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# On minimal spheres of area $4\pi$ and rigidity

Laurent Mazet, Harold Rosenberg

## Abstract

Let  $M$  be a complete Riemannian 3-manifold with sectional curvatures between 0 and 1. A minimal 2-sphere immersed in  $M$  has area at least  $4\pi$ . If an embedded minimal sphere has area  $4\pi$ , then  $M$  is isometric to the unit 3-sphere or to a quotient of the product of the unit 2-sphere with  $\mathbb{R}$ , with the product metric. We also obtain a rigidity theorem for the existence of hyperbolic cusps. Let  $M$  be a complete Riemannian 3-manifold with sectional curvatures bounded above by  $-1$ . Suppose there is a 2-torus  $T$  embedded in  $M$  with mean curvature one. Then the mean convex component of  $M$  bounded by  $T$  is a hyperbolic cusp; *i.e.*, it is isometric to  $T \times \mathbb{R}$  with the constant curvature  $-1$  metric:  $e^{-2t}d\sigma_0^2 + dt^2$  with  $d\sigma_0^2$  a flat metric on  $T$ .

*Keywords:* area of minimal sphere, rigidity of 3-manifolds, hyperbolic cusp.

## 1 Introduction

Consider a smooth ( $C^\infty$ ) complete metric on the 2-sphere  $S$  whose curvature is between 0 and 1. It is well known that a simple closed geodesic in  $S$  has length at least  $2\pi$  (see [4] or Klingenberg's theorem in higher dimension [3, 2]). It is less well known that when such an  $S$  has a simple closed geodesic of length exactly  $2\pi$ , then  $S$  is isometric to the unit 2-sphere  $\mathbb{S}_1^2$ . This result is proved in [1], and the authors attribute the theorem to E. Calabi.

With this in mind, we consider what happens in a complete 3-manifold  $M$  with sectional curvatures between 0 and 1 (henceforth we suppose this curvature condition on  $M$ , unless stated otherwise).

Let  $\Sigma$  be an embedded minimal 2-sphere in  $M$ . Then the Gauss-Bonnet theorem and the Gauss equation tells us that the area of  $S$  is at least  $4\pi$ : indeed we have

$$4\pi = \int_{\Sigma} \bar{K}_{\Sigma} = \int \det(A) + K_{T\Sigma} \leq \int_{\Sigma} 1 = A(\Sigma) \quad (1)$$

with  $\det(A)$  the determinant of the shape operator which is non positive. We prove in Theorem 1, that when the area of  $\Sigma$  equals  $4\pi$ , then  $M$  is isometric to the unit 3-sphere  $\mathbb{S}_1^3$  or to a quotient of the product of the unit 2-sphere with  $\mathbb{R}$ ,  $\mathbb{S}_1^2 \times \mathbb{R}$ , with the product metric.

We remark that Theorem 1 does not hold for embedded minimal tori. Given  $\varepsilon$  greater than zero, there are Berger spheres with curvatures between 0 and 1, which contain embedded minimal tori of area less than  $\varepsilon$ . But a minimal sphere always has area at least  $4\pi$ .

It would be interesting to know what happens in higher dimensions. In the unit  $n$ -sphere  $\mathbb{S}_1^n$ , a compact minimal hyper-surface  $\Sigma$  always has volume at least the volume of the equatorial  $n - 1$  sphere  $\mathbb{S}_1^{n-1}$ . Is there a rigidity theorem when one allows metrics on  $\mathbb{S}^n$  ( $= M$ ), of sectional curvatures between 0 and 1? Two questions arise. First, does an embedded minimal hyper-sphere  $\Sigma$  in  $M$  have volume at least the volume of  $\mathbb{S}_1^{n-1}$ . If this is so, and if  $\Sigma$  is an embedded minimal hyper-sphere with volume exactly the volume of  $\mathbb{S}_1^{n-1}$ , is  $M$  isometric to  $\mathbb{S}_1^n$  or to  $\mathbb{S}_1^{n-1} \times \mathbb{R}$ ?

In the same spirit as Theorem 1, we prove a rigidity theorem for hyperbolic cusps. We recall that a 3 dimensional hyperbolic cusp is a manifold of the form  $T \times \mathbb{R}$  with  $T$  a 2-torus and the hyperbolic metric  $e^{-2t}d\sigma_0^2 + dt^2$  with  $d\sigma_0^2$  a flat metric on  $T$ . In Theorem 2, we prove that if  $M$  is a complete Riemannian manifold with sectional curvatures bounded above by  $-1$  and  $T$  is a constant mean curvature 1 torus embedded in  $M$  then the mean convex side of  $T$  in  $M$  is isometric to a hyperbolic cusp.

## 2 Minimal spheres of area $4\pi$ and rigidity of 3-manifolds

In this section, we prove a rigidity result for a Riemannian 3-manifold  $M$  whose sectional curvatures are between 0 and 1. As explained in the introduction, any minimal sphere in such a manifold has area at least  $4\pi$ .

We denote by  $\mathbb{S}_1^n$  the sphere of dimension  $n$  with constant sectional curvature 1. We then have the following result.

**Theorem 1.** *Let  $M$  be a complete Riemannian 3-manifold whose sectional curvatures satisfy  $0 \leq K \leq 1$ . Assume that there exists an embedded minimal sphere  $\Sigma$  in  $M$  with area  $4\pi$ . Then the manifold  $M$  is isometric either to the sphere  $\mathbb{S}_1^3$  or to a quotient of  $\mathbb{S}_1^2 \times \mathbb{R}$ .*

*Proof.* Let  $\Phi$  be the map  $\Sigma \times \mathbb{R} \rightarrow M, (p, t) \mapsto \exp_p(tN(q))$  where  $N$  is a unit normal vector field along  $\Sigma$ . In the following, we focus on  $\Sigma \times \mathbb{R}_+$ ; by

symmetry of the configuration, the study is similar for  $\Sigma \times \mathbb{R}_-$ .

$\Sigma$  is compact, so there is an  $\varepsilon$  such that  $\Phi$  is an immersion and even an embedding on  $\Sigma \times [0, \varepsilon)$ . Let us define

$$\varepsilon_0 = \sup\{\varepsilon > 0 \mid \Phi \text{ is an immersion on } \Sigma \times [0, \varepsilon)\};$$

$\varepsilon_0$  can be equal to  $+\infty$ . Using  $\Phi$ , we pull back the Riemannian metric of  $M$  to  $\Sigma \times [0, \varepsilon_0)$ . This metric can be written  $ds^2 = d\sigma_t^2 + dt^2$  where  $d\sigma_t^2$  is a smooth family of metrics on  $\Sigma$ . With this metric,  $\Phi$  becomes a local isometry from  $\Sigma \times [0, \varepsilon_0)$  to  $M$  and  $(\Sigma \times [0, \varepsilon_0), ds^2)$  has sectional curvatures between 0 and 1. Moreover,  $\Sigma_0$  is minimal and has area  $4\pi$ . Actually, we will prove the following facts.

**Claim.** *The metric  $d\sigma_0^2$  has constant sectional curvature 1 so  $(\Sigma, d\sigma_0^2)$  is isometric to  $\mathbb{S}_1^2$ . Moreover, we have two cases*

1.  $\varepsilon_0 = \pi/2$  and  $d\sigma_t^2 = \sin^2 t d\sigma_0^2$  or
2.  $\varepsilon_0 = +\infty$  and  $d\sigma_t^2 = d\sigma_0^2$

Let us denote by  $\Sigma_t = \Sigma \times \{t\}$  the equidistant surfaces. We denote by  $H(p, t)$  the mean curvature of  $\Sigma_t$  at the point  $(p, t)$  with respect to the unit normal vector  $\partial_t$ . We also define  $\lambda(p, t) \geq 0$  such that  $H + \lambda$  and  $H - \lambda$  are the principal curvature of  $\Sigma_t$  at  $(p, t)$ . We notice that  $\lambda = 0$  if  $\Sigma_t$  is umbilical at  $(p, t)$ .

The surfaces  $\Sigma_t$  are spheres so, using the Gauss equation, the Gauss-Bonnet formula implies:

$$4\pi = \int_{\Sigma_t} \bar{K}_{\Sigma_t} = \int_{\Sigma_t} (H + \lambda)(H - \lambda) + K_t = \int_{\Sigma_t} H^2 - \lambda^2 + K_t$$

where  $\bar{K}_{\Sigma_t}$  is the intrinsic curvature of  $\Sigma_t$  and  $K_t$  is the sectional curvature of the ambient manifold of the tangent space to  $\Sigma_t$ . Since  $K_t \leq 1$ , we obtain the following inequality

$$\int_{\Sigma_t} \lambda^2 = \int_{\Sigma_t} H^2 + K_t - 4\pi \leq \int_{\Sigma_t} H^2 + A(\Sigma_t) - 4\pi \quad (2)$$

where  $A(\Sigma_t)$  is the area of  $\Sigma_t$ . In the following, we denote by  $F(t)$  the right hand side of this inequality.

**Claim 1.**  *$F$  is vanishing on  $[0, \varepsilon_0)$ .*

Since  $\Sigma_0$  is minimal and has area  $4\pi$ , we have  $F(0) = 0$ . We notice that this implies that  $\lambda(p, 0) = 0$  so  $\Sigma_0$  is umbilical and  $K_{T\Sigma_0} = 1$ . Thus  $(\Sigma_0, d\sigma_0)$  is isometric to  $\mathbb{S}_1^2$ .

We have the usual formula:

$$\frac{\partial}{\partial t}A(\Sigma_t) = - \int_{\Sigma_t} 2H \quad \text{and} \quad \frac{\partial H}{\partial t} = \frac{1}{2}(Ric(\partial_t) + |A_t|^2) \quad (3)$$

where  $A_t$  is the shape operator of  $\Sigma_t$  and  $Ric$  is the Ricci tensor of  $\Sigma \times [0, \varepsilon_0)$ . Since the sectional curvatures of  $M \times [0, \varepsilon_0)$  are non-negative,  $Ric$  is non-negative. So the second formula above implies that  $H$  is increasing and thus  $H \geq 0$  everywhere. Let us now compute and estimate the derivative of  $F$ :

$$\begin{aligned} F'(t) &= \int_{\Sigma_t} (2H \frac{\partial H}{\partial t} - 2H^3) - \int_{\Sigma_t} 2H \\ &= \int_{\Sigma_t} H(Ric(\partial_t) + |A_t|^2 - 2H^2 - 2) \\ &= \int_{\Sigma_t} H((Ric(\partial_t) - 2) + ((H + \lambda)^2 + (H - \lambda)^2 - 2H^2)) \\ &= \int_{\Sigma_t} H((Ric(\partial_t) - 2) + 2\lambda^2) \\ &\leq 2 \int_{\Sigma_t} H\lambda^2 \end{aligned}$$

where the last inequality comes from  $Ric(\partial_t) - 2 \leq 0$  because of the hypothesis on the sectional curvatures. If we choose  $\varepsilon < \varepsilon_0$ , there is a constant  $C \geq 0$  such that  $H \leq C$  on  $\Sigma \times [0, \varepsilon]$ . So for  $t \in [0, \varepsilon]$ , using the inequality (2), we get  $F'(t) \leq 2CF(t)$ . Then  $F(t) \leq F(0)e^{2Ct} = 0$  on  $[0, \varepsilon]$ . So  $F \leq 0$  on  $[0, \varepsilon_0)$  and, because of (2),  $F = 0$  on  $[0, \varepsilon_0)$ ; this finishes the proof of Claim 1.

The first consequence of Claim 1 is that all the equidistant surfaces  $\Sigma_t$  are umbilical (see inequality (2)); so  $\lambda \equiv 0$ . In the computation of the derivative of  $F$ , this implies that

$$\int_{\Sigma_t} H(Ric(\partial_t) - 2) = 0$$

Since  $H(Ric(\partial_t) - 2) \leq 0$  everywhere, we obtain

$$H(Ric(\partial_t) - 2) = 0 \quad \text{everywhere.} \quad (4)$$

Moreover the umbilicity and (3) implies that  $\frac{\partial H}{\partial t} = \frac{1}{2}Ric(\partial_t) + H^2$ . We now prove the following claim

**Claim 2.** *Let  $(p, t) \in \Sigma \times [0, \varepsilon_0)$  ( $t > 0$ ) be such that  $H(p, t) > 0$  then  $H(q, t) > 0$  for any  $q \in \Sigma$*

In other words, when the mean curvature is positive at a point of an equidistant, it is positive at any point of this equidistant. We recall that  $H$  is increasing in the  $t$  variable so when it becomes positive it stays positive.

So assume that  $H(p, t) > 0$  and consider  $\Omega = \{q \in \Sigma \mid H(q, t) > 0\}$  which is a nonempty open subset of  $\Sigma$ . Let  $q \in \Omega$ . Since  $H(q, t) > 0$ ,  $Ric(\partial_t)(q, t) = 2$  by (4). Thus  $Ric(\partial_t)(r, t) = 2$  for any  $r \in \bar{\Omega}$ . So if  $r \in \bar{\Omega}$ ,  $Ric(\partial_t)(r, s) > 0$  for  $s < t$ , close to  $t$  and, by (3), this implies that  $H(r, t) > 0$  and  $r \in \Omega$ . So  $\Omega$  is closed and  $\Omega = \Sigma$ . This finishes the proof of Claim 2.

Let us assume that there is an  $\varepsilon_1 > 0$  such that  $H(p, t) = 0$  for  $(p, t) \in \Sigma \times [0, \varepsilon_1]$  and  $H(p, t) > 0$  for any  $(p, t) \in \Sigma \times (\varepsilon_1, \varepsilon_0)$ . Because of the evolution equation of  $H$ , this implies that  $Ric(\partial_t) = 0$  on  $\Sigma \times [0, \varepsilon_1]$ . On  $\Sigma \times (\varepsilon_1, \varepsilon_0)$ , we have  $Ric(\partial_t) = 2$  because of (4). So by continuity of  $Ric(\partial_t)$ , we get a contradiction and then we have two possibilities

1.  $H = 0$  on  $\Sigma \times [0, \varepsilon_0)$  and  $Ric(\partial_t) = 0$  on  $\Sigma \times [0, \varepsilon_0)$ .
2.  $H > 0$  on  $\Sigma \times (0, \varepsilon_0)$  and  $Ric(\partial_t) = 2$  on  $\Sigma \times [0, \varepsilon_0)$ .

In the first case, this implies that the sectional curvature of any 2-plane orthogonal to  $\Sigma_t$  is zero. Thus  $d\sigma_t^2 = d\sigma_0^2$ . Since the map  $\Phi$  ceases to be an immersion only if  $d\sigma_t^2$  becomes singular this implies that  $\varepsilon_0 = +\infty$ . Thus  $\Sigma \times \mathbb{R}_+$  with the induced metric is isometric to  $\mathbb{S}_1^2 \times \mathbb{R}_+$  and  $\Phi$  is a local isometry from  $\mathbb{S}_1^2 \times \mathbb{R}_+$  to  $M$ .

In the second case, the sectional curvature of any 2-plane orthogonal to  $\Sigma_t$  is equal to 1. Thus  $d\sigma_t^2 = \sin^2 t d\sigma_0$  and  $\varepsilon_0 = \pi/2$ . This also implies that  $\Phi(p, \pi/2)$  is a point. So  $\Sigma \times [0, \pi/2]$  with the metric  $ds^2$  is isometric to a hemisphere of  $\mathbb{S}_1^3$  and the map  $\Phi$  is a local isometry from that hemisphere to  $M$ .

Doing the same study for  $\Sigma \times \mathbb{R}_-$ , we get in the first case a local isometry  $\Phi : \mathbb{S}_1^2 \times \mathbb{R} \rightarrow M$  and in the second case a local isometry  $\Phi : \mathbb{S}_1^3 \rightarrow M$ . Since  $\mathbb{S}_1^2 \times \mathbb{R}$  and  $\mathbb{S}_1^3$  are simply connected,  $\Phi$  is then the universal cover of  $M$  and  $M$  is then isometric to a quotient of  $\mathbb{S}_1^2 \times \mathbb{R}$  or  $\mathbb{S}_1^3$ . Since  $\Phi$  is injective on  $\Sigma$  this implies that in the second case,  $\Phi$  is actually injective and then a global isometry.  $\square$

*Remark 1.* In the proof, since  $\Phi$  is injective on  $\Sigma$ , the possible quotients of  $\mathbb{S}_1^2 \times \mathbb{R}$  are either  $\mathbb{S}_1^2 \times \mathbb{R}$  or its quotient by the subgroup generated by an isometry of the form  $\mathbb{S}_1^2 \times \mathbb{R} \rightarrow \mathbb{S}_1^2 \times \mathbb{R}; (p, t) \mapsto (\alpha(p), t + t_0)$  with  $\alpha$  an isometry of  $\mathbb{S}_1^2$  and  $t_0 \neq 0$ .

*Remark 2.* Something can be said about constant mean curvature  $H_0$  spheres in a Riemannian 3-manifold with sectional curvatures between 0 and 1. Indeed, the computation (1) implies that the area of  $\Sigma$  is larger than  $\frac{4\pi}{1+H_0^2}$ , which is the area of a geodesic sphere in  $\mathbb{S}_1^3$  of mean curvature  $H_0$ . Moreover, if  $\Sigma$  has area  $\frac{4\pi}{1+H^2}$ , the above proof can be adapted to prove that the mean convex side of  $\Sigma$  is isometric to a spherical cap of  $\mathbb{S}_1^3$  with constant mean curvature  $H_0$  (see Theorem 2 below, for a similar result in the hyperbolic case).

*Remark 3.* Let  $M$  be a Riemannian  $n$ -manifold whose sectional curvatures are between 0 and 1 and let  $\Sigma$  be a minimal 2-sphere in  $M$ . A computation similar to (1) proves also that the area of  $\Sigma$  is larger than  $4\pi$ . It also implies that, if  $\Sigma$  has area  $4\pi$ ,  $\Sigma$  is totally geodesic and isometric to  $\mathbb{S}_1^2$ .

### 3 Existence of hyperbolic cusps

Let  $(\mathbb{T}^2, g)$  be a flat 2 torus, the manifold  $\mathbb{T}^2 \times \mathbb{R}_+$  with the complete Riemannian metric  $e^{-2t}g + dt^2$  is a hyperbolic 3-dimensional cusp.  $\mathbb{T}^2 \times \mathbb{R}$  is actually isometric to the quotient of a horoball of  $\mathbb{H}^3$  by a  $\mathbb{Z}^2$  subgroup of isometries of  $\mathbb{H}^2$  leaving the horoball invariant. Any  $\mathbb{T}^2 \times \{t\}$  has constant mean curvature 1. The following theorem says that, in certain 3-manifolds, a constant mean curvature 1 torus is necessarily the boundary of a hyperbolic cusp.

**Theorem 2.** *Let  $M$  be a complete Riemannian 3-manifold with its sectional curvatures satisfying  $K \leq -1$ . Assume that there exists a constant mean curvature 1 torus  $T$  embedded in  $M$ . Then  $T$  separates  $M$  and its mean convex side is isometric to a hyperbolic cusp.*

As a consequence, the existence of this torus implies that  $M$  can not be compact. The proof uses the same ideas as in Theorem 1

*Proof.* Let us consider the map  $\Phi : T \times \mathbb{R}_+ \rightarrow M, (p, t) \mapsto \exp_p(tN(p))$  where  $N$  is the unit normal vector field normal to  $T$  such that  $N$  is the mean curvature vector of  $T$ . Let us define

$$\varepsilon_0 = \sup\{\varepsilon > 0 \mid \Phi \text{ is an immersion on } T \times [0, \varepsilon)\}.$$

Using  $\Phi$ , we pull back the Riemannian metric of  $M$  to  $T \times [0, \varepsilon_0)$ ; it can be written  $ds^2 = dt^2 + d\sigma_t^2$ . We define  $T_t = T \times \{t\}$  the equidistant surfaces to  $T_0$ . We also denote by  $H(p, t)$  the mean curvature of the equidistant surfaces

at  $(p, t)$  with respect to  $\partial_t$ . We finally define  $\lambda(p, t)$  such that  $H + \lambda$  and  $H - \lambda$  are the principal curvatures of  $T_t$  at  $(p, t)$ .

The surfaces  $T_t$  are tori so, by the Gauss equation and the Gauss-Bonnet formula, we have

$$0 = \int_{T_t} \bar{K}_{T_t} = \int_{T_t} H^2 - \lambda^2 + K_t$$

where  $K_t$  is the sectional curvature of the ambient manifold of the tangent space to  $T_t$ . Since  $K_t \leq -1$ , we obtain the inequality

$$\int_{T_t} \lambda^2 = \int_{T_t} H^2 + K_t \leq \int_{T_t} H^2 - A(T_t)$$

Let  $F(t)$  denote the right hand term of the above inequality. By hypothesis,  $H(p, 0) = 1$  so  $F(0) = 0$  and  $F(t) \geq 0$  for any  $t \geq 0$ . Let us compute the derivative of  $F$

$$\begin{aligned} F'(t) &= \int_{T_t} (2H \frac{\partial H}{\partial t} - 2H^3) + \int_{T_t} 2H \\ &= \int_{T_t} H(\text{Ric}(\partial_t) + |A_t|^2 - 2H^2 + 2) \\ &= \int_{T_t} H((\text{Ric}(\partial_t) + 2) + 2\lambda^2) \end{aligned}$$

Since  $H(p, 0) = 1$ , we can consider  $\varepsilon \in (0, \varepsilon_0)$  such that  $0 < H \leq C$  on  $T \times [0, \varepsilon]$ . Since  $\text{Ric}(\partial_t) + 2 \leq 0$  we get:

$$F'(t) \leq \int_{T_t} 2H\lambda^2 \leq 2CF(t)$$

Thus  $F(t) \leq F(0)e^{2Ct}$  for  $t \in [0, \varepsilon]$ ; this implies  $F(t) = 0$  on that segment. We then obtain  $\lambda = 0$  on  $T \times [0, \varepsilon]$  (the equidistant surfaces are umbilical) and  $\text{Ric}(\partial_t) = -2$  since  $H > 0$ . Thus  $H$  satisfies the differential equation  $\frac{\partial H}{\partial t} = -2 + 2H^2$ . This gives that  $H = 1$  on  $T \times [0, \varepsilon]$  since  $H = 1$  on  $T_0$ . Thus we can let  $\varepsilon$  tend to  $\varepsilon_0$  to obtain that  $F(t) = 0$  on  $[0, \varepsilon_0)$  and  $\text{Ric}(\partial_t) = -2$  and  $H = 1$  on  $T \times [0, \varepsilon_0)$ . Since  $0 = \int_{T_t} H^2 + K_t$  and  $K_t \leq -1$ , it follows that  $K_t = -1$  for all  $t$  in the interval. We then have proved that the sectional curvature of  $T \times [0, \varepsilon_0)$  with the metric  $ds^2$  is equal to  $-1$  for any 2-plane. Moreover, we get that  $d\sigma_0^2$  is flat and that  $d\sigma_t^2 = e^{-2t}d\sigma_0^2$ . This implies that  $\Phi$  is actually an immersion on  $T \times \mathbb{R}_+$  ( $\varepsilon_0 = +\infty$ ) and  $T \times \mathbb{R}_+$  is isometric to a hyperbolic cusp.  $\Phi$  is then a local isometry from this hyperbolic cusp to  $M$ .



To finish the proof, let us prove that  $\Phi$  is in fact injective. If this is not the case, let  $\varepsilon_1 > 0$  be the smallest  $\varepsilon$  such that  $\Phi$  is not injective on  $T \times [0, \varepsilon]$ . This implies that there exist  $p$  and  $q$  in  $T$  such that

- either  $\Phi(p, 0) = \Phi(q, \varepsilon_1)$
- or  $\Phi(p, \varepsilon_1) = \Phi(q, \varepsilon_1)$  (with  $p \neq q$  in this case).

Let  $U$  and  $V$  be respective neighborhoods of  $(p, 0)$  (or  $(p, \varepsilon_1)$ ) in  $T_0$  (or  $T_{\varepsilon_1}$ ) and  $(q, \varepsilon_1)$  in  $T_{\varepsilon_1}$  such that  $\Phi$  is injective on them. Since  $\varepsilon_1$  is the smallest one,  $\Phi(U)$  and  $\Phi(V)$  are two constant mean curvature 1 surfaces in  $M$  that are tangent at  $\Phi(q, \varepsilon_1)$ . Moreover, in the first case,  $\Phi(U)$  is included in the mean convex side of  $\Phi(V)$  so by the maximum principle  $\Phi(U) = \Phi(V)$ . Thus  $\Phi(T_0)$  would be equal to  $\Phi(T_{\varepsilon_1})$  which is impossible since these two surfaces do not have the same area. In the second case,  $\Phi(U)$  is included in the mean convex side of  $\Phi(V)$  and then  $\Phi$  is not injective on  $T_s$  for  $s$  near  $t$   $s < t$ , which is a contradiction.  $\square$

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