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A unified topological framework for digital imaging

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Abstract. In this article, a tractable *modus operandi* is proposed to model a (binary) digital image (*i.e.*, an image defined on $\mathbb{Z}^n$ and equipped with a standard pair of adjacencies) as an image defined in the space of cubical complexes ($\mathbb{F}^n$). In particular, it is shown that all the standard pairs of adjacencies in $\mathbb{Z}^n$ can then be correctly modelled in $\mathbb{F}^n$. Moreover, it is established that the digital fundamental group of a digital image in $\mathbb{Z}^n$ is isomorphic to the fundamental group of its corresponding image in $\mathbb{F}^n$, thus proving the topological correctness of the proposed approach. From these results, it becomes possible to establish links between topology-oriented methods developed either in classical digital spaces ($\mathbb{Z}^n$) or cubical complexes ($\mathbb{F}^n$).

Key words: digital imaging, digital topology, cubical complexes, homotopy, fundamental group.

1 Introduction

The rise of digital imaging, and the associated need of efficient image analysis tools, have provided a strong motivation to research related to the definition of sound digital topological models. Indeed, in order to process digital images, it is often fundamental to be able to preserve, get back or integrate topological information.

Basically, an n-dimensional (digital) binary image can be considered as a subset of $\mathbb{Z}^n$. However, the actual structures visualised in such images are generally continuous ones, corresponding to objects of the real world, *i.e.*, objects of $\mathbb{R}^n$, and not of $\mathbb{Z}^n$. In order to deal with this continuous/discrete issue, research efforts have essentially focused on specific and pragmatic questions related to topology, namely the definition of a notion of *adjacency* relation, and the induced notions of *connectivity* and *arcs*. These notions lead, in particular, to high-level concepts of topology, such as *homotopy*, with natural applications to "homotopy type-preserving" image processing.

The first solution proposed to model the topology of a digital image in $\mathbb{Z}^n$ was to consider that two points (also called *xels*) are adjacent if they are neighbours in the n-D cubic grid naturally induced by $\mathbb{Z}^n$. In this framework, partial solutions have been found to model as well as possible the above topological properties, for instance by defining *dual adjacencies* for the object and the background (composed of the xels of value 1 and 0, respectively), enabling to define, from these adjacency relations, the notions of connectivity [1] and of *digital fundamental group* [2], which permits to compare objects from a topological point of view. This approach is known as *digital topology* [3].

Other discrete spaces, enabling to model the continuous topological properties of digital images, have also been proposed. These alternative approaches of topology modelling are *connected ordered topological spaces* [4], *abstract cell complexes* [5] and *orders* [6]. Broadly speaking, they propose to put some "topological glue" between the xels of digital images to define the topological links with their continuous analogues.

By comparison to these (more sophisticated) approaches, digital topology may appear as the less satisfactory solution to deal with topological properties of binary images. Nevertheless, it remains the most commonly used framework for developing image processing tools dealing with topological issues. Indeed, since digital topology is directly defined in $\mathbb{Z}^n$, methods relying on it will also provide final results in $\mathbb{Z}^n$, which is a desired property in most applications. Moreover, a large literature has been devoted to homotopy-type preservation in digital topology, especially thanks to the concept of simple point [7]. In this context, very few methods have been based on alternative models while digital topology has led to the design of quite numerous ones (see, *e.g.*, [8,9]).

Because of this intensive use of digital topology, it may be important to guarantee that there exists an actual compatibility between digital topology and the other proposed topological approaches (and more generally with the "continuous" topology). This requires to be able to embed a binary image defined in $\mathbb{Z}^n$ into a richer space (while respecting the chosen adjacencies in $\mathbb{Z}^n$) while preserving certain topological characteristics of objects (see, *e.g.*, [10,11]).

The "richer space" that is used here is $\mathbb{F}^n$, namely the space of cubical complexes, which is together a connected ordered topological space, a cellular space and an order (*i.e.*, a poset). Though it is commonly admitted that there exists a strong link between digital topology and cubical complexes [12], since complexes are closed objects, the images handled in $\mathbb{F}^n$ correspond generally to images defined in $\mathbb{Z}^n$ with a $(3^n - 1, 2n)$-adjacency pair. In [13], a method has been proposed to retrieve and improve digital topology in the framework of posets, but the case of the $(6, 18)$- and $(18, 6)$-adjacency pairs was not considered. In [14], a way is described to embed digital pictures in a space of complexes according to the kind of connectivity chosen in $\mathbb{Z}^n$. However, there is no use of an intrinsic topology on complexes which are just a step between $\mathbb{Z}^n$ and $\mathbb{R}^n$, leading to define specific notions of connectedness and digital homotopy in $\mathbb{F}^n$. Thereby, in this paper, we propose a complete framework to correctly embed a binary digital image in the topological space $\mathbb{F}^n$, according to the choice of adjacencies which has been made in $\mathbb{Z}^n$.

The article is organised as follows. Sec. 2 recalls background notions. Sec. 3 describes the mapping enabling to associate a binary image defined on $\mathbb{Z}^n$ to an equivalent image defined in $\mathbb{F}^n$. Sec. 4 presents the main contribution of this work. It states that the connected components and the digital fundamental group of the digital images in $\mathbb{Z}^n$ are preserved in $\mathbb{F}^n$ when using the mapping described in Sec. 3. Sec. 5 concludes this article. By lack of space, no proofs are given in this article, they can be found in [15].

2 Background notions

This section provides the minimal set of background notions required to make this article globally self-contained, and then more comprehensible for the reader.

2.1 Posets

Let X be a set. A binary relation on X is a *partial order* if it is reflexive, antisymmetric, and transitive. A *partially ordered set* (or *poset*) is a ordered pair $(X, \leqslant)$ where the relation $\leqslant$ is a partial order on X. We write $x < y$ when $x \leqslant y$ and $x \neq y$. The relation $\geqslant$ defined on X by $x \geqslant y$ if $y \leqslant x$ is a partial order on X called the *dual order*. The *covering* relation $\prec$, associated to $\leqslant$, is defined by: $x \prec y$ (say "y covers x") if $x < y$ and there is no z such that $x < z < y$. We set: $x^{\uparrow} = \{y \in X \mid x \leqslant y\}$; $x^{\uparrow +} = \{y \in x^{\uparrow} \mid y^{\uparrow} = \{y\}\}$; $x^{\downarrow} = \{y \in X \mid y \leqslant x\}$; $x^{\downarrow \star} = x^{\downarrow} \setminus \{x\} = \{y \in X \mid y < x\}$; $x^{\prec} = \{y \in X \mid x \prec y\}$. An element $x \in X$ is *minimal* if $x^{\downarrow} = \{x\}$ and *maximal* if $x^{\uparrow} = \{x\}$. An element $x \in X$ is the *minimum* of X if $x^{\uparrow} = X$ and is the *maximum* of X if $x^{\downarrow} = X$. If the minimum (resp., the maximum) exists, then it is unique.

2.2 Topological spaces

Let X be a set, the elements of which will be called *points*. A *topology* on X is a collection $\mathcal{U}$ of subsets of X, called *open sets*, such that:

(i) $\emptyset, X$ are open sets;
(ii) any finite intersection of open sets is an open set;
(iii) any union of open sets is an open set.

The complement in X of an open set is called a *closed set*. From the above definition, any finite union of closed sets is a closed set and any intersection of closed sets is a closed set. A set of open sets is a *basis* for a topology if any open set is a union of open sets of this basis. A *neighbourhood* of a point $x \in X$ is a subset of X including an open set containing x.

Let $(X, \leqslant)$ be a poset. The set $\mathcal{U}$ defined by $\mathcal{U} = \{U \subseteq X \mid \forall x \in U, x^{\uparrow} \subseteq U\}$ is a topology on X which is called the *Alexandroff topology*. In this topology, the set $x^{\uparrow}$ is the smallest open set containing x (or the smallest neighbourhood of x, called the *star* of x) and the set $x^{\downarrow}$ is the smallest closed set containing x (the *closure* of x). Two points $x, y \in X$ are *adjacent* if $x \leqslant y$ or $y \leqslant x$. A sequence $(z_i)_{i=0}^{r}$ $(r \geq 0)$ of elements of X is an *arc* in X (*from* z_0 *to* z_r) if for all $i \in [\![1, r]\!]$, z_{i-1} and z_i are distinct and adjacent. A subset Y of X is *connected* if for all $x, y \in Y$, there exists an arc in Y from x to y. A *connected component* of a subset Y of X is a maximal (for inclusion) connected subset of Y.

The *closure* $Y^{\downarrow}$ of a subset $Y \subseteq X$ is the smallest closed set including Y. The *interior* Y° of a subset $Y \subseteq X$ is the largest open set included in Y (it is also the union of all open sets included in Y). Closure and interior are dual notions since $\neg(Y^{\circ}) = (\neg Y)^{\downarrow}$ and $\neg(Y^{\downarrow}) = (\neg Y)^{\circ}$ where $\neg Y = X \setminus Y$. An open set Y is a *regular open set* if $Y = (Y^{\downarrow})^{\circ}$ and a closed set is a *regular closed set* if $Y = (Y^{\circ})^{\downarrow}$. The complement of a regular open set is a regular closed set.

2.3 Cubical complexes

Let $\mathbb{Z}$ be the set of integers. Let $\mathbb{F}_0^1 = \{\{a\} \mid a \in \mathbb{Z}\}$ and $\mathbb{F}_1^1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$. Let $n \geq 1$.

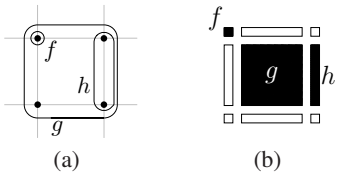


Fig. 1. Two representations of a set of faces $F = \{f, g, h\}$ in $\mathbb{Z}^2$ with, for instance, $f = \{0\} \times \{1\}$, $g = \{0, 1\} \times \{0, 1\}$ and $h = \{1\} \times \{0, 1\}$. The face g is a facet of F (and also of $\mathbb{F}^2$).

Let $f \subset \mathbb{Z}^n$. If f is the Cartesian product of m elements of $\mathbb{F}^1_1$ and $n - m$ elements of $\mathbb{F}^1_0$, we say that f is a *face* or an *m-face* (*of* $\mathbb{Z}^n$), m is the *dimension* of f, and we write $\dim(f) = m$ (some faces of $\mathbb{Z}^2$ are depicted in Fig. 1). We denote by $\mathbb{F}^n$ the set composed of all faces of $\mathbb{Z}^n$; this set is the (n-D) space of *cubical complexes*. We denote by $\mathbb{F}^n_k$ ($0 \leq k \leq n$) the set composed of all k-faces of $\mathbb{Z}^n$. The couple $(\mathbb{F}^n, \subseteq)$ is a poset. Let $F \subseteq \mathbb{F}^n$ be a set of faces. Let $f \in F$ be a face. The face f is a *facet* of F if f is maximal in F. In particular, if $x = (x_i)_{i=1}^n \in \mathbb{Z}^n$, the set $\dot{x} = \prod_{i=1}^n \{x_i, x_i + 1\}$ is a facet of $\mathbb{F}^n$.

2.4 Digital topology

Let $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$ be two points in $\mathbb{Z}^n$ (also called *n-xels*, or simply *xels*). The points x and y are *2n-adjacent* if $\sum_{i=1}^n |x_i - y_i| = 1$. They are $(3^n - 1)$-*adjacent* if $\max_{i=1}^n \{|x_i - y_i|\} = 1$. When $n = 3$, the points x and y are 18-*adjacent* if they are 26-adjacent and $\sum_{i=1}^n |x_i - y_i| \leq 2$. Let $\alpha \in \{2n, 3^n - 1\}$ (or possibly $\alpha = 18$ if $n = 3$). Any point in $\mathbb{Z}^n$ is α-adjacent to α other points. A sequence $\gamma = (z_i)_{i=0}^r$ ($r \geq 0$) of points in $X \subseteq \mathbb{Z}^n$ is a *(digital) α-path* (*from z_0 to z_r*) if for all $i \in [\![1, r]\!]$, z_{i-1} and z_i are α-adjacent. The integer r is the *length* of γ. A subset $X \subseteq \mathbb{Z}^n$ is *α-connected*, if for all $x, y \in X$, there exists a digital α-path from x to y in X. In order to retrieve some topological features in binary digital images (such as the notion of *hole*), it is necessary to use pairs of adjacencies, one for the *object* X and one for the *background* $\mathbb{Z}^n \setminus X$. The suitable pairs are $(2n, 3^n - 1)$ and $(3^n - 1, 2n)$ (plus, when $n = 3$, $(6, 18)$ and $(18, 6)$).

3 Connectivity: from $\mathbb{Z}^n$ to $\mathbb{F}^n$

A (digital) image λ on $\mathbb{Z}^n$ is a function from $\mathbb{Z}^n$ to $\{0, 1\}$. A (complex) image μ on $\mathbb{F}^n$ is a function from $\mathbb{F}^n$ to $\{0, 1\}$. The *object* (resp. the *background*) associated to the image $\theta : X \to \{0, 1\}$ (with $X = \mathbb{Z}^n$ or $\mathbb{F}^n$) is the set $\theta^{-1}(\{1\})$ (resp. $\theta^{-1}(\{0\})$).

If μ is a complex image, we write $\bigvee_{x \in X} \mu(x)$ (resp. $\bigwedge_{x \in X} \mu(x)$) for the maximum (resp. minimum) of the set $\{\mu(x) \mid x \in X\}$ and we also write $\mu(a) \vee \mu(b)$ (resp. $\mu(a) \wedge \mu(b)$) for $\bigvee_{x \in \{a,b\}} \mu(x)$ (resp. $\bigwedge_{x \in \{a,b\}} \mu(x)$).

The poset $(\mathbb{F}^n, \subseteq)$ is equipped with its Alexandroff topology.

3.1 One-to-one correspondence between images on $\mathbb{Z}^n$ and $\mathbb{F}^n$

When two faces $g, h \in \mathbb{F}^n$ cover a face $f \in \mathbb{F}^n$ and their smallest neighbourhoods do not intersect (*i.e.*, $g^\uparrow \cap h^\uparrow = \emptyset$), we say that their are *opposite* with respect to the face f. We denote $\mathrm{opp}(f)$ the set of all $\{g, h\}$ for g opposite to h w.r.t. f. Intuitively, the face f is required to "locally connect" the faces g and h. When f is a facet, we have $\mathrm{opp}(f) = \emptyset$.

Definition 1. (regular image) *Let $\varepsilon : [\![1, n]\!] \to \{-1, 1\}$ be a function called* connectivity function[1]. *A function $\mu : \mathbb{F}^n \to \{0, 1\}$ is an ε-*regular image *(or simply a* regular image*) if for all $m \in [\![1, n]\!]$ and $f \in \mathbb{F}^n_{m-1}$, we have, recursively*

$$\mu(f) = \begin{cases} \bigwedge_{\{a,b\} \in \mathrm{opp}(f)} \mu(a) \vee \mu(b) & \text{if } \varepsilon(m) = 1 \\ \bigvee_{\{a,b\} \in \mathrm{opp}(f)} \mu(a) \wedge \mu(b) & \text{if } \varepsilon(m) = -1 \end{cases}$$

For each connectivity function $\varepsilon : [\![1, n]\!] \to \{-1, 1\}$, we define the function $\zeta_\varepsilon : \{0, 1\}^{\mathbb{Z}^n} \to \{0, 1\}^{\mathbb{F}^n}$ which maps any digital image λ to the unique ε-regular image $\zeta_\varepsilon(\lambda)$ such that, for each $a \in \mathbb{Z}^n$, we have $\zeta_\varepsilon(\lambda)(\dot{a}) = \lambda(a)$. It is obvious that, for each ε, the function ζ_ε is a bijection between the set of digital images $\{0, 1\}^{\mathbb{Z}^n}$ and the subset of ε-regular images of $\{0, 1\}^{\mathbb{F}^n}$. Moreover, thanks to the choice of the connectivity function ε, we can accurately "carve" an image in $\mathbb{F}^n$ to model the desired connectivity in $\mathbb{Z}^n$ (see Fig. 2). In particular, we can get the usual pairs of adjacencies (see Figs. 2–4 and Table 1). In Section 4, the correspondences given in Table 1 will be justified by two theorems establishing that, by following these links, we preserve the connected components and fundamental groups.

When the function ε is constant, Definition 1 can be simplified. Note that the case $\varepsilon = -1$ corresponds to the $2n$-adjacency in $\mathbb{Z}^n$ while the case $\varepsilon = 1$ corresponds to the $(3^n - 1)$-adjacency in $\mathbb{Z}^n$.

Proposition 1. *Let $\mu : \mathbb{F}^n \to \{0, 1\}$ be an ε-regular image. Let f be a face of $\mathbb{F}^n$.*

(i) If $\forall m > \dim(f), \varepsilon(m) = -1$, then we have $\mu(f) = \bigwedge_{f<a} \mu(a) = \bigwedge_{a \in f^{\uparrow +}} \mu(a)$
(ii) If $\forall m > \dim(f), \varepsilon(m) = 1$, then we have $\mu(f) = \bigvee_{f<a} \mu(a) = \bigvee_{a \in f^{\uparrow +}} \mu(a)$

In particular, if $\varepsilon = -1$ then $\mu(f) = \bigwedge_{a \in f^{\uparrow +}} \mu(a)$ for all $f \in \mathbb{F}^n$, and if $\varepsilon = 1$ then $\mu(f) = \bigvee_{a \in f^{\uparrow +}} \mu(a)$ for all $f \in \mathbb{F}^n$.

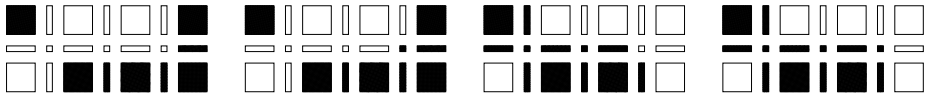
3.2 Duality

Let $\theta : X \to \{0, 1\}$ with ($X = \mathbb{Z}^n$ or $\mathbb{F}^n$) be an image. We define the *negative image* $\neg\theta : X \to \{0, 1\}$ of θ by $\neg\theta(x) = 1 - \theta(x)$, for all $x \in X$.

Proposition 2. *If $\mu : \mathbb{F}^n \to \{0, 1\}$ is an ε-regular image, then $\neg\mu$ is $(-\varepsilon)$-regular.*

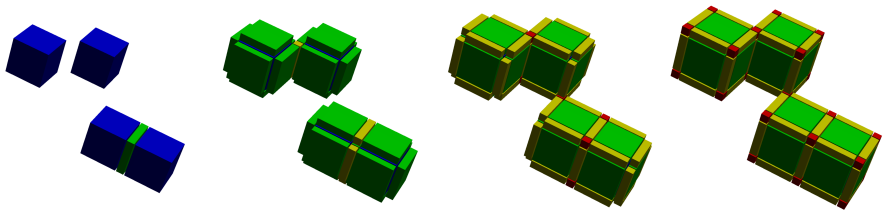
Let $\mu : \mathbb{F}^n \to \{0, 1\}$ be an ε-regular image. We define the image $-\mu : \mathbb{F}^n \to \{0, 1\}$ by $(-\mu)(f) = \mu(f)$ for all $f \in \mathbb{F}^n_n$ and μ is $(-\varepsilon)$-regular.

[1] We will often write $(\varepsilon(i))_{i=1}^n$ instead of ε, as done for instance in Fig. 2. For the sake of simplicity, we will also write $\varepsilon = 1$ (or $\varepsilon = -1$) when ε is a constant function.



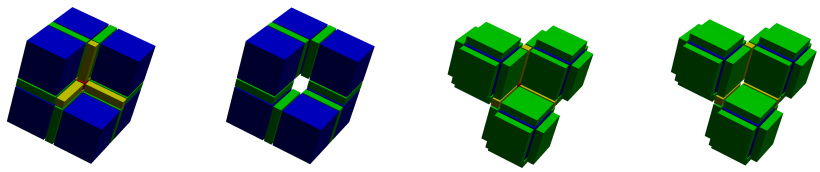
(a) $\varepsilon = -1$ (b) $\varepsilon = (1, -1)$ (c) $\varepsilon = (-1, 1)$ (d) $\varepsilon = 1$

Fig. 2. Images $\zeta_\varepsilon(\lambda) : \mathbb{F}^2 \to \{0, 1\}$ for some given $\lambda : \mathbb{Z}^2 \to \{0, 1\}$.



(a) (b) (c) (d)

Fig. 3. Images $\zeta_\varepsilon(\lambda) : \mathbb{F}^3 \to \{0, 1\}$ for some given $\lambda : \mathbb{Z}^3 \to \{0, 1\}$. In Figs. 3–4, the different colours are only used to distinguish the faces of the object $(\zeta_\varepsilon(\lambda))^{-1}(\{1\})$ (blue: 3-faces; green: 2-faces; yellow: 1-faces; red: 0-faces). (a) With $\varepsilon(3) = -1$, we obtain the 6-adjacency in $\mathbb{Z}^3$. (b) With $\varepsilon(3) = 1$ and $\varepsilon(2) = -1$, we obtain the 18-adjacency in $\mathbb{Z}^3$. (c, d) With $\varepsilon(3) = \varepsilon(2) = 1$, we obtain the 26-adjacency in $\mathbb{Z}^3$.



(a) (b) (c) (d)

Fig. 4. A torus built with six 3-faces illustrates how the two 6-adjacencies can be obtained. (a) With $\varepsilon = (\ .\ , 1, -1)$, the foreground is a horn-torus so we obtain the 6-adjacency associated to the 18-adjacency. (b) With $\varepsilon = (\ .\ , -1, -1)$, the foreground is a ring-torus, so we obtain the 6-adjacency associated to the 26-adjacency. (c,d) An object built from three facets with two connectivity functions which could *a priori* be used to model the 18-adjacency (see Fig. 3(b)). In (c), with $\varepsilon = (1, -1, 1)$, we can see a red 0-face between the three cubes. This is what is expected for the background must have a 6-adjacency. In (d), with $\varepsilon = (-1, -1, 1)$, there is a hole instead of the red 0-face, which is not correct in 18-adjacency.

Table 1. Correspondence between pairs of adjacencies in $\mathbb{Z}^n$ and connectivity functions.

Space dimension	$n = 2$		$n = 3$				$n \geq 4$ (actually, $n \in \mathbb{N}^*$)	
Adjacencies in $\mathbb{Z}^n$	$(4, 8)$	$(8, 4)$	$(6, 26)$	$(6, 18)$	$(18, 6)$	$(26, 6)$	$(2n, 3^n - 1)$	$(3^n - 1, 2n)$
ε	-1	1	-1	$(-1, 1, -1)$	$(1, -1, 1)$	1	-1	1

Proposition 3. *If $\mu : \mathbb{F}^n \to \{0,1\}$ is an ε-regular image, then we have $\neg(-\mu) = -(\neg\mu)$.*

From the above definitions and propositions, we straightforwardly derive the following result.

Proposition 4. *Let $\lambda : \mathbb{Z}^n \to \{0,1\}$ be a digital image. Let $\varepsilon : [\![1,n]\!] \to \{-1,1\}$ be a connectivity function. Then, we have $\neg(\zeta_\varepsilon(\lambda)) = \zeta_{-\varepsilon}(\neg\lambda)$.*

$$\begin{array}{ccc} \lambda & \xrightarrow{\neg} & \neg\lambda \\ {\scriptstyle \zeta_\varepsilon}\downarrow & & \downarrow{\scriptstyle \zeta_{-\varepsilon}} \\ \zeta_\varepsilon(\lambda) & \xrightarrow{\neg} & \zeta_{-\varepsilon}(\neg\lambda) \end{array}$$

Remark 1. This proposition establishes that, for a given connectivity function ε (and the associated pair of adjacencies (α,β)), all the properties valid for $\lambda^{-1}(\{1\})$ and $\mu^{-1}(\{1\})$ are also valid for $\lambda^{-1}(\{0\})$ and $\mu^{-1}(\{0\})$ for the opposite connectivity function $-\varepsilon$ (and the associated pair of adjacencies (β,α)). Broadly speaking, this means that the notions of object and background can be switched without loss of generality, provided that the pair of adjacencies (β,α) is also switched accordingly.

3.3 Computing values directly from facets

The aim of this section is to find the number of facets which must have the value 1 in the star of a face to ensure that this face also has value 1. In $\mathbb{F}^2$, the answer is straightforward. In $\mathbb{F}^3$, it requires to carefully study a particular configuration (depicted in Fig. 5), however, it can be answered, as stated hereafter. In higher dimensional spaces, the particular configurations to study are too numerous to get a useful result.

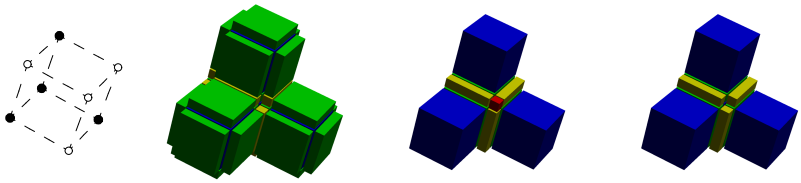
Let $f \in \mathbb{F}^n$, with $n \geq 3$. If $\dim(f) = n-3$, the poset $(f^{\uparrow}, \subseteq)$ has a unique minimum, namely f, and 8 maximal elements, namely the facets forming $f^{\uparrow+}$. From an adjacency point of view, these facets are geometrically organised as the 8 vertices of a cubical structure. When $f_1^{\uparrow+}$ (*i.e.*, the facets of $f^{\uparrow+}$ whose values are equal to 1) is organised as in the configuration depicted in Fig. 5(a) (up to rotations and symmetries), we say that $f_1^{\uparrow+}$ is a *trihedron*. We define $\mathrm{Card}^-(E) = 3$ and $\mathrm{Card}^+(E) = 5$, if E is a trihedron, and $\mathrm{Card}^-(E) = \mathrm{Card}^+(E) = \mathrm{Card}(E)$ otherwise.

For each connectivity function ε, we define recursively the function $\delta_\varepsilon : [\![0,n]\!] \to [\![1,2^n]\!]$ by $\delta_\varepsilon(0) = 1$, and for all $i > 0$, $\delta_\varepsilon(i+1) = 2\delta_\varepsilon(i) - 1$ if $\varepsilon(n-i) = 1$, and $\delta_\varepsilon(i+1) = 2\delta_\varepsilon(i)$ if $\varepsilon(n-i) = -1$. It is easy to check that, for all $m \in [\![0,n]\!]$, we have $\delta_\varepsilon(m) = 1 + \sum_{k=1}^{m}(1 - \varepsilon(n-k+1))2^{m-k-1}$.

In the sequel, we write $f_1^{\uparrow+}$ for the set of facets in the star of f which have value 1:$f_1^{\uparrow+} = \{g \in \mathbb{F}_n^n \mid f \leqslant g \text{ and } \mu(g) = 1\}$.

Proposition 5. *Let $\mu : \mathbb{F}^n \to \{0,1\}$ be an ε-regular image. Let f be a k-face of $\mathbb{F}^n$ (with $n-3 \leq k \leq n-1$).*

(i) If $\mathrm{Card}^-(f_1^{\uparrow+}) \geq \delta_\varepsilon(n-k)$, then $\mu(f) = 1$.
(ii) If $\mu(f) = 1$, then $\mathrm{Card}^+(f_1^{\uparrow+}) \geq \delta_\varepsilon(n-k)$.



(a) (b) $\varepsilon = (-1, -1, 1)$ (c) $\varepsilon = (1, 1, -1)$ (d) $\varepsilon = (-1, 1, -1)$

Fig. 5. (a) Symbolic representation of a trihedron related to a face $f \in \mathbb{F}^n$ such that $\dim(f) = n-3$. Black dots: $f_1^{\uparrow+}$; white dots $f^{\uparrow+} \setminus f_1^{\uparrow+}$. The dash lines represent the existence of a face of dimension $n-1$ forming the intersection between two facets of $f^{\uparrow+}$. (b–d) Examples of trihedra, with three connectivity functions ε (one of the blue facets is hidden).

Table 2. Necessary and sufficient conditions to obtain $\mu(f) = 1$ (see Corollary 1).

$\dim(f) = n-2$				
ε	$(\ldots, 1, 1)$	$(\ldots, -1, 1)$	$(\ldots, 1, -1)$	$(\ldots, -1, -1)$
C_ε	$\mathrm{Card}(f_1^{\uparrow+}) \geq 1$	$\mathrm{Card}(f_1^{\uparrow+}) \geq 2$	$\mathrm{Card}(f_1^{\uparrow+}) \geq 3$	$\mathrm{Card}(f_1^{\uparrow+}) \geq 4$

$\dim(f) = n-3$				
ε	$(\ldots, 1, 1, 1)$	$(\ldots, -1, 1, 1)$	$(\ldots, 1, -1, 1)$	$(\ldots, -1, -1, 1)$
C_ε	$\mathrm{Card}(f_1^{\uparrow+}) \geq 1$	$\mathrm{Card}(f_1^{\uparrow+}) \geq 2$	$\mathrm{Card}(f_1^{\uparrow+}) \geq 3$	$\mathrm{Card}(f_1^{\uparrow+}) \geq 4$ $f_1^{\uparrow+}$ not a trihedron
ε	$(\ldots, 1, 1, -1)$	$(\ldots, -1, 1, -1)$	$(\ldots, 1, -1, -1)$	$(\ldots, -1, -1, -1)$
C_ε	$\mathrm{Card}(f_1^{\uparrow+}) \geq 5$ or $f_1^{\uparrow+}$ a trihedron	$\mathrm{Card}(f_1^{\uparrow+}) \geq 6$	$\mathrm{Card}(f_1^{\uparrow+}) \geq 7$	$\mathrm{Card}(f_1^{\uparrow+}) = 8$

From Proposition 5 and Definition 1 (needed when $f_1^{\uparrow+}$ is a trihedron), we derive the following corollary.

Corollary 1. *Let $\mu : \mathbb{F}^n \to \{0, 1\}$ be an ε-regular image. Let f be a k-face of $\mathbb{F}^n$ (with $n-3 \leq k \leq n-1$). Then $\mu(f) = 1$ iff the set $f_1^{\uparrow+}$ satisfies the condition C_ε given in Table 2.*

3.4 Regular images and regular open/closed sets

The object (resp. the background) of a regular image $\mu : \mathbb{F}^n \to \{0, 1\}$, *i.e.*, the set $\mu^{-1}(\{1\})$ (resp. $\mu^{-1}(\{0\})$) is topologically regular, *i.e.*, it does not have thin parts nor thin holes (by "thin", we mean of lower dimension than the surrounding space).

Proposition 6. *Let $\mu : \mathbb{F}^n \to \{0, 1\}$ be an ε-regular image. Let $x \in \{0, 1\}$. Then $(\mu^{-1}(\{x\}))^\circ$ is a regular open set and $(\mu^{-1}(\{x\}))^\downarrow$ is a regular closed set.*

Corollary 2. *Let $\mu : \mathbb{F}^n \to \{0, 1\}$ be an ε-regular image. If $\varepsilon = -1$, then $\mu^{-1}(\{1\})$ (resp. $\mu^{-1}(\{0\})$) is a regular open (resp. closed) set. If $\varepsilon = 1$, then $\mu^{-1}(\{1\})$ (resp. $\mu^{-1}(\{0\})$) is a regular closed (resp. open) set.*

4 Paths and (digital) fundamental groups

In this section, we study how the functions ζ_ε behave relatively to the classical notions of path in $\mathbb{Z}^n$ and $\mathbb{F}^n$. Theorem 2 states that ζ_ε induces a bijection between the set of the connected components of the object (resp. background) associated to an image $\lambda : \mathbb{Z}^n \to \{0,1\}$, and the set of the connected components of the object (resp. background) associated to the regular image $\mu = \zeta_\varepsilon(\lambda)$, the function ε being chosen with respect to a given pair of adjacencies in $\mathbb{Z}^n$. Theorem 3 states that ζ_ε induces an isomorphism between the digital fundamental group of $\lambda^{-1}(\{1\})$ (resp. $\lambda^{-1}(\{0\})$) and the fundamental group of $\mu^{-1}(\{1\})$ (resp. $\mu^{-1}(\{0\})$).

4.1 Background notions on paths and arcs

The fundamental group of topological spaces Let X be a topological space. A path p in X is a continuous function $p : [0,1] \to X$. Two paths p, q in X are *equivalent* if they have the same extremities (*i.e.*, $p(0) = q(0)$ and $p(1) = q(1)$) and p can be continuously deformed to fit q, that is if there exists a continuous map $h : [0,1] \times [0,1] \to X$ such that, for all $t \in [0,1]$, $h(t,0) = p(t)$ and $h(t,1) = q(t)$, and, for all $u \in [0,1]$, $h(0,u) = p(0) = q(0)$ and $h(1,u) = p(1) = q(1)$ (the map h is called a *path-homotopy*). This relation on paths is actually an equivalence relation. We write $[p]$ for the equivalence class of p. If p, q are two paths in X such that $p(1) = q(0)$ we can define the product $p \cdot q$ by

$$(p \cdot q)(t) = \begin{cases} p(2t) & \text{if } t \in [0, \frac{1}{2}] \\ q(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

This product is well defined on equivalence classes by $[p] \cdot [q] = [p \cdot q]$. Let x be a point of X. A *loop* at x is a path in X which starts and ends at x. The product of two loops at x is a loop at x and the set $\pi(X, x)$ of equivalence classes of loops at x is a group for this product. It is called the *fundamental group* of X (with *basepoint* x).

Finite paths in posets In a poset X, a function $f : [0,1] \to X$ is a *step function* if there exist finitely many intervals $(I_i)_{i=0}^r$ $(r \geq 0)$ such that f is constant on each interval I_i and $[0,1] = \bigcup_{i=0}^r I_i$. A *finite path* in X is a path in X which is a step function. The sequence $(I_i)_{i=0}^r$ is called the *intervals sequence* of p and the sequence $(x_i)_{i=0}^r$ the *track* of p. A finite path is *regular* if there is no singleton in its intervals sequence.

The *product* of two arcs $(x_i)_{i=0}^r$ and $(y_i)_{i=0}^s$ is defined by $(x_i)_{i=0}^r.(y_i)_{i=0}^s = (x_0, \ldots, x_r, y_1, \ldots, y_s)$ provided that $x_r = y_0$.

An arc $\chi = (x_i)_{i=0}^r$ $(r \geq 2)$ is an *elementary stretching (in X)* of an arc χ' if for some $j \in [\![1, r-1]\!]$, $\chi' = (x_i)_{i=0, i\neq j}^r$ or $(x_{j-1} = x_{j+1}$ and $\chi' = (x_i)_{i=0, i\neq j-1, i\neq j}^r)$. An arc χ is a *deformation (in X)* of an arc χ' if there exists a sequence $(\chi_i)_{i=0}^s$ of arcs in X such that $\chi_0 = \chi, \chi_s = \chi'$ and for any $i \in [\![1, s]\!]$, either χ_i is an elementary stretching of χ_{i-1} or χ_{i-1} is an elementary stretching in X of χ_i.

Let x be a point in X. "Being a deformation or equal" is an equivalence relation in the set of arcs in X from x to x. The set of equivalence classes, denoted by $\rho(X, x)$, is a group for the arc product.

Theorem 1 ([16]). *Let $x \in X$. The fundamental group $\pi(X, x)$ of X with basepoint x is isomorphic to the group $\rho(X, x)$.*

The digital fundamental group of $\mathbb{Z}^n$ A discrete analogue of the concept of fundamental group has been proposed in digital topology [2]. Let $n \in \{2, 3\}$ and $X \subseteq \mathbb{Z}^n$. The definition of the product for digital paths is straightforward but not so is the notion of equivalence between digital paths or loops. Two paths in X with same extremities are *directly equivalent (in X)* if they differ only in a unit lattice cube of $\mathbb{Z}^n$ provided that, if $n = 3$ and the pair of adjacencies is $(6, 26)$, the cube must not contain two diametrically opposite points not in X. Finally, two paths in X, p_0, p_t $(t \geq 0)$, with same extremities are *equivalent (in X)* if there is a sequence $(p_i)_{i=0}^t$ of paths in X such that, for all $i \in [\![1, t]\!]$, p_i is directly equivalent in X to p_{i-1}.

4.2 Mapping paths in $\mathbb{Z}^n$ onto arcs in $\mathbb{F}^n$

Let χ and χ' be two arcs in $\mathbb{F}^n$. We write $\chi \leqslant \chi'$ if there exist two paths $p \leqslant p'$ in $\mathbb{F}^n$ whose tracks are χ and χ' (all paths in $\mathbb{F}^n$ considered in the sequel are regular finite paths).

Definition 2. *Let ω be an adjacency relation on $\mathbb{Z}^n$ and $\gamma = (p_i)_{i=0}^r$ $(r \geq 0)$ be an ω-path in $\mathbb{Z}^n$, given in its reduced form ($p_i \neq p_{i-1}$ for all $i \in [\![1, r]\!]$). We define the arc $\zeta(\gamma)$ in $\mathbb{F}^n$ by $\zeta(\gamma) = (q_j)_{j=0}^{2r}$ with $q_j = p_{\frac{j}{2}}$ if j is even and $q_j = q_{j-1} \cap q_{j+1}$ if j is odd, for all $j \in [\![0, 2r]\!]$.*

It is obvious that the sequence of faces $\zeta(\gamma)$ defined above is actually an arc in $\mathbb{F}^n$ which is itself the track of a regular finite path in $\mathbb{F}^n$ [16].

The following proposition states that ζ associates to a path in the object (resp. in the background), of a digital image λ, an arc in the object (resp. in the background) of the complex image $\zeta_\varepsilon(\lambda)$ under the condition that the connectivity function ε has been well chosen. The main consequence of this proposition is that the images of the connected components of the digital object (resp. background) are included in the connected components of the image of the object (resp. background).

Proposition 7. *Let (α, β) be a pair of adjacencies on $\mathbb{Z}^n$. Let ε be the connectivity function associated to (α, β). Let $x \in \{0, 1\}$. Let $\omega = \alpha$ if $x = 1$ and $\omega = \beta$ if $x = 0$. Let $\lambda : \mathbb{Z}^n \to \{0, 1\}$ be an image in $\mathbb{Z}^n$ and $\mu = \zeta_\varepsilon(\lambda)$ be the corresponding image in $\mathbb{F}^n$. Let $\gamma = (p_i)_{i=0}^r$ $(r \geq 0)$ be an ω-path in $\lambda^{-1}(\{x\})$. Then, $\zeta(\gamma)$ is an arc in $\mu^{-1}(\{x\})$.*

The following proposition is straightforward.

Proposition 8. *Let ω be an adjacency relation on $\mathbb{Z}^n$. The corresponding function ζ is a homomorphism for the paths product and the arc product: for all ω-paths $\gamma, \gamma' \in \mathbb{Z}^n$, $\zeta(\gamma.\gamma') = \zeta(\gamma).\zeta(\gamma')$.*

The injectivity of ζ is obvious since two distinct n-xels $a, b \in \mathbb{Z}^n$ are associated to distinct facets $\dot{a}, \dot{b} \in \mathbb{F}^n$. Proposition 9 establishes the surjectivity up to deformations: any arc χ from $\dot{a}$ to $\dot{b}$ in an object (resp. in the background) of the complex image is the

deformation of an arc $\zeta(\gamma)$ for some path γ from a to b of the object (resp. background) of its associated digital image (if the complex image is associated to such a digital image).

Proposition 9. *Let (α,β) be a pair of adjacencies on $\mathbb{Z}^n$. Let ε be the connectivity function associated to (α,β). Let $x \in \{0,1\}$. Let $\omega = \alpha$ if $x = 1$ and $\omega = \beta$ if $x = 0$. Let $\lambda : \mathbb{Z}^n \to \{0,1\}$ be an image in $\mathbb{Z}^n$ and $\mu = \zeta_\varepsilon(\lambda)$ be the corresponding image in $\mathbb{F}^n$. Let $a, b \in \mathbb{Z}^n$. Let χ be an arc from the facet $\dot{a}$ to the facet $\dot{b}$ in $\mu^{-1}(\{x\})$. Then, there exists an ω-path γ from a to b in $\lambda^{-1}(\{x\})$ such that $\zeta(\gamma)$ is a deformation in $\mu^{-1}(\{x\})$ of χ. Moreover, if p is a path in $\mu^{-1}(\{x\})$ whose track is χ, there exists a path q in $\mu^{-1}(\{x\})$, $q \geqslant p$, whose track is $\zeta(\gamma)$.*

Theorem 2. *Let (α,β) be a pair of adjacencies in $\mathbb{Z}^n$. Let ε be the connectivity function associated to (α,β). Let $\lambda : \mathbb{Z}^n \to \{0,1\}$ be a digital image and $\zeta_\varepsilon(\lambda) = \mu : \mathbb{F}^n \to \{0,1\}$ be the corresponding complex image. Let $x \in \{0,1\}$. Then the function ζ_ε induces a one-to-one correspondence between the connected components of $\lambda^{-1}(\{x\})$ and the connected components of $\mu^{-1}(\{x\})$.*

4.3 Fundamental groups

The aim of this section is to compare the digital fundamental group defined by Kong [2] for subsets of $\mathbb{Z}^n$, $n \in \{2,3\}$, with the fundamental group of subspaces of $\mathbb{F}^n$. Thanks to Theorem 1, we can use arcs as well as paths in $\mathbb{F}^n$ in order to perform this comparison.

Proposition 10. *Let (α,β) be a pair of adjacencies on $\mathbb{Z}^n$. Let ε be the connectivity function associated to (α,β). Let $\lambda : \mathbb{Z}^n \to \{0,1\}$ be a digital image and $\zeta_\varepsilon(\lambda) = \mu : \mathbb{F}^n \to \{0,1\}$ be the corresponding complex image. Let $x \in \{0,1\}$. Let $\omega = \alpha$ if $x = 1$ and $\omega = \beta$ if $x = 0$. Two ω-paths in $\lambda^{-1}(\{x\})$, γ and γ', are equivalent in $\lambda^{-1}(\{x\})$ iff the arc $\zeta(\gamma)$ is equal to or is a deformation in $\mu^{-1}(\{x\})$ of the arc $\zeta(\gamma')$.*

Let $a \in \lambda^{-1}(\{x\})$. Let $\pi_D(\lambda^{-1}(\{x\}), a)$ be the digital fundamental group of $\lambda^{-1}(\{x\})$ with basepoint a and $\rho(\mu^{-1}(\{x\}), \dot{a})$ be the group of arcs in $\mu^{-1}(x)$ from $\dot{a}$ to $\dot{a}$, up to deformations.

From Propositions 7 and 10, we know that the function $\dot{\zeta}$ defined by

$$\left| \begin{array}{lccc} \dot{\zeta} : & \pi_D(\lambda^{-1}(x), a) & \to & \rho(\mu^{-1}(x), \dot{a}) \\ & [\gamma] & \mapsto & [\zeta(\gamma)] \end{array} \right.$$

where $[y]$ denotes the equivalence class of y (for the equivalence relation on digital paths of $\mathbb{Z}^n$ on the left side and for the deformation on arcs of $\mathbb{F}^n$ on the right side), is well-defined. Proposition 8 then states that $\dot{\zeta}$ is a morphism. Propositions 9 and 10 give the surjectivity and the the injectivity of $\dot{\zeta}$, respectively. We conclude that the groups $\pi_D(\lambda^{-1}(\{x\}), a)$ and $\rho(\mu^{-1}(\{x\}), \dot{a})$ are isomorphic and, since $\rho(\mu^{-1}(\{x\}), \dot{a})$ and $\pi(\mu^{-1}(\{x\}), \dot{a})$ are isomorphic (Theorem 1), the following theorem holds.

Theorem 3. *Let (α,β) be a pair of adjacencies on $\mathbb{Z}^n$. Let ε be the connectivity function associated to (α,β). Let $\lambda : \mathbb{Z}^n \to \{0,1\}$ be an image in $\mathbb{Z}^n$ and $\mu = \zeta_\varepsilon(\lambda)$ be the corresponding image in $\mathbb{F}^n$. For any $a \in \lambda^{-1}(\{x\})$, the digital fundamental group of $\lambda^{-1}(\{x\})$ with basepoint a is isomorphic to the fundamental group of the poset $(\mu^{-1}(\{x\}), \subseteq)$ with base point $\dot{a}$.*

5 Conclusion

In this article, a *modus operandi* has been proposed to embed digital binary images, equipped with a pair of standard adjacencies, in the space of cubical complexes. In particular, it has been proved that it preserves the connected components of both object and background and, more generally, preserves also the (digital) fundamental groups.

These results, associated to those of [16], justify the soundness of all the contributions previously devoted to design homotopy type-preserving binary image processing methods, especially concerning the correctness of their behaviour with respect to the "continuous" topology of the handled digital objects. They also permit to establish links between image processing methods developed either in classical digital spaces ($\mathbb{Z}^n$) or cubical complexes ($\mathbb{F}^n$), and to potentially unify some of them. An extended version of this work (with proofs of the propositions and theorems) will be proposed soon [15].

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