



**HAL**  
open science

# Replicator dynamics in public goods games with reward funds

Tatsuya Sasaki, Tatsuo Unemi

► **To cite this version:**

Tatsuya Sasaki, Tatsuo Unemi. Replicator dynamics in public goods games with reward funds. *Journal of Theoretical Biology*, 2011, 287, pp.109. 10.1016/j.jtbi.2011.07.026 . hal-00730278

**HAL Id: hal-00730278**

**<https://hal.science/hal-00730278v1>**

Submitted on 9 Sep 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Author's Accepted Manuscript

Replicator dynamics in public goods games with reward funds

Tatsuya Sasaki, Tatsuo Unemi

PII: S0022-5193(11)00376-6  
DOI: doi:10.1016/j.jtbi.2011.07.026  
Reference: YJTBI6567

To appear in: *Journal of Theoretical Biology*

Received date: 14 March 2011  
Revised date: 23 July 2011  
Accepted date: 26 July 2011

Cite this article as: Tatsuya Sasaki and Tatsuo Unemi, Replicator dynamics in public goods games with reward funds, *Journal of Theoretical Biology*, doi:10.1016/j.jtbi.2011.07.026

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting galley proof before it is published in its final citable form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



[www.elsevier.com/locate/jtbi](http://www.elsevier.com/locate/jtbi)

1 Replicator dynamics in public goods games with reward funds

2  
3 Tatsuya Sasaki<sup>1,2,\*</sup>

4 Email address: sasakit@iiasa.ac.at

5  
6 <sup>1</sup>*Evolution and Ecology Program*  
7 *International Institute for Applied Systems Analysis (IIASA)*  
8 *Schlossplatz 1, Laxenburg A-2361, Austria*

9 <sup>2</sup>*Graduate School of Engineering, Soka University*  
10 *Hachioji, Tokyo 192-8577, Japan*

11  
12 Tatsuo Unemi<sup>3</sup>

13 Email address: unemi@iss.soka.ac.jp

14 <sup>3</sup>*Department of Information Systems Science, Soka University*  
15 *Hachioji, Tokyo 192-8577, Japan*

16  
17 \*Corresponding Author at:

18 *Evolution and Ecology Program*  
19 *International Institute for Applied Systems Analysis (IIASA)*  
20 *Schlossplatz 1, Laxenburg A-2361, Austria*

21 Tel: +43-2236-807

22 Fax: +43-2236-71313

23  
24 14 Mar 2011

25 For the submission to *the Journal of Theoretical Biology*

26 (The revised version on 23 Jul 2011)

27

## 1 **Abstract**

2 Which punishment or rewards are most effective at maintaining cooperation in public  
3 goods interactions and deterring defectors who are willing to freeload on others'  
4 contribution? The sanction system is itself a public good and can cause problematic  
5 “second-order free riders” who do not contribute to the provisions of the sanctions and thus  
6 may subvert the cooperation supported by sanctioning. Recent studies have shown that  
7 public goods games with punishment can lead to a coercion-based regime if participation in  
8 the game is optional. Here, we reveal that even with compulsory participation, rewards can  
9 maintain cooperation within an infinitely large population. We consider three strategies for  
10 players in a standard public goods game: to be a cooperator or a defector in a standard  
11 public goods game, or to be a rewarder who contributes to the public good and to a fund  
12 that rewards players who contribute during the game. Cooperators do not contribute to the  
13 reward fund and are therefore classified as second-order free riders. The replicator  
14 dynamics for the three strategies exhibit a rock-scissors-paper cycle, and can be analyzed  
15 fully, despite the fact that the expected payoffs are nonlinear. The model does not require  
16 repeated interaction, spatial structure, group selection, or reputation. We also discuss a  
17 simple method for second-order sanctions, which can lead to a globally stable state where  
18 100% of the population are rewarders.

19 **Keywords:** evolutionary game theory; cooperation; sanction; second-order social dilemma;  
20 rock-scissors-paper cycle

## 1 **1. Introduction**

2 An enduring conundrum in the biological and social sciences is how cooperation can  
3 emerge and be maintained in a sizable group containing exploiters. The conundrum is the  
4 so-called social dilemma [1, 2] because its nature is described as follows: groups of  
5 cooperators outperform groups of defectors, whereas in a mixed group defectors always  
6 outperform cooperators. This represents common conflicts between a social optimum and  
7 individual interests very well, and it has traditionally been modeled as the public goods  
8 game in many experimental and theoretical studies [3].

9 In the public goods game (PGG), cooperators confer benefits on others with some cost to  
10 themselves, whereas defectors exploit the benefits without such contribution to others.  
11 Defection is the selfish choice that results in a decrease in the total benefit to the group, but  
12 defection is rational from the evolutionary viewpoint because of a higher individual payoff,  
13 with no cost. Thus, natural selection will often drive elimination of cooperation. Classical  
14 and evolutionary game studies have, however, identified supportive mechanisms under  
15 which cooperation is nonetheless sustained, such as repeated interactions [4, 5], reputation  
16 [6, 7], spatial structure [8, 9], and group selection [10, 11].

17 Punishment of defectors and rewards for cooperators are also major factors that maintain  
18 cooperation between self-interested individuals, as suggested by growing experimental and  
19 theoretical evidence [12–32]. However, sanctions are costly, and therefore pose the next  
20 conundrum: how costly sanctioning can subsist in the presence of those who freeload on  
21 others' contributions to sanctions. This issue is the “second-order social dilemma” [12, 14],  
22 which has been particularly well addressed, in the case of costly punishment. One of  
23 possible solutions is to punish second-order freeloaders as well [13, 15, 24, 32].

24 At the same time, there is the issue of how costly punishment can emerge [21, 33]. In a  
25 population of defectors, a rare punisher suffers enormous costs because of the need to  
26 continuously punish defectors. However, recent studies have shown that punishment-based

1 cooperation can emerge if participation in the PGG is optional rather than compulsory [20,  
2 21, 26, 32]. We note that optional participation is another way to maintain cooperation  
3 [33–39], which can lead to “rock-scissors-paper”-type cyclic domination, well-known in  
4 evolutionary game theory [40, 41], among cooperators, defectors, and loners who earn a  
5 small but fixed payoff, instead of participating in the PGG [37–39]. Interestingly, Sigmund  
6 *et al.* [32] have found that when it comes to punishing second-order freeloaders, natural  
7 selection favors pool-punishment rather than peer-punishment. Peer-punishment is a  
8 sanctioning technique which has been the most widely used form of punishment in PGGs in  
9 which players decide whether to impose fines on exploiters *after* the PGG. By contrast, in  
10 pool-punishment, players have to decide whether to contribute to a punishment fund *before*  
11 the PGG [14], analogous to forming a volunteer band of watchmen in advance.

12 While optional participation could be required for a population to evolve from a stalemate  
13 where everybody defects to a coercion-based regime, there problems associated with opting  
14 out of a public goods project, such as global environmental issues, remain [21]. When  
15 participation is compulsory, peer-rewarding can cause cyclical dynamics in infinite  
16 populations if reputation alone is important (for pair-wise interactions see Sigmund *et al.*  
17 [16]; for interactions of arbitrary size see Hauert [30]). In contrast, reputation is given less  
18 weight in finite populations [29].

19 In this work, we explore the effects of pool-rewarding in compulsory PGGs with infinite  
20 populations. Similar to pool-punishment, players first decide whether to contribute to a  
21 reward fund. After a one-shot PGG among all group members, the common fund is divided  
22 equally among those players who contributed, irrespective of their contribution to the fund.  
23 While the list of real-world examples of reward funds is too long to list, we shall consider a  
24 generous voluntary fund which may be threatened with collapse by second-order  
25 freeloaders. We propose a minimalistic model for infinite populations that does not require  
26 repeated interactions, reputation, spatial structure, group selection, or optional participation.  
27 We also compare two types of benefit-sharing models, which differ on whether or not a

1 contributor in the PGG may oneself benefit, thus corresponding to “weak altruism” and  
 2 “strong altruism” [42, 43]. The evolution of cooperation is investigated by means of the  
 3 replicator dynamics [40, 41].

## 4 **2. The game-theoretical model**

5 Consider an infinitely large, well-mixed population of constant size. From time to time, a  
 6 group of  $N$  players is randomly formed from the population (where  $N \geq 2$ ). The PGG is  
 7 of a one-shot version. Each player is asked to contribute  $c_1 > 0$  to the public good. The  
 8 contributions are then distributed in the following different ways: in the case of weak  
 9 altruism (**WA**), the contribution,  $c_1$ , will be multiplied by  $r_1 > 1$  and then equally shared  
 10 among all  $N$  players in the group, but in the case of strong altruism (**SA**), it will be  
 11 shared among  $N-1$  other co-players only. In both cases, if all group members contribute,  
 12 they obtain a payoff of  $(r_1 - 1)c_1 > 0$ . The PGG with **SA** is a social dilemma for any rate of  
 13  $r_1$ , and the PGG with **WA**, also for  $r_1 < N$ . Indeed, in each case, a player that does not  
 14 contribute to the public goods can get an improved payoff by  $c_1$  with **SA**, and by  
 15  $c_1(1 - r_1/N) > 0$  with **WA**, no matter what the other players do. For the PGG with **WA**,  
 16 we assume  $r_1 < N$ , as the social dilemma would otherwise be completely relaxed due to  
 17 the benefits by switching to a contributor.

18 Next, we introduce the following pool-rewarding mechanism. Before participating in the  
 19 PGG, each player is first asked to contribute  $c_2 > 0$  to a fund to reward cooperative  
 20 behaviors in the PGG. The integrated contribution to the reward fund is multiplied by  
 21  $r_2 > 1$ , and after the PGG distributed equally to those who have contributed to the public  
 22 good, if any.

23 We consider the following three strategies: rewarders (**R**) who contribute both to the PGG  
 24 and to the reward fund, cooperators (**C**) who contribute to the PGG but not to the reward  
 25 fund, and defectors (**D**) who contribute neither to the PGG nor to the reward. If all  $S$   
 26 contributors in the PGG are **R**-players, they each obtain a net reward of  $(r_2 - 1)c_2 > 0$ , and  
 27 if all of them are **C**-players, they obtain nothing. The rewarding system is a second-order  
 28 social dilemma for  $r_2 < S$  because withdrawing one’s contribution to the reward fund can  
 29 increase individual payoff by  $c_2(1 - r_2/S) > 0$ .

30 We note that pool-rewarding itself is another case of weak altruism: an **R**-player is allowed  
 31 to obtain a return from contributing to the reward fund. We do not eliminate a return for  
 32 individuals who choose to contribute to rewards. **R**-players would be more likely to evolve  
 33 with it than without it. In the latter case **D**-players dominate (See Appendix A.1 for details).

1 Nevertheless, it is not clear whether or not such weakly altruistic, reward system can subsist  
 2 in the presence of second-order freeloaders. Indeed, the funding stage is set up before the  
 3 PGG and thus R-players cannot avoid the risk of being exploited by C-players.

4 We denote the expected payoff values for R-, C-, and D-players with  $P_R$ ,  $P_C$ , and  $P_D$ ,  
 5 respectively. The frequencies of the three strategies are expressed as  $x$ ,  $y$ , and  $z$   
 6 ( $x + y + z = 1$ ). The average payoff for the population is given by  $\bar{P} = xP_R + yP_C + zP_D$ . The  
 7 strategy's expected payoff is supposed to be the sum of the payoff from the PGG and from  
 8 the reward fund. The replicator equations are written as

$$9 \quad \dot{x} = x \left( P_R - \bar{P} \right), \quad \dot{y} = y \left( P_C - \bar{P} \right), \quad \dot{z} = z \left( P_D - \bar{P} \right). \quad (1)$$

10 We first calculate the expected payoffs from the PGG. In the case of **WA**, a D-player in a  
 11 group with  $S$  contributors obtains a benefit of  $r_1 c_1 S / N$  ( $0 \leq S \leq N-1$ ). Hence, the  
 12 expected payoff is given by

$$13 \quad P_D^1 = \sum_{S=0}^{N-1} \binom{N-1}{S} (1-z)^S z^{N-S-1} \frac{r_1 c_1 S}{N}$$

$$14 \quad = r_1 c_1 \left( 1 - \frac{1}{N} \right) (1-z), \quad (2a)$$

15 where  $\binom{N-1}{S} (1-z)^S z^{N-S-1}$  is the probability that  $S$  of  $N-1$  co-players in the PGG  
 16 are contributors. In the case of **SA**, a D-player in the group obtains a benefit of  
 17  $r_1 c_1 S / (N-1)$ , and calculating the expected payoff as in Eq. (2a),

$$18 \quad P_D^1 = r_1 c_1 (1-z). \quad (2b)$$

19 Both the expected payoffs for R- and C-players (denoted by  $P_R^1$ , resp.  $P_C^1$ ) are reduced  
 20 from  $P_D^1$ , by the cost for a contributor  $\sigma$ :  $\sigma = c_1 (1 - r_1 / N)$  in the case of **WA** and  $\sigma = c_1$   
 21 in the case of **SA**.

22 Regarding the reward system, the expected payoff for D-players is  $P_D^2 = 0$ . A C-player in a  
 23 group with  $S$  contributors and  $n_R$  R-players (and thus  $S - n_R$  C-players) receives a  
 24 reward of  $r_2 c_2 n_R / S$  ( $0 \leq n_R \leq S-1$ ). Hence, the expected reward for a C-player in a group  
 25 with  $S$  contributors is

$$26 \quad P_C^2(S) = \sum_{n_R=0}^{S-1} \binom{S-1}{n_R} \left( \frac{x}{1-z} \right)^{n_R} \left( \frac{y}{1-z} \right)^{S-n_R-1} \frac{r_2 c_2 n_R}{S},$$

$$27 \quad = r_2 c_2 \left( 1 - \frac{1}{S} \right) \left( \frac{x}{1-z} \right),$$



1 (3)

2 where  $\binom{S-1}{n_R} \left(\frac{x}{1-z}\right)^{n_R} \left(\frac{y}{1-z}\right)^{S-n_R-1}$  is the probability that  $n_R$  of the other  $S-1$   
 3 contributors are R-players. Consequently, the expected reward for a C-player is

$$4 \quad P_C^2 = \sum_{S=1}^N \binom{N-1}{S-1} (1-z)^{S-1} z^{N-S} P_C^2(S)$$

$$5 \quad = r_2 c_2 \left(1 - \frac{1-z^N}{N(1-z)}\right) \left(\frac{x}{1-z}\right)$$

6 (4)

7 Among  $S$  contributors, switching from R to C yields  $c_2(1-r_2/S)$ . Thus, the expected  
 8 net reward for an R-player,  $P_R^2$ , is reduced from  $P_C^2$  by

$$9 \quad c_2 \sum_{S=1}^N \binom{N-1}{S-1} (1-z)^{S-1} z^{N-S} \left(1 - \frac{r_2}{S}\right) = c_2 \left(1 - \frac{r_2}{N} \frac{1-z^N}{1-z}\right)$$

$$10 \quad =: F(z). \quad (5)$$

11  $F(z)$  has a unique root  $\hat{z}$  in the open interval  $(0,1)$  if, and only if,  $1 < r_2 < N$ , because  
 12  $F(z)$  is monotonic,  $F(0) = c_2(1-r_2/N) > 0$ , and  $F(1) = c_2(1-r_2) < 0$ . Therefore, the  
 13 advantage C-players have over R-players will change from positive to negative as  $z$   
 14 increases across  $\hat{z}$ .

15 Integrating the above results, we can determine that  $P_R = P_R^1 + P_R^2$ ,  $P_C = P_C^1 + P_C^2$ , and  
 16  $P_D = P_D^1 + P_D^2$ , and obtain a simple expression for the average payoff for the population

$$17 \quad \bar{P} = c_1(r_1-1)(1-z) + c_2(r_2-1)x, \quad (6)$$

18 both for the **WA** and **SA** cases.

### 19 **3. Dynamics**

20 The evolutionary dynamics of the three strategies take place in the state space  
 21  $S_3 = \{(x, y, z) : x, y, z \geq 0, x + y + z = 1\}$ . The three homogeneous states in which 100% of  
 22 the population are R-players ( $x = 1$ ), C-players ( $y = 1$ ), and D-players ( $z = 1$ ) correspond to  
 23 three vertices of the simplex  $S_3$  (which we denote by R, C, and D, respectively). These  
 24 are obviously fixed points for the replicator system Eq. (1). There are no other fixed points  
 25 on the boundary of  $S_3$  for non-degenerate cases. Indeed, on the edge C-D:  $x = 0$ ,

26  $z = (P_D - P_C)z(1-z) = \sigma z(1-z) > 0$ , where  $\sigma = c_1(1-r_1/N)$  in the case of **WA** and  
 27  $\sigma = c_1$  in the case of **SA**. Thus, the evolution on the edge C-D is unidirectional from C to D.

1 On the edge R-C:  $z=0$  and on the edge D-R:  $y=0$ , resulting in  
 2  $y=(P_C - P_R)y(1-y) = c_2(1-r_2/N)y(1-y)$  and  
 3  $x=(P_R - P_D)x(1-x) = [c_2(r_2-1) - \sigma]x(1-x)$ , respectively. The evolution on both edges is  
 4 unidirectional and its direction depends on the magnitude of the relationship between  $r_2$   
 5 and  $N$ , and between  $c_2(r_2-1)$  and  $\sigma$ , respectively.

6 To analyze the dynamics in the interior of  $S_3$ , let us introduce a new variable  
 7  $f = x/(1-z)$ , which represents the fraction of contributors in the PGG that are also  
 8 rewarders. This yields

$$9 \quad f = -\frac{xy}{(1-z)^2}(P_C - P_R) = -f(1-f)F(z). \quad (7)$$

10 Substituting  $x = f(1-z)$  and Eq. (6) into  $z = z(P_D - P)$  yields

$$11 \quad z = -z(1-z)[c_2(r_2-1)f - \sigma]. \quad (8)$$

### 12 **3.1. The global attractor D**

13 Supposing  $c_2(r_2-1) - \sigma < 0$ , then the direction of evolution on the edge D-R is from R to  
 14 D. Eq. (8) yields  $z > 0$  in the interior of  $S_3$ . Thus, there is no interior fixed point and all  
 15 interior orbits converge to the vertex D, which is a global attractor (Fig. 1a). If  $r_2 < N$ , the  
 16 direction of evolution on the edge R-C is from R to C; if  $r_2 > N$  and otherwise, it is from  
 17 C to R; and when  $r_2 = N$ , the edge R-C consists of unstable fixed points. We note that if  
 18  $r_2 < 1$ , then  $c_2(r_2-1) - \sigma < 0$  holds. In the boundary case that  $c_2(r_2-1) - \sigma = 0$ ,  $z = 0$   
 19 holds when  $f = 1$  and thus, the edge D-R is a line of fixed points. If  $r_2 < N$ , the edge is  
 20 separated into an unstable segment ( $0 \leq z < z^*$ ) and a stable one ( $z^* < z \leq 1$ ). Since  $z > 0$   
 21 holds in the interior of  $S_3$ , all interior orbits converge to the stable segment (Fig. 1b). If  
 22  $r_2 \geq N$ , then the edge D-R has no unstable segment. Random drift and occasional invasion  
 23 of the missing C-player will eventually send the state within the stable segment to the  
 24 vertex D, in the long run.

25 [Fig. 1 is around here]

### 26 **3.2. The global attractor R**

1 Supposing  $c_2(r_2 - 1) - \sigma > 0$  and  $r_2 > N$ , then the direction of evolution on the edge D-R  
 2 is from D to R, and from C to R on the edge R-C. The fact that  $F(z) < 0$  in the open  
 3 interval  $(0, 1)$  yields  $x > 0$  in the interior of  $S_3$ . Thus, there is no interior fixed point and  
 4 all interior orbits converge to the vertex R, which is a global attractor (Fig. 2). If  $r_2 = N$ ,  
 5 then the edge R-C is a line of fixed points, which consists of an unstable segment  
 6  $(0 \leq x < x_{RC})$  and a stable one  $(x_{RC} < x \leq 1)$ , where  $x_{RC}$  is given by  $\sigma / [c_2(r_2 - 1)]$  as a  
 7 non-trivial solution of Eq. (8). The fact that all interior states satisfy  $x > 0$  leads the  
 8 population to evolve towards the stable segment. Thus, random drift and occasional  
 9 invasion of the missing D-player will eventually bring the population to the vertex R, in the  
 10 long run.  
 11 [Fig. 2 is around here]

### 12 3.3. The mixture equilibrium of the three strategies

13 Supposing that  $c_2(r_2 - 1) - \sigma > 0$  and  $1 < r_2 < N$ , the direction of evolution on the edge  
 14 D-R is from D to R, and from R to C on the edge R-C. Thus, the three edges of  $S_3$  form a  
 15 heteroclinic cycle of a rock-scissors-paper type. We now have a unique interior root  $\hat{z}$  of  
 16  $F(z)$  and  $0 < \hat{f} := \sigma / [c_2(r_2 - 1)] < 1$ . From Eqs. (7) and (8), we see that there is a unique  
 17 interior fixed point  $Q = (\hat{x}, \hat{y}, \hat{z})$ , with  
 18 
$$\hat{x} = \hat{f} \left( 1 - \hat{z} \right), \quad \hat{y} = (1 - \hat{f}) \left( 1 - \hat{z} \right). \quad (9)$$

19 The mixture equilibrium,  $Q$ , is a center, i.e., it is neutrally stable and surrounded by closed  
 20 orbits that fill the interior of  $S_3$  (Fig. 3). This results because the equations of Eqs. (7) and  
 21 (8) can be expressed in the form of a Hamiltonian system,  $H$ , and now  $H$  has a strict  
 22 maximum at the unique fixed point  $(\hat{f}, \hat{z})$  corresponding to  $Q$  (See Appendix A.2 and  
 23 [30] for details).

24 Given  $c_1$ ,  $r_1$ , and  $N$ , which are all original parameters for the PGG, the location of  $Q$  can  
 25 be determined by the remaining parameters,  $c_2$  and  $r_2$ . According to Eq. (9),  $Q$  lies on  
 26 the line  $y = (1/\hat{f} - 1)x$ , independent of the group size,  $N$ . As  $N$  increases,  $Q$  moves  
 27 toward the vertex D along the line  $y$  and  $\hat{z} \rightarrow 1$  as  $N \rightarrow \infty$ . On the other hand, as  $N$   
 28 decreases,  $Q$  moves in the opposite direction and  $\hat{z}$  decreases to  $2/r_2 - 1 > 0$ , which

1 occurs when  $N = 2$ . In other extreme cases, where  $r_2 = 1$ ,  $r_2 = N$ ,  $c_2(r_2 - 1) = \sigma$ , and  
 2  $c_2 = \infty$ ,  $Q$  arrives at the vertex D, the edges R-C, D-R, and C-D, respectively.  
 3 [Fig. 3 is around here]

#### 4 **4. Discussion**

5 Conflict between contributors and freeloaders in public-goods interactions is inevitable.  
 6 How can we avoid conflict between contributors and freeloaders? An effective solution is  
 7 to set up a reward fund for cooperative behaviors. The key conditions for the reward system  
 8 necessary to maintain cooperation with free riders in public goods games (PGGs) are given  
 9 by

$$10 \quad c_2(r_2 - 1) > \sigma, \quad (10)$$

11 where  $\sigma = c_1(1 - r_1/N)$  in the case of weak altruism and  $\sigma = c_1$  in the case of strong  
 12 altruism. Eq. (10) means that the optimum group reward should exceed the cost for a  
 13 contributor in the PGG, which is relaxed by a self-returning benefit of  $r_1 c_1 / N$  in the case  
 14 of weak altruism. In infinite populations, it has been determined that peer-rewarding is a  
 15 potent motivator, but only if reputation is important [16, 30]. However, in pool-rewarding,  
 16 this is not the case. With such attractive rewards, cooperative investments in both the PGG  
 17 and the reward fund can subsist, even when second-order freeloaders can dominate the  
 18 rewarding system, i.e., for  $r_2 < N$ . In the case, the replicator dynamics exhibit a  
 19 rock-scissors-paper cycle among the three strategies: defectors who never contribute  
 20 (first-order freeloaders), cooperators who contribute only in the PGG (second-order  
 21 freeloaders), and rewarders who contribute to both.

22 The cyclical evolutionary scenario can be described as follows. If most players are  
 23 rewarders, the reward system is actually a second-order social dilemma and thus  
 24 cooperators spread. If cooperators are prevalent, it is better to become a defector due to the  
 25 social dilemma. If most players are defectors, the number of beneficiaries of the reward is  
 26 usually small enough to subvert cooperator dominance over rewarders, and thus the number  
 27 of rewarders increases. If the number of rewarders increases sufficiently, then the  
 28 second-order dilemma returns. In this scenario, traditional defectors play a pivotal role in  
 29 maintaining the cyclic domination among the three strategies. The moderate advantage  
 30 defectors have over cooperators, given by  $\sigma$ , prevents the second-order dilemma from  
 31 eliminating rewarders and then ensures that rewarders, not cooperators, dominate.

1 Global environmental and energy issues often appear to be compulsory public goods  
 2 projects, such that in the short-term cooperation will yield only very little benefit and the  
 3 social optimum is not to cooperate. The situation is not a social dilemma, and has thus  
 4 remained outside the scope of studies on the evolution of cooperation in large groups. In  
 5 our model, this may correspond to the case where  $0 \leq r_1 < 1$ . We remark that the results  
 6 shown hold even when  $0 \leq r_1 < 1$ , and thus pool-rewarding is applicable to a broader range  
 7 of public-goods interactions.

8 We note that in the extreme case where  $r_1 = 0$ , our model is significantly similar to an  
 9 earlier public goods game with optional participation [37, 38, 39]. Indeed, the PGG  
 10 degenerates into a game in which there is no longer benefit from contribution  $c_1$ . Each  
 11 player therefore seems to have the option to avoid the participation fee of  $c_1$ , instead of  
 12 taking part in another PGG with a cost of  $c_2$  and a multiplier of  $r_2$ . This is just an  
 13 implementation of the inverse form of the loner's option.

14 A fascinating extension of this work is to consider second-order sanctions [13, 15, 24, 32].  
 15 Indeed, in our model, it looks practical for the rewarding system to mete out punishment on  
 16 cooperators (second-order freeloaders) in such a way that will reduce rewards for those [12].  
 17 Let us see how, for instance, reducing rewards to cooperators by  $a\%$  changes the  
 18 dynamics. According to preliminary numerical simulations, the existing interior fixed point  
 19  $Q$  is destabilized (Fig. 4), and for discount rates  $a$  higher than a threshold value, the  
 20 population can converge to a state of 100% rewarders, irrespective of the initial conditions  
 21 (Fig. 4b). As increasing  $a$  crosses the threshold, a new mixture equilibrium  $P$ , of  
 22 cooperators and rewarders, enters the state space  $S_3$  and is unstable within the  
 23 rewarder-cooperator boundary (See Appendix A.3 for details). If defectors (first-order  
 24 freeloaders) are absent, the population cannot avoid the resulting coordination problem:  
 25 depending on the initial condition, the population evolves to become either 100% rewarders  
 26 or 100% cooperators. Otherwise, interestingly, the population can make an end run around  
 27 the bistability and establish the social optimum. It would be a rather intriguing issue for  
 28 future research to theoretically analyze the result that reward-based cooperation will  
 29 necessarily become globally stable, whenever it cannot be invaded by second-order  
 30 freeloaders. By contrast, in the case of pool-punishment, punishment-based cooperation can  
 31 never become globally stable, even if second-order sanctions are assumed, because a state  
 32 of 100% first-order freeloaders remain stable [44].

33 [Fig. 4 is around here]

34 One important issue we left out is the effects of economies and diseconomies of scale on  
 35 the provision of sanctions. So far we have focused on linear cost-benefit functions for  
 36 rewarding, whereby any group of rewarders generates the same per capita group benefit.  
 37 According to Mathew and Boyd [33], the existing interior fixed point of the optional public  
 38 goods game becomes an attractor for decreasing returns and a repeller for increasing returns.

1 In practice, the rich dynamics afforded by scale would provide many options for the proper  
2 design of sanctioning systems to support the evolution of cooperation.

3

#### 4 **Appendix**

##### 5 **A.1. The strongly altruistic rewarding**

6 We here turn to a strongly altruistic variant of pool-rewarding, in which the rewards  
7 resulting from an R-player will be shared among *other* contributors only. We assume that if  
8 there exists no other contributor, the investment to the incentive from a single R-player will  
9 be exactly refunded to her. The expected reward for a C-player turns into

$$10 \quad P_C^2 = r_2 c_2 \left( \frac{x}{1-z} \right) (1-z^{N-1}),$$

11 and that for an R-player is reduced from  $P_C^2$  by the expected incentive cost  $c_2(1-z^{N-1})$ .

12 Eqs. (7) and (8) turn into

$$13 \quad f = -c_2 f (1-f) (1-z^{N-1}),$$

$$14 \quad z = -z(1-z)[c_2(r_2-1)f(1-z^{N-1}) - \sigma].$$

15 Since  $f$  is negative in the interior of the state space  $S_3$ ,  $int S_3$ , there is no interior fixed  
16 point. If  $c_2(r_2-1) - \sigma \leq 0$ , then  $z > 0$  holds in  $int S_3$ , and thus, all interior orbits  
17 converge to the vertex D.

18 If  $c_2(r_2-1) - \sigma > 0$ , the system has a new equilibrium at  $(f, z) = \left( 1, \left( 1 - \frac{\sigma}{c_2(r_2-1)} \right)^{\frac{1}{N-1}} \right)$

19 on the edge D-R, which is a source. The vertex D is a sink, while the vertex R still remains  
20 a saddle. We consider the  $z$ -isocline that is the set where  $z = 0$ : in  $int S_3$ , this is the set

21 where  $f = \frac{\sigma}{c_2(r_2-1)(1-z^{N-1})}$ . The interior component forms a curve that connects the

22 new fixed point and the point  $(f, z) = \left( \frac{\sigma}{c_2(r_2-1)}, 0 \right)$  on the edge R-C, and divides  $int S_3$

1 to two regions: one region where  $z < 0$  and the other where  $z > 0$ . The last one includes  
 2 the vicinity of the edge C-D given by  $x = 0$ . Since  $f < 0$  holds in  $\text{int } S_3$ , any interior  
 3 orbit which starts in the state with  $z < 0$  has to travel to the region where  $z > 0$ . Hence,  
 4 all interior orbits converge to the vertex D.

## 5 **A.2. The Hamiltonian System**

6 Divide the right-hand side of Eqs. (7) and (8) by the function  $f(1-f)z(1-z)$ , which is  
 7 positive for any  $(f, z)$  in the interior of the unit square  $[0, 1]^2$ . Hence,

$$8 \quad f = \frac{-F(z)}{z(1-z)} =: -g(z), \quad z = \frac{\sigma - c_2(r_2 - 1)f}{f(1-f)} =: -l(f).$$

9 This transformation corresponds to a change in velocity and does not affect orbit. Let us  
 10 introduce  $H(f, z) := G(z) + L(f)$ , where  $G(z)$  and  $L(f)$  are primitives of  $g(z)$  and  
 11  $l(f)$ , respectively:

$$12 \quad G(z) = c_2 \left( 1 - \frac{r_2}{N} \right) \log z + c_2 (r_2 - 1) \log(1 - z) + R(z),$$

$$13 \quad L(f) = \sigma \log f + [c_2 (r_2 - 1) - \sigma] \log(1 - f).$$

14 with  $R(z)$  bounded on  $[0, 1]$ . Thus, we obtain the Hamiltonian system

$$15 \quad \dot{f} = -\frac{\partial H}{\partial z}, \quad \dot{z} = \frac{\partial H}{\partial f}.$$

16 Because the system is conservative and the Hamiltonian attains a strict global maximum at  
 17  $(\hat{f}, \hat{z})$  if  $c_2(r_2 - 1) - \sigma > 0$  and  $1 < r_2 < N$ , the interior equilibrium  $Q$  is a stable point  
 18 surrounded by closed orbits. Indeed, all interior orbits are closed:  $G(z) \rightarrow -\infty$  as  $z \rightarrow 0, 1$   
 19 if  $1 < r_2 < N$  and  $L(f) \rightarrow -\infty$  as  $f \rightarrow 0, 1$  if  $0 < \sigma < c_2(r_2 - 1)$ . Hence,  $H \rightarrow -\infty$   
 20 uniformly near the boundary of  $[0, 1]^2$  and thus all constant level sets of  $H$  are closed  
 21 curves around  $(\hat{f}, \hat{z})$ . The solutions have to remain on the constant level sets and thus  
 22 return to their starting points.

23

24

25

## 26 **A.3. The second-order sanctioning**

27 We examine an extensive model in which rewards for cooperators (second-order  
 28 freeloaders) will be reduced by  $100\alpha\%$  ( $0 \leq \alpha \leq 1$ ), under the assumptions that

1  $c_2(r_2 - 1) > \sigma$  and  $1 < r_2 < N$ . In the extension, the expected payoff for a cooperator is  
 2 given by

$$3 \quad P_C^2 = (1 - \alpha)r_2c_2 \left( 1 - \frac{1 - z^N}{N(1 - z)} \right) \left( \frac{x}{1 - z} \right),$$

4 and, Eqs. (7) and (8) turn to

$$5 \quad f = -f(1 - f)[F(z) - \alpha(c_2(r_2 - 1) + F(z))f],$$

$$6 \quad z = -z(1 - z)[c_2(r_2 - 1)f - \sigma - \alpha(c_2(r_2 - 1) + F(z))f(1 - f)].$$

7 In the interior of  $S_3$ , there exists at most one fixed point  $Q = (f_Q, z_Q)$  such that

$$8 \quad f_Q = \frac{\sigma}{\alpha\sigma + (1 - \alpha)c_2(r_2 - 1)} \quad \text{and} \quad F(z_Q) = c_2(r_2 - 1) \frac{\alpha f_Q}{1 - \alpha f_Q}.$$

9 The fact that  $F(z)$  is monotonically decreasing and  $F(z_Q) \geq 0$  yields that  $0 < z_Q \leq \hat{z}$ ,

10 where  $\hat{z}$  is the unique solution of  $F(z) = 0$ .  $f_Q$  increases and  $z_Q$  decreases, with  
 11 increasing  $\alpha$ . This implies that as  $\alpha$  increases,  $Q$  moves towards the edge R-C. As  $\alpha$  crosses

12 a threshold  $\alpha_p$  given by  $\frac{F(0)}{c_2(r_2 - 1) + F(0)}$ , a new equilibrium with

13  $(f, z) = \left( \frac{N - r_2}{\alpha r_2 (N - 1)}, 0 \right)$  enters the edge R-C through the vertex R, which then turns into a

14 sink. The boundary equilibrium,  $P$ , is a saddle point, unstable within the edge and stable to  
 15 invasion of defectors. As  $\alpha$  further increases,  $P$  moves towards the vertex C, and when  $\alpha$

16 crosses another threshold  $\alpha_Q$  given by  $\frac{F(0)}{\sigma + F(0)}$ ,  $Q$  exits  $S_3$  through  $P$ , which then

17 turns into a source. For larger values of  $\alpha$ ,  $S_3$  has no interior equilibrium but  $P$  still

18 remains within the edge. Preliminary numerical simulations imply that  $Q$  is a source for  $\alpha >$

19 0, and all interior orbits converge, if  $0 < \alpha < \alpha_p$ , to a heteroclinic cycle on the boundary of

20  $S_3$ , and if  $\alpha_p < \alpha \leq 1$ , to the vertex R.



**References**

- [1] R.M. Dawes, Social dilemmas, *Ann. Rev. Psychol.* 31 (1980) 169–193.
- [2] P. Kollock, Social dilemmas: the anatomy of cooperation, *Annu. Rev. Sociol.* 24 (1998) 183–214.
- [3] J.O. Ledyard, Public Goods: A Survey of Experimental Research, in: J.H. Kagel, A.E. Roth (Eds.), *The Handbook of Experimental Economics*, Princeton University Press, Princeton, NJ, 1995, pp. 111–194.
- [4] Q. Trivers, The evolution of reciprocal altruism, *Rev. Biol.* 46 (1971) 35–57.
- [5] R. Axelrod, W.D. Hamilton, The evolution of cooperation, *Science* 211 (1981) 1390–1396.
- [6] M.A. Nowak, K. Sigmund, Evolution of indirect reciprocity by image scoring, *Nature* 393 (1998) 573–577.
- [7] M. Milinski, D. Semmann, H.-J. Krambeck, Reputation helps to solve the 'tragedy of the commons', *Nature* 415 (2002) 424–426.
- [8] M.A. Nowak, R.M. May, Evolutionary games and spatial chaos, *Nature* 359 (1992) 826–829.
- [9] T. Killingback, M. Doebeli, N. Knowlton, Variable investment, the Continuous Prisoner's Dilemma, and the origin of cooperation, *Proc. R. Soc. B.* 266 (1999) 1723–1728.
- [10] D.S. Wilson, E. Sober, Reintroducing group selection to the human behavioral sciences, *Behav. Brain Sci.* 17 (1994) 585–654.
- [11] A. Traulsen, M.A. Nowak, Evolution of cooperation by multilevel selection, *Proc. Natl. Acad. Sci. U.S.A.* 29 (2006) 10952–10955.
- [12] P. Oliver, Rewards and punishments as selective incentives for collective action: theoretical investigations, *Am. J. Sociol.* 85 (1980) 1356–1375.
- [13] R. Axelrod, An evolutionary approach to norms, *Am. Polit. Sci. Rev.* 80 (1986) 1095–1111.
- [14] T. Yamagishi, The provision of a sanctioning system as a public good, *J. Pers. Soc. Psychol.* 51 (1986) 110–116.

- 1 [15] R. Boyd, P.J. Richerson, Punishment allows the evolution of cooperation (or anything  
2 else) in sizable groups, *Ethol. Sociobiol.* 13 (1992) 171–195.
- 3 [16] K. Sigmund, C. Hauert, M.A. Nowak, Reward and punishment, *Proc. Natl. Acad. Sci.*  
4 *U.S.A.* 98 (2001) 10757–10762.
- 5 [17] E. Fehr, S. Gächter, Altruistic punishment in humans, *Nature* 415 (2002) 137–140.
- 6 [18] J. Andreoni, W.T. Harbaugh, L. Vesterlund, The carrot or the stick: rewards,  
7 punishments, and cooperation, *Am. Econ. Rev.* 93 (2003) 893–902.
- 8 [19] A. Gardner, S.A. West, Cooperation and punishment, especially in humans, *Am. Nat.*  
9 164 (2004) 753–764.
- 10 [20] J. Fowler, Altruistic punishment and the origin of cooperation, *Proc. Natl. Acad. Sci.*  
11 *U.S.A.* 102, (2005) 7047–7049.
- 12 [21] C. Hauert, A. Traulsen, H. Brandt, M.A. Nowak, K. Sigmund, Via freedom to  
13 coercion: the emergence of costly punishment, *Science* 316 (2007) 1905–1907.
- 14 [22] M. Sefton, R. Shupp, J. Walker, The effects of rewards and sanctions in provision of  
15 public goods, *Econ. Inquiry* 45 (2007) 671–690.
- 16 [23] K. Sigmund, Punish or Perish? Retaliation and Collaboration among humans, *Trends*  
17 *Ecol. Evol.* 22 (2007) 593–600.
- 18 [24] T. Kiyonari, P. Barclay, Cooperation in the social dilemma: free riding may be  
19 thwarted by second-order reward rather than by punishment, *J. Pers. Soc. Psychol.* 95  
20 (2008) 826–842.
- 21 [25] M. Shinada, T. Yamagishi, Bringing back Leviathan into social dilemmas, in: A. Biel,  
22 D. Eek, T. Gärling (Eds.), *New Issues and Paradigms in Research on Social*  
23 *Dilemmas*. Springer-Verlag, Berlin, Germany, 2008, pp. 93–123.
- 24 [26] H. De Silva, C. Hauert, A. Traulsen, K. Sigmund, Freedom, enforcement, and the  
25 social dilemma of strong altruism, *J. Evol. Econ.* 20 (2009) 203–217.
- 26 [27] M. Nakamaru, U. Dieckmann, Runaway selection for cooperation and  
27 strict-and-severe punishment, *J. Theor. Biol.* 257 (2009) 1–8.
- 28 [28] R. Boyd, H. Gintis, S. Bowles, Coordinated punishment of defectors sustains  
29 cooperation and can proliferate when rare, *Science* 328 (2010) 617–620.

- 1 [29] P.A.I. Forsyth, C. Hauert, Public goods games with reward in finite populations, J.  
2 Math. Biol. (2010) Published Online First: 24 September 2010. Doi:  
3 10.1007/s00285-010-0363-7
- 4 [30] C. Hauert, Replicator dynamics of reward & reputation in public goods games, J.  
5 Theor. Biol. 267 (2010) 22–28.
- 6 [31] C. Hilbe, K. Sigmund, Incentives and opportunism: from the carrot to the stick, Proc.  
7 R. Soc. B. 277 (2010) 2427–2433.
- 8 [32] K. Sigmund, H. De Silva, C. Hauert, A. Traulsen, Social learning promotes  
9 institutions for governing the commons, Nature 466 (2010) 861–863.
- 10 [33] S. Mathew, R. Boyd, When does optional participation allow the evolution of  
11 cooperation?, Proc. R. Soc. B. 276 (2009) 1167–1174.
- 12 [34] J.H. Orbell, R.M. Dawes, Social welfare, cooperator’s advantage, and the option of not  
13 playing the game, Am. Sociol. Rev. 58 (1993) 787–800.
- 14 [35] C.A. Aktipis, When to walk away and when to stay: cooperation evolves when agents  
15 can leave unproductive partners and groups, J. Theor. Biol. 231 (2004) 249–260.
- 16 [36] T. Sasaki, T. Unemi, Probabilistic participation in public goods games. Proc. R. Soc.  
17 B. 274 (2007) 2639–2642.
- 18 [37] C. Hauert, S. De Monte, J. Hofbauer, K. Sigmund, Volunteering as red queen  
19 mechanism for cooperation in public goods games, Science 296 (2002) 1129–1132.
- 20 [38] C. Hauert, S. De Monte, J. Hofbauer, K. Sigmund, Replicator dynamics for optional  
21 public goods games, J. Theor. Biol. 218 (2002) 187–194.
- 22 [39] D. Semmann, H.-J. Krambeck, M. Milinski, Volunteering leads to rock-paper-scissors  
23 dynamics in a public goods game, Nature 425 (2003) 390–393.
- 24 [40] J. Hofbauer, K. Sigmund, Evolutionary Games and Population Dynamics. Cambridge  
25 Univ. Press, Cambridge, 1998.
- 26 [41] K. Sigmund, M.A. Nowak, Evolutionary dynamics of biological games, Science 303  
27 (2004) 793–798.
- 28 [42] J.A. Fletcher, M. Doebeli, A simple and general explanation for the evolution of  
29 altruism, Proc. R. Soc. B. 276 (2009) 13–19.

- 1 [43] J.A. Fletcher, M. Zwick, The evolution of altruism: game theory in multilevel  
2 selection and inclusive fitness, *J. Theor. Biol.* 245 (2007) 26–36.
- 3 [44] K. Sigmund, C. Hauert, A. Traulsen, H. De Silva, Social control and the social  
4 contract: the emergence of sanctioning systems for collective action, *Dyn. Games.*  
5 *Appl.* 1 (2011) 149–171.
- 6

Accepted manuscript

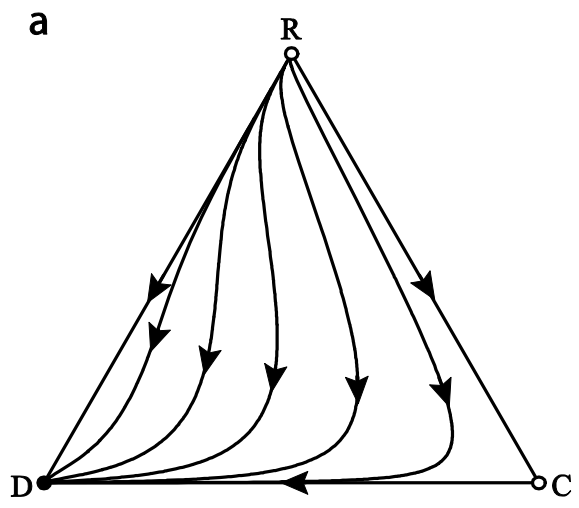
## 1 Figure Captions

- 2 **Figure 1.** Defectors (first-order freeloaders) prevail. Oscillations do not occur and the  
 3 interior state space has no fixed point. (a) All interior states evolve towards the vertex D.  
 4 (b) In the boundary case that  $c_2(r_2 - 1) - \sigma = 0$ , the edge D-R is a line of fixed points. All  
 5 interior orbits converge to a stable (lower) segment of the edge. Random drift and  
 6 occasional invasion of the missing C-player will eventually send the state to the vertex D.  
 7 Parameters:  $N = 5$ ;  $r_1 = 3$ ;  $c_2 = 1$ ;  $r_2 = 1.2$  (a) or 1.4 (b);  $\sigma = 0.4$ ; and  $c_1 = 1$  (in the  
 8 case of **WA**),  $c_1 = 0.4$  (in the case of **SA**).
- 9 **Figure 2.** Rewarders prevail. Oscillations do not occur and all interior states evolve  
 10 towards the vertex R. The interior state space has no fixed point. Parameters:  $N = 5$ ;  
 11  $r_1 = 3$ ;  $c_2 = 1$ ;  $r_2 = 5.5$ ;  $\sigma = 0.4$ ; and  $c_1 = 1$  (in the case of **WA**),  $c_1 = 0.4$  (in the case  
 12 of **SA**).
- 13 **Figure 3.** Rock-scissors-paper cycles. All three corners of the simplex  $S_3$  are saddle  
 14 points and the boundary of  $S_3$  represents a heteroclinic cycle. The interior of  $S_3$  has a  
 15 unique fixed point  $Q$ , which is a center surrounded by closed orbits. Parameters:  $N = 5$ ;  
 16  $r_1 = 3$ ;  $c_2 = 1$ ;  $r_2 = 3$ ;  $\sigma = 0.4$ ; and  $c_1 = 1$  (in the case of **WA**),  $c_1 = 0.4$  (in the case of  
 17 **SA**).
- 18 **Figure 4.** The effects of second-order sanctions. (a) The existing interior fixed point  $Q$   
 19 turns into a repeller by cutting off  $a\%$  rewards for cooperators. The population converges  
 20 to a heteroclinic cycle on the boundary of  $S_3$ . (b) For a sufficiently high  $a$ , the vertex R  
 21 can be a global attractor. At the same time,  $S_3$  has a boundary fixed point  $P$ , which  
 22 divides the basins of attraction of rewarders and cooperators on the edge R-C and is stable  
 23 if there is an invasion of defectors. Parameters:  $N = 5$ ;  $r_1 = 3$ ;  $c_2 = 1$ ;  $r_2 = 3$ ;  $\sigma = 0.4$ ;  
 24 and  $c_1 = 1$  (in the case of **WA**),  $c_1 = 0.4$  (in the case of **SA**). The rewards are cut by the  
 25 following percentages (a)  $a = 10$  and (b)  $a = 20$ .  
 26

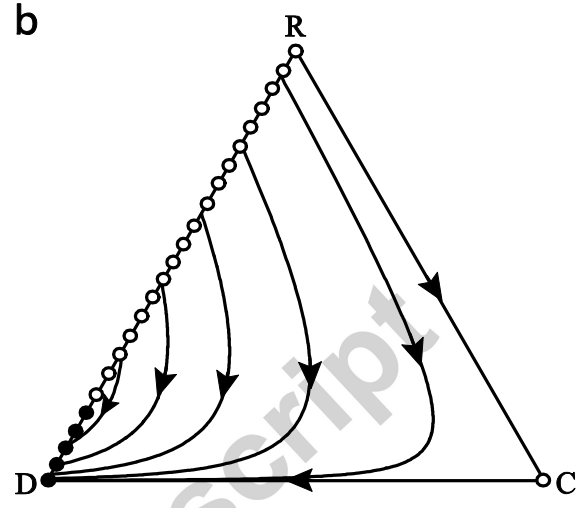
1 Fig 1

2

a



b



3

4

5

6

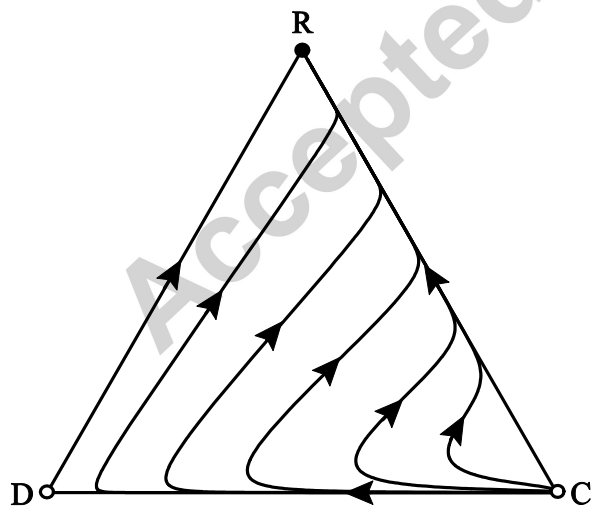
7

8

9

10 Fig 2

11



12

13

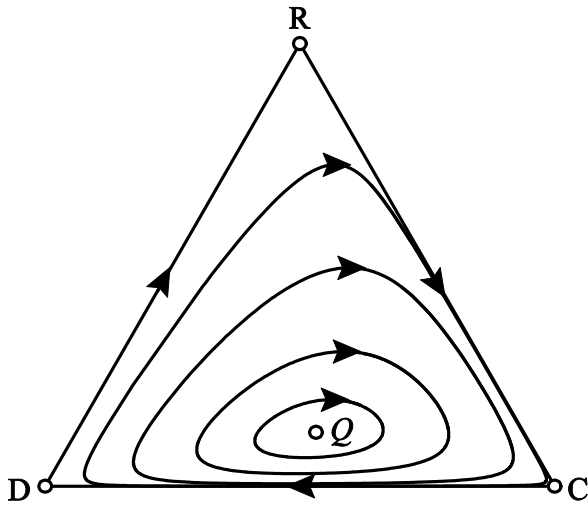
14

15

16

1

2 Fig 3



3

4

5

6

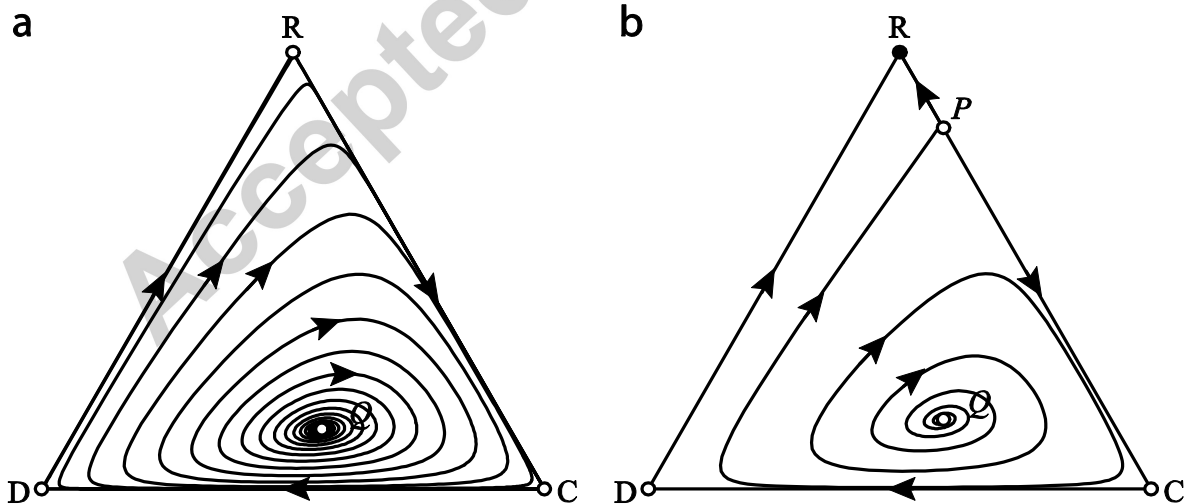
7

8

9

10 Fig 4

11



12

13

1  
2  
3  
4  
5  
6  
7  
8

**Highlights**

- > We investigate how a reward fund affects the evolution of cooperation.
- > The reward fund can evolve in the presence of second-order free riders.
- > Rewarders, second-, and first-order free riders can form a rock-scissors-paper cycle.
- > The replicator dynamics can fully be analyzed despite the nonlinear payoff functions.
- > The model does not require iterated game, reputation, or optional participation.

Accepted manuscript