) locally at each element. Nonetheless, we should remark that this is not always possible as the exact value of  $\sigma$  is unknown in the vast majority of cases, especially for the dual problem.

## 6.1 Problem 1: Thick-wall cylinder subjected to internal pressure.

The geometrical model for this problem and the initial mesh are represented in Figure 2. Due to symmetry, only 1/4 of the section is modelled. The domain of interest (DoI)  $\Omega_i$  is denoted by a yellow square whereas the contour of interest  $\Gamma_i$  is the internal surface of radius  $a$ . Young's modulus is  $E = 1000$ , Poisson's ratio is  $\nu = 0.3$ ,  $a = 5$ ,  $b = 20$  and the load  $P = 1$ .

The exact solution for the radial displacement assuming plane strain conditions is given by

$$u_r(r) = \frac{P(1+\nu)}{E(c^2-1)} \left( r(1-2\nu) + \frac{b^2}{r} \right) \quad (43)$$

where  $c = b/a$ ,  $r = \sqrt{x^2 + y^2}$  and  $\phi = \arctan(y/x)$ . Stresses in cylindrical coordinates are

$$\sigma_r(r) = \frac{P}{c^2-1} \left( 1 - \frac{b^2}{r^2} \right) \quad \sigma_\theta(r) = \frac{P}{c^2-1} \left( 1 + \frac{b^2}{r^2} \right) \quad \sigma_z(r, \theta) = 2\nu \frac{P}{c^2-1} \quad (44)$$

Three linear quantities of interest have been considered for this problem: the mean radial displacements along  $\Gamma_i$ , the mean displacements  $u_x$  in the domain of interest  $\Omega_i$  and the mean stresses  $\sigma_x$  in  $\Omega_i$ .

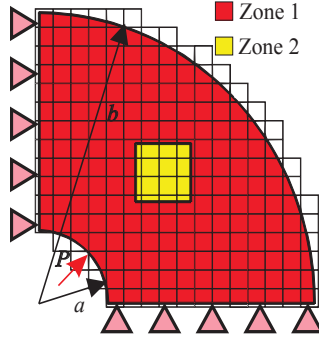


Fig. 2: Thick-wall cylinder subjected to an internal pressure. Model and analytical solution, the domain of interest  $\Omega_i$  is indicated by a yellow square (Zone 2).

### 6.1.1 Problem 1.a.: Mean radial (or normal) displacements along $\Gamma_i$

Let  $Q(\mathbf{u})$  be the functional that evaluates the mean normal displacement  $\bar{u}_n$  along a given part of the boundary  $\Gamma_i$  such that:

$$Q(\mathbf{u}) = \bar{u}_n = \frac{1}{|\Gamma_i|} \int_{\Gamma_i} (\mathbf{R}\mathbf{u})^T \mathbf{c}_u d\Gamma \quad (45)$$

where  $\mathbf{R}$  is the standard rotation matrix for the displacements that aligns the coordinate system with the boundary  $\Gamma_i$  and  $\mathbf{c}_u = \{1, 0\}^T$  is the extraction vector that selects the normal component.

Let us evaluate as QoI the normal displacements along the surface corresponding to the inner radius of the cylinder. Thus, the exact value of the QoI given by (43) for  $r = a$  is  $\bar{u}_n = -7.106 \cdot 10^{-3}$ .

Figure 3 shows the set of meshes resulting from the  $h$ -adaptive process guided by the error estimate in this QoI.

In Table 1 we show the results for the error estimate  $\mathcal{E}_1$  and the exact error  $Q(\mathbf{e})$ . The recovery technique accurately captures the exact error and provides good effectivities  $\theta$ . The evolution of the effectivity index for the proposed technique and the standard SPR is represented in Figure 4. In this case, the SPR-CX converges faster to the optimal value  $\theta = 1$  and shows a high level of accuracy.

Note that in this case, the dual problem corresponds to a traction  $\mathbf{t} = \mathbf{R}^T \mathbf{c} / |\Gamma_i|$  that represents a constant traction normal to the internal boundary. Therefore, the solution to the dual problem can be evaluated exactly using (43) and (44) where  $P$  takes the value of this constant traction.

Since the exact solution for the dual problem is known for this particular quantity of interest, we can represent the local effectivity  $D$  in (42) to evaluate the performance of the proposed recovery technique to estimate the error in the quantity of interest.

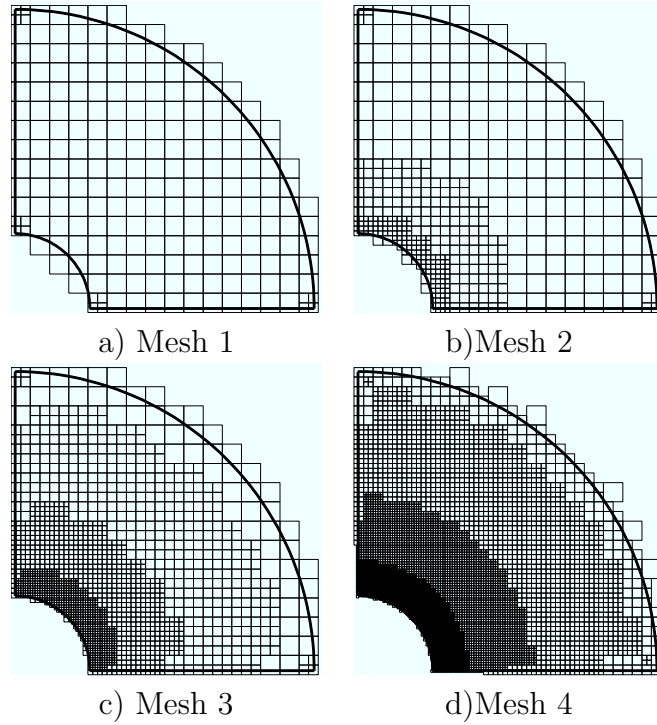


Fig. 3: Problem 1.a. Sequence of meshes for the mean displacement in a domain of interest  $\Gamma_i$ .

Tab. 1: Problem 1.a. The table shows the values for the error estimate  $\mathcal{E}_1$  and its effectivities.

dof	$Q(\mathbf{e}_{es}) = \mathcal{E}_1$	$Q(\mathbf{e})$	$\theta$	$\theta_{QoI}$
488	$-7.761841 \cdot 10^{-5}$	$-7.892325 \cdot 10^{-5}$	0.98346696	0.99981639
996	$-1.598034 \cdot 10^{-5}$	$-1.621484 \cdot 10^{-5}$	0.98553793	0.99996700
3,826	$-3.448937 \cdot 10^{-6}$	$-3.463751 \cdot 10^{-6}$	0.99572309	0.99999792
16,466	$-7.498724 \cdot 10^{-7}$	$-7.504639 \cdot 10^{-7}$	0.99921188	0.99999992
72,094	$-1.645847 \cdot 10^{-7}$	$-1.645499 \cdot 10^{-7}$	1.00021162	1.00000000

In Figure 5 we show the local effectivity of the error in the quantity of interest at the element level of the first three meshes for the standard SPR, in Figure 6 we represent the same values for the SPR-CX. We can see that the enhanced recovery improves the solution, especially along the boundaries, which is in agreement with previous results for this type of recovery enhancement when considering the error in energy norm. As the  $h$ -adaptive procedures use local information to refine the mesh, it is good to provide an accurate local error estimate to the adaptive algorithm. For this reason, the local performance of the proposed technique indicates

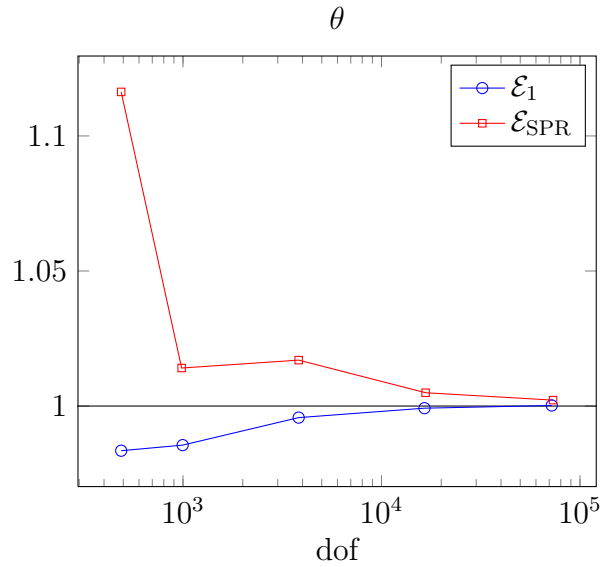


Fig. 4: Problem 1.a. Evolution of the effectivity index  $\theta$  considering locally equilibrated,  $\mathcal{E}_1$ , and non-equilibrated recovery,  $\mathcal{E}_{\text{SPR}}$ .

that it could result more useful to guide the  $h$ -adaptive process, even in cases in which the global effectivity is similar to the effectivity of the standard SPR.

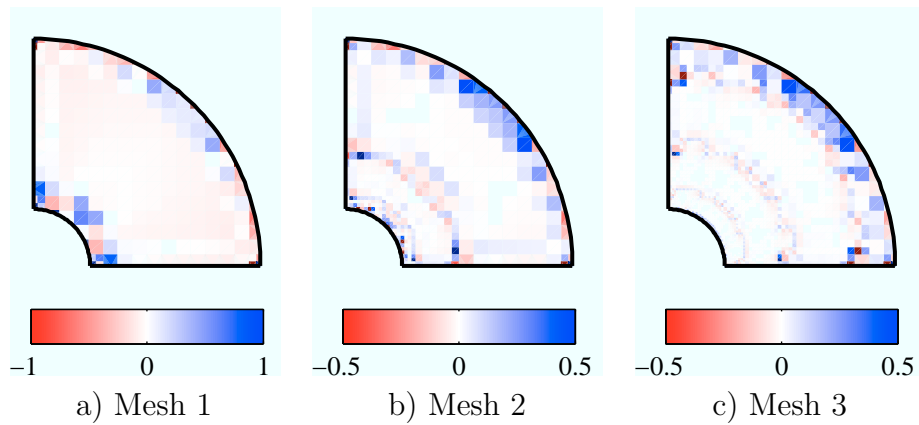


Fig. 5: Problem 1.a. Local effectivity for the error in the QoI using the standard SPR recovery technique.



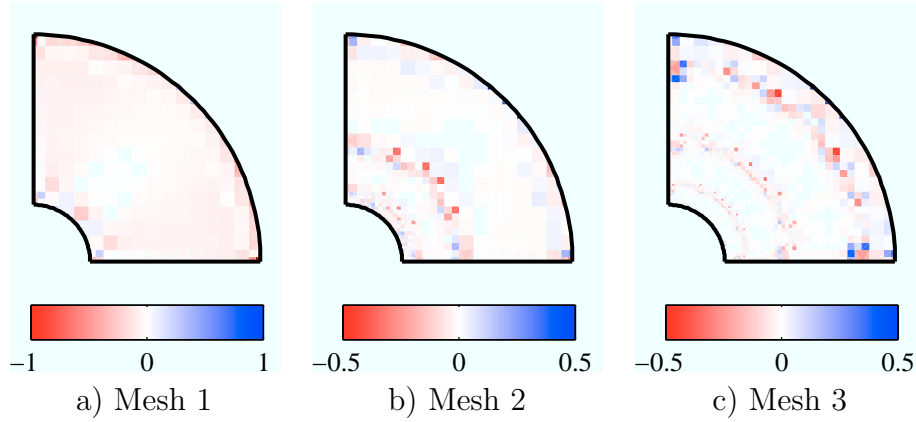


Fig. 6: Problem 1.a. Local effectivity for the error in the QoI using the locally equilibrated SPR-CX recovery technique.

### 6.1.2 Problem 1.b.: Mean displacements $\bar{u}_x$ in $\Omega_i$

Let us consider the mean displacement  $\bar{u}_x$  in  $\Omega_i$  as the quantity of interest. The objective is to evaluate the error when evaluating  $\bar{u}_x$  defined by the functional:

$$Q(\mathbf{u}) = \bar{u}_x = \frac{1}{|\Omega_i|} \int_{\Omega_i} u_x d\Omega \quad (46)$$

The exact value of the QoI can be computed for this problem and is  $\bar{u}_x = 0.002238239291713$ . Figure 7 shows the first four meshes used in the  $h$ -adaptive refinement process guided by the error estimate for the QoI.

Figure 8 shows the relative error (in percentage) for the error estimates in (16) evaluated using the proposed recovery technique. The exact error  $Q(\mathbf{e})$  is also represented for comparison. For all the curves the relative error decreases monotonically when increasing the number of degrees of freedom (dof), showing that the  $h$ -adaptive process has a stable convergence. The most accurate estimation is obtained for the estimate  $\mathcal{E}_1$  (see (16)), which practically coincides with the exact relative error. The other estimates tend to overestimate the exact error, although strictly speaking they do not have bounding properties, which is in agreement with the observations made previously in Section 3.2. Figure 9 represents the evolution of the effectivity index as we increase the number of dof. The curves are in consonance with the results in Figure 8 and show good values for the effectivity index, especially for  $\mathcal{E}_1$ .

Table 2 shows the estimated error in the QoI  $Q(\mathbf{e}_{es})$ , the exact error  $Q(\mathbf{e})$ , the effectivity in the quantity of interest  $\theta_{QoI}$  and the effectivity of the error estimator  $\theta$  for the sharp error estimate  $\mathcal{E}_1$ . Comparing  $Q(\mathbf{e})$  and  $Q(\mathbf{e}_{es})$  we can notice that both values decrease as we refine and that the estimate  $\mathcal{E}_1$  gives a good approximation of the exact error. The effectivity of the error estimator  $\theta$  converges and is very close to the theoretical value  $\theta = 1$ . As expected from these results,

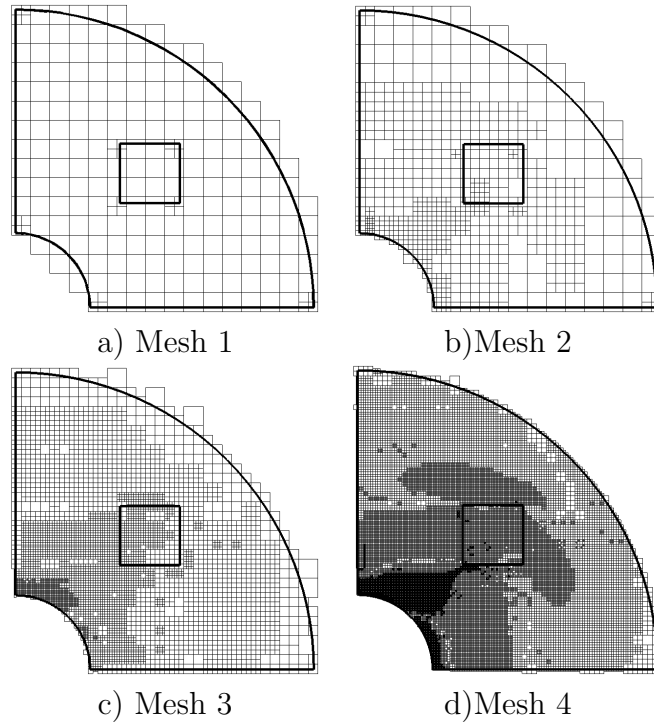


Fig. 7: Problem 1.b. Sequence of  $h$ -adaptive refined meshes.

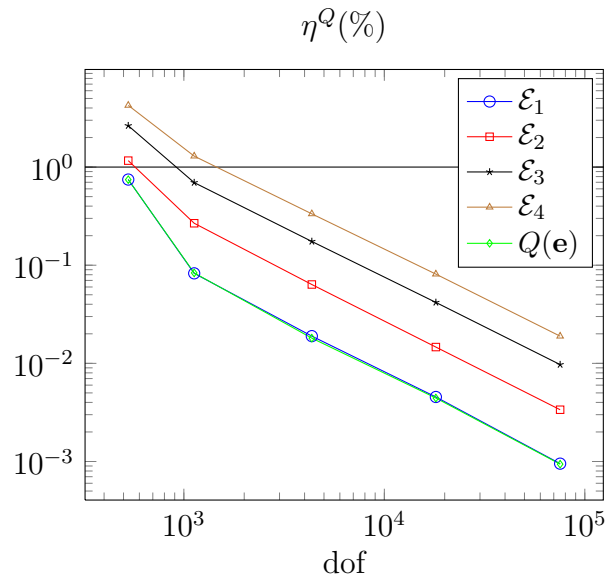


Fig. 8: Problem 1.b. Evolution of the relative error  $\eta^Q$  considering the error estimates in (16) and the exact error  $Q(\mathbf{e})$ .

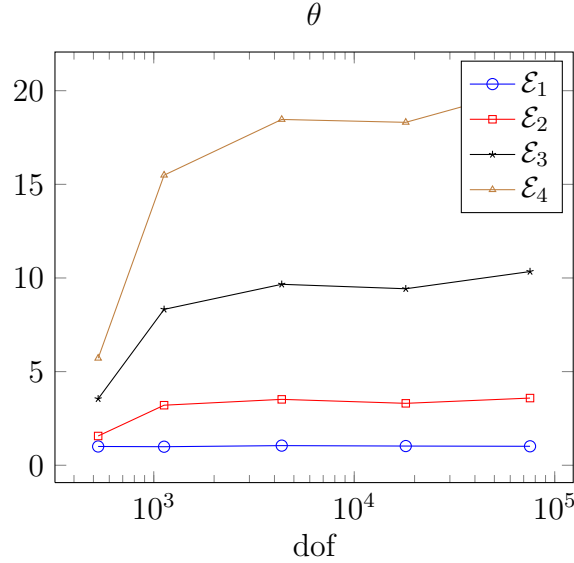


Fig. 9: Problem 1.b. Evolution of the effectivity index  $\theta$  considering the error estimates in (16)

Tab. 2: Problem 1.b. Error estimate  $\mathcal{E}_1$  and its effectivities.

dof	$Q(\mathbf{e}_{es}) = \mathcal{E}_1$	$Q(\mathbf{e})$	$\theta$	$\theta_{QoI}$
528	$1.670429 \cdot 10^{-5}$	$1.666535 \cdot 10^{-5}$	1.00233710	1.00001740
1,126	$1.849599 \cdot 10^{-6}$	$1.868635 \cdot 10^{-6}$	0.98981317	0.99999150
4,348	$4.240859 \cdot 10^{-7}$	$4.046075 \cdot 10^{-7}$	1.04814162	1.00000870
18,076	$1.017070 \cdot 10^{-7}$	$9.922496 \cdot 10^{-8}$	1.02501376	1.00000111
75,328	$2.128196 \cdot 10^{-8}$	$2.102331 \cdot 10^{-8}$	1.01230285	1.00000012

the effectivity  $\theta_{QoI}$  is very accurate as well.

If in (16a) we consider the case of the non-equilibrated superconvergent patch recovery procedure, resembling the averaging error estimators presented in [19], we obtain the results shown in Figure 10. This figure shows that, in this case, the effectivity of the error estimator provided by the standard SPR technique is similar to the effectivity obtained with the SPR-CX technique here proposed, although the later results are more accurate in the coarsest mesh.

### 6.1.3 Problem 1.c.: Mean stress $\bar{\sigma}_x$ in $\Omega_i$

Consider as QoI the mean stress value  $\bar{\sigma}_x$  given in (29). The model and domain of interest  $\Omega_i$  are those shown in Figure 2. Figure 11 shows the four first meshes of bilinear elements used in the refinement process guided by the error estimated

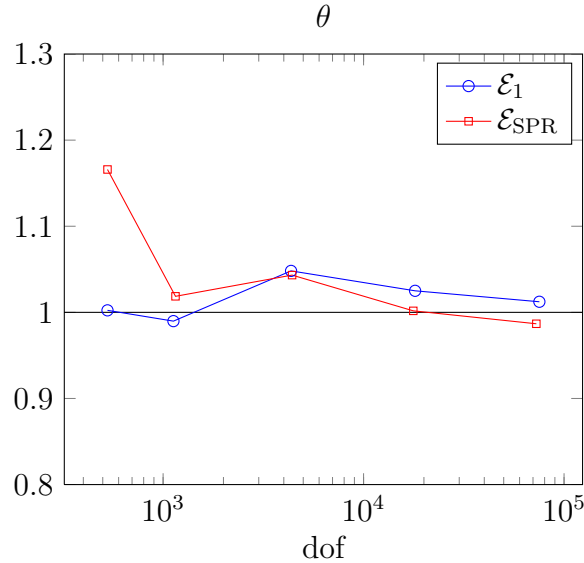


Fig. 10: Problem 1.b. Evolution of the effectivity index  $\theta$  considering equilibrated  $\mathcal{E}_1$  and non-equilibrated recovery,  $\mathcal{E}_{\text{SPR}}$ .

for this QoI.

The evolution of the relative error for the estimates presented in (16) using the proposed recovery technique and the exact error is shown in Figure 12. Similar to the observations done for the previous examples, the most accurate results are obtained when considering the estimate  $\mathcal{E}_1$ . In this case, the other estimates considerably overestimate the true error. Figure 13 shows the effectivity index for  $\mathcal{E}_1$ , which uses the locally equilibrated SPR-CX recovery technique, together with the effectivity obtained with the non-equilibrated SPR technique ( $\mathcal{E}_{\text{SPR}}$  curve). This graph clearly shows the improvement obtained with the SPR-CX recovery technique, with effectivities very close to 1.

Table 3 shows the estimate  $Q(\mathbf{e}_{es}) = \mathcal{E}_1$ , the exact error  $Q(\mathbf{e})$ , the effectivity index for the QoI  $\theta_{\text{QoI}}$  and the effectivity of the error estimator  $\theta$ . For this problem the exact value of the QoI is  $\bar{\sigma}_x = 0.06$ . Table 3 indicates that the equilibrated recovery procedure (SPR-CX) provides very accurate estimations of the error in the QoI and in the value of the QoI itself.

## 6.2 Problem 2: L-Shape plate

Let us consider the singular problem of a finite portion of an infinite domain with a reentrant corner. The model is loaded on the boundary with the tractions corresponding to the first symmetric term of the asymptotic expansion that describes the exact solution under mode I and mode II loading conditions around the singular vertex, see Figure 14. The exact values of boundary tractions on the boundaries

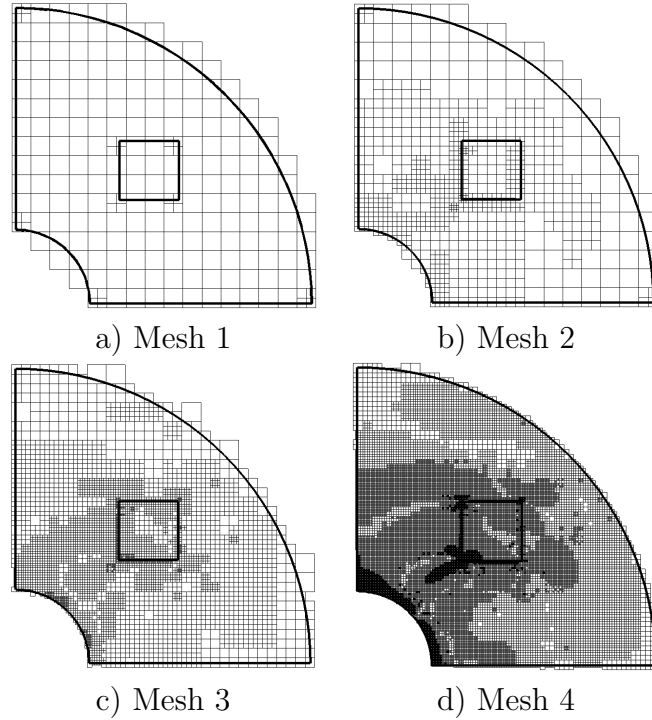


Fig. 11: Problem 1.c. Sequence of meshes for the mean stress in a domain of interest.

Tab. 3: Problem 1.c. Values for the error estimate  $\mathcal{E}_1$  and effectivities.

dof	$Q(\mathbf{e}_{es}) = \mathcal{E}_1$	$Q(\mathbf{e})$	$\theta$	$\theta_{QoI}$
528	$1.659077 \cdot 10^{-3}$	$1.738923 \cdot 10^{-3}$	0.95408307	1.00119769
1,350	$8.225557 \cdot 10^{-5}$	$7.687113 \cdot 10^{-5}$	1.07004501	1.00008077
5,131	$2.678276 \cdot 10^{-5}$	$3.016986 \cdot 10^{-5}$	0.88773219	0.99994919
19,573	$1.108891 \cdot 10^{-5}$	$1.131773 \cdot 10^{-5}$	0.97978268	0.99999657
79,442	$1.655151 \cdot 10^{-6}$	$1.613254 \cdot 10^{-6}$	1.02597049	1.00000063

represented by discontinuous thick lines have been imposed in the FE analyses.

The exact displacement and stress fields for this singular elasticity problem can be found in [26]. Exact values of the generalised stress intensity factors (GSIF) [26] have been taken as  $K_I = 1$  and  $K_{II} = 0$ . The material parameters are Young's modulus  $E = 1000$ , and Poisson's ratio  $\nu = 0.3$ . As the analytical solution of this problem is singular at the reentrant corner of the plate, for the recovery of the dual and primal fields we apply the *singular+smooth* decomposition of the stresses as explained in Section 5. We utilise a domain integral method based on extraction functions to obtain an approximation of the recovered singular part as

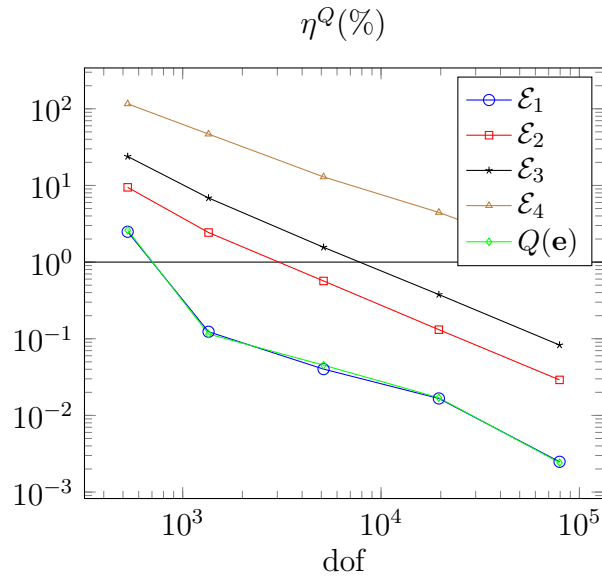


Fig. 12: Problem 1.c. Evolution of the relative error  $\eta^Q$  considering the error estimates in (16) and the exact error  $Q(\mathbf{e})$ .

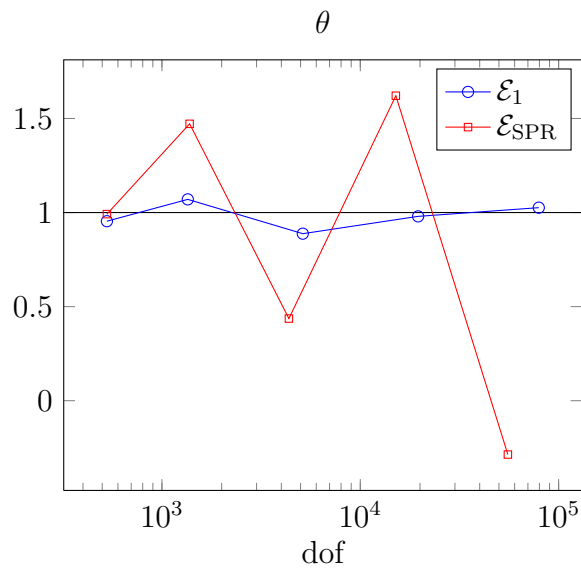


Fig. 13: Problem 1.c. Evolution of the effectivity index  $\theta$  considering equilibrated and non-equilibrated recovery,  $\mathcal{E}_{\text{SPR}}$ .

explained in [13].

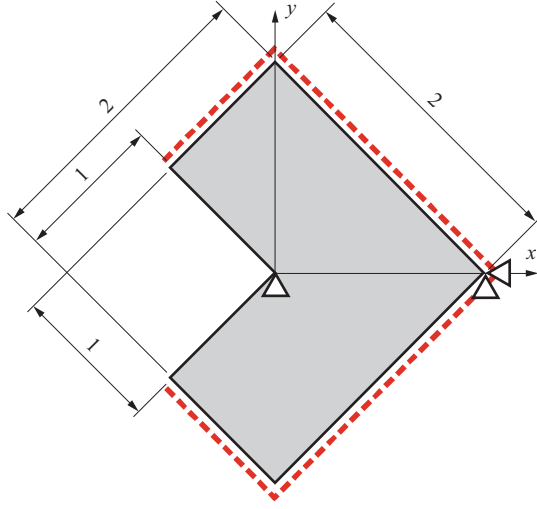


Fig. 14: Problem 2. L-shaped domain.

### 6.2.1 Generalised stress intensity factor.

In this example, we consider the problem of evaluating the GSIF that characterises the singular solution in problems with reentrant corners and cracks as the quantity of interest.

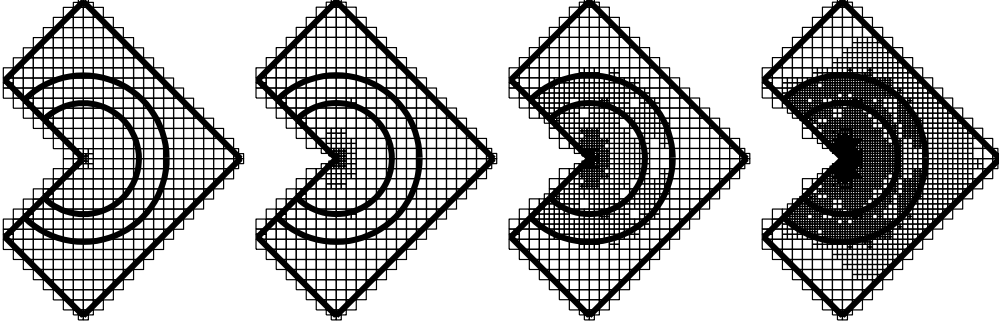
From [26, 30] we take the expression to evaluate the GSIF, which reads:

$$K^{(1,2)} = -\frac{1}{C} \int_{\Omega_i} \left( \sigma_{jk}^{(1)} u_k^{(2)} - \sigma_{jk}^{(2)} u_k^{(1)} \right) \frac{\partial q}{\partial x_j} d\Omega \quad (47)$$

where  $u^{(1)}$ ,  $\sigma^{(1)}$  are the displacement and stress fields from the FE solution,  $u^{(2)}$ ,  $\sigma^{(2)}$  are the auxiliary fields used to extract the GSIFs in mode I or mode II and  $C$  is a constant that is dependent on the geometry.  $q$  is an arbitrary  $C^0$  function used to define the extraction zone which must take the value of 1 at the singular point and 0 at the boundaries represented by discontinuous thick lines, shown in Figure 14. The FE approximation to the evaluation of (47) provides the equivalent forces vector used in the dual problem to extract the GSIFs values.

Figure 15 shows the Cartesian meshes used to solve the primal and dual problems for the mode I case. For the dual problem, we use the same Dirichlet conditions as shown in Figure 14 and the set of nodal forces used to extract the QoI in the annular domain  $\Omega_i$ , defined by the selected plateau function  $q$ , shown in Figure 15.

In order to impose equilibrium conditions during the recovery of the stresses we use the following approach. For the primal solution we enforce internal equilibrium and compatibility in  $\Omega$ , and boundary equilibrium all along the Neumann boundary. For the dual problem, note that the expressions for the extraction fields  $\mathbf{b}_d$  and  $\mathbf{t}_d$  associated to  $K^{(1,2)}$  are not directly available. Therefore, in this case

Fig. 15: Problem 2. Cartesian meshes with  $h$ -adaptive refinement.Tab. 4: Problem 2. Stress intensity factor  $K_I$  as QoI.

dof	$Q(\mathbf{e}_{es}) = \mathcal{E}_1$	$Q(\mathbf{e})$	$\theta$	$\theta_{QoI}$
1,107	$8.304661 \cdot 10^{-3}$	$9.566751 \cdot 10^{-3}$	0.86807538	0.99873791
2,955	$2.115235 \cdot 10^{-3}$	$2.106131 \cdot 10^{-3}$	1.00432253	1.00000910
10,335	$5.155068 \cdot 10^{-4}$	$5.204793 \cdot 10^{-4}$	0.99044626	0.99999503
37,177	$1.319637 \cdot 10^{-4}$	$1.325887 \cdot 10^{-4}$	0.99528650	0.99999938

we enforce internal equilibrium within all the domain except in  $\Omega_i$ . Compatibility is enforced within all the domain.

Table 4 shows the results for the L-Shape problem under mode I loading conditions. Similarly to the results for other QoIs, we observe that the proposed technique provides an accurate estimate  $Q(\mathbf{e}_{es})$  of the exact error  $Q(\mathbf{e})$ . The effectivity index  $\theta$  is always close to the optimal value  $\theta = 1$  and the worst effectivity measured in this case is  $\theta = 0.868$  for 1107 dof. As a result, an in agreement with previous results, the effectivity  $\theta_{QoI}$  is highly accurate. Table 5 shows the same results for the problem under mode II loading conditions. Again, despite of the fact that equilibrium has not been imposed in  $\Omega_i$ , we observe a good behaviour of the error indicator and very accurate effectivities, both for the error estimate and for the QoI.

Tab. 5: Problem 2. Stress intensity factor  $K_{II}$  as QoI.

dof	$Q(\mathbf{e}_{es}) = \mathcal{E}_1$	$Q(\mathbf{e})$	$\theta$	$\theta_{QoI}$
1,107	$1.707817 \cdot 10^{-3}$	$1.797716 \cdot 10^{-3}$	0.94999310	0.99991010
2,941	$4.451169 \cdot 10^{-4}$	$4.235498 \cdot 10^{-4}$	1.05091987	1.00002157
9,743	$9.956510 \cdot 10^{-5}$	$1.009319 \cdot 10^{-4}$	0.98645854	0.99999863
34,651	$2.706030 \cdot 10^{-5}$	$2.639920 \cdot 10^{-5}$	1.02504270	1.00000066



Figure 16 shows the evolution of the relative error with respect to the number of dof for the two loading conditions. Figures 17 and 18 show the evolution of the effectivity index of the error estimators obtained with the locally equilibrated SPR-CX technique, and with the non-equilibrated recovery as we increase the number of dof. The results indicate that the proposed methodology accurately evaluates the error in the QoI, giving values of  $\theta$  close to 1 and considerably improving the results obtained with the original SPR technique.

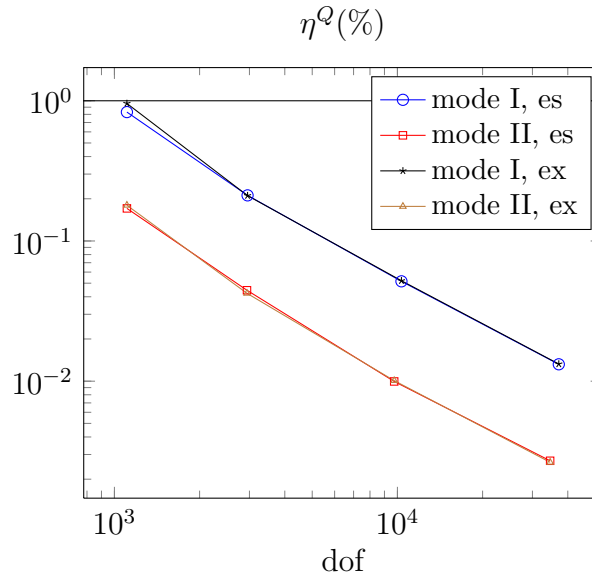


Fig. 16: Problem 2. Evolution of the relative error  $\eta^Q$  for the SIF.

## 7 CONCLUSIONS AND FUTURE WORK.

In this paper, we presented an *a posteriori* recovery-based strategy that aims to control the error in quantities of interest. The proposed technique is based on the use of a recovery procedure that considers locally equilibrated stress fields for both the primal and the dual problem.

To recover the solution for the primal problem the formulation is based on [13]. In order to enforce equilibrium conditions when evaluating the recovered stress fields for the dual problem, we have considered an approach that allows to express the functional which defines some QoI in terms of body loads and boundary tractions applied to the dual problem. The proposed technique has been tested on different quantities of interest: mean displacements on a domain of interest and along a boundary, mean stresses in a domain of interest and the generalised stress intensity factor in the singular problem of a reentrant corner. The proposed technique has provided accurate global and local evaluations of the error in the

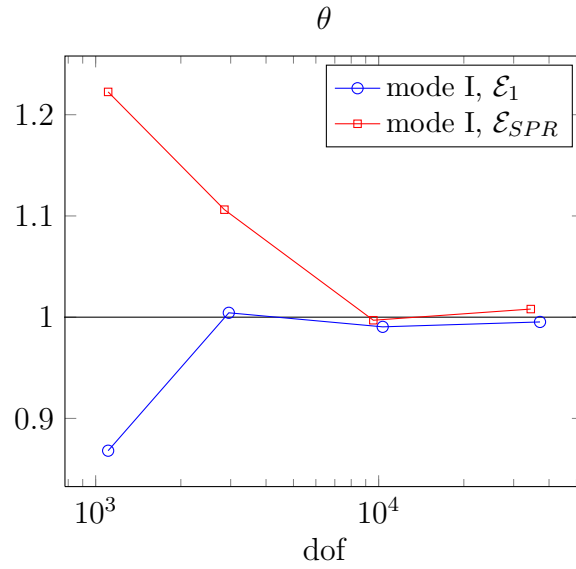


Fig. 17: Problem 2. Evolution of the effectivity index  $\theta$  for  $K_I$

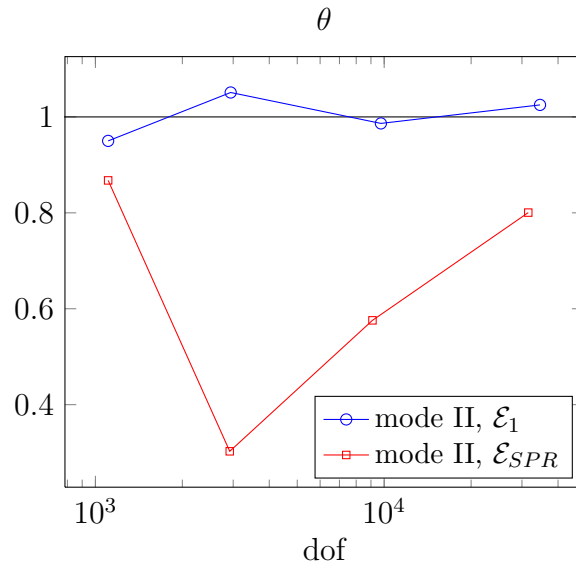


Fig. 18: Problem 2. Evolution of the effectivity index  $\theta$  for  $K_{II}$

different quantities of interest analysed, improving the results obtained with the original SPR technique.

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