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## To cite this version:

Laurent Mazet, Harold Rosenberg. On minimal spheres of area $4 \pi$ and rigidity. Commentarii Mathematici Helvetici, 2014, pp.921-928. 10.4171/CMH/338 . hal-00730816

## HAL Id: hal-00730816

## https://hal.science/hal-00730816

Submitted on 21 Feb 2023

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# On minimal spheres of area $4 \pi$ and rigidity 

Laurent Mazet, Harold Rosenberg


#### Abstract

Let $M$ be a complete Riemannian 3-manifold with sectional curvatures between 0 and 1. A minimal 2 -sphere immersed in $M$ has area at least $4 \pi$. If an embedded minimal sphere has area $4 \pi$, then $M$ is isometric to the unit 3 -sphere or to a quotient of the product of the unit 2 -sphere with $\mathbb{R}$, with the product metric. We also obtain a rigidity theorem for the existence of hyperbolic cusps. Let $M$ be a complete Riemannian 3-manifold with sectional curvatures bounded above by -1 . Suppose there is a 2 -torus $T$ embedded in $M$ with mean curvature one. Then the mean convex component of $M$ bounded by $T$ is a hyperbolic cusp;,i.e., it is isometric to $T \times \mathbb{R}$ with the constant curvature -1 metric: $e^{-2 t} d \sigma_{0}^{2}+d t^{2}$ with $d \sigma_{0}^{2}$ a flat metric on $T$.


Keywords: area of minimal sphere, rigidity of 3-manifolds, hyperbolic cusp.

## 1 Introduction

Consider a smooth $\left(C^{\infty}\right)$ complete metric on the 2-sphere $S$ whose curvature is between 0 and 1 . It is well known that a simple closed geodesic in $S$ has length at least $2 \pi$ (see [4] or Klingenberg's theorem in higher dimension $[3,2])$. It is less well known that when such an $S$ has a simple closed geodesic of length exactly $2 \pi$, then $S$ is isometric to the unit 2 -sphere $\mathbb{S}_{1}^{2}$. This result is proved in [1], and the authors attribute the theorem to E. Calabi.

With this in mind, we consider what happens in a complete 3 -manifold $M$ with sectional curvatures between 0 and 1 (henceforth we suppose this curvature condition on $M$, unless stated otherwise).

Let $\Sigma$ be an embedded minimal 2 -sphere in $M$. Then the Gauss-Bonnet theorem and the Gauss equation tells us that the area of $S$ is at least $4 \pi$ : indeed we have

$$
\begin{equation*}
4 \pi=\int_{\Sigma} \bar{K}_{\Sigma}=\int \operatorname{det}(A)+K_{T \Sigma} \leq \int_{\Sigma} 1=A(\Sigma) \tag{1}
\end{equation*}
$$

with $\operatorname{det}(A)$ the determinant of the shape operator which is non positive. We prove in Theorem 1, that when the area of $\Sigma$ equals $4 \pi$, then $M$ is isometric to the unit 3 -sphere $\mathbb{S}_{1}^{3}$ or to a quotient of the product of the unit 2 -sphere with $\mathbb{R}, \mathbb{S}_{1}^{2} \times \mathbb{R}$, with the product metric.

We remark that Theorem 1 does not hold for embedded minimal tori. Given $\varepsilon$ greater than zero, there are Berger spheres with curvatures between 0 and 1 , which contain embedded minimal tori of area less than $\varepsilon$. But a minimal sphere always has area at least $4 \pi$.

It would be interesting to know what happens in higher dimensions. In the unit $n$-sphere $\mathbb{S}_{1}^{n}$, a compact minimal hyper-surface $\Sigma$ always has volume at least the volume of the equatorial $n-1$ sphere $\mathbb{S}_{1}^{n-1}$. Is there a rigidity theorem when one allows metrics on $\mathbb{S}^{n}(=M)$, of sectional curvatures between 0 and 1? Two questions arise. First, does an embedded minimal hyper-sphere $\Sigma$ in $M$ have volume at least the volume of $\mathbb{S}_{1}^{n-1}$. If this is so, and if $\Sigma$ is an embedded minimal hyper-sphere with volume exactly the volume of $\mathbb{S}_{1}^{n-1}$, is $M$ isometric to $\mathbb{S}_{1}^{n}$ or to $\mathbb{S}_{1}^{n-1} \times \mathbb{R}$ ?

In the same spirit as Theorem 1, we prove a rigidity theorem for hyperbolic cusps. We recall that a 3 dimensional hyperbolic cusp is a manifold of the form $T \times \mathbb{R}$ with $T$ a 2-torus and the hyperbolic metric $e^{-2 t} d \sigma_{0}^{2}+d t^{2}$ with $d \sigma_{0}^{2}$ a flat metric on $T$. In Theorem 2, we prove that if $M$ is a complete Riemannian manifold with sectional curvatures bounded above by -1 and $T$ is a constant mean curvature 1 torus embedded in $M$ then the mean convex side of $T$ in $M$ is isometric to a hyperbolic cusp.

## 2 Minimal spheres of area $4 \pi$ and rigidity of 3manifolds

In this section, we prove a rigidity result for a Riemannian 3-manifold $M$ whose sectional curvatures are between 0 and 1 . As explained in the introduction, any minimal sphere in such a manifold has area at least $4 \pi$.

We denote by $\mathbb{S}_{1}^{n}$ the sphere of dimension $n$ with constant sectional curvature 1 . We then have the following result.

Theorem 1. Let $M$ be a complete Riemannian 3-manifold whose sectional curvatures satisfy $0 \leq K \leq 1$. Assume that there exists an embedded minimal sphere $\Sigma$ in $M$ with area $4 \pi$. Then the manifold $M$ is isometric either to the sphere $\mathbb{S}_{1}^{3}$ or to a quotient of $\mathbb{S}_{1}^{2} \times \mathbb{R}$.

Proof. Let $\Phi$ be the map $\Sigma \times \mathbb{R} \rightarrow M,(p, t) \mapsto \exp _{p}(t N(q))$ where $N$ is a unit normal vector field along $\Sigma$. In the following, we focus on $\Sigma \times \mathbb{R}_{+}$; by
symmetry of the configuration, the study is similar for $\Sigma \times \mathbb{R}_{-}$.
$\Sigma$ is compact, so there is an $\varepsilon$ such that $\Phi$ is an immersion and even an embedding on $\Sigma \times[0, \varepsilon)$. Let us define

$$
\varepsilon_{0}=\sup \{\varepsilon>0 \mid \Phi \text { is an immersion on } \Sigma \times[0, \varepsilon)\}
$$

$\varepsilon_{0}$ can be equal to $+\infty$. Using $\Phi$, we pull back the Riemannian metric of $M$ to $\Sigma \times\left[0, \varepsilon_{0}\right)$. This metric can be written $d s^{2}=d \sigma_{t}^{2}+d t^{2}$ where $d \sigma_{t}^{2}$ is a smooth family of metrics on $\Sigma$. With this metric, $\Phi$ becomes a local isometry from $\Sigma \times\left[0, \varepsilon_{0}\right)$ to $M$ and $\left(\Sigma \times\left[0, \varepsilon_{0}\right), d s^{2}\right)$ has sectional curvatures between 0 and 1 . Moreover, $\Sigma_{0}$ is minimal and has area $4 \pi$. Actually, we will prove the following facts.

Claim. The metric $d \sigma_{0}^{2}$ has constant sectional curvature 1 so $\left(\Sigma, d \sigma_{0}^{2}\right)$ is isometric to $\mathbb{S}_{1}^{2}$. Moreover, we have two cases

1. $\varepsilon_{0}=\pi / 2$ and $d \sigma_{t}^{2}=\sin ^{2} t d \sigma_{0}^{2}$ or
2. $\varepsilon_{0}=+\infty$ and $d \sigma_{t}^{2}=d \sigma_{0}^{2}$

Let us denote by $\Sigma_{t}=\Sigma \times\{t\}$ the equidistant surfaces. We denote by $H(p, t)$ the mean curvature of $\Sigma_{t}$ at the point $(p, t)$ with respect to the unit normal vector $\partial_{t}$. We also define $\lambda(p, t) \geq 0$ such that $H+\lambda$ and $H-\lambda$ are the principal curvature of $\Sigma_{t}$ at $(p, t)$. We notice that $\lambda=0$ if $\Sigma_{t}$ is umbilical at $(p, t)$.

The surfaces $\Sigma_{t}$ are spheres so, using the Gauss equation, the GaussBonnet formula implies:

$$
4 \pi=\int_{\Sigma_{t}} \bar{K}_{\Sigma_{t}}=\int_{\Sigma_{t}}(H+\lambda)(H-\lambda)+K_{t}=\int_{\Sigma_{t}} H^{2}-\lambda^{2}+K_{t}
$$

where $\bar{K}_{\Sigma_{t}}$ is the intrinsic curvature of $\Sigma_{t}$ and $K_{t}$ is the sectional curvature of the ambient manifold of the tangent space to $\Sigma_{t}$. Since $K_{t} \leq 1$, we obtain the following inequality

$$
\begin{equation*}
\int_{\Sigma_{t}} \lambda^{2}=\int_{\Sigma_{t}} H^{2}+K_{t}-4 \pi \leq \int_{\Sigma_{t}} H^{2}+A\left(\Sigma_{t}\right)-4 \pi \tag{2}
\end{equation*}
$$

where $A\left(\Sigma_{t}\right)$ is the area of $\Sigma_{t}$. In the following, we denote by $F(t)$ the right hand side of this inequality.

Claim 1. $F$ is vanishing on $\left[0, \varepsilon_{0}\right)$.

Since $\Sigma_{0}$ is minimal and has area $4 \pi$, we have $F(0)=0$. We notice that this implies that $\lambda(p, 0)=0$ so $\Sigma_{0}$ is umbilical and $K_{T \Sigma_{0}}=1$. Thus $\left(\Sigma_{0}, d \sigma_{0}\right)$ is isometric to $\mathbb{S}_{1}^{2}$.

We have the usual formula:

$$
\begin{equation*}
\frac{\partial}{\partial t} A\left(\Sigma_{t}\right)=-\int_{\Sigma_{t}} 2 H \quad \text { and } \quad \frac{\partial H}{\partial t}=\frac{1}{2}\left(\operatorname{Ric}\left(\partial_{t}\right)+\left|A_{t}\right|^{2}\right) \tag{3}
\end{equation*}
$$

where $A_{t}$ is the shape operator of $\Sigma_{t}$ and Ric is the Ricci tensor of $\Sigma \times\left[0, \varepsilon_{0}\right)$. Since the sectional curvatures of $M \times\left[0, \varepsilon_{0}\right)$ are non-negative, Ric is nonnegative. So the second formula above implies that $H$ is increasing and thus $H \geq 0$ everywhere. Let us now compute and estimate the derivative of $F$ :

$$
\begin{aligned}
F^{\prime}(t) & =\int_{\Sigma_{t}}\left(2 H \frac{\partial H}{\partial t}-2 H^{3}\right)-\int_{\Sigma_{t}} 2 H \\
& =\int_{\Sigma_{t}} H\left(\operatorname{Ric}\left(\partial_{t}\right)+\left|A_{t}\right|^{2}-2 H^{2}-2\right) \\
& =\int_{\Sigma_{t}} H\left(\left(\operatorname{Ric}\left(\partial_{t}\right)-2\right)+\left((H+\lambda)^{2}+(H-\lambda)^{2}-2 H^{2}\right)\right) \\
& =\int_{\Sigma_{t}} H\left(\left(\operatorname{Ric}\left(\partial_{t}\right)-2\right)+2 \lambda^{2}\right) \\
& \leq 2 \int_{\Sigma_{t}} H \lambda^{2}
\end{aligned}
$$

where the last inequality comes from $\operatorname{Ric}\left(\partial_{t}\right)-2 \leq 0$ because of the hypothesis on the sectional curvatures. If we choose $\varepsilon<\varepsilon_{0}$, there is a constant $C \geq 0$ such that $H \leq C$ on $\Sigma \times[0, \varepsilon]$. So for $t \in[0, \varepsilon]$, using the inequality $(2)$, we get $F^{\prime}(t) \leq 2 C F(t)$. Then $F(t) \leq F(0) e^{2 C t}=0$ on $[0, \varepsilon]$. So $F \leq 0$ on $\left[0, \varepsilon_{0}\right)$ and, because of $(2), F=0$ on $\left[0, \varepsilon_{0}\right)$; this finishes the proof of Claim 1.

The first consequence of Claim 1 is that all the equidistant surfaces $\Sigma_{t}$ are umbilical (see inequality $(2)$ ); so $\lambda \equiv 0$. In the computation of the derivative of $F$, this implies that

$$
\int_{\Sigma_{t}} H\left(\operatorname{Ric}\left(\partial_{t}\right)-2\right)=0
$$

Since $H\left(\operatorname{Ric}\left(\partial_{t}\right)-2\right) \leq 0$ everywhere, we obtain

$$
\begin{equation*}
H\left(\operatorname{Ric}\left(\partial_{t}\right)-2\right)=0 \text { everywhere. } \tag{4}
\end{equation*}
$$

Moreover the umbilicity and (3) implies that $\frac{\partial H}{\partial t}=\frac{1}{2} \operatorname{Ric}\left(\partial_{t}\right)+H^{2}$. We now prove the following claim

Claim 2. Let $(p, t) \in \Sigma \times\left[0, \varepsilon_{0}\right)(t>0)$ be such that $H(p, t)>0$ then $H(q, t)>0$ for any $q \in \Sigma$

In other words, when the mean curvature is positive at a point of an equidistant, it is positive at any point of this equidistant. We recall that $H$ is increasing in the $t$ variable so when it becomes positive it stays positive.

So assume that $H(p, t)>0$ and consider $\Omega=\{q \in \Sigma \mid H(q, t)>0\}$ which is a nonempty open subset of $\Sigma$. Let $q \in \Omega$. Since $H(q, t)>0$, $\operatorname{Ric}\left(\partial_{t}\right)(q, t)=2$ by (4). Thus $\operatorname{Ric}\left(\partial_{t}\right)(r, t)=2$ for any $r \in \bar{\Omega}$. So if $r \in \bar{\Omega}$, $\operatorname{Ric}\left(\partial_{t}\right)(r, s)>0$ for $s<t$, close to $t$ and, by (3), this implies that $H(r, t)>0$ and $r \in \Omega$. So $\Omega$ is closed and $\Omega=\Sigma$. This finishes the proof of Claim 2.

Let us assume that there is an $\varepsilon_{1}>0$ such that $H(p, t)=0$ for $(p, t) \in$ $\Sigma \times\left[0, \varepsilon_{1}\right]$ and $H(p, t)>0$ for any $(p, t) \in \Sigma \times\left(\varepsilon_{1}, \varepsilon_{0}\right)$. Because of the evolution equation of $H$, this implies that $\operatorname{Ric}\left(\partial_{t}\right)=0$ on $\Sigma \times\left[0, \varepsilon_{1}\right]$. On $\Sigma \times\left(\varepsilon_{1}, \varepsilon_{0}\right)$, we have $\operatorname{Ric}\left(\partial_{t}\right)=2$ because of (4). So by continuity of $\operatorname{Ric}\left(\partial_{t}\right)$, we get a contradiction and then we have two possibilities

1. $H=0$ on $\Sigma \times\left[0, \varepsilon_{0}\right)$ and $\operatorname{Ric}\left(\partial_{t}\right)=0$ on $\Sigma \times\left[0, \varepsilon_{0}\right)$.
2. $H>0$ on $\Sigma \times\left(0, \varepsilon_{0}\right)$ and $\operatorname{Ric}\left(\partial_{t}\right)=2$ on $\Sigma \times\left[0, \varepsilon_{0}\right)$.

In the first case, this implies that the sectional curvature of any 2-plane orthogonal to $\Sigma_{t}$ is zero. Thus $d \sigma_{t}^{2}=d \sigma_{0}^{2}$. Since the map $\Phi$ ceases to be an immersion only if $d \sigma_{t}^{2}$ becomes singular this implies that $\varepsilon_{0}=+\infty$. Thus $\Sigma \times \mathbb{R}_{+}$with the induced metric is isometric to $\mathbb{S}_{1}^{2} \times \mathbb{R}_{+}$and $\Phi$ is a local isometry from $\mathbb{S}_{1}^{2} \times \mathbb{R}_{+}$to $M$.

In the second case, the sectional curvature of any 2-plane orthogonal to $\Sigma_{t}$ is equal to 1 . Thus $d \sigma_{t}^{2}=\sin ^{2} t d \sigma_{0}$ and $\varepsilon_{0}=\pi / 2$. This also implies that $\Phi(p, \pi / 2)$ is a point. So $\Sigma \times[0, \pi / 2]$ with the metric $d s^{2}$ is isometric to a hemisphere of $\mathbb{S}_{1}^{3}$ and the map $\Phi$ is a local isometry from that hemisphere to $M$.

Doing the same study for $\Sigma \times \mathbb{R}_{-}$, we get in the first case a local isometry $\Phi: \mathbb{S}_{1}^{2} \times \mathbb{R} \rightarrow M$ and in the second case a local isometry $\Phi: \mathbb{S}_{1}^{3} \rightarrow M$. Since $\mathbb{S}_{1}^{2} \times \mathbb{R}$ and $\mathbb{S}_{1}^{3}$ are simply connected, $\Phi$ is then the universal cover of $M$ and $M$ is then isometric to a quotient of $\mathbb{S}_{1}^{2} \times \mathbb{R}$ or $\mathbb{S}_{1}^{3}$. Since $\Phi$ is injective on $\Sigma$ this implies that in the second case, $\Phi$ is actually injective and then a global isometry.

Remark 1. In the proof, since $\Phi$ is injective on $\Sigma$, the possible quotients of $\mathbb{S}_{1}^{2} \times \mathbb{R}$ are either $\mathbb{S}_{1}^{2} \times \mathbb{R}$ or its quotient by the subgroup generated by an isometry of the form $\mathbb{S}_{1}^{2} \times \mathbb{R} \rightarrow \mathbb{S}_{1}^{2} \times \mathbb{R} ;(p, t) \mapsto\left(\alpha(p), t+t_{0}\right)$ with $\alpha$ an isometry of $\mathbb{S}_{1}^{2}$ and $t_{0} \neq 0$.

Remark 2. Something can be said about constant mean curvature $H_{0}$ spheres in a Riemannian 3 -manifold with sectional curvatures between 0 and 1 . Indeed, the computation (1) implies that the area of $\Sigma$ is larger than $\frac{4 \pi}{1+H_{0}^{2}}$, which is the area of a geodesic sphere in $\mathbb{S}_{1}^{3}$ of mean curvature $H_{0}$. Moreover, if $\Sigma$ has area $\frac{4 \pi}{1+H^{2}}$, the above proof can be adapted to prove that the mean convex side of $\Sigma$ is isometric to a spherical cap of $\mathbb{S}_{1}^{3}$ with constant mean curvature $H_{0}$ (see Theorem 2 below, for a similar result in the hyperbolic case).
Remark 3. Let $M$ be a Riemannian $n$-manifold whose sectional curvatures are between 0 and 1 and let $\Sigma$ be a minimal 2 -sphere in $M$. A computation similar to (1) proves also that the area of $\Sigma$ is larger than $4 \pi$. It also implies that, if $\Sigma$ has area $4 \pi, \Sigma$ is totally geodesic and isometric to $\mathbb{S}_{1}^{2}$.

## 3 Existence of hyperbolic cusps

Let $\left(\mathbb{T}^{2}, g\right)$ be a flat 2 torus, the manifold $\mathbb{T}^{2} \times \mathbb{R}_{+}$with the complete Riemannian metric $e^{-2 t} g+d t^{2}$ is a hyperbolic 3 -dimensional cusp. $\mathbb{T}^{2} \times \mathbb{R}$ is actually isometric to the quotient of a horoball of $\mathbb{H}^{3}$ by a $\mathbb{Z}^{2}$ subgroup of isometries of $\mathbb{H}^{2}$ leaving the horoball invariant. Any $\mathbb{T}^{2} \times\{t\}$ has constant mean curvature 1 . The following theorem says that, in certain 3 -manifolds, a constant mean curvature 1 torus is necessarily the boundary of a hyperbolic cusp.

Theorem 2. Let $M$ be a complete Riemannian 3-manifold with its sectional curvatures satisfying $K \leq-1$. Assume that there exists a constant mean curvature 1 torus $T$ embedded in $M$. Then $T$ separates $M$ and its mean convex side is isometric to a hyperbolic cusp.

As a consequence, the existence of this torus implies that $M$ can not be compact. The proof uses the same ideas as in Theorem 1

Proof. Let us consider the map $\Phi: T \times \mathbb{R}_{+} \rightarrow M,(p, t) \mapsto \exp _{p}(t N(p))$ where $N$ is the unit normal vector field normal to $T$ such that $N$ is the mean curvature vector of $T$. Let us define

$$
\varepsilon_{0}=\sup \{\varepsilon>0 \mid \Phi \text { is an immersion on } T \times[0, \varepsilon)\} .
$$

Using $\Phi$, we pull back the Riemannian metric of $M$ to $T \times\left[0, \varepsilon_{0}\right)$; it can be written $d s^{2}=d t^{2}+d \sigma_{t}^{2}$. We define $T_{t}=T \times\{t\}$ the equidistant surfaces to $T_{0}$. We also denote by $H(p, t)$ the mean curvature of the equidistant surfaces
at $(p, t)$ with respect to $\partial_{t}$. We finally define $\lambda(p, t)$ such that $H+\lambda$ and $H-\lambda$ are the principal curvatures of $T_{t}$ at $(p, t)$.

The surfaces $T_{t}$ are tori so, by the Gauss equation and the Gauss-Bonnet formula, we have

$$
0=\int_{T_{t}} \bar{K}_{T_{t}}=\int_{T_{t}} H^{2}-\lambda^{2}+K_{t}
$$

where $K_{t}$ is the sectional curvature of the ambient manifold of the tangent space to $T_{t}$. Since $K_{t} \leq-1$, we obtain the inequality

$$
\int_{T_{t}} \lambda^{2}=\int_{T_{t}} H^{2}+K_{t} \leq \int_{T_{t}} H^{2}-A\left(T_{t}\right)
$$

Let $F(t)$ denote the right hand term of the above inequality. By hypothesis, $H(p, 0)=1$ so $F(0)=0$ and $F(t) \geq 0$ for any $t \geq 0$. Let us compute the derivative of $F$

$$
\begin{aligned}
F^{\prime}(t) & =\int_{T_{t}}\left(2 H \frac{\partial H}{\partial t}-2 H^{3}\right)+\int_{T_{t}} 2 H \\
& =\int_{T_{t}} H\left(\operatorname{Ric}\left(\partial_{t}\right)+\left|A_{t}\right|^{2}-2 H^{2}+2\right) \\
& =\int_{T_{t}} H\left(\left(\operatorname{Ric}\left(\partial_{t}\right)+2\right)+2 \lambda^{2}\right)
\end{aligned}
$$

Since $H(p, 0)=1$, we can consider $\varepsilon \in\left(0, \varepsilon_{0}\right)$ such that $0<H \leq C$ on $T \times[0, \varepsilon]$. Since $\operatorname{Ric}\left(\partial_{t}\right)+2 \leq 0$ we get:

$$
F^{\prime}(t) \leq \int_{T_{t}} 2 H \lambda^{2} \leq 2 C F(t)
$$

Thus $F(t) \leq F(0) e^{2 C t}$ for $t \in[0, \varepsilon]$; this implies $F(t)=0$ on that segment. We then obtain $\lambda=0$ on $T \times[0, \varepsilon]$ (the equidistant surfaces are umbilical) and $\operatorname{Ric}\left(\partial_{t}\right)=-2$ since $H>0$. Thus $H$ satisfies the differential equation $\frac{\partial H}{\partial t}=-2+2 H^{2}$. This gives that $H=1$ on $T \times[0, \varepsilon]$ since $H=1$ on $T_{0}$. Thus we can let $\varepsilon$ tend to $\varepsilon_{0}$ to obtain that $F(t)=0$ on $\left[0, \varepsilon_{0}\right)$ and $\operatorname{Ric}\left(\partial_{t}\right)=-2$ and $H=1$ on $T \times\left[0, \varepsilon_{0}\right)$. Since $0=\int_{T_{t}} H^{2}+K_{t}$ and $K_{t} \leq-1$, it follows that $K_{t}=-1$ for all $t$ in the interval. We then have proved that the sectional curvature of $T \times\left[0, \varepsilon_{0}\right)$ with the metric $d s^{2}$ is equal to -1 for any 2-plane. Moreover, we get that $d \sigma_{0}^{2}$ is flat and that $d \sigma_{t}^{2}=e^{-2 t} d \sigma_{0}^{2}$. This implies that $\Phi$ is actually an immersion on $T \times \mathbb{R}_{+}\left(\varepsilon_{0}=+\infty\right)$ and $T \times \mathbb{R}_{+}$is isometric to a hyperbolic cusp. $\Phi$ is then a local isometry from this hyperbolic cusp to $M$.

To finish the proof, let us prove that $\Phi$ is in fact injective. If this is not the case, let $\varepsilon_{1}>0$ be the smallest $\varepsilon$ such that $\Phi$ is not injective on $T \times[0, \varepsilon]$. This implies that there exist $p$ and $q$ in $T$ such that

- either $\Phi(p, 0)=\Phi\left(q, \varepsilon_{1}\right)$
- or $\Phi\left(p, \varepsilon_{1}\right)=\Phi\left(q, \varepsilon_{1}\right)$ (with $p \neq q$ in this case).

Let $U$ and $V$ be respective neighborhoods of $(p, 0)$ (or $\left(p, \varepsilon_{1}\right)$ ) in $T_{0}$ (or $T_{\varepsilon_{1}}$ ) and $\left(q, \varepsilon_{1}\right)$ in $T_{\varepsilon_{1}}$ such that $\Phi$ is injective on them. Since $\varepsilon_{1}$ is the smallest one, $\Phi(U)$ and $\Phi(V)$ are two constant mean curvature 1 surfaces in $M$ that are tangent at $\Phi\left(q, \varepsilon_{1}\right)$. Moreover, in the first case, $\Phi(U)$ is included in the mean convex side of $\Phi(V)$ so by the maximum principle $\Phi(U)=\Phi(V)$. Thus $\Phi\left(T_{0}\right)$ would be equal to $\Phi\left(T_{\varepsilon_{1}}\right)$ which is impossible since these two surfaces do not have the same area. In the second case, $\Phi(U)$ is included in the mean convex side of $\Phi(V)$ and then $\Phi$ is not injective on $T_{s}$ for $s$ near $t$ $s<t$, which is a contradiction.

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