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# (Nearly-)Tight Bounds on the Linearity and Contiguity of Cographs

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EXTENDED ABSTRACT

**Introduction.** Linearity and contiguity are graph parameters introduced to obtain efficient codings of neighborhoods in graphs, by decomposing each neighborhood as a union of  $p$  intervals chosen in one or several orders on the vertices [1]. Indeed, storing an order of the vertices as well as a pair of pointers for each of the  $p$  intervals of this order (one pointer for the beginning of the interval and one for the end), with fixed  $p$ , allows to store the graph in  $O(n)$  space (instead of  $O(n + m)$  with adjacency lists) and access the neighborhood of any vertex  $v$  in  $O(d)$  time (instead of  $O(n)$  with adjacency matrices), where  $d$  is the degree of  $v$ .

More formally, a *closed  $p$ -interval-model* of a graph  $G = (V, E)$  is a linear order  $\sigma$  on  $V$  such that  $\forall v \in V, \exists (I_1, \dots, I_p) \in (2^V)^p$  such that  $\forall i \in \int 1, p, I_i$  is an interval of  $\sigma$  and  $N[v] = \bigcup_{1 \leq i \leq p} I_i$ . The *closed contiguity* of  $G$ , denoted by  $cont(G)$ , is the minimum integer  $p$  such that there exists a closed  $p$ -interval-model of  $G$ . A *closed  $p$ -line-model* of a graph  $G = (V, E)$  is a tuple  $(\sigma_1, \dots, \sigma_p)$  of linear orders on  $V$  such that  $\forall v \in V, \exists (I_1, \dots, I_p) \in (2^V)^p$  such that  $\forall i \in \int 1, p, I_i$  is an interval of  $\sigma_i$  and  $N[v] = \bigcup_{1 \leq i \leq p} I_i$ . The *closed linearity* of  $G$ , denoted by  $lin(G)$ , is the minimum  $p$  such that there exists a closed  $p$ -line-model of  $G$ .

Not much is known about these parameters, which cannot be bounded by a constant even in very restricted graph classes, like interval or permutation graphs [1]. We focus here on the contiguity and linearity of cographs (graphs without induced  $P_4$  subgraphs), whose very constrained structure can be represented by their *cotree*, a rooted tree with two kinds of nodes labeled by  $P$  and  $S$ , giving a tight upper bound for the asymptotic contiguity of cographs and an upper bound for their linearity. To this aim, we first establish a min-max theorem on the link between the rank of rooted trees and their decompositions into paths.

**A min-max theorem on the rank of a tree.** The *rank* [2, 3] of a tree  $T$  is the maximal height of a complete binary tree obtained from  $T$  by edge contractions, that is  $rank(T) = \max\{h(T') \mid T' \text{ complete binary tree, minor of } T\}$ .

A *path partition* of a tree  $T$  is a partition  $\{P_1, \dots, P_k\}$  of  $V(T)$  such that for any  $i$ , the subgraph  $T[P_i]$  of  $T$  induced by  $P_i$  is a path, as shown in Figure 1(a). The *partition tree* of a path partition  $\mathcal{P}$ , denoted by  $T_p(\mathcal{P})$  and illustrated in Figure 1(b), is the tree whose nodes are  $P_i$ 's and where the node of  $T_p(\mathcal{P})$  corresponding to  $P_i$  is the parent of the node corresponding to  $P_j$  iff some node of  $P_i$  is the parent in  $T$  of the root of  $P_j$ . The height of a path partition  $\mathcal{P}$  of a tree  $T$ , denoted by  $h(\mathcal{P})$ , is the height  $h(T_p(\mathcal{P}))$  of its partition tree. The *path-height* of  $T$  is the minimal height of a path partition of  $T$ , that is  $ph(T) = \min\{h(\mathcal{P}) \mid \mathcal{P} \text{ path partition of } T\}$ .

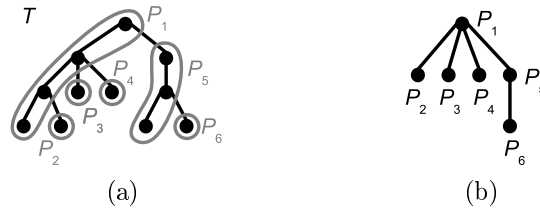


Figure 1: A tree  $T$  and a path partition  $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$  of  $T$  (a), as well as the partition tree of  $\mathcal{P}$  (b).

**Lemma 1** For a rooted complete binary tree  $T$ ,  $\text{rank}(T) = \text{ph}(T) = h(T)$ .

**Theorem 2** For any rooted tree  $T$ , we have  $\text{rank}(T) = \text{ph}(T)$ .

**Upper bounds for contiguity and linearity of cographs.** We now combine the results of the previous section with a decomposition of the cotree of the input cograph into paths, in order to obtain a constructive proof that the contiguity of any cograph is at most  $O(\log n)$ . This decomposition is obtained recursively, using a root-path decomposition of the cotree, thanks to the Caterpillar Composition Lemma below.

A *root-path decomposition* (see Fig. 2) of a rooted tree  $T$  is a set  $\{T_1, \dots, T_p\}$  of disjoint subtrees of  $T$ , with  $p \geq 2$ , such that every leaf of  $T$  belongs to some  $T_i$ , with  $i \in [1..p]$ , and the sets of parents in  $T$  of the roots of  $T_i$ 's is a path containing the root of  $T$ .

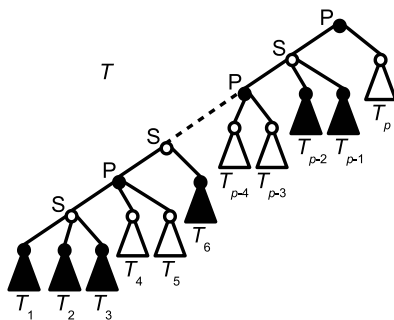


Figure 2: The root-path decomposition  $\{T_1, \dots, T_p\}$  of a rooted tree  $T$ .

**Lemma 3 (Caterpillar Composition Lemma)** Given a cograph  $G = (V, E)$  and a root-path decomposition  $\{T_i\}_{1 \leq i \leq p}$  of its cotree, where  $X_i$  is the set of leaves of  $T_i$ ,  $\text{cont}(G) \leq 2 + \max_{i \in [1..p]} \text{cont}(G[X_i])$ .

**Lemma 4** Given a rooted tree  $T$  such that  $\text{rank}(T) = k \geq 1$ , there exists a root-path decomposition  $\{T_1, \dots, T_p\}$  of  $T$  such that for each  $i \in [1..p]$ ,  $\text{rank}(T_i) \leq k - 1$ .

**Lemma 5** Let  $G$  be a cograph and  $T$  its cotree. We have  $\text{cont}(G) \leq 2\text{rank}(T) + 1$ .

**Theorem 6** The closed contiguity of a cograph is at most logarithmic in its number of vertices, or more formally, if  $G = (V, E)$  is a cograph, then  $\text{cont}(G) \leq 2 \log_2 |V| + 1$ .

**Lower bounds for contiguity and linearity of cographs.** Finally, we focus on cographs whose cotrees are complete binary trees, and obtain a tight lower bound for their asymptotic contiguity as well as a lower bound for their asymptotic linearity.

**Theorem 7** Let  $G$  be a cograph whose cotree is a complete binary tree. Then,  $\text{cont}(G) = \Omega(\log n)$  and  $\text{lin}(G) = \Omega(\log n / \log \log n)$ .

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