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Ronan Le Guével, Jacques Lévy-Vehel, Lining Liu

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Abstract

We compare two definitions of multistable Lévy motions. Such processes are extensions of classical Lévy motion where the stability index is allowed to vary in time. We show that the two multistable Lévy motions have distinct properties: in particular, one is a pure-jump Markov process, while the other one satisfies neither of these properties. We prove that both are semimartingales and provide semimartingale decompositions.

Keywords: Lévy motion, multistable process, semimartingale.

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1 Background on multistable Lévy motions

The class of *multistable processes* was introduced in [4] and further developed in [5, 6, 7, 8, 10]. These processes extend the well-known stable processes (see, *e.g.* [11]) by letting the stability index α evolve in "time". Such models are useful in various applications where the data display jumps with varying intensity, such as financial records, EEG or natural terrains. Three paths have been explored so far to define multistable processes: the first one uses a field of stable processes $X(t, \alpha)$, and obtains a multistable process by considering a "diagonal" $Y(t) = X(t, \alpha(t))$ on this field. This is the approach of [4, 6]. In [5], multistable processes are obtained from moving average processes. Finally, in [8, 10], multistable measures are used for this purpose.

In this work, we shall be interested only in *multistable Lévy motions*, which are the simplest examples of multistable processes. These multistable

extensions of $\alpha-$stable Lévy motion were constructed in [4, 6] using respectively Poisson and Fergusson-Klass-LePage representations, while similar processes were defined in [8, 10] by their characteristic functions. In order to give precise definitions, let us set the following notations, which will be used throughout the paper:

- $\alpha : \mathbb{R} \to [c, d] \subset (0, 2)$ is a continuous function.
- $a : \mathbb{R} \to \mathbb{R}$ is a C^1 function.
- Π is a Poisson process on $[0, T] \times \mathbb{R}$, $T > 0$, with mean measure the Lebesgue measure $\mathcal{L}$.
- $(\Gamma_i)_{i\geq 1}$ is a sequence of arrival times of a Poisson process with unit arrival time.
- $(V_i)_{i\geq 1}$ is a sequence of i.i.d. random variables with uniform distribution on $[0, T]$.
- $(\gamma_i)_{i\geq 1}$ is a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$.

The three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are assumed to be independent.

The following process, that we shall call *field-based multistable Lévy motion*, is considered in [4]:

$$L_{FB}(t) = a(t)C_{\alpha(t)}^{1/\alpha(t)} \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathbf{1}_{[0,t]}(\mathsf{X})\mathsf{Y}^{<-1/\alpha(t)>} \quad (t \in [0, T]), \tag{1}$$

where $Y^{<-1/\alpha(t)>} := \text{sign}(Y)|Y|^{-1/\alpha(t)}$ and

$$C_u = \left(\int_0^\infty x^{-u} \sin(x) dx \right)^{-1}. \tag{2}$$

Note that, when $\alpha(t)$ equals the constant α for all t, L_{FB} is simply the Poisson representation of $\alpha-$stable Lévy motion. If one considers instead the Fergusson-Klass-LePage representation of Lévy motion, then it is natural to define L_{FB} as follows:

$$L_{FB}(t) = a(t)C_{\alpha(t)}^{1/\alpha(t)} T^{1/\alpha(t)} \sum_{i=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} 1_{[0,t]}(V_i) \quad (t \in [0, T]). \tag{3}$$

This is the approach of [6], where it is proven that (1) and (3) indeed define the same process, and that the joint characteristic function of L_{FB} equals:

$$\mathbb{E}\left(e^{i\sum_{j=1}^{m}\theta_j L_{FB}(t_j)}\right) = \exp\left(-2\int_{[0,T]}\int_0^{+\infty}\sin^2\left(\sum_{j=1}^{m}\theta_j a(t_j)\frac{C_{\alpha(t_j)}^{1/\alpha(t_j)}}{2y^{1/\alpha(t_j)}}1_{[0,t_j]}(x)\right)\,dy\,dx\right) \tag{4}$$

where $m \in \mathbb{N}, (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m, (t_1, \ldots, t_m) \in \mathbb{R}^m$.

In [8, 10] another path is followed to define a multistable extension of Lévy motion. Considering the characteristic function of $\alpha-$stable Lévy motion L:

$$\mathbb{E}(e^{i\theta L(t)}) = \exp\left(-t|\theta|^{\alpha}\right),$$

one defines the process L_{II} by its joint characteristic function as follows:

$$\mathbb{E}\left(\exp\left(i\sum_{j=1}^{d}\theta_j L_{II}(t_j)\right)\right) = \exp\left(-\int\left|\sum_{j=1}^{d}a(t_j)\theta_j 1_{[0,t_j]}(s)\right|^{\alpha(s)}ds\right). \tag{5}$$

It is clear from this definition that L_{II} has independent increments (this is why we call this version the *independent increments multistable Lévy motion*). This is a strong difference with L_{FB}, which is not even a Markov process. The construction based on multistable measures thus offers the advantage of retaining some properties of the classical Lévy motion: obviously, increments stationarity cannot hold anymore, but we still deal with an additive process. This is not the case for the field-based Lévy motion. However, a drawback of (5) is that, while L_{FB} coincides, at each fixed time, with a Lévy motion, this is not true of L_{II}.

The aim of this work is to elucidate the links between L_{FB} and L_{II} (section 2), to prove that both processes are semimartingales and to give semimartingales decompositions of these processes (section 3).

2 Series representations of independent increments multistable Lévy Motion

While three characterizations of L_{FB} are known, namely the Poisson and Fergusson-Klass-LePage representations and its characteristic function, only the latter is available for L_{II}. In this section, we provide the series representation of independent increments multistable Lévy motion.

2.1 Poisson series representation

Proposition 1. *L_{II} admits the following representation in law for $t \in [0, T]$:*

$$L_{II}(t) = a(t)\sum_{(X,Y)\in\Pi} C_{\alpha(X)}^{1/\alpha(X)}\mathbf{1}_{[0,t]}(X)Y^{<-1/\alpha(X)>}. \tag{6}$$

Proof. To simplify, we take $a(t) \equiv 1$ in this proof. Call $\tilde{L}$ the process on the right hand side of (6). Proposition 4.2 in [4] entails that the series converges, and that its marginal characteristic function reads:

$$\mathbb{E}(e^{i\theta\tilde{L}(t)}) = \exp\left(-2\int\int \sin^2\left(\frac{1}{2}\theta C_{\alpha(s)}^{1/\alpha(s)}\mathbf{1}_{[0,t]}(s)|y|^{-1/\alpha(s)}\right)dsdy\right),$$

(an integral sign without bounds means integration over the whole domain). We first prove that this quantity is equal to $\exp\left(-\int_0^t |\theta|^{\alpha(s)}ds\right)$. One computes:

$$\begin{aligned}\int 2\sin^2\left(\frac{1}{2}\theta C_{\alpha(s)}^{1/\alpha(s)}\mathbf{1}_{[0,t]}(s)|y|^{-1/\alpha(s)}\right)dy &= \int\left(1-\cos(\theta C_{\alpha(s)}^{1/\alpha(s)}\mathbf{1}_{[0,t]}(s)|y|^{-1/\alpha(s)})\right)dy \\ &= \alpha(s)|\theta|^{\alpha(s)}C_{\alpha(s)}\mathbf{1}_{[0,t]}(s)\int\frac{(1-\cos(z))}{|z|^{\alpha(s)+1}}dz \\ &= |\theta\mathbf{1}_{[0,t]}(s)|^{\alpha(s)},\end{aligned}$$

where we have used the change of variables $z = \theta C_{\alpha(s)}^{1/\alpha(s)}\mathbf{1}_{[0,t]}(s)y^{-1/\alpha(s)}$ and the fact that

$$\alpha(s)\int\frac{(1-\cos(z))}{|z|^{\alpha(s)+1}}dz = \frac{1}{C_{\alpha(s)}}.$$

Thus,

$$\begin{aligned}\mathbb{E}(e^{i\theta\tilde{L}(t)}) &= \exp\left(-\int\int 2\sin^2\left(\frac{1}{2}\theta C_{\alpha(s)}^{1/\alpha(s)}\mathbf{1}_{[0,t]}(s)y^{-1/\alpha(s)}\right)dsdy\right) \\ &= \exp\left(-\int|\theta\mathbf{1}_{[0,t]}(s)|^{\alpha(s)}ds\right) \\ &= \exp\left(-\int_0^t|\theta|^{\alpha(s)}ds\right).\end{aligned}$$

The process defined by the series on the right hand side of (6) clearly has independent increments. This fact and the above computation ensures that it coincides in law with L_{II}.

□

2.2 Fergusson-Klass-Le Page series representation

To simplify the notation, we take $T = 1$ in this section.

Proposition 2. *L_{II} admits the following representation in law:*

$$L_{II}(t) = a(t)\sum_{i=1}^{\infty} C_{\alpha(V_i)}^{1/\alpha(V_i)}\gamma_i\Gamma_i^{-1/\alpha(V_i)}\mathbf{1}_{(V_i\leq t)}. \tag{7}$$

Proof. We set again $a(t) \equiv 1$. Let us first prove that the marginal characteristic function of the right hand side Y of (7) is

$$\mathbb{E}(e^{i\theta Y(t)}) = \exp\left(-\int_0^t |\theta|^{\alpha(u)} du\right). \tag{8}$$

Set

$$f(X, m_X) = C_{\alpha(U)}^{1/\alpha(U)} \gamma X^{-1/\alpha(U)} \mathbf{1}_{(U \leq t)}$$

where $m_X = (\gamma, U)$. Using the Marking Theorem (see [3, p. 55]), since $\mathbb{P}(\gamma = 1) = \mathbb{P}(\gamma = -1) = 1/2$, one computes:

$$\begin{aligned}
\mathbb{E}(e^{-i\theta Y(t)}) &= \exp\left(\int\int (1 - e^{-f(x,m)}) dx \mathbb{P}(x, dm)\right) \\
&= \exp\left(\int\int (1 - e^{-i\theta C_{\alpha(u)}^{1/\alpha(u)} \gamma x^{-1/\alpha(u)} \mathbf{1}_{(u \leq t)}}) dx \mathbb{P}(x, dm)\right) \\
&= \exp\left(-\frac{1}{2}\int_0^\infty \int_0^\infty \left(\left(1 - e^{i\theta x^{-1/\alpha(u)} \mathbf{1}_{(u \leq t)}}\right) + \left(1 - e^{-i\theta x^{-1/\alpha(u)} \mathbf{1}_{(u \leq t)}}\right)\right) dx du\right) \\
&= \exp\left(-\int_0^\infty \int_0^\infty \left(1 - \frac{e^{i\theta x^{-1/\alpha(u)} \mathbf{1}_{(u \leq t)}} + e^{-i\theta x^{-1/\alpha(u)} \mathbf{1}_{(u \leq t)}}}{2}\right) dx du\right) \\
&= \exp\left(-\int_0^\infty \int_0^\infty (1 - \cos(\theta C_{\alpha(u)}^{1/\alpha(u)} \mathbf{1}_{(u \leq t)} x^{-1/\alpha(u)})) dx du\right) \qquad (9) \\
&= \exp\left(-\int_0^t |\theta|^{\alpha(u)} du\right) \qquad (10)
\end{aligned}$$

where the passage from (9) to (10) follows along the same lines as in the proof of Proposition 1. Now,

$$\begin{aligned}
\mathbb{E}\left(e^{-i\theta(Y(t) - Y(s))}\right) &= \exp\left(\int\int (1 - e^{-i\theta C_{\alpha(u)}^{1/\alpha(u)} \gamma x^{-1/\alpha(u)} \mathbf{1}_{(s < u \leq t)}}) dx \mathbb{P}(x, dm)\right) \\
&= \exp\left(-\frac{1}{2}\int_0^\infty \int_0^\infty \left(\left(1 - e^{i\theta x^{-1/\alpha(u)} \mathbf{1}_{(s < u \leq t)}}\right) + \left(1 - e^{-i\theta x^{-1/\alpha(u)} \mathbf{1}_{(s < u \leq t)}}\right)\right) dx du\right) \\
&= \exp\left(-\int_0^\infty \int_0^\infty \left(1 - \frac{e^{i\theta x^{-1/\alpha(u)} \mathbf{1}_{(s < u \leq t)}} + e^{-i\theta x^{-1/\alpha(u)} \mathbf{1}_{(s < u \leq t)}}}{2}\right) dx du\right) \\
&= \exp\left(-\int_0^\infty \int_0^\infty (1 - \cos(\theta C_{\alpha(u)}^{1/\alpha(u)} \mathbf{1}_{(s < u \leq t)} x^{-1/\alpha(u)})) dx du\right) \\
&= \exp\left(-\int_0^\infty |\mathbf{1}_{(s,t]}(u)\theta|^{\alpha(u)} du\right) \\
&= \exp\left(-\int_s^t |\theta|^{\alpha(u)} du\right).
\end{aligned}$$

This entails independence of the increments and ends the proof. □

3 Multistable Lévy motions are semimartingales

3.1 Case of independent increments multistable Lévy motion

Proposition 3. *L_{II} is a semimartingale.*

Proof. We use Theorem 4.14 in chapter II of [9]. Indeed, L_{II} is an independent increments process, and it suffices to verify that the function $t \mapsto \mathbb{E}(e^{i\theta Y(t)})$ has finite variations on finite intervals for each θ. This is obvious in view of (8). □

It is easy to check that the system of generating triplets (in the notation of [12]) of L_{II} is $(0, \nu, 0)$, with $\nu(dx, dz) = |z|^{-\alpha(x)-1} dz dx$. If α ranges in $[c, d] \subset (0, 1)$, then L_{II} is a finite variation process, while if α ranges in $[c, d] \subset (1, 2)$, then $\int_0^1 \int_1^\infty |z| \nu(dx, dz) = \int_0^1 \int_1^\infty |z|^{-\alpha(x)} dx dz < \infty$, which entails that L_{II} is a martingale. The proof is a simple adaptation of the one for stable processes and is left to the reader. In general, the following decomposition holds:

Proposition 4.

$$L_{II}(t) = A(t) + M(t),$$

where

$$A(t) = a(t) \sum_{(X,Y) \in \Pi, |Y| < 1} C_{\alpha(X)}^{1/\alpha(X)} \mathbf{1}_{[0,t]}(X) Y^{<-1/\alpha(X)>}$$

is a finite variation process and

$$M(t) = a(t) \sum_{(X,Y) \in \Pi, |Y| \geq 1} C_{\alpha(X)}^{1/\alpha(X)} \mathbf{1}_{[0,t]}(X) Y^{<-1/\alpha(X)>}$$

is a martingale.

Proof. That A has finite variation is a direct consequence of the fact that it is almost surely composed of a finite number of (jump) terms.
To prove that M is a martingale, it is sufficient to show that it is in $L_1(\Omega)$ for all t. Note that the jumps of M are bounded by $K = \sup_{t \in [0,T]} a(t) \sup_{b \in [c,d]} C_b^{1/b}$. As a consequence, with obvious notations,

$$\int_0^t \int_1^\infty |z| \nu_M(dx, dz) \leq \int_0^t \int_1^K |z| \nu(dx, dz) = 2 \int_0^t \int_1^K |z|^{-\alpha(x)} dx \, dz < \infty.$$

This entails that, for all t, $M(t)$ has finite mean (see, *e.g.* [12, Theorem 25.3]. □

Remarks

- A corresponding decomposition holds of course for the Fergusson-Klass-LePage representation, *i.e.*:

$$L_{II}(t) = A'(t) + M'(t),$$

where

$$A'(t) = a(t) \sum_{i=1,\Gamma_i<1}^{\infty} C_{\alpha(U_i)}^{1/\alpha(U_i)} \gamma_i \Gamma_i^{-1/\alpha(U_i)} \mathbf{1}_{(U_i \le t)}$$

has finite variation and

$$M'(t) = a(t) \sum_{i=1,\Gamma_i\ge 1}^{\infty} C_{\alpha(U_i)}^{1/\alpha(U_i)} \gamma_i \Gamma_i^{-1/\alpha(U_i)} \mathbf{1}_{(U_i \le t)}$$

is a martingale.

- As is well-known, the decomposition above is not unique. Another decomposition of interest is the following:

$$L_{II}(t) = M_1(t) + A_1(t)$$

with

$$M_1(t) = a(t) \sum_{i=1,\alpha(U_i)>\frac{1}{i}}^{\infty} C_{\alpha(U_i)}^{1/\alpha(U_i)} \gamma_i \Gamma_i^{-1/\alpha(U_i)} \mathbf{1}_{(U_i \le t)}$$

is a martingale and

$$A_1(t) = a(t) \sum_{i=1,\alpha(U_i)<\frac{1}{i}}^{\infty} C_{\alpha(U_i)}^{1/\alpha(U_i)} \gamma_i \Gamma_i^{-1/\alpha(U_i)} \mathbf{1}_{(U_i \le t)}$$

is an adapted process with finite variations.

3.2 Case of field-based multistable Lévy motion

L_{FB} is not an independent increments process, and, contrarily to what its definition might suggest at first sight, it is not a pure jump process (see below for a more precise statement). Thus it is not immediately obvious that it is indeed a semimartingale. We shall use the characterization of semimartingales as "good integrators" (see, *e.g.* [2, 13]) to prove this fact. More precisely, fix $t > 0$ and consider simple predictable processes ξ of the form

$$\xi(u) = \xi_0 \mathbf{1}_{\{0\}}(u) + \sum_{k=1}^{n} \xi_k \mathbf{1}_{(s_k,t_k]}(u)$$

where $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq ... \leq s_n < t_n = t$, $\xi_k \in \mathcal{F}_{s_k}$ and $|\xi_k| \leq 1$ a.s. for all $0 \leq k \leq n$. The integral of ξ with respect to a process Y is:

$$I_Y(\xi) = \sum_{k=1}^{n} \xi_k (Y_{t_k} - Y_{s_k}).$$

It is well known that Y is a semimartingale if and only if the family $\{I_Y(\xi), |\xi| \leq 1, \xi \text{ is a simple predictable process}\}$ is bounded in probability.

In fact, we shall establish the semimartingale property for a more general class of processes defined through Ferguson-Klass-LePage representations as follows:

$$X(t) = a(t) C_{\alpha(t)}^{1/\alpha(t)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} f(t, V_i), \tag{11}$$

where the function f is such that, for all t, $\int_0^T |f(t,x)|^{\alpha(t)}\, dx < \infty$, and give conditions on the kernel f ensuring that X is a semimartingale.

Theorem 5. *Let X be defined by (11), with α a C^1 function. Assume that X is a càdlàg adapted process, and that there exists L_∞ a positive constant such that for all $(t,x) \in \mathbb{R} \times [0,T]$, $|f(t,x)| \leq L_\infty$. Assume in addition that, for all $x \in E$, the function $u \mapsto f(u,x)$ has finite variation on finite intervals, with total variation $Vf(.,x)$ verifying*

$$\int_0^T |Vf(.,x)|^p\, dx < +\infty \tag{12}$$

for some $p \in (d,2)$. Then X is a semimartingale.

Proof. We will show that $\frac{X(t)}{a(t) C_{\alpha(t)}^{1/\alpha(t)}}$ is a semimartingale. Both processes $t \mapsto \gamma_1 \Gamma_1^{-1/\alpha(t)}$ and $t \mapsto f(t, V_1)$ have finite variation, thus the same holds for $t \mapsto \gamma_1 \Gamma_1^{-1/\alpha(t)} f(t, V_1)$. We shall prove that $Y(t) = \sum_{i=2}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} f(t, V_i)$ is a semimartingale.

One computes

$$\begin{aligned} I_Y(\xi) &= \sum_{k=1}^{n} \xi_k (Y_{t_k} - Y_{s_k}) \\ &= \sum_{k=1}^{n} \xi_k \sum_{i=2}^{+\infty} \gamma_i \left(\Gamma_i^{-1/\alpha(t_k)} f(t_k, V_i) - \Gamma_i^{-1/\alpha(s_k)} f(s_k, V_i) \right) \\ &= \sum_{k=1}^{n} \xi_k \sum_{i=2}^{+\infty} \gamma_i \left(\Gamma_i^{-1/\alpha(t_k)} - \Gamma_i^{-1/\alpha(s_k)} \right) f(t_k, V_i) \\ &\quad + \sum_{k=1}^{n} \xi_k \sum_{i=2}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(s_k)} \left(f(t_k, V_i) - f(s_k, V_i) \right) \\ &=: A + B. \end{aligned}$$

Let $K > 0$. We need to control $P(|I_Y(\xi)| > K)$.
For the term A, we shall use the mean value theorem: there exists a sequence of random variables $w_i^k \in (s_k, t_k)$ such that

$$\Gamma_i^{-1/\alpha(t_k)} - \Gamma_i^{-1/\alpha(s_k)} = (t_k - s_k) \frac{\alpha'(w_i^k)}{\alpha^2(w_i^k)} (\log \Gamma_i) \Gamma_i^{-1/\alpha(w_i^k)}.$$

One has

$$\begin{aligned} & P\left(\left| \sum_{k=1}^{n} \xi_k \sum_{i=2}^{+\infty} \gamma_i \left(\Gamma_i^{-1/\alpha(t_k)} - \Gamma_i^{-1/\alpha(s_k)} \right) f(t_k, V_i) \right| > K \right) \\ = \; & P\left(\left| \sum_{i=2}^{+\infty} \gamma_i \sum_{k=1}^{n} \xi_k \left((t_k - s_k) \frac{\alpha'(w_i^k)}{\alpha^2(w_i^k)} (\log \Gamma_i) \Gamma_i^{-1/\alpha(w_i^k)} \right) f(t_k, V_i) \right|^p > K^p \right) \\ \leq \; & \frac{1}{K^p} E\left[\left| \sum_{i=2}^{+\infty} \gamma_i \sum_{k=1}^{n} \xi_k \left((t_k - s_k) \frac{\alpha'(w_i^k)}{\alpha^2(w_i^k)} (\log \Gamma_i) \Gamma_i^{-1/\alpha(w_i^k)} \right) f(t_k, V_i) \right|^p \right]. \end{aligned}$$

As in the proof of Proposition 4.9 of [7], we use Theorem 2 of [1] (note that random variables w_i^k are independent of the $(\gamma_j)_j$). Since p<2,

$$
\begin{aligned}
& P\left(\left|\sum_{k=1}^{n}\xi_k\sum_{i=2}^{+\infty}\gamma_i\left(\Gamma_i^{-1/\alpha(t_k)}-\Gamma_i^{-1/\alpha(s_k)}\right)f(t_k,V_i)\right|>K\right)\\
\leq\ & \frac{2}{K^p}\sum_{i=2}^{+\infty}E\left[\left|\gamma_i\sum_{k=1}^{n}\xi_k\left((t_k-s_k)\frac{\alpha'(w_i^k)}{\alpha^2(w_i^k)}(\log\Gamma_i)\Gamma_i^{-1/\alpha(w_i^k)}\right)f(t_k,V_i)\right|^p\right]\\
\leq\ & \frac{2}{K^p}\sum_{i=2}^{+\infty}E\left[\left(\sum_{k=1}^{n}(t_k-s_k)\frac{\sup_{s\in[0,t]}|\alpha'(s)|}{c^2}\left|\log\Gamma_i\right|(\Gamma_i^{-1/c}+\Gamma_i^{-1/d})L_\infty\right)^p\right]\\
\leq\ & 2\left(\frac{L_\infty t\sup_{s\in[0,t]}|\alpha'(s)|}{Kc^2}\right)^p\ \sum_{i=2}^{+\infty}E\left[\left|\log\Gamma_i(\Gamma_i^{-1/c}+\Gamma_i^{-1/d})\right|^p\right],
\end{aligned}
$$

where the infinite sum in the last line above converges since $p>d$. Let us now consider the second term B of $I_Y(\xi)$:

$$
\begin{aligned}
& P\left(\left|\sum_{k=1}^{n}\xi_k\sum_{i=2}^{+\infty}\gamma_i\Gamma_i^{-1/\alpha(s_k)}\left(f(t_k,V_i)-f(s_k,V_i)\right)\right|>K\right)\\
\leq\ & \frac{1}{K^p}E\left[\left|\sum_{i=2}^{+\infty}\gamma_i\sum_{k=1}^{n}\xi_k\Gamma_i^{-1/\alpha(s_k)}\left(f(t_k,V_i)-f(s_k,V_i)\right)\right|^p\right].
\end{aligned}
$$

We use again Theorem 2 of [1] :

$$
\begin{aligned}
& P\left(\left|\sum_{k=1}^{n}\xi_k\sum_{i=2}^{+\infty}\gamma_i\Gamma_i^{-1/\alpha(s_k)}\left(f(t_k,V_i)-f(s_k,V_i)\right)\right|>K\right)\\
\leq\ & \frac{2}{K^p}\sum_{i=2}^{+\infty}E\left[\left|\sum_{k=1}^{n}\xi_k\Gamma_i^{-1/\alpha(s_k)}\left(f(t_k,V_i)-f(s_k,V_i)\right)\right|^p\right]\\
\leq\ & \frac{2}{K^p}\sum_{i=2}^{+\infty}E\left[\left(\sum_{k=1}^{n}\left(\Gamma_i^{-1/c}+\Gamma_i^{-1/d}\right)\left|f(t_k,V_i)-f(s_k,V_i)\right|\right)^p\right]\\
\leq\ & \frac{2}{K^p}\sum_{i=2}^{+\infty}E\left[\left(\Gamma_i^{-1/c}+\Gamma_i^{-1/d}\right)^p\left(\sum_{k=1}^{n}\left|f(t_k,V_i)-f(s_k,V_i)\right|\right)^p\right]\\
\leq\ & \frac{2}{K^p}\sum_{i=2}^{+\infty}E\left[\left(\Gamma_i^{-1/c}+\Gamma_i^{-1/d}\right)^p\left|Vf(.,V_i)\right|^p\right]\\
=\ & \frac{2}{K^p}\sum_{i=2}^{+\infty}E\left[\left(\Gamma_i^{-1/c}+\Gamma_i^{-1/d}\right)^p\right]E\left[\left|Vf(.,V_i)\right|^p\right]\\
=\ & \frac{2}{K^p}E\left[\left|Vf(.,V_1)\right|^p\right]\sum_{i=2}^{+\infty}E\left[\left(\Gamma_i^{-1/c}+\Gamma_i^{-1/d}\right)^p\right],
\end{aligned}
$$

where, in the last line above, the first expectation is finite by assumption (12) and the infinite sum converges converges because $p > d$.

We have thus shown that $P(|I_Y(\xi)| > K) \leq \frac{C}{K^p}$ for some constant C, as required. □

Corollary 6. *Assume α is a C^1 function. Then the field-based multistable Lévy motion is a semimartingale.*

The result above does not give a semimartingale decomposition of L_{FB}. The following theorem does so. In addition, it elucidates the links between the two multistable Lévy motions:

Theorem 7. *Assume α is a C^1 function. Almost surely, for all t,*

$$L_{FB}(t) = A(t) + L_{II}(t), \tag{13}$$

where A is the finite variation process defined by:

$$A(t) = \int_0^t \sum_{i=1}^{+\infty} \gamma_i \frac{d\left(C_{\alpha(.)}^{1/\alpha(.)} \Gamma_i^{-1/\alpha(.)}\right)}{dt}(s) \mathbf{1}_{[0,s[}(V_i) ds. \tag{14}$$

From a heuristic point of view, this result states that, at all jumps points, both Lévy multistable motions vary by the same amount. "In-between" jumps, however, L_{II} does not change (it is a pure jump process), while L_{FB} moves in a continuous fashion (and thus is not a pure jump process).

Proof. For the sake of notational simplicity, let:

$$b(t) = C_{\alpha(t)}^{1/\alpha(t)},$$

$$g_i(t) = C_{\alpha(t)}^{1/\alpha(t)} \Gamma_i^{-1/\alpha(t)} = b(t) \Gamma_i^{-1/\alpha(t)},$$

$$d_i(t) = g_i'(t) = b'(t) \Gamma_i^{-1/\alpha(t)} + \frac{\alpha'(t) b(t)}{\alpha^2(t)} \Gamma_i^{-1/\alpha(t)} \ln \Gamma_i,$$

$$e_i(t) = b'(t) i^{-1/\alpha(t)} + \frac{b(t)\alpha'(t)}{\alpha^2(t)} i^{-1/\alpha(t)} (\ln i).$$

We wish to prove that (again, we set $a(t) \equiv 1$):

$$C_{\alpha(t)}^{1/\alpha(t)} \sum_{i=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} \mathbf{1}_{[0,t]}(V_i) = \int_0^t \sum_{i=1}^{+\infty} \gamma_i g_i'(s) \mathbf{1}_{[0,s[}(V_i) ds + \sum_{i=1}^{+\infty} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} \Gamma_i^{-1/\alpha(V_i)} \mathbf{1}_{[0,t]}(V_i).$$

Let

$$D_N(s) = \sum_{i=1}^{N} \gamma_i d_i(s) \mathbf{1}_{[0,s[}(V_i),$$

and

$$D(s) = \sum_{i=1}^{+\infty} \gamma_i d_i(s) \mathbf{1}_{[0,s[}(V_i).$$

Finally, write:

$$E_N(s) = \sum_{i=1}^{N} \gamma_i e_i(s) \mathbf{1}_{[0,s[}(V_i),$$

and

$$E(s) = \sum_{i=1}^{+\infty} \gamma_i e_i(s) \mathbf{1}_{[0,s[}(V_i).$$

The function $s \mapsto \gamma_i d_i(s) \mathbf{1}_{[0,s[}(V_i)$ is continuous and differentiable on $(V_i, 1)$, and vanishes on $[0, V_i]$.
Choose $A \in \mathbb{R}$ such that $\forall t \in (0,1)$, $|b'(t)| \leq A$, $|\frac{\alpha'(t)b(t)}{\alpha^2(t)}| \leq A$ and $|\alpha'(t)| \leq A$.
Case $d < 1$**:**
Fix $1 > \hat{d} > d$ and $i_0 \in \mathbb{N}$ such that, for $i \geq i_0$, $\frac{(1+\ln i)}{i^{1/d}} \leq \frac{1}{i^{1/\hat{d}}}$. Then, for all $s \in [0,1]$ and $p \geq p_0$,

$$\begin{aligned}
\sum_{i=p}^{q} \gamma_i e_i(s) \mathbf{1}_{[0,s[}(V_i) &\leq \sum_{i=p}^{q} A \frac{(1+\ln i)}{i^{1/d}} \\
&\leq \sum_{i=p}^{q} A \frac{1}{i^{1/\hat{d}}} \\
&\leq A \sum_{i=p}^{+\infty} \frac{1}{i^{1/\hat{d}}} \\
&\leq A p^{1-1/\hat{d}}.
\end{aligned}$$

The uniform Cauchy criterion entails that $E_N(s)$ converges uniformly on $[0,1]$ to $E(s)$.
Case $d \geq 1$**:**
Set

$$E_\alpha^j(s) = \sum_{i=2^j}^{2^{j+1}-1} \gamma_i e_i(s) \mathbf{1}_{[0,s[}(V_i),$$

and $b(j) = j\frac{A(1+(j+1)\ln 2)}{2^{j/d}}\sqrt{2^j}$. Consider

$$\begin{aligned} E(j) &= \left\{ \sup_{s\in[0,1]} |E^j_\alpha(s)| \leq j\frac{A(1+(j+1)\ln 2)}{2^{j/d}}\sqrt{2^j} \right\} \\ &= \left\{ \sup_{s\in[0,1]} |E^j_\alpha(s)| \leq b(j) \right\}. \end{aligned}$$

Fix $\hat{d} > d \geq 1$ and $i_0 \in \mathbb{N}$ such that, for $i \geq i_0$, $\frac{(1+\ln i)\ln i}{i^{1/d}} \leq \frac{1}{i^{1/\hat{d}}}$. Let $\delta > \frac{1}{2} + \frac{1}{d} - \frac{1}{\hat{d}}$. Define the step function $\alpha_j(s) = \sum_{k=0}^{[2^{j\delta}]-1} \alpha(\frac{k}{[2^{j\delta}]})\mathbf{1}_{[\frac{k}{[2^{j\delta}]},\frac{k+1}{[2^{j\delta}]}[}(s)$. For all $s \in [0,1]$,

$$\begin{aligned} |\alpha_j(s) - \alpha(s)| &= \sum_{k=0}^{[2^{j\delta}]-1} |\alpha(\frac{k}{[2^{j\delta}]}) - \alpha(s)|\mathbf{1}_{[\frac{k}{[2^{j\delta}]},\frac{k+1}{[2^{j\delta}]}[}(s) \\ &\leq \frac{A}{[2^{j\delta}]}, \end{aligned}$$

and thus

$$\sup_{s\in[0,1]} |\alpha_j(s) - \alpha(s)| \leq \frac{A}{[2^{j\delta}]}.$$

Set

$$E^j_{\alpha_j}(s) = a'(s)\sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha_j(s)}\mathbf{1}_{[0,s[}(V_i) + \frac{a(s)\alpha'(s)}{\alpha^2(s)}\sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha_j(s)}(\ln i)\mathbf{1}_{[0,s[}(V_i).$$

Then,

$$\begin{aligned} E^j_\alpha(s) - E^j_{\alpha_j}(s) &= a'(s)\sum_{i=2^j}^{2^{j+1}-1} \gamma_i \left(i^{-1/\alpha(s)} - i^{-1/\alpha_j(s)}\right)\mathbf{1}_{[0,s[}(V_i) \\ &\quad + \frac{a(s)\alpha'(s)}{\alpha^2(s)}\sum_{i=2^j}^{2^{j+1}-1} \gamma_i(\ln i)\left(i^{-1/\alpha(s)} - i^{-1/\alpha_j(s)}\right)\mathbf{1}_{[0,s[}(V_i). \end{aligned}$$

The finite increments theorem entails that there exists $\alpha^i(s) \in [\alpha_j(s), \alpha(s)]$ (or in $[\alpha(s), \alpha_j(s)]$) such that

$$\begin{aligned} E^j_\alpha(s) - E^j_{\alpha_j}(s) &= a'(s)(\alpha(s) - \alpha_j(s))\sum_{i=2^j}^{2^{j+1}-1} \gamma_i \frac{\ln i}{(\alpha^i(s))^2} i^{-1/\alpha^i(s)}\mathbf{1}_{[0,s[}(V_i) + \\ &\quad \frac{a(s)\alpha'(s)}{\alpha^2(s)}(\alpha(s) - \alpha_j(s))\sum_{i=2^j}^{2^{j+1}-1} \gamma_i \frac{(\ln i)^2}{(\alpha^i(s))^2} i^{-1/\alpha^i(s)}\mathbf{1}_{[0,s[}(V_i), \end{aligned}$$

and thus

$$\begin{aligned}
|E^j_\alpha(s) - E^j_{\alpha_j}(s)| &\leq \frac{A^2}{[2^{j\delta}]} \sum_{i=2^j}^{2^{j+1}-1} \frac{\ln i(1+\ln i)}{d^2} i^{-1/d} \\
&\leq \frac{A^2}{[2^{j\delta}]d^2} \sum_{i=2^j}^{2^{j+1}-1} \frac{1}{i^{1/\hat{d}}} \\
&\leq \frac{A^2}{[2^{j\delta}]d^2} \frac{2^j}{2^{j/\hat{d}}}.
\end{aligned}$$

Choose $J_0 \in \mathbb{N}$ such that, for all $j \geq J_0$,

$$\sup_{s\in[0,1]} |E^j_\alpha(s) - E^j_{\alpha_j}(s)| \leq \frac{b(j)}{2}.$$

Then, for $j \geq J_0$,

$$\begin{aligned}
\mathbb{P}\left(\overline{E(j)}\right) &= \mathbb{P}\left(\sup_{s\in[0,1]} |E^j_\alpha(s)| > b(j)\right) \\
&\leq \mathbb{P}\left(\frac{b(j)}{2} + \sup_{s\in[0,1]} |E^j_{\alpha_j}(s)| > b(j)\right) \\
&= \mathbb{P}\left(\sup_{s\in[0,1]} |E^j_{\alpha_j}(s)| > \frac{b(j)}{2}\right).
\end{aligned}$$

Denote $\mathcal{A}^j$ the set of possible values of α_j. Then $Card(\mathcal{A}^j) = [2^{j\delta}]$. For each random drawing of the $(V_i)_i$, $E^j_{\alpha_j}(s)$ is composed of a sum of n terms of the form $\gamma_i l_i^{-1/\alpha_0}$ where $\alpha_0 \in \mathcal{A}^j$. There are 2^j possible values for n, and $[2^{j\delta}]$

possible values for α_0. One thus has the following estimates:

$$\begin{aligned}
\mathbb{P}\left(\overline{E(j)}\right) &\leq \mathbb{P}\left(\sup_{s\in[0,1]}|E^j_{\alpha_j}(s)|>\frac{b(j)}{2}\right)\\
&\leq \mathbb{P}\left(\cup_{l_1,...,l_{2^j}\in[\![2^j,2^{j+1}-1]\!]}\{V_{l_1}<V_{l_2}<...<V_{l_{2^j}}\}\cap\{\sup_{\alpha_0\in\mathcal{A}^j}\sup_{k=1}^{2^j}|\sum_{i=1}^k\gamma_{l_i}l_i^{-1/\alpha_0}|>\frac{b(j)}{2}\}\right)\\
&\leq \sum_{l_1,...,l_{2^j}\in[\![2^j,2^{j+1}-1]\!]}\mathbb{P}\left(\{V_{l_1}<V_{l_2}<...<V_{l_{2^j}}\}\cap\{\sup_{\alpha_0\in\mathcal{A}^j}\sup_{k=1}^{2^j}|\sum_{i=1}^k\gamma_{l_i}l_i^{-1/\alpha_0}|>\frac{b(j)}{2}\}\right)\\
&\leq \sum_{l_1,...,l_{2^j}\in[\![2^j,2^{j+1}-1]\!]}\mathbb{P}\left(V_{l_1}<V_{l_2}<...<V_{l_{2^j}}\right)\mathbb{P}\left(\sup_{\alpha_0\in\mathcal{A}^j}\sup_{k=1}^{2^j}|\sum_{i=1}^k\gamma_{l_i}l_i^{-1/\alpha_0}|>\frac{b(j)}{2}\right)\\
&\leq \sum_{l_1,...,l_{2^j}\in[\![2^j,2^{j+1}-1]\!]}\frac{1}{(2^j)!}\mathbb{P}\left(\cup_{\alpha_0\in\mathcal{A}^j}\cup_{k=1}^{2^j}|\sum_{i=1}^k\gamma_{l_i}l_i^{-1/\alpha_0}|>\frac{b(j)}{2}\right)\\
&\leq \frac{1}{(2^j)!}\sum_{l_1,...,l_{2^j}\in[\![2^j,2^{j+1}-1]\!]}\sum_{\alpha_0\in\mathcal{A}^j}\sum_{k=1}^{2^j}\mathbb{P}\left(|\sum_{i=1}^k\gamma_{l_i}l_i^{-1/\alpha_0}|>\frac{b(j)}{2}\right)\\
&\leq \frac{1}{(2^j)!}\sum_{l_1,...,l_{2^j}\in[\![2^j,2^{j+1}-1]\!]}\sum_{\alpha_0\in\mathcal{A}^j}\sum_{k=1}^{2^j}2e^{-\frac{j^2}{8}}\\
&\leq 2.2^j[2^{j\delta}]e^{-\frac{j^2}{8}},
\end{aligned}$$

where Stute's lemma [14] was used in the end. As a consequence,

$$\mathbb{P}\left(\liminf_j E(j)\right)=1.$$

Let now $p\in\mathbb{N}$, and set $j_p=[\frac{\ln p}{\ln 2}]$. Define

$$E^{m,p}_\alpha(s)=\sum_{i=m}^{2^{j_p+1}-1}\gamma_i e_i(s)\mathbf{1}_{[0,s[}(V_i),$$

and

$$E^{m,p}_{\alpha_{j_p}}(s)=a'(s)\sum_{i=m}^{2^{j_p+1}-1}\gamma_i i^{-1/\alpha_{j_p}(s)}\mathbf{1}_{[0,s[}(V_i)+\frac{a(s)\alpha'(s)}{\alpha^2(s)}\sum_{i=m}^{2^{j_p+1}-1}\gamma_i i^{-1/\alpha_{j_p}(s)}(\ln i)\mathbf{1}_{[0,s[}(V_i).$$

As above,

$$\sup_{s\in[0,1]}|\alpha_{j_p}(s)-\alpha(s)|\leq\frac{A}{[2^{j_p\delta}]}$$

and

$$\begin{aligned}|E_\alpha^{m,p}(s) - E_{\alpha_{j_p}}^{m,p}(s)| &\leq \frac{A^2}{[2^{j_p\delta}]d^2}\sum_{i=m}^{2^{j_p+1}-1}\frac{1}{i^{1/\hat{d}}} \\ &\leq \frac{A^2}{[2^{j_p\delta}]d^2}\sum_{i=2^{j_p}}^{2^{j_p+1}-1}\frac{1}{i^{1/\hat{d}}} \\ &\leq \frac{A^2}{[2^{j_p\delta}]d^2}\frac{2^{j_p}}{2^{j_p/\hat{d}}}.\end{aligned}$$

Fix $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$,

$$\sup_{m\in[\![2^{j_p},2^{j_p+1}-1]\!]}\sup_{s\in[0,1]}|E_\alpha^{m,p}(s) - E_{\alpha_{j_p}}^{m,p}(s)| \leq \frac{b(j_p)}{2},$$

and consider

$$E(p) = \left\{\sup_{m\in[\![2^{j_p},2^{j_p+1}-1]\!]}\sup_{s\in[0,1]}|E_\alpha^{m,p}(s)| \leq b(j_p)\right\}.$$

For $p \geq p_0$,

$$\begin{aligned}\mathbb{P}\left(\overline{E(p)}\right) &\leq \mathbb{P}\left(\sup_{m\in[\![2^{j_p},2^{j_p+1}-1]\!]}\sup_{s\in[0,1]}|E_{\alpha_{j_p}}^{m,p}(s)| > \frac{b(j_p)}{2}\right) \\ &\leq \sum_{m\in[\![2^{j_p},2^{j_p+1}-1]\!]}\mathbb{P}\left(\sup_{s\in[0,1]}|E_{\alpha_{j_p}}^{m,p}(s)| > \frac{b(j_p)}{2}\right) \\ &\leq \sum_{m\in[\![2^{j_p},2^{j_p+1}-1]\!]}2.2^{j_p}[2^{j_p\delta}]e^{-\frac{j_p^2}{8}} \\ &\leq 2.2^{2j_p}[2^{j_p\delta}]e^{-\frac{j_p^2}{8}}.\end{aligned}$$

As a consequence,

$$\mathbb{P}\left(\liminf_p E(p)\right) = 1.$$

Reasoning along the same lines and setting

$$E_\alpha^{m,q}(s) = \sum_{i=2^{j_q}}^{m}\gamma_i e_i(s)\mathbf{1}_{[0,s[}(V_i),$$

and

$$E(q) = \left\{ \sup_{m\in[\![2^{j_q},2^{j_q+1}-1]\!]} \sup_{s\in[0,1]} |E_\alpha^{m,q}(s)| \le b(j_q) \right\},$$

one gets

$$\mathbb{P}\left(\liminf_q E(q)\right) = 1.$$

Finally, let $p, q \in \mathbb{N}$, with $q \ge p$: for all $s \in [0,1]$,

$$\begin{aligned}
|E_q(s) - E_p(s)| &= \left| \sum_{i=p}^{2^{j_p+1}-1} \gamma_i e_i(s) \mathbf{1}_{[0,s[}(V_i) + \sum_{j=j_p+1}^{j_q-1} \sum_{i=2^j}^{2^{j+1}-1} \gamma_i e_i(s) \mathbf{1}_{[0,s[}(V_i) + \sum_{i=2^{j_q}}^{q} \gamma_i e_i(s) \mathbf{1}_{[0,s[}(V_i) \right| \\
&\le b(j_p) + \sum_{j=j_p+1}^{j_q-1} b(j) + b(j_q) \\
&\le 2b(j_p) + \sum_{j=j_p}^{+\infty} b(j).
\end{aligned}$$

Again, the uniform Cauchy criterion entails that $E_N(s)$ converges uniformly on $[0,1]$ to $E(s)$.

Convergence of the difference term:

Let $q \ge p$ and denote $R_{p,q}(s) = \sum_{i=p}^{q} \gamma_i (d_i(s) - e_i(s)) \mathbf{1}_{[0,s[}(V_i)$.

Fix p_0 such that, for all $p \ge p_0$,

$$R_{p,q}(s) = \sum_{i=p}^{q} \gamma_i (d_i(s) - e_i(s)) \mathbf{1}_{[0,s[}(V_i) \mathbf{1}_{\frac{1}{2} \le \frac{\Gamma_i}{i} \le 2}.$$

One computes:

$$\begin{aligned}
R_{p,q}(s) &\le \sum_{i=p}^{q} |d_i(s) - e_i(s)| \mathbf{1}_{\frac{1}{2} \le \frac{\Gamma_i}{i} \le 2} \\
&\le A \sum_{i=p}^{q} |i^{-1/\alpha(t)} - \Gamma_i^{-1/\alpha(t)}| \mathbf{1}_{\frac{1}{2} \le \frac{\Gamma_i}{i} \le 2} + A \sum_{i=p}^{q} |i^{-1/\alpha(t)} \ln i - \Gamma_i^{-1/\alpha(t)} \ln \Gamma_i| \mathbf{1}_{\frac{1}{2} \le \frac{\Gamma_i}{i} \le 2} \\
&\le A \sum_{i=p}^{q} \frac{1}{i^{1/d}} |1 - (\frac{\Gamma_i}{i})^{-1/\alpha(t)}| \mathbf{1}_{\frac{1}{2} \le \frac{\Gamma_i}{i} \le 2} + A \sum_{i=p}^{q} \frac{\ln i}{i^{1/d}} |1 - \frac{\ln \Gamma_i}{\ln i} (\frac{\Gamma_i}{i})^{-1/\alpha(t)}| \mathbf{1}_{\frac{1}{2} \le \frac{\Gamma_i}{i} \le 2} \\
&\le A K_{c,d} \sum_{i=p}^{q} \frac{(1 + \ln i)}{i^{1/d}} \left| \frac{\Gamma_i}{i} - 1 \right|.
\end{aligned}$$

The series $\sum_i \frac{(1+\ln i)}{i^{1/d}} \left|\frac{\Gamma_i}{i} - 1\right|$ converges almost surely. The uniform Cauchy criterion thus applies to the effect that $D_N(s) - E_N(s)$ converges uniformly. As a consequence, $D_N(s)$ converges uniformly to $D(s)$.

Thus, almost surely,

$$\lim_{N\to+\infty} \int_0^t D_N(s)ds = \int_0^t D(s)ds = \int_0^t \sum_{i=1}^{+\infty} \gamma_i g_i'(s) \mathbf{1}_{[0,s[}(V_i)ds.$$

Besides,

$$\begin{aligned}
\int_0^t D_N(s)ds &= \sum_{i=1}^{N} \gamma_i \int_0^t g_i'(s)\mathbf{1}_{[0,s[}(V_i)ds \\
&= \sum_{i=1}^{N} \gamma_i \left(\int_{V_i}^t d_i(s)ds \right) \mathbf{1}_{[0,t]}(V_i) \\
&= \sum_{i=1}^{N} \gamma_i \left(g_i(t) - g_i(V_i) \right) \mathbf{1}_{[0,t]}(V_i) \\
&= \sum_{i=1}^{N} \gamma_i g_i(t)\mathbf{1}_{[0,t]}(V_i) - \sum_{i=1}^{N} \gamma_i g_i(V_i)\mathbf{1}_{[0,t]}(V_i)
\end{aligned}$$

which entails finally that

$$\lim_{N\to+\infty} \int_0^t D_N(s)ds = \sum_{i=1}^{+\infty} \gamma_i g_i(t)\mathbf{1}_{[0,t]}(V_i) - \sum_{i=1}^{+\infty} \gamma_i g_i(V_i)\mathbf{1}_{[0,t]}(V_i).$$

This is (13).

That A has finite variation follows from the fact that it is an absolutely continuous function. □

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