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Abstract

This note shows that there are close connections between the determinacy of a stationary state equilibrium and its stability under learning whenever agents try to estimate both the law of motion of the state variable and the stationary state value.

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1 Introduction

Although early temporary equilibrium literature highlights that determinacy can be closely related to stability in the dynamics with learning (Grandmont and Laroque (1986), (1990), Guesnerie (1992), Guesnerie and Woodford (1991), Marcet and Sargent (1989) or Moore (1993)), many recent examples, following Grandmont (1991), suggest a lack of link between these properties (Duffy (1994), Evans and Honkapohja (1992), Grandmont and Laroque (1991)). The purpose of this note is to show that determinacy is in fact a benchmark for stability under learning of a stationary state rational expectations equilibrium as soon as economic agents try to discover this equilibrium.

2 A preliminary example

I consider a simple model in which the current state of the system is a real number x_t that depends on the common forecast of the next state x^e_{t+1} and on the predetermined state x_{t-1} through the temporary equilibrium relation

$$\gamma x^e_{t+1} + x_t + \delta x_{t-1} = 0, \tag{1}$$

where $\gamma \neq 0$, so that expectations matter. In order to close the model (1), I first assume perfect foresight, i.e., that the forecast x^e_{t+1} equals the actual realization x_{t+1} in (1). The resulting dynamics writes

$$\gamma x_{t+1} + x_t + \delta x_{t-1} = 0. \tag{2}$$

It is governed by the two perfect foresight roots λ_1 and λ_2 of the characteristic polynomial associated with (2). Let $|\lambda_1| < |\lambda_2|$ by definition. Let also $1+\delta+\gamma \neq 0$, which ensures that the stationary state $x_t = \bar{x} \equiv 0$ is regular. This equilibrium is then said to be a determinate source if $|\lambda_1| > 1$, a determinate saddle if $|\lambda_1| < 1 < |\lambda_2|$, and an indeterminate sink if $|\lambda_2| < 1$ (see, e.g., Grandmont and Laroque (1991)).

As Grandmont (1991) first emphasized, agents can learn an indeterminate stationary state when they try to discover the law of motion of the state variable, i.e., the value of some perfect foresight root. Assume indeed that agents *a priori* believe that $x_t = g(t)x_{t-1}$ (where $g(t)$ represents the time t estimate of some perfect foresight root) and accordingly expect x^e_{t+1} to be equal to $g(t)x_t$. Reintroducing such a forecast into (1) determines the actual state as a function of the past state,

$$x_t = [-\delta/(1+\gamma g(t))]\, x_{t-1}, \tag{3}$$

provided that $1+\gamma g(t) \neq 0$. A simple myopic learning rule, which fits the iterative version of the expectational stability criterion used by Evans and Honkapohja (1992), consists to take the time t actual growth rate in (3) as time $(t+1)$ growth rate estimate, that is,

$$g(t+1) = -\delta/(1+\gamma g(t)). \tag{4}$$

The fixed points of (4) are such that $g(t+1) = g(t)$ and coincide consequently with the two roots λ_1 and λ_2. Whether agents can locally learn λ_i $(i = 1, 2)$ depends on the stability properties of the local dynamics with learning, obtained by linearizing (4) at point $g(t) = \lambda_i$ whatever t is,

$$(g(t+1) - \lambda_i) = (\lambda_i/\lambda_j)(g(t) - \lambda_i), \tag{5}$$

for $j = 1, 2$ and $j \neq i$ (I used the identities $1/\gamma = -(\lambda_1 + \lambda_2)$ and $\delta/\gamma = \lambda_1\lambda_2$ to get (5)). It follows from standard algebra that λ_i is locally stable in the learning dynamics (5) as soon as $|\lambda_i/\lambda_j| < 1$, so that λ_1 is locally stable while λ_2 is locally unstable in this dynamics. Therefore, in the long run, the system will evolve according to the law $x_t = \lambda_1 x_{t-1}$ and will converge towards the stationary state $\bar{x} \equiv 0$ if $|\lambda_1| < 1$, which clearly covers the indeterminate sink configuration of (2).

My aim is to show, however, that close connections between determinacy and stability under learning reappear whenever agents try to discover not only some perfect foresight root but also the stationary state value, i.e., they believe that

$x_t = g(t)x_{t-1} + \bar{x}(t)$, where $\bar{x}(t)$ stands for their time t estimate of the stationary state value. Given this belief, their forecast x^e_{t+1} is equal to $g(t)x_t + \bar{x}(t)$, and, as described above, the actual dynamics, which comes from reintroducing the agents' forecast into (1), writes now

$$\gamma(g(t)x_t + \bar{x}(t)) + x_t + \delta x_{t-1} = 0$$

$$\Leftrightarrow x_t = \left[-\delta/(1+\gamma g(t))\right] x_{t-1} + \left[-\gamma\bar{x}(t)/(1+\gamma g(t))\right], \tag{6}$$

provided that $1 + \gamma g(t) \neq 0$. According to the myopic learning scheme, agents make their time $(t+1)$ estimates $(g(t+1), \bar{x}(t+1))$ equal to the time t actual realizations of these two variables in (6), namely,

$$g(t+1) = -\delta/(1+\gamma g(t)), \tag{7}$$

$$\text{and} \quad \bar{x}(t+1) = -\gamma\bar{x}(t)/(1+\gamma g(t)). \tag{8}$$

Since (4) and (7) are identical, agents locally discover λ_1 and the system converges towards the stationary state only if $|\lambda_1| < 1$. Nevertheless, unlike the previous case where the learning dynamics was in fact reduced to (7), agents have also to learn the stationary state value in (8), with $g(t) = \lambda_1$ (observe that $\bar{x}(t) = \bar{x}(t+1) \Rightarrow \bar{x}(t) = \bar{x}$ in (8)). They succeed in learning it if and only if

$$|-\gamma/(1+\gamma\lambda_1)| < 1 \Leftrightarrow |1/(\lambda_1 + 1/\gamma)| = |1/\lambda_2| < 1. \tag{9}$$

This shows that the system converges towards the stationary state if and only if both $|\lambda_1| < 1$ (which comes from (7)) and $|\lambda_2| > 1$ (which comes from (8)), i.e., if the stationary state is a determinate saddle. The issue I tackle in the next section is whether this equivalence between saddle determinacy and stability under learning may arise in a more general framework than (1). Actually this connection still holds true if $L \geq 1$ predetermined variables enter the temporary equilibrium relation.

3 One-Step Forward Looking Models

I shall deal with the model

$$\gamma x^e_{t+1} + x_t + \sum_{l=1}^{L} \delta_l x_{t-l} = 0. \tag{10}$$

The perfect foresight dynamics corresponding to (10) is now governed by $(L+1)$ perfect foresight roots $\lambda_1, \ldots, \lambda_{L+1}$ (with $|\lambda_1| < \cdots < |\lambda_{L+1}|$). If the stationary state ($\bar{x} \equiv 0$) of this dynamics is regular, i.e., $1+\delta_1+\cdots+\delta_L+\gamma \neq 0$, then it is said to be a determinate source when $|\lambda_L| > 1$, a determinate saddle when $|\lambda_L| < 1 < |\lambda_{L+1}|$ and an indeterminate sink when $|\lambda_{L+1}| < 1$. In the sequel, however, I will assume that agents are not aware of information that would ensure perfect foresight. Instead, as in the previous section (where $L = 1$), they believe that

$$x_t = \sum_{l=1}^{L} g_l(t) x_{t-l} + \bar{x}(t), \tag{11}$$

where the coefficients $g_l(t)$ $(l = 1, \ldots, L)$ are the time t estimates of the law of motion of the state variable, and $\bar{x}(t)$ stands for the time t estimate of the stationary state value. Given this belief, agents form their forecast by iterating once (11),

$$x^e_{t+1} = \sum_{l=1}^{L} g_l(t) x_{t-l+1} + \bar{x}(t). \tag{12}$$

Recall that, if agents use a myopic learning scheme, their new estimates $(g_1(t+1), \ldots, g_L(t+1), \bar{x}(t+1))$ are the coefficients that govern the actual dynamics, obtained by reintroducing the forecast (12) into the temporary equilibrium relation (10). It is simple to verify that the current state x_t is actually related to x_{t-l} $(l = 1, \ldots, L)$ according to the law

$$x_t = -\sum_{l=1}^{L} \left[(\gamma g_{l+1}(t) + \delta_l)/(1 + \gamma g_1(t)) \right] x_{t-l} + \left[-\gamma \bar{x}(t)/(1 + \gamma g_1(t)) \right], \tag{13}$$

with $g_{L+1}(t) = 0$, and provided that $1 + \gamma g_1(t) \neq 0$. The myopic learning dynamics then follows directly from (13),

$$g_l(t+1) = -(\gamma g_{l+1}(t) + \delta_l)/(1 + \gamma g_1(t)), \quad \text{for } l = 1, \ldots, L, \tag{14}$$

$$\text{and} \quad \bar{x}(t+1) = -\gamma \bar{x}(t)/(1+\gamma g_1(t)). \tag{15}$$

As it was already the case for (7) and (8), learning the law of motion, i.e., a fixed point $(g_1^*, \ldots, g_L^*)$ of the L-dimensional system (14), is independent of learning the stationary state value in (15). Since the homogenous part of (11) evaluated at $(g_1^*, \ldots, g_L^*)$ governs the evolution of the state variable restricted to some L-dimensional subspace of the dynamics with perfect foresight, the roots of the characteristic polynomial associated with (11) for $g_l = g_l^*$ are any L different perfect foresight roots among $(L+1)$, so that (14) admits in fact $(L+1)$ fixed points. It is shown in Gauthier (1999) that the set of coefficients $(g_1^*, \ldots, g_L^*)$ corresponding to the L perfect foresight roots of lowest modulus $\lambda_1, \ldots, \lambda_L$ is the only one to be locally stable in the learning dynamics (14). The economic system is consequently stable only if $|\lambda_L| < 1$. As a result, should agents be aware of the stationary state value, then (15) would disappear from the learning dynamics and the system could converge towards an indeterminate sink stationary state. Here, however, and this is the new point, equation (15) matters. This difference turns out to be of importance as the following result shows.

Proposition. *Consider a linear one-step forward looking economy with an arbitrary, but fixed, number of predetermined variables. Then a regular stationary state equilibrium is locally stable in the dynamics with learning* (14)-(15) *if and only it is a determinate saddle in the dynamics with perfect foresight corresponding to* (10)*, i.e., if and only if* $|\lambda_L| < 1 < |\lambda_{L+1}|$.

Proof. For the stationary state value $\bar{x}(t) = \bar{x}(t+1) = 0$ of the first order linear difference (15) to be locally stable in the dynamics with learning, it must be the case that

$$\left| -\frac{\gamma}{1+\gamma g_1^*} \right| < 1. \tag{16}$$

For t large enough, agents will discover the set of coefficients $(g_1^*, \ldots, g_L^*)$ associated with the perfect foresight roots $\lambda_1, \ldots, \lambda_L$. For this set of coefficients, $\lambda_1, \ldots, \lambda_L$ are the L roots of the characteristic polynomial associated with the homogenous part of

(11), with $g_l(t) = g_l^*$ $(l = 1, \ldots, L)$. It is then straightforward to show that

$$g_1^* = \lambda_1 + \cdots + \lambda_L. \tag{17}$$

By the same way, given that $\lambda_1, \ldots, \lambda_{L+1}$ are the $(L+1)$ roots of the characteristic polynomial associated with the perfect foresight dynamics obtained from (10), one gets

$$1/\gamma = -(\lambda_1 + \cdots + \lambda_{L+1}). \tag{18}$$

Thus condition (16) rewrites

$$\left| -\frac{\gamma}{1 + \gamma g_1^*} \right| = \left| \frac{1}{(1/\gamma) + g_1^*} \right| = \left| \frac{1}{\lambda_{L+1}} \right| < 1,$$

which shows the result. ■

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