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Nonparametric estimation of the local Hurst function of multifractional Gaussian processes

Joint paper with Donatas Surgailis (Lithuania)

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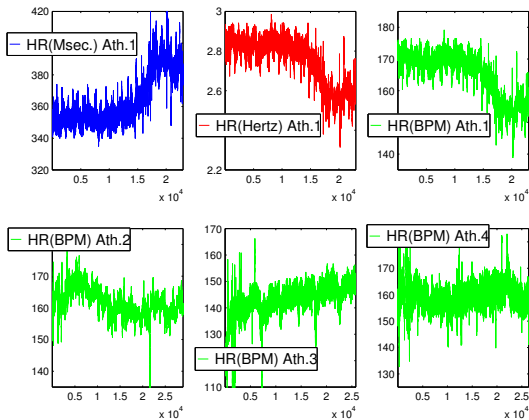
Outline

- 1 Introduction
- 2 A new nonparametric estimator of the local Hurst function
 - First definition, assumptions and limit theorems
 - Second definition and limit theorems
 - Case of the General multifractional Brownian motion
- 3 Numerical comparison with the quadratic variations estimators
 - Definition and limit theorems
 - Numerical comparisons

Outline

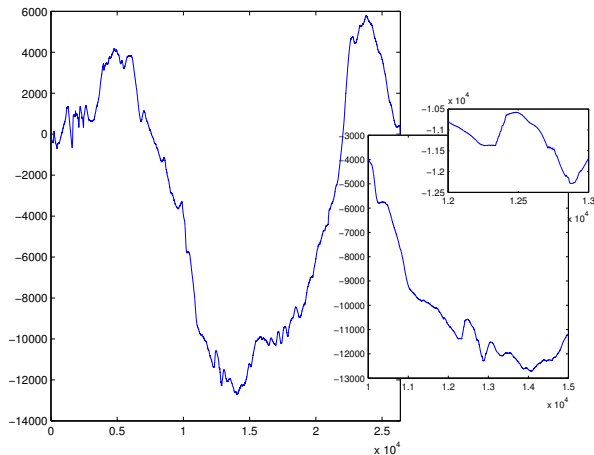
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An athletic example...



Heart beat of athletes during Marathon de Paris 2004

A fact...



Successive zooms on the aggregated series of one athlete

The fractional Brownian motion

- $B^H = \{B_t^H, t \in \mathbb{R}\}$ **fractional Brownian motion** (FBM) with $H \in [0, 1]$:

B^H is a Gaussian centered process with stationary increments and
 $\text{Var}(B_t^H) = \sigma^2 |t|^{2H}$.

Property

*A Gaussian process X having stationary increments **H -selfsimilar***

$\iff X$ *F.B.M. with parameter H*

- A trajectory of B^H is a.s. **α -Hölderian** for any $\alpha < H$:

$\implies H$ measures the **local smoothness** of B^H also called **Hurst parameter**

Two trajectories of FBM

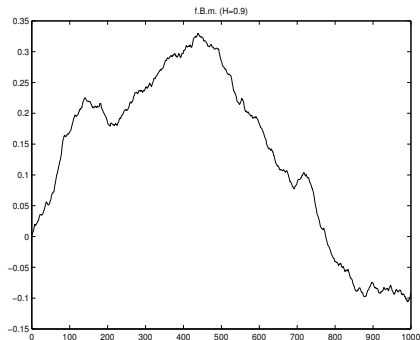
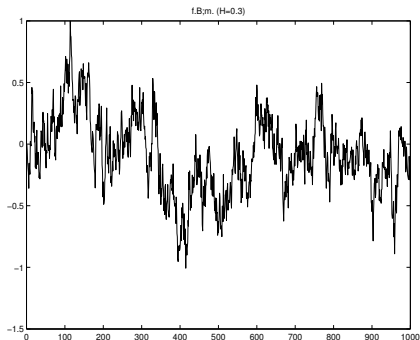


Figure: Trajectories of FBM with $H = 0.3$ (left) and $H = 0.9$ (right)

Other definitions of FBM

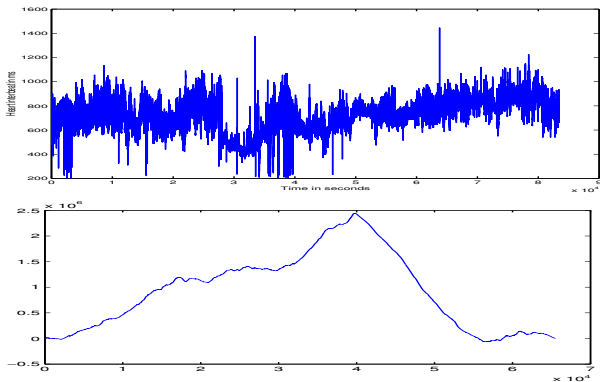
- Harmonizable representation:

$$B_t^H = C_1(H) \sigma^2 \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{2H+1}} \widehat{W}(d\xi) \quad t \in \mathbb{R}$$

- Temporal representation:

$$B_t^H = C_2(H) \sigma^2 \int_{\mathbb{R}} ((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}) W(du) \quad t \in \mathbb{R}$$

Another example



Heartbeats during 24h.

 $\Rightarrow H$ depending on t !

Multifractional Brownian motion

Two first versions:

- **Harmonizable representation** (Benassi *et al.*, 1997):

$$X_t = C_1(H(t)) \sigma^2 \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{2H(t)+1}} \widehat{W}(d\xi) \quad t \in \mathbb{R}$$

- **Temporal representation** (Peltier et Lévy-Véhel, 1995):

$$X_t = C_2(H(t)) \sigma^2 \int_{\mathbb{R}} ((t-u)_+^{H(t)-1/2} - (-u)_+^{H(t)-1/2}) W(du) \quad t \in \mathbb{R}$$

Aims

From an observed trajectory $(X_{\frac{1}{n}}, X_{\frac{2}{n}}, \dots, X_1)$,

- Define a **new non-parametric estimator** $H(\cdot)$;
- **Asymptotic properties** of this estimator.
- Compare this estimator with the well-known **quadratic variations estimator**.

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First definition

For $t_0 \in (0, 1)$ and $\alpha \in (0, 1)$, define:

$$\widehat{H}_{n,\alpha}^{(IR)}(t_0) := \Lambda^{-1} \left(\frac{1}{2n^{1-\alpha}} \sum_{k=\lfloor nt_0 - n^{1-\alpha} \rfloor}^{\lfloor nt_0 + n^{1-\alpha} \rfloor} \frac{|\Delta_n^k X + \Delta_n^{k+1} X|}{|\Delta_n^k X| + |\Delta_n^{k+1} X|} \right)$$

with

- $\Delta_n^k X = X_{\frac{k+2}{n}} - 2X_{\frac{k+1}{n}} + X_{\frac{k}{n}}$
- $\Lambda(h) = \mathbb{E} \left[\frac{|\Delta_1^0 B^h + \Delta_1^1 B^h|}{|\Delta_1^0 B^h| + |\Delta_1^1 B^h|} \right]$ for $h \in (0, 1)$
 $(= \frac{1}{\pi} \arccos(-\rho_2(h)) + \frac{1}{\pi} \sqrt{\frac{1+\rho_2(h)}{1-\rho_2(h)}} \log \left(\frac{2}{1+\rho_2(h)} \right) \text{ with } \rho_2(h) = \frac{-3^{2h} + 2^{2h+2} - 7}{8 - 2^{2h+1}}).$

⇒ Explanation...

Multifractional Gaussian processes?

Assumptions: $X = (X_t)_t$ is a centered Gaussian process such as:

- (A) _{κ} There exist η -Hölderian functions $0 < H(t) < 1$ and $c(t) > 0$ for $t \in (0, 1)$ such that for any $0 < \varepsilon < 1/2$ and $j \in \mathbb{Z}$,

$$\max_{[n\varepsilon] \leq k \leq [(1-\varepsilon)n]} n^\kappa \left| \frac{\text{Cov}(\Delta_n^k X, \Delta_n^{k+j} X)}{\text{Cov}(\Delta_n^k B^{H(k/n)}, \Delta_n^{k+j} B^{H(k/n)})} - c\left(\frac{k}{n}\right) \right| \xrightarrow{n \rightarrow \infty} 0.$$

- (B) _{α} There exist $C > 0$, $\gamma > 1/2$ and $0 \leq \theta < \gamma/2$ such that for any $n \in \mathbb{N}^*$ and $1 \leq k, k' < n - q$

$$\left| \text{Cor}(\Delta_n^k X, \Delta_n^{k'} X) \right| \leq C n^{(1-\alpha)\theta} (|k' - k| \wedge n^{1-\alpha})^{-\gamma}.$$

Limit theorems for multifractional Gaussian processes

Theorem

Under Assumptions **(A)** $_{\kappa}$ and **(B)** $_{\alpha}$, for all $t_0 \in (0, 1)$,

- If $0 < \alpha < \frac{\gamma-2\theta}{2(\gamma-\theta)}$,

$$\widehat{H}_{n,\alpha}^{(IR)}(t_0) \xrightarrow[n \rightarrow \infty]{a.s.} H(t_0).$$

- If $\kappa \geq \mu$ and $\frac{2\gamma-2\theta}{3\gamma-2\theta+4\gamma(\eta \wedge 2)} \leq \alpha < \frac{2\gamma-2\theta}{3\gamma-2\theta}$ then for any $\epsilon > 0$

$$\sup_{\epsilon < t < 1-\epsilon} |\widehat{H}_{n,\alpha}^{(IR)}(t) - H(t)| = O_p(n^{-\mu}).$$

- If $\kappa \geq \mu_1$ and $\frac{\gamma-2\theta}{3\gamma-2\theta+4\gamma(\eta \wedge 2)} \leq \alpha < \frac{\gamma-2\theta}{3\gamma-2\theta}$ then for any $\epsilon > 0$, $\delta > 0$

$$\sup_{\epsilon < t < 1-\epsilon} |\widehat{H}_{n,\alpha}^{(IR)}(t) - H(t)| = O(n^{-(\mu_1-\delta)}) \quad a.s.$$

Central Limit theorem for multifractional Gaussian processes

Theorem

Let $Z = (Z(t))_{t \in (0,1)}$ be a zero-mean Gaussian process satisfying $(\mathbf{A})_\kappa$ and $(\mathbf{B})_\alpha$, with $\alpha > \frac{1}{1+2(\eta \wedge 2)}$, $\kappa \geq \frac{1-\alpha}{2}$ and $\theta = 0$. Then for $0 < t_1 < \dots < t_u < 1$,

$$\sqrt{2n^{1-\alpha}} \left(\widehat{H}_{n,\alpha}^{(IR)}(t_i) - H(t_i) \right)_{1 \leq i \leq u} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \left(W^{(IR)}(t_i) \right)_{1 \leq i \leq u},$$

where $W^{(IR)}(t_i)$, $i = 1, \dots, u$ are independent centered Gaussian r.v.'s such as

$$\mathbb{E}[W^{(IR)}(t_i)]^2 := \left[\frac{\partial}{\partial x} (\Lambda_2)^{-1} (\Lambda_2(H(t_i))) \right]^2 \sigma^2(H(t_i))$$

where
$$\sigma^2(H) := \sum_{k \in \mathbb{Z}} \text{Cov}(\psi(\Delta_1^0 B_H, \Delta_1^1 B_H), \psi(\Delta_1^k B_H, \Delta_1^{k+1} B_H)).$$

Proof: based on a CLT of triangular arrays of functional of Gaussian vectors

Theorem (Bardet et Surgailis, 2012)

Let $(\mathbf{Y}_n(k))_{1 \leq k \leq n, n \in \mathbb{N}}$ be a triangular array of standard Gaussian \mathbb{R}^ν -vectors.

- For $m \geq 1$, there exists $\rho : \mathbb{N} \rightarrow \mathbb{R}$ such as for $1 \leq p, q \leq \nu$,

$$\forall(j, k), \quad \left| \mathbb{E} Y_n^{(p)}(j) Y_n^{(q)}(k) \right| \leq |\rho(j-k)| \quad \text{with} \quad \sum_{j \in \mathbb{Z}} |\rho(j)|^m < \infty;$$

- For $\tau \in [0, 1]$ and $J \in \mathbb{N}^*$, with $(\mathbf{W}_\tau(j))_{j \in \mathbb{Z}}$ a stationary Gaussian process

$$(\mathbf{Y}_n([n\tau] + j))_{-J \leq j \leq J} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} (\mathbf{W}_\tau(j))_{-J \leq j \leq J};$$

- If $\tilde{f}_{k,n} \in \mathbb{L}_0^2(\mathbf{X})$ ($n \geq 1, 1 \leq k \leq n$) with Hermite rank $\geq m$ and $\exists \tilde{\phi}_\tau, \tau \in [0, 1]$

such as

$$\sup_{\tau \in [0, 1]} \|\tilde{f}_{[n\tau], n} - \tilde{\phi}_\tau\|^2 = \sup_{\tau \in [0, 1]} \mathbb{E}(\tilde{f}_{[n\tau], n}(\mathbf{X}) - \tilde{\phi}_\tau(\mathbf{X}))^2 \xrightarrow[n \rightarrow \infty]{} 0.$$

Theorem

Then with $\sigma^2 = \int_0^1 d\tau \left(\sum_{j \in \mathbb{Z}} \mathbb{E} [\tilde{\phi}_\tau(\mathbf{W}_\tau(0)) \tilde{\phi}_\tau(\mathbf{W}_\tau(j))] \right) < \infty$,

$$n^{-1/2} \sum_{k=1}^n \tilde{f}_{k,n}(\mathbf{Y}_n(k)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2).$$

Second definition

For $t, \alpha \in (0, 1)$ and $j = 1, \dots, p$,

$$\widehat{H}_{n,\alpha,j}^{(IR)}(t_0) := \Lambda_j^{-1} \left(\frac{1}{2n^{1-\alpha}} \sum_{k=[nt_0-n^{1-\alpha}]}^{[nt_0+n^{1-\alpha}]} \frac{|\Delta_n^k X + \Delta_n^{k+j} X|}{|\Delta_n^k X| + |\Delta_n^{k+j} X|} \right).$$

Theorem

Under Assumptions $(\mathbf{A})_\kappa$ and $(\mathbf{B})_\alpha$, for all $t_0 \in (0, 1)$,

$$n^{(1-\alpha)/2} \left(\widehat{H}_{n,\alpha,j}^{(IR)}(t_0) - H(t_0) \right)_{1 \leq j \leq p} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Gamma(H(t_0)))$$

Second definition (end)

Let $\widehat{\Gamma} := \Gamma(\widehat{H}_{n,\alpha}^{(IR)})$.

A new nonparametric estimator of $H(\cdot)$ using Pseudo-Generalized least squares:

$$\widetilde{H}_{n,\alpha}^{(IR)}(t_0) := (\mathbb{1}'_p \widehat{\Gamma}^{-1} \mathbb{1}_p)^{-1} \mathbb{1}'_p \widehat{\Gamma}^{-1} (\widehat{H}_{n,\alpha,j}^{(IR)}(t_0))_{1 \leq j \leq p}.$$

Theorem

Under Assumptions $(\mathbf{A})_\kappa$ and $(\mathbf{B})_\alpha$, for all $t_0 \in (0, 1)$,

$$n^{(1-\alpha)/2} (\widetilde{H}_{n,\alpha}^{(IR)}(t_0) - H(t_0)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}\left(0, (\mathbb{1}'_p \Gamma^{-1}(H) \mathbb{1}_p)^{-1}\right)$$

Definition (Stoev and Taqqu, 2006)

On pose

$$Y_{(a^+, a^-)}(t) := K(H(t)) \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{H(t) + \frac{1}{2}}} U_{(a^+, a^-)}(H(t), x) \widehat{W}(dx),$$

with for $h \in (0, 1)$

$$\begin{aligned} U_{(a^+, a^-)}(h, x) &:= \frac{(a^+ e^{-i \operatorname{sign}(x)(h + \frac{1}{2})\frac{\pi}{2}} + a^- e^{i \operatorname{sign}(x)(h + \frac{1}{2})\frac{\pi}{2}})}{((a^+)^2 + (a^-)^2 - 2a^+ a^- \sin(\pi h))^{1/2}}. \\ &:= \frac{1}{\sqrt{\pi}} \quad \text{if } h = 1/2 \text{ and } a^+ = a^-. \end{aligned}$$

Estimation of $H(\cdot)$

Theorem

For a trajectory of the process $(Y_{(a^+, a^-)}(t))_t$,

- If $\max(0, 1 - 4((\eta \wedge 2) - H(t))) < \alpha < \frac{1}{2}$,

$$\tilde{H}_{n,\alpha}^{(IR)}(t_0) \xrightarrow[n \rightarrow \infty]{a.s.} H(t_0)$$

- If $\max\left\{\frac{1}{1 + 2(\eta \wedge 2)}, 1 - 4(\eta \wedge 2 - H(t_0))\right\} < \alpha < 1$,

$$n^{(1-\alpha)/2} (\tilde{H}_{n,\alpha}^{(IR)}(t_0) - H(t_0)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}\left(0, (\mathbb{I}'_p \Gamma^{-1}(H) \mathbb{I}_p)^{-1}\right)$$

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Generalized quadratic variations estimator of $H(\cdot)$

Define (see Istas and Lang, 1994, Benassi et al., 1998, Coeurjolly, 2005):

$$\widehat{H}_{n,\alpha}^{(QV)}(t_0) := \frac{1}{2} \frac{A'}{A'A} \left(\log \left(\frac{1}{2n^{1-\alpha}} \sum_{k=[nt_0-n^{1-\alpha}]}^{[nt_0+n^{1-\alpha}]} |X_{\frac{k+2j}{n}} - 2X_{\frac{k+j}{n}} + X_{\frac{k}{n}}|^2 \right) \right)'_{1 \leq j \leq p}$$

with $A := (\log i - \frac{1}{p} \sum_{j=1}^p \log j)_{1 \leq i \leq p} \in \mathbb{R}^p$.

Theorem

Mutatis mutandis, the limit theorems obtained for $\widetilde{H}_{n,\alpha}^{(IR)}(t_0)$ also hold for $\widehat{H}_{n,\alpha}^{(QV)}(t_0)$.

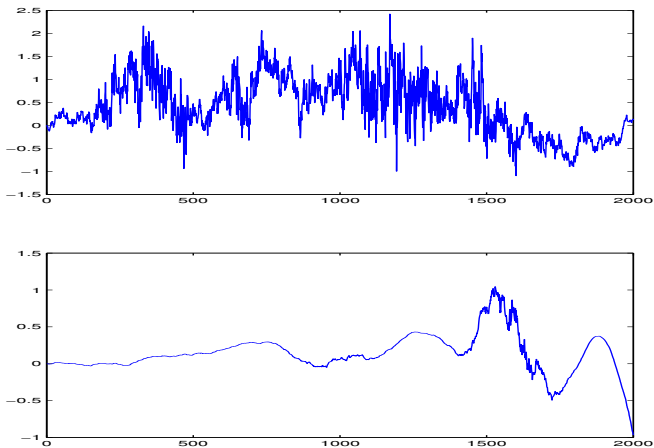


Figure: Examples of MBM trajectories (up, $H \in C^{\eta-}$ with $\eta = 0.6$, down $H(t) = 0.1 + 0.8(1 - t) \sin^2(10t)$)

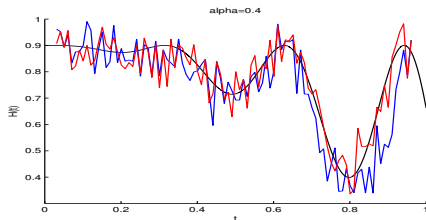
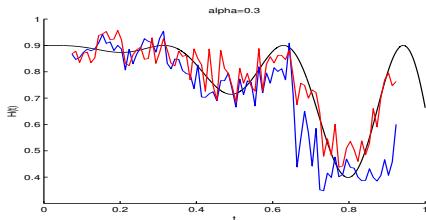
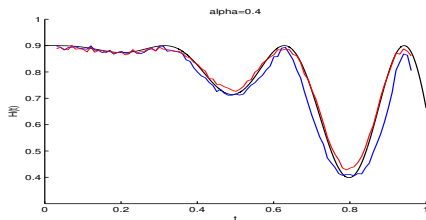
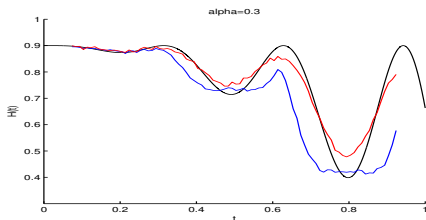


Figure: Estimation of $H_4(t) = 0.1 + 0.8(1-t)\sin^2(10t)$ for $n = 6000$, $\alpha = 0.3$ (left) and 0.4 (right)

| | | | | | |
|---------------------------------------|--|-------|-------|-------|-------|
| $H_1(t) = 0.6$ | α | 0.2 | 0.3 | 0.4 | 0.5 |
| $n = 2000$ | $\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV)}$ | 0.041 | 0.051 | 0.069 | 0.096 |
| | $\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR)}$ | 0.061 | 0.077 | 0.106 | 0.145 |
| $n = 6000$ | $\sqrt{\text{MISE}}$ for $\tilde{H}_{n,\alpha}^{(QV)}$ | 0.025 | 0.033 | 0.050 | 0.074 |
| | $\sqrt{\text{MISE}}$ for $\tilde{H}_{n,\alpha}^{(IR)}$ | 0.037 | 0.049 | 0.076 | 0.115 |
| $H_2(t) = 0.1 + 0.8t$ | α | 0.2 | 0.3 | 0.4 | 0.5 |
| $n = 2000$ | $\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV)}$ | 0.170 | 0.073 | 0.072 | 0.096 |
| | $\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR)}$ | 0.059 | 0.071 | 0.098 | 0.135 |
| $n = 6000$ | $\sqrt{\text{MISE}}$ for $\tilde{H}_{n,\alpha}^{(QV)}$ | 0.114 | 0.044 | 0.048 | 0.070 |
| | $\sqrt{\text{MISE}}$ for $\tilde{H}_{n,\alpha}^{(IR)}$ | 0.036 | 0.046 | 0.069 | 0.103 |
| $H_3(t) = 0.5 + 0.4 \sin(5t)$ | α | 0.2 | 0.3 | 0.4 | 0.5 |
| $n = 2000$ | $\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV)}$ | 0.362 | 0.123 | 0.080 | 0.096 |
| | $\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR)}$ | 0.093 | 0.071 | 0.091 | 0.124 |
| $n = 6000$ | $\sqrt{\text{MISE}}$ for $\tilde{H}_{n,\alpha}^{(QV)}$ | 0.260 | 0.077 | 0.052 | 0.072 |
| | $\sqrt{\text{MISE}}$ for $\tilde{H}_{n,\alpha}^{(IR)}$ | 0.057 | 0.047 | 0.065 | 0.097 |
| $H_4(t) = 0.1 + 0.8(1-t) \sin^2(10t)$ | α | 0.2 | 0.3 | 0.4 | 0.5 |
| $n = 2000$ | $\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV)}$ | 0.320 | 0.164 | 0.117 | 0.112 |
| | $\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR)}$ | 0.165 | 0.098 | 0.091 | 0.112 |
| $n = 6000$ | $\sqrt{\text{MISE}}$ for $\tilde{H}_{n,\alpha}^{(QV)}$ | 0.251 | 0.135 | 0.071 | 0.078 |
| | $\sqrt{\text{MISE}}$ for $\tilde{H}_{n,\alpha}^{(IR)}$ | 0.148 | 0.062 | 0.067 | 0.091 |

Table: Estimators $\tilde{H}_{n,\alpha}^{(QV)}$ et $\tilde{H}_{n,\alpha}^{(IR)}$ when $H(\cdot)$ is a C^∞ function.

| $H \in \mathcal{C}^{1.5}(0, 1)$ | α | 0.2 | 0.3 | 0.4 | 0.5 |
|---------------------------------|---|-------|-------|-------|-------|
| $n = 2000$ | $\sqrt{\widehat{MISE}}$ for $\tilde{H}_{n,\alpha}^{(QV)}$ | 0.261 | 0.112 | 0.085 | 0.098 |
| | $\sqrt{\widehat{MISE}}$ for $\tilde{H}_{n,\alpha}^{(IR)}$ | 0.098 | 0.077 | 0.093 | 0.128 |
| $n = 6000$ | $\sqrt{\widehat{MISE}}$ for $\tilde{H}_{n,\alpha}^{(QV)}$ | 0.164 | 0.066 | 0.053 | 0.070 |
| | $\sqrt{\widehat{MISE}}$ for $\tilde{H}_{n,\alpha}^{(IR)}$ | 0.054 | 0.047 | 0.066 | 0.098 |
| $H \in \mathcal{C}^{0.6}(0, 1)$ | α | 0.2 | 0.3 | 0.4 | 0.5 |
| $n = 2000$ | $\sqrt{\widehat{MISE}}$ for $\tilde{H}_{n,\alpha}^{(QV)}$ | 0.140 | 0.086 | 0.081 | 0.094 |
| | $\sqrt{\widehat{MISE}}$ for $\tilde{H}_{n,\alpha}^{(IR)}$ | 0.088 | 0.078 | 0.096 | 0.135 |
| $n = 6000$ | $\sqrt{\widehat{MISE}}$ for $\tilde{H}_{n,\alpha}^{(QV)}$ | 0.130 | 0.067 | 0.056 | 0.071 |
| | $\sqrt{\widehat{MISE}}$ for $\tilde{H}_{n,\alpha}^{(IR)}$ | 0.066 | 0.052 | 0.067 | 0.103 |

Table: Estimators $\tilde{H}_{n,\alpha}^{(QV)}$ et $\tilde{H}_{n,\alpha}^{(IR)}$ when $H(\cdot)$ is a $\mathcal{C}^{\eta-}$ function.

Figures

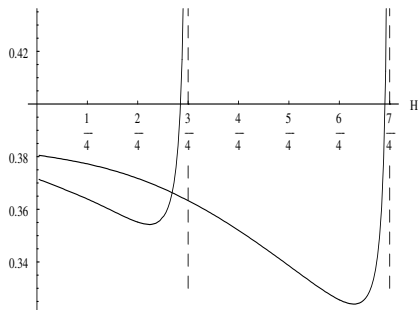
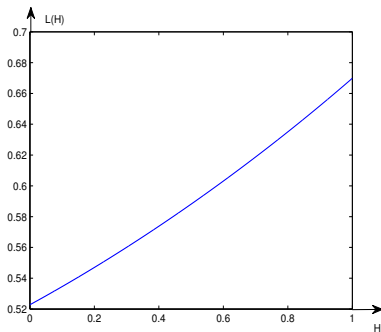


Figure: Graphs of functions $H \mapsto \Lambda(H)$ and $H \mapsto \sigma(H)$