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# COSMOLOGICAL PERTURBATION THEORY REVISITED

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## Abstract

Increasingly accurate observations are driving theoretical cosmology toward the use of more sophisticated descriptions of matter and the study of nonlinear perturbations of Friedmann-Lemaitre cosmologies, whose governing equations are notoriously complicated. Our goal in this paper is to formulate the governing equations for linear perturbation theory in a particularly simple and concise form in order to facilitate the extension to nonlinear perturbations. Our approach has several novel features. We show that the use of so-called *intrinsic gauge invariants* has two advantages. It naturally leads to: (i) a physically motivated choice of a gauge invariant associated with the matter density, and (ii) two distinct and complementary ways of formulating the evolution equations for scalar perturbations, associated with the work of Bardeen and of Kodama and Sasaki. In the first case the perturbed Einstein tensor gives rise to a second order (in time) linear differential operator, and in the second case to a pair of coupled first order (in time) linear differential operators. These operators are of fundamental importance in cosmological perturbation theory, since they provide the leading order terms in the governing equations for nonlinear perturbations.

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## 1 Introduction

The analysis of linear perturbations of Friedmann-Lemaitre (FL) cosmologies was initiated by Lifshitz (1946) in a paper of far-reaching importance. Working in the so-called synchronous gauge, this paper showed that an arbitrary linear perturbation can be written as the sum of three modes, a scalar mode that describes perturbations in the matter density, a vector mode that describes vorticity and a tensor mode that describes gravitational waves. For many years, however, the theory was plagued by gauge problems, *i.e.* by the fact that the behaviour of the scalar mode depends significantly on the choice of gauge. A major step in alleviating this difficulty was taken by Bardeen (1980), who reformulated the linearized Einstein field equations in terms of a set of *gauge-invariant* variables, as an alternative to the traditional use of the synchronous gauge. Central to Bardeen's paper are two gauge-invariant equations that govern the behaviour of scalar perturbations. The first of these governs the evolution in time of a gauge-invariant gravitational (*i.e.* metric) potential and the second determines a gauge-invariant perturbation of the matter density in terms of the spatial Laplacian of the gravitational potential. Since this potential continues to play a central role in the study of scalar perturbations, it seems appropriate to refer to it as the *Bardeen potential*. Bardeen's paper makes clear, however, that there is no unique way of constructing gauge-invariant variables.

From our perspective, one drawback of Bardeen's paper is that he performs a harmonic decomposition of the variables *ab initio*, with the result that the mathematical structure of the governing equations is somewhat obscured. In a subsequent paper, Brandenberger, Khan and Press (1983) address this deficiency by giving a new derivation of Bardeen's gauge-invariant equations. They do not perform a harmonic decomposition, with the result that their evolution equation is a partial differential equation rather than an ordinary differential equation as in Bardeen's paper. However, unlike Bardeen they restrict consideration to a spatially flat Robertson-Walker (RW) background.<sup>1</sup>

In subsequent developments the status of the Bardeen potential was further enhanced by the appearance of the major review paper by Mukhanov *et al* (1992), which contains a simplified derivation of the Bardeen potential and the evolution equation for scalar perturbations, without performing a harmonic decomposition. However, the treatment in Mukhanov *et al* (1992) is less general than that of Bardeen (1980) and Brandenberger *et al* (1983) in two respects. First, they assume the anisotropic stresses are zero, and second, they make a specific choice of gauge invariants *a priori*, namely those associated with the so-called longitudinal gauge.

Currently, increasingly accurate observations are driving theoretical cosmology towards more sophisticated models of matter and the study of possible nonlinear deviations from FL cosmology. Motivated by this state of affairs, our long term goal is to provide a general but concise description of nonlinear perturbations of FL cosmologies that will reveal the mathematical structure of the governing equations and enable one to make the transition between different gauge-invariant formula-

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<sup>1</sup>We follow the nomenclature of Wainwright and Ellis (1997) where an FL cosmology is a RW geometry that satisfies Einstein's field equations.

tions, thereby simplifying and relating the different approaches that have been used to date.<sup>2</sup> In pursuing this objective we have found it necessary to revisit linear perturbation theory, even though it is by now a mature discipline.<sup>3</sup> Our intent in the present paper is to formulate the governing equations for the linear theory in a particularly simple and concise form in order to facilitate the extension to nonlinear perturbations.

Based on earlier work by Bruni *et al* (1997) on gauge-invariant higher order perturbation theory, Nakamura (2003) introduced a geometrical method for constructing gauge invariants for linear and nonlinear (second order) perturbations which he later applied to derive the governing equations (see Nakamura (2006) and Nakamura (2007)). In the present paper we use a dimensionless version of Nakamura's method for constructing gauge invariants, but we complement it with the observation that gauge invariants are of two distinct types: *intrinsic gauge invariants*, i.e., gauge invariants that can be constructed from a given tensor alone, and *hybrid gauge invariants*, i.e. gauge invariants that are constructed from more than one tensor.

In Nakamura's approach, the linear perturbation of any tensor is written as the sum of a gauge-invariant quantity and a gauge-variant quantity, which is the Lie derivative of the zero order tensor with respect to a suitably chosen vector field  $X$ . A choice of  $X$  yields a set of gauge-invariant variables that are associated with a specific fully fixed gauge. We will show that for the metric tensor there exist two natural complementary choices of  $X$  that yield intrinsic metric gauge invariants. One choice, used in all of Nakamura's papers, leads to the two gauge-invariant metric potentials of Bardeen (1980), which are associated with the so-called *Poisson gauge*.<sup>4</sup> The other choice leads to the two gauge-invariant metric potentials of Kodama and Sasaki (1984), which are associated with the so-called *uniform curvature gauge*.<sup>5</sup> We will show that these two preferred choices lead to two distinct ways in which to present the linearized Einstein field equations: with the Bardeen choice the evolution of linear scalar perturbations is governed by a *second order* (in time) linear partial differential operator, while with the Kodama-Sasaki choice the evolution is governed by two coupled temporal *first order* linear operators.

The plan of the paper is as follows. In Section 2 we discuss the geometrical construction of gauge-invariants: we focus on the metric tensor and, with the Einstein tensor and the stress-energy tensor in mind, on mixed rank two tensors. In Section 3 we use intrinsic gauge invariants to derive the general governing equations for linear perturbations in two gauge-invariant forms associated with the Poisson and the uniform curvature gauges. The required expressions for the Einstein gauge invariants are derived efficiently in Appendix B. One of the ingredients in our derivation is the so-called Replacement Principle, which is formulated in Appendix A. Another ingredient is a general formula that expresses the Riemann gauge invariants in terms of the metric gauge invariants. In Section 4 we give an interpretation of the intrinsic

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<sup>2</sup>See, for example, Noh and Hwang (2004), Nakamura (2007) and Malik (2007).

<sup>3</sup>For some recent reviews and books, see, for example, Tsagas *et al* (2008), Malik and Wands (2009), Mukhanov (2005), Weinberg (2008), Durrer (2008) and Lyth and Liddle (2009).

<sup>4</sup>The Poisson gauge, which was introduced by Bertschinger (1996) (see his equation (4.46)), is a generalization of the longitudinal gauge, which only applies to scalar perturbations.

<sup>5</sup>See, for example, Malik and Wands (2009), page 20, and other references given there.

matter gauge invariants and specialize our equations to the cases of a perfect fluid and a scalar field. Section 5 contains a brief discussion of future developments.

## 2 Geometrical definition of gauge invariants

### 2.1 General formulation

Following standard cosmological perturbation theory (see for example, Chapter 7.5 in Wald (1984)), we consider a 1-parameter family of spacetimes  $g_{ab}(\epsilon)$ , where  $g_{ab}(0)$ , the unperturbed metric, is a RW metric, and  $\epsilon$  is referred to as the *perturbation parameter*.<sup>6</sup> We assign physical dimension *length* to the scale factor  $a$  of the RW metric and  $(\text{length})^2$  to  $g_{ab}(\epsilon)$ . Then the conformal transformation

$$g_{ab}(\epsilon) = a^2 \bar{g}_{ab}(\epsilon), \quad (1)$$

yields a dimensionless metric  $\bar{g}_{ab}(\epsilon)$ . Our reason for making this choice<sup>7</sup> concerning the allocation of physical dimensions is that it enables one to create dimensionless quantities by multiplying by the appropriate power of  $a$ , leading to simple perturbation equations that do not contain  $a$  explicitly. We refer to Appendix B, where this process is applied.

The Riemann tensor associated with the metric  $g_{ab}(\epsilon)$  is a function of  $\epsilon$ , denoted  $R^{ab}_{cd}(\epsilon)$ , as is the Einstein tensor,  $G^a_b(\epsilon)$ . The stress-energy tensor of the matter distribution is also assumed to be a function of  $\epsilon$ , denoted  $T^a_b(\epsilon)$ . We include all these possibilities by considering a 1-parameter family of tensor fields  $A(\epsilon)$ , which we assume can be expanded in powers of  $\epsilon$ , *i.e.* as a Taylor series:

$$A(\epsilon) = {}^{(0)}A + \epsilon {}^{(1)}A + \frac{1}{2}\epsilon^2 {}^{(2)}A + \dots \quad (2)$$

The coefficients are given by<sup>8</sup>

$${}^{(0)}A = A(0), \quad {}^{(1)}A = \left. \frac{\partial A}{\partial \epsilon} \right|_{\epsilon=0}, \quad {}^{(2)}A = \left. \frac{\partial^2 A}{\partial \epsilon^2} \right|_{\epsilon=0}, \quad \dots, \quad (3)$$

where  ${}^{(0)}A$  is called the *unperturbed value*,  ${}^{(1)}A$  is called the *first order (linear) perturbation* and  ${}^{(2)}A$  is called the *second order perturbation* of  $A(\epsilon)$ .

The primary difficulty in cosmological perturbation theory is that the perturbations of a tensor field  $A(\epsilon)$  depend on the choice of gauge, and hence cannot be directly related to observations. It is therefore desirable to formulate the theory in terms of *gauge-invariant* quantities, *i.e.* to replace the gauge-variant perturbations

<sup>6</sup>We use Latin letters  $a, b, \dots, f$  to denote abstract spacetime indices.

<sup>7</sup>An alternative choice in cosmology is to make  $a$  dimensionless and let the spacetime coordinates of  $\bar{g}_{ab}(0)$  have dimension *length* (see, for example, Malik and Wands (2009), page 48). This choice is unsuitable for our purposes since it does not lead naturally to perturbative equations involving dimensionless quantities. For discussions about dimensions and their uses, see for example, Eardley (1974), Martin-Garcia and Gundlach (2002), Wiesenfeld (2001), and Heinzle *et al* (2003).

<sup>8</sup>The notation  $A(\epsilon)$  should be viewed as shorthand for  $A(x, \epsilon)$ , indicating that the tensor fields are functions of the spacetime coordinates, which necessitates the use of partial differentiation with respect to  $\epsilon$ .

${}^{(1)}A, {}^{(2)}A, \dots$  of  $A(\epsilon)$  by gauge-invariant quantities. In this paper we restrict our attention to first order, *i.e.* linear, perturbations, but with a view to subsequently working with higher order perturbations we use a method pioneered by Nakamura (2003), and adapt it so as to create quantities that are gauge-invariant *and* dimensionless.

Given a family of tensor fields  $A(\epsilon)$  the change induced in the first order perturbation  ${}^{(1)}A$  by a gauge transformation is generated by a dimensionless vector field  $\xi^a$  on the background according to

$$\Delta{}^{(1)}A = \mathcal{L}_\xi{}^{(0)}A, \quad (4)$$

where  $\mathcal{L}_\xi$  denotes the Lie derivative (see, for example, Bruni *et al* (1997), equation (1.2)). We now introduce an as yet arbitrary dimensionless vector field  $X$  on the background which we use to define the dimensionless object

$${}^{(1)}\mathbf{A}[X] := a^n ({}^{(1)}A - \mathcal{L}_X{}^{(0)}A), \quad (5)$$

where we assume that  $A(\epsilon)$  is such that  $a^n A(\epsilon)$  is dimensionless. It follows from (4) and (5) that

$$\Delta{}^{(1)}\mathbf{A}[X] = a^n (\mathcal{L}_\xi{}^{(0)}A - \mathcal{L}_{\Delta X}{}^{(0)}A) = a^n \mathcal{L}_{\xi - \Delta X}{}^{(0)}A. \quad (6)$$

The key step is to choose an  $X$  that satisfies

$$\Delta X^a = \xi^a, \quad (7)$$

under a gauge transformation. With this choice, (6) implies that  $\Delta{}^{(1)}\mathbf{A}[X] = 0$ , *i.e.*,  ${}^{(1)}\mathbf{A}[X]$  is gauge-invariant. We say that  ${}^{(1)}\mathbf{A}[X]$  is the *gauge invariant associated with*  ${}^{(1)}A$  *by*  $X$ -*compensation*. Equations (4), (5) and (7) are central to our version of Nakamura's method for constructing gauge invariants associated with the first order perturbation of a tensor  $A$  (see Nakamura (2007), equations (2.19), (2.23) and (2.26)). In what follows we will drop the superscript  ${}^{(1)}$  on  $\mathbf{A}$  for convenience since in this paper we are dealing only with first order perturbations.

The above 'gauge compensating vector field'  $X$ , which we for brevity shall refer to as the *gauge field*, requires comment. Unlike the geometric and matter tensor fields such as  $g_{ab}(\epsilon)$  and  $T^a_b(\epsilon)$  it is not the perturbation of a corresponding quantity on the background spacetime. Instead it should be viewed as a vector field on the background spacetime that is constructed from the linear perturbations of the geometric and matter tensors in such a way that (7) holds. We will construct specific examples of  $X$  in section 2.2.

Before continuing we briefly digress to point out that associated with a tensor  $A$  there are of two distinct types of gauge invariants: those that are solely constructed from components of  ${}^{(1)}A$  and  ${}^{(0)}A$  are called *intrinsic* gauge invariants, while those that depend on the components of another perturbed tensor are called *hybrid* gauge invariants. In particular if the gauge field  $X$  is formed solely from components of  ${}^{(1)}A$  and  ${}^{(0)}A$ , then  $\mathbf{A}[X]$  is an intrinsic gauge invariant; otherwise,  $\mathbf{A}[X]$  is a hybrid gauge invariant.

In the following sections we will calculate the quantities in equations (4) and (5) for various geometric objects  $A$ . To do this it is necessary to use the well known

formulae for the Lie derivative. The formula for a tensor of type  $(1, 1)$ , which we now give, establishes the pattern:

$$\mathcal{L}_\xi A^a_b = A^a_{b,c} \xi^c + \xi^c_{,b} A^a_c - \xi^a_{,c} A^c_b, \quad (8)$$

where  $,$  denotes partial differentiation. In a formula such as (8) one can replace the partial derivatives by covariant derivatives. For our purposes it is convenient to use the covariant derivative  ${}^0\bar{\nabla}_a$  associated with the *unperturbed conformal metric*  $\bar{g}_{ab}(0)$ :

$$\mathcal{L}_\xi A^a_b = ({}^0\bar{\nabla}_c A^a_b) \xi^c + ({}^0\bar{\nabla}_b \xi^c) A^a_c - ({}^0\bar{\nabla}_c \xi^a) A^c_b. \quad (9)$$

We also need to work in a coordinate frame so that we can calculate time and spatial components separately. We thus introduce local coordinates<sup>9</sup>  $x^\mu = (\eta, x^i)$ , with  $\eta$  being the usual conformal time coordinate<sup>10</sup> for the RW metric  $g_{ab}(0)$ , and such that the unperturbed conformal metric  $\gamma_{ab} := \bar{g}_{ab}(0)$  has components

$$\gamma_{00} = -1, \quad \gamma_{0i} = 0, \quad \gamma_{ij}, \quad (10)$$

where  $\gamma_{ij}$  is the metric of a spatial geometry of constant curvature. The curvature index of the RW metric, denoted  $K$ , determines the sign of the curvature of the spatial geometry, and if non-zero can be scaled to be  $+1$  or  $-1$  (see, for example, Plebanski and Krasinski (2006), page 261).

The spacetime covariant derivative  ${}^0\bar{\nabla}_a$  determines a temporal derivative  ${}^0\bar{\nabla}_0 A = \partial_\eta A$ , where  $\partial_\eta$  denotes partial differentiation with respect to  $\eta$ , and a spatial covariant derivative  ${}^0\bar{\nabla}_i$  that is associated with the spatial metric  $\gamma_{ij}$ . We introduce the notation

$$\mathbf{D}_i A := {}^0\bar{\nabla}_i A. \quad (11)$$

The derivative operators  $\partial_\eta$  and  $\mathbf{D}_i$  will be used throughout this paper once local coordinates have been introduced. However, for simplicity we shall denote the derivative of a function  $f(\eta)$  that depends only on  $\eta$  by  $f'(\eta)$ .

With our present allocation of dimensions, the scalar  $\mathcal{H}$  defined by

$$\mathcal{H} := \frac{a'}{a} = aH, \quad (12)$$

where  $H$  is the Hubble scalar,<sup>11</sup> is dimensionless. We shall refer to it as the *dimensionless Hubble scalar*. The use of this scalar, e.g. by Mukhanov *et al* (1992) (see page 218), is essential in eliminating  $a$  from the perturbation equations.

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<sup>9</sup>We use Greek letters to denote spacetime coordinate indices on the few occasions that they occur, and we use Latin letters  $i, j, k, m$  to denote spatial coordinate indices, which are lowered and raised using  $\gamma_{ij}$  and its inverse  $\gamma^{ij}$ , respectively.

<sup>10</sup>Since we assigned  $a$  to have physical dimension *length*, the conformal time  $\eta$  and the conformal spatial line-element  $\gamma_{ij} dx^i dx^j$  are dimensionless. We choose the  $x^i$  to be dimensionless, which implies that the  $\gamma_{ij}$  are also dimensionless.

<sup>11</sup>Recall that  $H := \frac{1}{a} \frac{da}{dt}$ , where  $t$  is cosmic time, and that  $\frac{dt}{d\eta} = a$ .

## 2.2 Metric gauge invariants

We expand  $\bar{g}_{ab}(\epsilon)$ , defined by equation (1), in powers of  $\epsilon$ :

$$\bar{g}_{ab}(\epsilon) = {}^{(0)}\bar{g}_{ab} + \epsilon {}^{(1)}\bar{g}_{ab} + \dots,$$

and label the unperturbed metric and (linear) metric perturbation according to

$$\gamma_{ab} := {}^{(0)}\bar{g}_{ab} = \bar{g}_{ab}(0), \quad f_{ab} := {}^{(1)}\bar{g}_{ab} = \frac{\partial \bar{g}_{ab}}{\partial \epsilon}(0), \quad (13)$$

which is consistent with (3). Applying the general transformation law (4) to the metric tensor  $g_{ab}(\epsilon) = a^2 \bar{g}_{ab}(\epsilon)$  we obtain

$$\Delta^{(1)}g_{ab} = \mathcal{L}_\xi^{(0)}g_{ab}, \quad \text{or, equivalently,} \quad \Delta f_{ab} = a^{-2} \mathcal{L}_\xi(a^2 \gamma_{ab}), \quad (14)$$

in terms of the notation (13). The gauge invariant  $\mathbf{f}_{ab}[X]$  associated with the metric perturbation  $f_{ab}$  by  $X$ -compensation, given by (5), assumes the form

$$\mathbf{f}_{ab}[X] = f_{ab} - a^{-2} \mathcal{L}_X(a^2 \gamma_{ab}). \quad (15)$$

Introducing local coordinates and using (8) and (9) adapted to a  $(0, 2)$  tensor, equations (14) and (15) lead to

$$\Delta f_{00} = -2(\partial_\eta + \mathcal{H})\xi^0, \quad \mathbf{f}_{00}[X] = f_{00} + 2(\partial_\eta + \mathcal{H})X^0, \quad (16a)$$

$$\Delta f_{0i} = -\mathbf{D}_i \xi^0 + \partial_\eta \xi_i, \quad \mathbf{f}_{0i}[X] = f_{0i} + \mathbf{D}_i X^0 - \partial_\eta X_i, \quad (16b)$$

$$\Delta f_{ij} = 2\mathcal{H} \xi^0 \gamma_{ij} + 2\mathbf{D}_{(i} \xi_{j)}, \quad \mathbf{f}_{ij}[X] = f_{ij} - 2\mathcal{H} X^0 \gamma_{ij} - 2\mathbf{D}_{(i} X_{j)}. \quad (16c)$$

In order to construct a gauge field  $X$  that satisfies (7), *using only the metric*, we need to decompose the metric perturbation  $f_{ab}$  into scalar, vector and tensor modes.<sup>12</sup> We introduce the notation<sup>13</sup>

$$f_{00} = -2\varphi, \quad (17a)$$

$$f_{0i} = \mathbf{D}_i B + B_i, \quad (17b)$$

$$f_{ij} = -2\psi \gamma_{ij} + 2\mathbf{D}_i \mathbf{D}_j C + 2\mathbf{D}_{(i} C_{j)} + 2C_{ij}, \quad (17c)$$

where the vectors  $B_i$  and  $C_i$  and the tensor  $C_{ij}$  satisfy

$$\mathbf{D}^i B_i = 0, \quad \mathbf{D}^i C_i = 0, \quad C^i_i = 0, \quad \mathbf{D}^i C_{ij} = 0.$$

<sup>12</sup>In order to guarantee that the functions  $B, B_i, C, C_i$  and  $C_{ij}$  in (17) are uniquely determined by  $f_{0i}$  and  $f_{ij}$  we need to assume that the inverses of  $\mathbf{D}^2$ ,  $\mathbf{D}^2 + 2K$  and  $\mathbf{D}^2 + 3K$  exist. See the proposition in Appendix B.1. See also Nakamura (2007), following equation (4.15), for a helpful discussion of this matter.

<sup>13</sup>We are denoting the scalar mode functions by  $\varphi, B, C$  and  $\psi$ , in agreement with Mukhanov *et al* (1992) (see equation (2.10), but note the different signature) and Malik and Wands (2009) (see equations (2.7)-(2.12)), with the difference that we use  $C$  instead of  $E$ . Bardeen (1980) used the notation  $A, -B, H_T$  and  $-H_L + \frac{1}{3}\mathbf{D}^2 H_T$  for these functions, the choice of the fourth one being motivated by harmonic decomposition. Bardeen's notation has been used by subsequent authors, for example, Kodama and Sasaki (1984) and Durrer (1994), although the latter author replaced  $-B$  by  $B$ .

The vector  $\xi$  is also decomposed into a scalar mode and a vector mode with components

$$\xi^0, \quad \xi^i = \mathbf{D}^i \xi + \tilde{\xi}^i. \quad (18)$$

It follows from (16), (17) and (18) that

$$\Delta\varphi = (\partial_\eta + \mathcal{H})\xi^0, \quad \Delta B = -\xi^0 + \partial_\eta \xi, \quad \Delta C = \xi, \quad \Delta\psi = -\mathcal{H}\xi^0, \quad (19a)$$

$$\Delta B_i = \partial_\eta \tilde{\xi}_i, \quad \Delta C_i = \tilde{\xi}_i, \quad (19b)$$

$$\Delta C_{ij} = 0. \quad (19c)$$

We can draw two immediate conclusions. First, it follows from (19b) and (19c) that  $B_i - C'_i$  and  $C_{ij}$  are gauge invariants. We introduce the following bold-face notation:

$$\mathbf{B}_i := B_i - \partial_\eta C_i, \quad \mathbf{C}_{ij} := C_{ij}. \quad (20)$$

Second, by inspection of (18), (19a) and (19b) we obtain

$$\Delta(\mathbf{D}_i C + C_i) = \xi_i, \quad \Delta\chi = \Delta\left(\frac{\psi}{\mathcal{H}}\right) = -\xi^0, \quad (21)$$

where we have introduced the notation

$$\chi := B - \partial_\eta C. \quad (22)$$

We are now in a position to satisfy the requirement (7). Firstly, referring to (21), we can satisfy the *spatial part*  $\Delta X^i = \xi^i$  of the requirement by choosing

$$X_i = \mathbf{D}_i C + C_i, \quad (23)$$

which we will take to be our default choice for  $X_i$ . With this choice, the expressions (16) for the components of the gauge invariant  $\mathbf{f}_{ab}[X]$ , when combined with (17), assume the form

$$\mathbf{f}_{00}[X] = -2\Phi[X], \quad (24a)$$

$$\mathbf{f}_{0i}[X] = \mathbf{D}_i \mathbf{B}[X] + \mathbf{B}_i, \quad (24b)$$

$$\mathbf{f}_{ij}[X] = -2\Psi[X]\gamma_{ij} + 2\mathbf{C}_{ij}. \quad (24c)$$

where

$$\Phi[X] := \varphi - (\partial_\eta + \mathcal{H})X^0, \quad \Psi[X] := \psi + \mathcal{H}X^0, \quad \mathbf{B}[X] := \chi + X^0, \quad (24d)$$

and  $\mathbf{B}_i$ ,  $\mathbf{C}_{ij}$  and  $\chi$  are given by (20) and (22), respectively.

Secondly, referring to (21), we can satisfy the *timelike part*  $\Delta X^0 = \xi^0$  of the requirement (7) in two obvious ways, by choosing

$$X^0 = X_p^0 := -\chi, \quad \text{or} \quad X^0 = X_c^0 := -\frac{\psi}{\mathcal{H}}, \quad (25)$$

which leads to the metric gauge invariants associated with the *Poisson gauge*, or the *uniform curvature gauge*, respectively. On substituting these choices into (24d) we obtain the conditions

$$\mathbf{B}[X_p] = 0 \quad \text{and} \quad \Psi[X_c] = 0, \quad (26)$$

which characterize these two gauge choices.

### The Poisson gauge invariants

On substituting the first of equations (25) into (24) we obtain

$$\mathbf{f}_{00}[X_p] := -2\Phi, \quad \mathbf{f}_{0i}[X_p] := \mathbf{B}_i, \quad \mathbf{f}_{ij}[X_p] := -2\Psi\gamma_{ij} + 2\mathbf{C}_{ij}, \quad (27)$$

where

$$\Phi := \Phi[X_p] = \varphi + (\partial_\eta + \mathcal{H})\chi, \quad \Psi := \Psi[X_p] = \psi - \mathcal{H}\chi. \quad (28)$$

Here  $\Phi$  and  $\Psi$  are the scalar metric gauge invariants associated with the Poisson gauge,<sup>14</sup> and  $\Psi$  is the Bardeen potential.

### The uniform curvature gauge invariants

On substituting the second of equations (25) into (24) we obtain

$$\mathbf{f}_{00}[X_c] = -2\mathbf{A}, \quad \mathbf{f}_{0i}[X_c] = \mathbf{D}_i\mathbf{B} + \mathbf{B}_i, \quad \mathbf{f}_{ij}[X_c] = 2\mathbf{C}_{ij}, \quad (29)$$

where

$$\mathbf{A} := \Phi[X_c] = \varphi + (\partial_\eta + \mathcal{H})\frac{\psi}{\mathcal{H}}, \quad \mathbf{B} := \mathbf{B}[X_c] = \chi - \frac{\psi}{\mathcal{H}}. \quad (30)$$

Here  $\mathbf{A}$  and  $\mathbf{B}$  are the scalar metric gauge invariants associated with the uniform curvature gauge,<sup>15</sup> introduced by Kodama and Sasaki (1984).<sup>16</sup>

In concluding this section we note that the gauge fields  $X$  used to construct the above gauge invariants have the same spatial components  $X^i$  given by (23) in both cases, leading to (24), with the vector and tensor modes described by the gauge invariants  $\mathbf{B}_i$  and  $\mathbf{C}_{ij}$ , respectively. The difference lies in the scalar metric gauge invariants which are related according to<sup>17</sup>

$$\mathbf{A} = \Phi + (\partial_\eta + \mathcal{H})\frac{\Psi}{\mathcal{H}}, \quad \mathbf{B} = -\frac{\Psi}{\mathcal{H}}, \quad (31)$$

as follows from (28) and (30). In both cases the gauge invariants are *intrinsic* since the gauge field  $X$  depends only on the metric.

A reader of this paper should be aware of the lack of agreement in the literature on labelling the scalar metric gauge invariants associated with the Poisson gauge. Our choice of  $(\Phi, \Psi)$  in (28) is the one initiated by Mukhanov *et al* (1992), and subsequently used by Nakamura (see, for example, Nakamura (2006)) and Malik and Wands (2009). On the other hand Durrer (2008) and Liddle and Lyth (2000) reverse the roles and use  $(\Psi, \Phi)$ , while Kodama and Sasaki (1984) use  $(\Psi, -\Phi)$ . Bardeen's original notation is  $(\Phi_A, -\Phi_H)$ .

<sup>14</sup>The gauge-fixing conditions for the Poisson gauge are  $B = C = 0, C_i = 0$  in (17).

<sup>15</sup>The gauge-fixing conditions for the uniform curvature gauge are  $\psi = C = 0, C_i = 0$  in (17).

<sup>16</sup>See equations (3.4) and (3.5), noting that  $H_L + n^{-1}H_T \equiv -\psi$  and  $B - k^{-1}H'_T \equiv \chi$ .

<sup>17</sup>These relations have recently been given by Christopherson *et al* (2011). See their equations (4.22) and (4.23).

### 2.3 Gauge invariants for mixed rank 2 tensors

In this subsection we consider a rank two tensor  $A^a_b$ , such that  $A_{ab}$  is symmetric and  $a^2 A^a_b$  is dimensionless. We expand  $A^a_b$  in a Taylor series in  $\epsilon$  as in (2), and assume that  ${}^{(0)}A^a_b$  obeys the background symmetries, which means it is spatially homogeneous and isotropic:

$$\mathbf{D}_i {}^{(0)}A^\alpha_\beta = 0, \quad {}^{(0)}A^0_i = {}^{(0)}A^i_0 = 0, \quad {}^{(0)}A^i_j = \frac{1}{3} \delta^i_j {}^{(0)}A^k_k. \quad (32)$$

We introduce the notation

$$\mathcal{A}_A := \frac{1}{2} a^2 ({}^{(0)}A^0_0 + \frac{1}{3} {}^{(0)}A^k_k), \quad \mathcal{C}_A^2 := -\frac{({}^{(0)}A^k_k)'}{3({}^{(0)}A^0_0)'}, \quad (33)$$

where as before ' denotes differentiation with respect to  $\eta$ . We further assume that  $A^a_b$  satisfies the conservation law  $\nabla_a A^a_b = 0$ . It follows that in the background

$$a^2 ({}^{(0)}A^0_0)' = 3a^2 \mathcal{H} ({}^{(0)}A^0_0 + \frac{1}{3} {}^{(0)}A^k_k) = 6\mathcal{H}\mathcal{A}_A, \quad (34)$$

which, in conjunction with (33), implies that

$$\mathcal{A}'_A = -(1 + 3\mathcal{C}_A^2)\mathcal{H}\mathcal{A}_A. \quad (35)$$

We can now calculate the gauge invariants  $\mathbf{A}^a_b[X]$  associated with  ${}^{(1)}A^a_b$  by  $X$ -compensation, as defined by equation (5) with  $n = 2$ . It is convenient to decompose  ${}^{(1)}A^i_j$  into its trace  ${}^{(1)}A^k_k$  and tracefree part defined by

$${}^{(1)}\hat{A}^i_j := {}^{(1)}A^i_j - \frac{1}{3} {}^{(1)}A^k_k \delta^i_j. \quad (36)$$

A straightforward calculation using (5), (8), (9) and (32) leads to<sup>18</sup>

$$\mathbf{A}^0_0[X] = a^2 {}^{(1)}A^0_0 - 6\mathcal{H}\mathcal{A}_A X^0 \quad (37a)$$

$$\mathbf{A}^0_i[X] = a^2 {}^{(1)}A^0_i + 2\mathcal{A}_A \mathbf{D}_i X^0, \quad (37b)$$

$$\mathbf{A}^k_k[X] = a^2 {}^{(1)}A^k_k + 18\mathcal{H}\mathcal{A}_A \mathcal{C}_A^2 X^0, \quad (37c)$$

$$\hat{\mathbf{A}}^i_j[X] = a^2 {}^{(1)}\hat{A}^i_j. \quad (37d)$$

In deriving these equations we have used (33) and (34) to express  ${}^{(0)}A^0_0$ ,  ${}^{(0)}A^k_k$  and their derivatives in terms of  $\mathcal{A}_A$  and  $\mathcal{C}_A^2$ .

Equation (37d) implies that  $\hat{\mathbf{A}}^i_j[X]$  is an *intrinsic* gauge invariant since it is constructed solely from the components of  ${}^{(1)}A^a_b$ . We denote this quantity by

$$\hat{\mathbf{A}}^i_j := \hat{\mathbf{A}}^i_j[X] = a^2 {}^{(1)}\hat{A}^i_j. \quad (38)$$

One can form two additional intrinsic gauge invariants by taking suitable combinations of  $\mathbf{A}^0_0[X]$ ,  $\mathbf{A}^0_i[X]$  and  $\mathbf{A}^k_k[X]$ . Indeed it follows from (37) that

$$\mathbf{A} := \mathcal{C}_A^2 \mathbf{A}^0_0[X] + \frac{1}{3} \mathbf{A}^k_k[X] = a^2 (\mathcal{C}_A^2 {}^{(1)}A^0_0 + \frac{1}{3} {}^{(1)}A^k_k), \quad (39a)$$

$$\mathbf{A}_i := -(\mathbf{D}_i \mathbf{A}^0_0[X] + 3\mathcal{H}\mathbf{A}^0_i[X]) = -a^2 (\mathbf{D}_i {}^{(1)}A^0_0 + 3\mathcal{H} {}^{(1)}A^0_i), \quad (39b)$$

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<sup>18</sup>We do not include the  ${}^{(1)}A^i_0$  components since they can be expressed in terms of the other components and the metric perturbation, due to the assumed symmetry.

which implies that  $\mathbf{A}$  and  $\mathbf{A}_i$  are *intrinsic gauge-invariants*.

In summary, the tensor  $A^a_b$  can be described by the three intrinsic gauge invariants  $\hat{\mathbf{A}}^i_j$ ,  $\mathbf{A}$ , and  $\mathbf{A}_i$ , given by (38), (39a) and (39b), and one hybrid gauge invariant  $\mathbf{A}^0_i[X]$ , given by (37b). In section 3.1 we will use these objects, constructed in terms of the Einstein tensor and the stress-energy tensor, to give a concise derivation of the governing equations in gauge-invariant form for linear perturbations of FL.

### 3 Linearized governing equations

#### 3.1 General formulation

In this section we work with the linear perturbations of the Einstein tensor and the stress-energy tensor, denoted by  ${}^{(1)}G^a_b$  and  ${}^{(1)}T^a_b$ , and defined via equation (3). The corresponding unperturbed quantities are labelled by a superscript  ${}^{(0)}$ .

We begin by imposing the background Einstein equations  ${}^{(0)}G^a_b = {}^{(0)}T^a_b$ . The non-zero components are given by<sup>19</sup>

$$a^2 {}^{(0)}G^0_0 = -3(\mathcal{H}^2 + K) = -a^2 {}^{(0)}\rho = a^2 {}^{(0)}T^0_0, \quad (40a)$$

$$a^2 {}^{(0)}G^i_j = -(2\mathcal{H}' + \mathcal{H}^2 + K)\delta^i_j = a^2 {}^{(0)}p \delta^i_j = a^2 {}^{(0)}T^i_j, \quad (40b)$$

where  $\mathcal{H}$  is given by (12) and  $K$  is the curvature index. It follows from (40), (33) and (34), with  $A$  replaced by  $G$  and  $T$ , respectively, that

$$\mathcal{A}_G = -\mathcal{H}' + \mathcal{H}^2 + K, \quad \mathcal{A}_T = \frac{1}{2}a^2({}^{(0)}\rho + {}^{(0)}p), \quad (41a)$$

$$\mathcal{A}'_G = -(1 + 3\mathcal{C}_G^2)\mathcal{H}\mathcal{A}_G, \quad \mathcal{C}_T^2 = \frac{{}^{(0)}p'}{{}^{(0)}\rho'}. \quad (41b)$$

The conservation law (34), with  $A$  replaced by  $T$ , gives

$$a^2({}^{(0)}\rho)' = -6\mathcal{H}\mathcal{A}_T = -3\mathcal{H}a^2({}^{(0)}\rho + {}^{(0)}p). \quad (42)$$

The background Einstein equations imply that  $\mathcal{A}_G = \mathcal{A}_T$  and  $\mathcal{C}_G^2 = \mathcal{C}_T^2$ . We denote the common values by  $\mathcal{A}$  and  $\mathcal{C}^2$ :

$$\mathcal{A} = \mathcal{A}_G = \mathcal{A}_T, \quad \mathcal{C}^2 = \mathcal{C}_G^2 = \mathcal{C}_T^2. \quad (43)$$

The linearized Einstein field equations are given by

$${}^{(1)}G^a_b = {}^{(1)}T^a_b. \quad (44)$$

In simplifying the linearized field equations we will make use of the intrinsic gauge invariants associated with the Einstein tensor and with the stress-energy tensor, which are given, in analogy with (38), (39a) and (39b), by

$$\hat{\mathbf{G}}^i_j = a^2 {}^{(1)}\hat{G}^i_j, \quad \hat{\mathbf{T}}^i_j = a^2 {}^{(1)}\hat{T}^i_j \quad (45a)$$

$$\mathbf{G}_i = -a^2 (\mathbf{D}_i {}^{(1)}G^0_0 + 3\mathcal{H} {}^{(1)}G^0_i), \quad \mathbf{T}_i = -a^2 (\mathbf{D}_i {}^{(1)}T^0_0 + 3\mathcal{H} {}^{(1)}T^0_i), \quad (45b)$$

$$\mathbf{G} = a^2 (\mathcal{C}_G^2 {}^{(1)}G^0_0 + \frac{1}{3} {}^{(1)}G^k_k), \quad \mathbf{T} = a^2 (\mathcal{C}_T^2 {}^{(1)}T^0_0 + \frac{1}{3} {}^{(1)}T^k_k), \quad (45c)$$

<sup>19</sup>See, for example, Mukhanov *et al* (1992), equation (4.2), noting the difference in signature.

where

$${}^{(1)}\hat{G}_j^i = {}^{(1)}G_j^i - \frac{1}{3}\delta_j^i {}^{(1)}G^k_k, \quad {}^{(1)}\hat{T}_j^i = {}^{(1)}T_j^i - \frac{1}{3}\delta_j^i {}^{(1)}T^k_k. \quad (46)$$

We also need the hybrid gauge invariants  $\mathbf{G}_i^0[X]$  and  $\mathbf{T}_i^0[X]$ , which are given by (37b) with  $A$  replaced by  $G$  and  $T$ :

$$\mathbf{G}_i^0[X] = a^2 {}^{(1)}G_i^0 + 2\mathcal{A}_G \mathbf{D}_i X^0, \quad \mathbf{T}_i^0[X] = a^2 {}^{(1)}T_i^0 + 2\mathcal{A}_T \mathbf{D}_i X^0. \quad (47)$$

Since the gauge invariants (45) and (47) are linear in  ${}^{(1)}G_b^a$  and  ${}^{(1)}T_b^a$  with coefficients depending on  ${}^{(0)}G_b^a$  and  ${}^{(0)}T_b^a$ , respectively, it follows that the linearized Einstein field equations immediately imply the following relations:

$$\hat{\mathbf{G}}_j^i - \hat{\mathbf{T}}_j^i = 0, \quad \mathbf{G}_i - \mathbf{T}_i = 0, \quad \mathbf{G} - \mathbf{T} = 0, \quad (48a)$$

$$\mathbf{G}_i^0[X] - \mathbf{T}_i^0[X] = 0. \quad (48b)$$

Expressions for the Einstein gauge invariants  $\hat{\mathbf{G}}_j^i$ ,  $\mathbf{G}_i$ ,  $\mathbf{G}$  and  $\mathbf{G}_i^0[X]$  in terms of the metric gauge invariants, decomposed into scalar, vector, and tensor modes, are given in equations (115) and (119) in Appendix B. To proceed we likewise decompose the matter gauge invariants  $\hat{\mathbf{T}}_j^i$ ,  $\mathbf{T}_i$ ,  $\mathbf{T}$  and  $\mathbf{T}_i^0[X]$  into scalar, vector, and tensor modes and label them as follows:<sup>20</sup>

$$\hat{\mathbf{T}}_j^i = \mathbf{D}_j^i \Pi + 2\gamma^{ik} \mathbf{D}_{(k} \Pi_{j)} + \Pi_j^i, \quad (49a)$$

$$\mathbf{T}_i = \mathbf{D}_i \Delta + \Delta_i, \quad (49b)$$

$$\mathbf{T} = \Gamma, \quad (49c)$$

$$\mathbf{T}_i^0[X] = 2(\mathbf{D}_i V[X] + V_i), \quad (49d)$$

where

$$\mathbf{D}^i \Pi_i = 0, \quad \Pi_k^k = 0, \quad \mathbf{D}_i \Pi_j^i = 0, \quad \mathbf{D}^i \Delta_i = 0, \quad \mathbf{D}^i V_i = 0, \quad (49e)$$

and

$$\mathbf{D}_{ij} := \mathbf{D}_{(i} \mathbf{D}_{j)} - \frac{1}{3}\gamma_{ij} \mathbf{D}^2, \quad \mathbf{D}^2 := \mathbf{D}^i \mathbf{D}_i. \quad (49f)$$

We stress that in making this decomposition we are not making any assumptions about the physical nature of the stress-energy tensor. By inspecting (115), (119) and (49) one concludes that equations (48) decompose into a scalar mode, a vector mode and a tensor mode, which we label as follows:

$$\mathbf{D}_{ij} \mathbb{A} + \mathbf{D}_{(i} \mathbb{A}_{j)} + \mathbb{A}_{ij} = 0,$$

$$\mathbf{D}_i \mathbb{B} + \mathbb{B}_i = 0,$$

$$\mathbb{C} = 0,$$

$$\mathbf{D}_i \mathbb{E}[X] + \mathbb{E}_i = 0.$$

Since we are assuming that the inverses of the operators  $\mathbf{D}^2$ ,  $\mathbf{D}^2 + 2K$  and  $\mathbf{D}^2 + 3K$  exist we can use the proposition in Appendix B.1 to write the linearized field equations concisely as

$$\textit{Scalar mode: } \mathbb{A} = 0, \quad \mathbb{B} = 0, \quad \mathbb{C} = 0, \quad \mathbb{E}[X] = 0. \quad (50a)$$

$$\textit{Vector mode: } \mathbb{A}_i = 0, \quad \mathbb{B}_i = 0, \quad \mathbb{E}_i = 0. \quad (50b)$$

$$\textit{Tensor mode: } \mathbb{A}_{ij} = 0. \quad (50c)$$

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<sup>20</sup>In subsection 4.1 we comment on the choice of the symbols  $\Pi$ ,  $\Gamma$ ,  $\Delta$  and  $V$ .

### 3.2 Scalar mode

In this subsection we give the governing equations (50a) for the scalar mode, first expressing them in terms of the uniform curvature gauge invariants  $\mathbf{A} = \Phi[X_c]$  and  $\mathbf{B} = \mathbf{B}[X_c]$  (see (30)). The scalars  $\mathbb{A}, \mathbb{B}$  and  $\mathbb{C}$  in (50a) are obtained without any calculation by taking the differences of equations (115) and (49) and reading off the scalar part. The scalar  $\mathbb{E}[X]$  is obtained in a similar manner from (119) and (49d) with  $X = X_p$ . The resulting equations are<sup>21</sup>

$$(\partial_\eta + 2\mathcal{H})\mathbf{B} + \mathbf{A} = -\Pi \quad (51a)$$

$$\mathcal{H} [(\partial_\eta + \mathcal{B}\mathcal{H})\mathbf{A} + \mathcal{C}_G^2 \mathbf{D}^2 \mathbf{B}] = \frac{1}{2}\Gamma + \frac{1}{3}\mathbf{D}^2 \Pi, \quad (51b)$$

$$\mathcal{H} (\mathbf{D}^2 + 3K)\mathbf{B} = -\frac{1}{2}\Delta, \quad (51c)$$

$$\mathcal{H}\mathbf{A} + (\mathcal{A}_G - K)\mathbf{B} = -V, \quad (51d)$$

where

$$\mathcal{B} = \frac{2\mathcal{H}'}{\mathcal{H}^2} + 1 + 3\mathcal{C}_G^2, \quad (52)$$

(see equation (116) in Appendix B), and  $V = V[X_p]$ . We shall refer to these equations as the *uniform curvature form* of the governing equations for the scalar mode.

We now give the governing equations in terms for the Poisson gauge invariants  $\Psi$  and  $\Phi$ . We eliminate  $\mathbf{A}$  in (51b) using (51a) and in (51d) using (31), and eliminate  $\mathbf{B}$  using  $\mathcal{H}\mathbf{B} = -\Psi$ . The resulting equations are

$$\Psi - \Phi = \Pi, \quad (53a)$$

$$(\mathcal{L} - \mathcal{C}_G^2 \mathbf{D}^2)\Psi = \frac{1}{2}\Gamma + \left(\frac{1}{3}\mathbf{D}^2 + \mathcal{H}(\partial_\eta + \mathcal{B}\mathcal{H})\right)\Pi, \quad (53b)$$

$$(\mathbf{D}^2 + 3K)\Psi = \frac{1}{2}\Delta, \quad (53c)$$

$$\partial_\eta \Psi + \mathcal{H}\Phi = -V, \quad (53d)$$

where the differential operator  $\mathcal{L}$  is defined by

$$\mathcal{L}(\bullet) := \mathcal{H}(\partial_\eta + \mathcal{B}\mathcal{H})(\partial_\eta + 2\mathcal{H})\left(\frac{\bullet}{\mathcal{H}}\right), \quad (54)$$

and  $\mathcal{B}$  is given by (52). Expanding the brackets yields<sup>22</sup>

$$\mathcal{L} = \partial_\eta^2 + 3(1 + \mathcal{C}_G^2)\mathcal{H}\partial_\eta + \mathcal{H}^2\mathcal{B} - (1 + 3\mathcal{C}_G^2)K. \quad (55)$$

We shall refer to the above equations as the *Poisson form* of the governing equations for the scalar mode, and to the evolution equation (53b) as the *Bardeen equation*.

Equations (51) and (53), which are linked by the factorization property (54), constitute one of the main results of this paper. Either system of equations determine the behaviour of linear scalar perturbations of an FL cosmology with arbitrary stress-energy content whose scalar mode is described by the gauge invariants  $\Gamma, \Pi, \Delta$  and  $V$ . The structure of these two systems of equations differs in a significant way.

<sup>21</sup>In deriving (51b) we use (51a) to replace  $(\partial_\eta + 2\mathcal{H})\mathbf{B} + \mathbf{A}$  by  $-\Pi$ .

<sup>22</sup>Referring to (41) to express  $\mathcal{H}'$  in terms of  $\mathcal{A}_G$  and then use the equation for  $\mathcal{A}'_G$ .

In the system (51) the time dependence is governed by two *first order* differential operators  $\partial_\eta + \mathcal{B}\mathcal{H}$  and  $\partial_\eta + 2\mathcal{H}$ , while in the system (53) the time dependence is governed by the *second order* linear differential operator  $\mathcal{L}$ . A key point is that *the coefficients in these operators depend only on the background RW geometry*, and this dependence manifests itself through the appearance of  $\mathcal{H}, \mathcal{H}', \mathcal{H}''$  and  $K$ . This property is significant since it means that these operators will have the same form irrespective of the nature of the source in the FL background model, *e.g.* whether it is a perfect fluid with  $p = p(\rho)$ , or a scalar field with potential  $V(\phi)$ . What will differ, however, is the functional dependence of  $\mathcal{H}(\eta)$ , which is determined by solving the Einstein equations in the background RW geometry, and hence depends on the source. Furthermore these differential operators will also appear in the linearized field equations in any geometrical theory of gravity, whose field equations depend in some way on the Einstein tensor.

To the best of our knowledge equations (51) have not been given in the literature, although if one performs a harmonic decomposition one obtains a system of first order ordinary differential equations closely related to that given by Kodama and Sasaki (1984) (see Chapter 2, equations (4.6a-d)). Likewise, the governing equations in Poisson form (53) have not appeared in the literature in the above fully general form. The use of the Poisson gauge invariants was initiated by Bardeen (1980), and the evolution equation (53b) for  $\Psi$  is now commonly used, although it is written in a variety of different forms, as a partial or ordinary differential equation with the coefficients usually expressed in terms of the matter variables of the background FL model. In contrast we have written the Bardeen equation in a fully general form in terms of the purely geometric differential operator  $\mathcal{L}$ , which is defined by the factorization property (54). We can relate our form of the equation to the literature by expanding  $\mathcal{L}$  as in (55) and expressing the coefficients in terms of the matter variables. If the matter content is a barotropic perfect fluid and a cosmological constant and one imposes the background Einstein field equations then the geometric coefficients  $\mathcal{C}_G^2$  and  $\mathcal{B}$  can be written as

$$\mathcal{C}_G^2 = c_s^2, \quad \mathcal{H}^2\mathcal{B} = (c_s^2 - w)\rho a^2 + (1 + c_s^2)\Lambda a^2 - (1 + 3c_s^2)K, \quad (56)$$

using (40), (43) and (78). The form in the literature that is closest to the purely geometric form (55) is that given by Mukhanov *et al* (1992), equation (5.22), who replace  $\mathcal{C}_G^2$  by the matter quantity  $c_s^2$  as in (56) but retain  $\mathcal{H}$  and  $\mathcal{H}'$ . Nakamura (2007) gives the same expression (see his equation (5.30)). A more common form in the literature has  $\mathcal{B}$ , in addition to  $\mathcal{C}_G^2$ , expressed in terms of the background matter variables as in (56). The earliest occurrence of which we are aware is Harrison (1967), equation (182), followed by Bardeen (1980), equation (5.30), after making the appropriate changes of notation and setting  $\Lambda = 0$ . See also Ellis, Hwang and Bruni (1989), equation (31) and Hwang and Vishniac (1990), equation (105).<sup>23</sup>

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<sup>23</sup>In these two references, the evolution equation in question arises in the 1 + 3 gauge-invariant approach to perturbations of FL, and the unknown is a vector quantity that is related to the scalar  $\Psi$ .

### 3.3 Vector and tensor modes

First, we give the governing equations (50b) for the *vector mode*. The vectors  $\mathbb{A}_i$  and  $\mathbb{B}_i$  in (50b) are obtained without any calculation by taking the differences of equations (115) and (49) and reading off the vector part. The vector  $\mathbb{E}_i$  is obtained in a similar manner from (119) and (49d). The resulting equations are

$$(\partial_\eta + 2\mathcal{H})\mathbf{B}_i = -2\Pi_i, \quad (57a)$$

$$(\mathbf{D}^2 + 2K)\mathbf{B}_i = 4V_i, \quad (57b)$$

as well as the relation  $\Delta_i = 6\mathcal{H}V_i$ , which is satisfied identically (see equation (67)). If  $\Pi_i$  is specified and can be regarded as a source term, the evolution equation (57a) is a first order linear ordinary differential equation that determines  $\mathbf{B}_i$ , which in turn determines  $V_i$  by differentiation using (57b).

Second, we give the governing equations (50c) for the *tensor mode*. The tensor  $\mathbb{A}_{ij}$  in (50c) is obtained without any calculation by taking the differences of equations (115) and (49) and reading off the tensor part, leading to

$$(\partial_\eta^2 + 2\mathcal{H}\partial_\eta + 2K - \mathbf{D}^2)\mathbf{C}_{ij} = \Pi_{ij}. \quad (58)$$

If  $\Pi_{ij}$  is specified and can be regarded as source term, this is a second order linear partial differential equation that determines  $\mathbf{C}_{ij}$ .

## 4 Interpretations and examples

### 4.1 Interpretation of the matter gauge invariants

In this section we give the physical interpretation of the gauge invariants  $\Pi, \Gamma, \Delta$  and  $V[X]$  associated with the scalar mode of the stress-energy tensor.

We begin with the decomposition of a stress-energy tensor with respect to a unit timelike vector field  $u^a$ , which is given by

$$T^a_b = (\rho + p)u^a u_b + p\delta^a_b + (q^a u_b + u^a q_b) + \pi^a_b, \quad (59)$$

where

$$u^a q_b = 0, \quad \pi^a_a = 0, \quad u_a \pi^a_b = 0. \quad (60)$$

We choose  $u^a$  to be the timelike eigenvector of  $T^a_b$ , which implies  $q^a = 0$ , *i.e.* we are using the so-called energy frame (see for example, Bruni *et al* (1992), page 37).

Assuming that the unperturbed stress-energy tensor  ${}^{(0)}T^a_b$  has the isotropy and homogeneity properties of the RW geometry, the expansion (2) to linear order for  $\rho, p, u_a$  and  $\pi^a_b$  has the form:<sup>24</sup>

$$\rho = {}^{(0)}\rho + \epsilon {}^{(1)}\rho, \quad p = {}^{(0)}p + \epsilon {}^{(1)}p, \quad (61a)$$

$$\pi^0_0 = 0 = \pi^0_i, \quad \pi^i_j = 0 + \epsilon {}^{(1)}\pi^i_j, \quad (61b)$$

$$u_0 = -a(1 + \epsilon\varphi), \quad u_i = a(0 + \epsilon v_i). \quad (61c)$$

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<sup>24</sup>The form of  $u_0$  is determined by the requirement that  $u^a$  is a unit vector. Recall that  $\varphi$  is one of the metric potentials in (17).

Decomposing  $v_i$  into a scalar and vector mode yields

$$v_i = \mathbf{D}_i v + \tilde{\mathbf{v}}_i, \quad \mathbf{D}^i \tilde{\mathbf{v}}_i = 0. \quad (62)$$

We use boldface in writing  $\tilde{\mathbf{v}}_i$  in view of the fact that this quantity is a dimensionless gauge invariant, as can be verified by applying (4) to  $u_a$ .

For ease of comparison with other work, we note that the expansion of  $u^a = g^{ab}u_b$  to linear order, expressed in terms of  $v$ ,  $\tilde{\mathbf{v}}^i$  and the linearly perturbed metric, is given by

$$u^0 = a^{-1}(1 - \epsilon\varphi), \quad u^i = a^{-1} [0 + \epsilon (\mathbf{D}^i(v - B) + (\tilde{\mathbf{v}}^i - B^i))]. \quad (63)$$

We digress briefly to mention that our expansion of the four-velocity differs from the usual approach in the literature in that we use the *covariant* vector  $u_a$  to define the perturbed three-velocity instead of the contravariant vector  $u^a$ , since we find that this leads to a number of simplifications.<sup>25</sup> For example, Malik and Wands (2009) (see equation (4.4)) have

$$u^i = a^{-1}[0 + \epsilon (\mathbf{D}^i v_{MW} + \tilde{v}_{MW}^i)],$$

so that

$$v_{MW} = v - B, \quad \tilde{v}_{MW}^i = \tilde{\mathbf{v}}^i - B^i.$$

From (59) and (61), and making use of (3), we obtain the following expressions for the components of the linear perturbation of the stress-energy tensor:

$${}^{(1)}T^0_0 = -{}^{(1)}\rho, \quad {}^{(1)}T^k_k = 3{}^{(1)}p, \quad {}^{(1)}T^0_i = ({}^{(0)}\rho + {}^{(0)}p)v_i, \quad {}^{(1)}\hat{T}^i_j = {}^{(1)}\pi^i_j. \quad (64)$$

It follows from (45), (49) and (64), in conjunction with (41) and (42), that the matter gauge invariants are determined by

$$a^2{}^{(1)}\pi^i_j = \mathbf{D}^i_j \Pi + 2\gamma^{ik} \mathbf{D}_{(k} \Pi_{j)} + \Pi^i_j, \quad (65a)$$

$$\Gamma = a^2(-\mathcal{C}_T^2 {}^{(1)}\rho + {}^{(1)}p), \quad (65b)$$

$$\Delta = a^2 ({}^{(1)}\rho + ({}^{(0)}\rho)' v), \quad (65c)$$

$$V[X] = \mathcal{A}_T(v + X^0), \quad V_i = \mathcal{A}_T \tilde{\mathbf{v}}_i. \quad (65d)$$

Before continuing we derive an additional relation. It follows from (39b) with  $A$  replaced by  $T$  that

$$\mathbf{T}_i = -\mathbf{D}_i \mathbf{T}^0_0[X] - 3\mathcal{H}\mathbf{T}^0_i[X]. \quad (66)$$

On substituting from (49b) and (49d) into this equation, we conclude that

$$\Delta = -\mathbf{T}^0_0[X] - 6\mathcal{H}V[X], \quad \Delta_i = -6\mathcal{H}V_i. \quad (67)$$

We can now give the physical interpretation of the matter gauge invariants. First, the gauge invariants  $\Pi$ ,  $\Pi_i$  and  $\Pi_{ij}$  represent the anisotropic stresses. The interpretation of  $\Gamma$  is given in the context of a perfect fluid in the next section. Next, the gauge invariants  $V = V[X_p]$  and  $V_i$  play a role in determining the shear

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<sup>25</sup>The source of these simplifications is the fact that  $u_i$  is invariant under purely spatial gauge transformations while  $u^i$  is not.

and vorticity of  $u_a$ . The relevant formulae are given in (127) in Appendix B.3. In particular,  $V[X_p]$  determines the scalar mode of the shear according to

$$\mathbf{D}^j \boldsymbol{\sigma}_j^i = \frac{2}{3} \mathcal{A}_T^{-1} \mathbf{D}^2 (\mathbf{D}^2 + 3K) V[X_p], \quad (68)$$

as follows from (127) in conjunction with (65d) with  $X = X_p$  and the identity (125e). We will hence use  $V := V[X_p]$  as our standard choice for the gauge invariant  $V[X]$ . However, since the choice  $V[X_c]$  is also of interest we note that

$$V[X_c] - V[X_p] = \mathcal{A}_T \mathbf{B}, \quad (69)$$

as follows from (65d), (25) and (30).

Finally, in order to interpret  $\Delta$  we need to make a small digression. For any scalar field  $A$  with the property that  $a^n A$  is dimensionless we can define a dimensionless gauge invariant  $\mathbf{A}[X]$  according to<sup>26</sup>

$$\mathbf{A}[X] = a^n \left( {}^{(1)}A - ({}^{(0)}A)' X^0 \right). \quad (70)$$

For the matter density  $\rho$  we denote the gauge invariant by  $\boldsymbol{\rho}[X]$ :

$$\boldsymbol{\rho}[X] = a^2 \left( {}^{(1)}\rho - ({}^{(0)}\rho)' X^0 \right). \quad (71)$$

On choosing  $X = X_v$  with  $X_v^0 := -v$  it follows from (65c) that  $\Delta = \boldsymbol{\rho}[X_v]$ . By comparing (71) with equation (3.13) in Bardeen (1980),<sup>27</sup> we conclude that  $\boldsymbol{\rho}[X_v]$ , and hence  $\Delta$ , equals the well-known Bardeen gauge-invariant density perturbation  $\epsilon_m$ , up to a factor of  $a^2 ({}^{(0)}\rho)$ . The specific relation is

$$\Delta = (a^2 ({}^{(0)}\rho)) \epsilon_m. \quad (72)$$

We note that the choice  $X_v^0 = -v$ , in conjunction with our default choice (23) for the spatial components of  $X$ , is associated with the so-called *total matter gauge* (see, for example, Malik and Wands (2009), pages 23-24). Thus  $\Delta$  is the density perturbation in the total matter gauge. In addition it turns out that  $\Delta$  is closely related to the 1 + 3 gauge-invariant approach to perturbations of FL, pioneered by Ellis and collaborators (see for example, Ellis and Bruni (1989), Ellis *et al* (1989)), in which the spatial gradient of the matter density orthogonal to  $u^a$  plays a key role. To elucidate the relation we define the dimensionless spatial density gradient<sup>28</sup>

$$\mathcal{D}_a = a^2 h_a^b \nabla_b \rho, \quad h_a^b = \delta_a^b + u_a u^b. \quad (73)$$

A straight-forward calculation shows that to linear order

$$\mathcal{D}_0 = 0, \quad \mathcal{D}_i = \mathbf{D}_i \Delta - 6\mathcal{H}V_i, \quad (74)$$

<sup>26</sup>This is equation (5) specialized to the case of a scalar field.

<sup>27</sup>One has to take into account differences in notation, the conservation equation (42), and the fact that Bardeen has performed a harmonic decomposition.

<sup>28</sup>Our  $\mathcal{D}_a$  differs from that in Bruni, Dunsby and Ellis (1992) by a factor of  $\rho a^2$  (see their equation (24)).

from which we conclude that  $\Delta$  equals the scalar mode of the spatial density gradient. In addition it follows from (49b) and (67) that  $\mathbf{T}_i = \mathcal{D}_i$ , giving a physical interpretation of the intrinsic gauge-invariant  $\mathbf{T}_i$ .

To end this section we comment on our choice of notation. In using the symbols  $\Pi, \Gamma, \Delta$  and  $V$  for the matter gauge invariants we are following Kodama and Sasaki (1984) with the difference that we scale the variables as follows:

$$\Pi = a^2 p \Pi_{KS}, \quad \Gamma = a^2 p \Gamma_{KS}, \quad \Delta = a^2 \rho \Delta_{KS}, \quad V = \mathcal{A}_T V_{KS}, \quad (75)$$

where  $p$  and  $\rho$  refer to the background. Our choice of scalings simplify the equations considerably.

## 4.2 Perfect fluid

For a perfect fluid the matter gauge invariants are restricted according to

$$\Pi = 0, \quad \Pi_i = 0, \quad \Pi_j^i = 0. \quad (76)$$

In addition it follows from (41b) and (65b) that

$$\Gamma = 0 \quad \text{if and only if} \quad p = p(\rho), \quad (77)$$

*i.e.* if and only if the equation of state is barotropic. In this case it is customary to introduce the notation

$$c_s^2 := \mathcal{C}_T^2, \quad w := \frac{{}^{(0)}p}{{}^{(0)}\rho}, \quad (78)$$

where  $c_s^2 = w$  if  $w$  is constant, as follows from (41b).

On account of (76) the governing equations in the Poisson form (53) for scalar perturbations imply that  $\Psi - \Phi = 0$ , which (when imposing the background field equations) reduces the governing equations for the scalar mode in the perfect fluid case to

$$(\mathcal{L} - c_s^2 \mathbf{D}^2)\Psi = \frac{1}{2}\Gamma, \quad (79a)$$

$$(\mathbf{D}^2 + 3K)\Psi = \frac{1}{2}\Delta, \quad (79b)$$

$$\Psi' + \mathcal{H}\Psi = -V, \quad (79c)$$

where  $\mathcal{L}$  is given by (55) with  $\mathcal{C}_G^2 = \mathcal{C}_T^2 = c_s^2$  and  $\mathcal{B}$  is expressed in terms of the background matter variables according to (56).

## 4.3 Scalar field

For a minimally coupled scalar field we show in Appendix C that the matter gauge invariants are given by

$$\Gamma = (1 - \mathcal{C}_T^2)\Delta, \quad (80a)$$

$$V[X] = -\frac{1}{2}{}^{(0)}\phi' \phi[X], \quad V_i = 0, \quad (80b)$$

$$\Pi = 0, \quad \Pi_i = 0, \quad \Pi_j^i = 0, \quad (80c)$$

where  $\phi[X]$  is the gauge invariant associated with  ${}^{(1)}\phi$  by X-replacement, given by<sup>29</sup>

$$\phi[X] = {}^{(1)}\phi - {}^{(0)}\phi' X^0. \quad (81)$$

Note that  $\mathcal{A}_T$  and  $\mathcal{C}_T^2$  are given by (132). The governing equations (53) in Poisson form imply that  $\Psi - \Phi = 0$ , and then reduce to

$$(\mathcal{L} - \mathcal{C}^2 \mathbf{D}^2) \Psi = \frac{1}{2}(1 - \mathcal{C}^2) \Delta, \quad (82a)$$

$$(\mathbf{D}^2 + 3K) \Psi = \frac{1}{2} \Delta, \quad (82b)$$

$$\Psi' + \mathcal{H} \Psi = \frac{1}{2} {}^{(0)}\phi' \phi_p, \quad (82c)$$

where  $\phi_p := \phi[X_p]$ , and where we have used  $\mathcal{C}_G^2 = \mathcal{C}_T^2 = \mathcal{C}^2$ . By combining (82a) and (82b) we obtain an evolution equation for  $\Psi$  without a source term:

$$(\mathcal{L} - 3(1 - \mathcal{C}^2)K - \mathbf{D}^2) \Psi = 0, \quad (83)$$

where  $\mathcal{L}$  is given by (55). Having solved this equation one can calculate  $\phi_p$  and  $\Delta$  from (82). If one expresses  $\mathcal{C}^2$  in  $\mathcal{L}$  in terms of the unperturbed scalar field and its derivatives (see (132)) and sets  $K = 0$ , equation (83) coincides with equation (6.48) in Mukhanov *et al* (1992). For the generalization to arbitrary  $K$ , see Nakamura (2007), equation (5.39).<sup>30</sup>

One can also use the governing equations (51) in uniform curvature form, obtaining equations equivalent to those derived by Malik (2007) (see equations (2.20)-(2.23), noting that he is considering multiple scalar fields).

## 5 Discussion

We have given a systematic account of the gauge-invariant quantities that are associated with a linearly perturbed RW geometry and stress-energy tensor, emphasizing the role of intrinsic dimensionless gauge invariants. First, we have shown that there are two distinct choices of dimensionless intrinsic gauge invariants for the perturbed metric, which are the gauge invariants associated with the Poisson gauge and the uniform curvature gauge, through the work of Bardeen (1980) and Kodama and Sasaki (1984), respectively. Second, we have introduced dimensionless intrinsic gauge invariants for the Einstein tensor and the stress-energy tensor, which we used to derive a particularly simple and concise form of the governing equations for linear perturbations of FL models. The specific form of the governing equations for the scalar mode depends on the choice of intrinsic gauge invariants for the perturbed metric. The Kodama-Sasaki choice leads to a coupled system of two *first order* (in time) linear differential operators that govern the evolution of the uniform curvature metric gauge invariants (see equations (51)). On going over to the Poisson picture, the product of these two operators yields the *second order* linear differential operator  $\mathcal{L}$  that governs the evolution of the Bardeen potential (see equation (55)), thereby providing a link between the two forms of the governing equations. A common feature of both systems is the appearance of the physically motivated gauge-invariant

<sup>29</sup>This is a special case of equation (70).

<sup>30</sup>We note a minor typo: a factor of 2 multiplying  $\partial_\eta^2$  should be deleted.

density perturbation  $\Delta$  that is one of the intrinsic gauge invariants associated with the stress-energy tensor (see equations (51c) and (53c)).

The mathematical structure of the governing equations for linear perturbations that we have elucidated here has in fact a much wider significance. Indeed, as one might expect on the basis of elementary perturbation theory, the governing equations for second order (nonlinear) perturbations have precisely the same form, apart from the inclusion of a source term that depends quadratically on the linear metric perturbation.<sup>31</sup> As an illustration of this we give the form of the equations that govern second order scalar perturbations using the metric gauge invariants associated with the Poisson gauge:

$${}^{(2)}\Psi - {}^{(2)}\Phi = {}^{(2)}\Pi + S_{aniso}({}^{(1)}\mathbf{f}), \quad (84a)$$

$$(\mathcal{L} - \mathcal{C}_G^2 \mathbf{D}^2) {}^{(2)}\Psi = \frac{1}{2} {}^{(2)}\Gamma + \left(\frac{1}{3} \mathbf{D}^2 + \mathcal{H}(\partial_\eta + \mathcal{B}\mathcal{H})\right) {}^{(2)}\Pi + S_{evol}({}^{(1)}\mathbf{f}), \quad (84b)$$

$$(\mathbf{D}^2 + 3K) {}^{(2)}\Psi = \frac{1}{2} {}^{(2)}\Delta + S_{matter}({}^{(1)}\mathbf{f}), \quad (84c)$$

$$\partial_\eta {}^{(2)}\Psi + \mathcal{H} {}^{(2)}\Phi = -{}^{(2)}V + S_{velocity}({}^{(1)}\mathbf{f}), \quad (84d)$$

where  $S_\bullet({}^{(1)}\mathbf{f})$  is a *source term* that depends quadratically on the first order gauge-invariant metric perturbation  ${}^{(1)}\mathbf{f}_{ab} \equiv \mathbf{f}_{ab}$  in equation (27). The key point is that, apart from the source terms, equations (84) have the same form as equations (53), with the variables  ${}^{(2)}\Psi$  and  ${}^{(2)}\Phi$  being the metric gauge invariants at second order determined by the Nakamura procedure. The second order matter terms  ${}^{(2)}\Pi$ ,  ${}^{(2)}\Gamma$ ,  ${}^{(2)}\Delta$  and  ${}^{(2)}V$  are defined in analogy with the first order terms  $\Pi$ ,  $\Gamma$ ,  $\Delta$  and  $V$  after expanding the stress-energy tensor  $T^a_b$  to second order in powers of  $\epsilon$ . All the complications lie in the source terms, whose explicit form has to be found by calculating the Riemann tensor to second order. In order to solve the above second order equations the source terms, which include scalar, vector and tensor modes, first have to be obtained by solving the governing equations for the scalar, vector and tensor linear perturbations. In a subsequent paper we will derive both the above Poisson form and the corresponding uniform curvature form of the governing equations for second order perturbations, relating our formulation to other recent work.

In this paper we have focussed exclusively on using the linearized Einstein field equations to describe the dynamics of scalar perturbations. There are, however, two alternatives to the direct use of the linearized Einstein equations. First, one can use the linearized conservation equations for the stress-energy tensor, and second, one can use the 1 + 3 gauge-invariant formalism,<sup>32</sup> in which the evolution equations are obtained from the Ricci identities. An advantage of using the first approach independently of the Einstein equations is that the results are applicable to theories of gravity other than general relativity. An advantage of the second approach is that one initially derives exact nonlinear evolution equations, which are then subsequently linearized. Both of these approaches lead to a system of first order partial differential equations that describe the evolution of scalar perturbations. An additional aspect of the dynamics of scalar perturbations that we have likewise not touched on

<sup>31</sup>This behaviour has been noted in general terms by Nakamura (2006), equations (38)-(39).

<sup>32</sup>See Bruni *et al* (1992) for a comprehensive treatment.

in this paper is that under certain conditions (*i.e.* in the long wavelength regime) the governing equations admit so-called conserved quantities, *i.e.* quantities that remain approximately constant during a restricted epoch. These quantities, which are related to both the linearized Einstein equations and the linearized conservation equations, have been found to be useful in analyzing the dynamics of scalar perturbations during inflation. We refer to Uggla and Wainwright (2011), where we discuss the above aspects of the dynamics of scalar perturbations within the framework of the present paper.

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## A The Replacement Principle

The expression for the perturbation of the Riemann tensor given in equation (103) in Appendix B, can be written symbolically in the form:

$$a^{2(1)}R^{ab}{}_{cd} = \mathbf{L}^{ab}{}_{cd}(f), \quad (85)$$

where  $\mathbf{L}^{ab}{}_{cd}$  is a linear operator and  $f$  is shorthand for  $f_{ab}$ . The Replacement Principle for the Riemann curvature states that the gauge invariants associated with  ${}^{(1)}R^{ab}{}_{cd}$  and with  $f_{ab}$  by  $X$ -compensation are related by *the same* linear operator:

$$\mathbf{R}^{ab}{}_{cd}[X] = \mathbf{L}^{ab}{}_{cd}(\mathbf{f}[X]), \quad (86)$$

where  $\mathbf{f}[X]$  is shorthand for  $\mathbf{f}_{ab}[X]$ .

This result is adapted from more general results given by Nakamura (2005) (see in particular, his equations (3.12), (3.15) and (3.23)). Similar results hold for the Einstein and Weyl tensors. Use of the Replacement Principle in Appendix B makes the transition from gauge-variant to gauge-invariant equations particularly easy and transparent.

## B Derivation of the curvature formula

In this appendix we derive expressions for the Einstein gauge invariants, namely, the three intrinsic gauge invariants  $\hat{\mathbf{G}}^i_j$ ,  $\mathbf{G}_i$  and  $\mathbf{G}$ , and the single hybrid gauge invariant  ${}^{(1)}\mathbf{G}^0_i[X]$ , defined by equations (45) and (47). Our strategy incorporates the following ideas:

- i) *Conformal structure.* We adapt to the conformal structure of the background geometry, determined by the scale factor  $a$  of the RW metric, from the outset. In particular we create dimensionless quantities by multiplying with appropriate powers of  $a$ , which simplifies the equations considerably.

- ii) *Index conventions.* We represent tensors of even rank, apart from the metric tensor, with equal numbers of covariant and contravariant indices. This makes contractions trivial to perform and ensures that the components of the tensor have the same physical dimension as the associated contracted scalar.
- iii) *Timing of specialization.* We defer performing the decomposition into scalar, vector and tensor modes as long as possible, and do not make harmonic decompositions. This strategy helps to reveal structure in the equations and serves to reduce the amount of calculation.

### Calculation of $R^{ab}_{cd}(\epsilon)$

We begin by deriving an exact expression for the Riemann tensor<sup>33</sup>  $R^{ab}_{cd}(\epsilon)$  of the metric  $g_{ab}(\epsilon)$  in terms of the covariant derivative of the conformal background metric  $\gamma_{ab}$ . We thus relate the covariant derivative of  $g_{ab}(\epsilon)$ , denoted  ${}^\epsilon\nabla_a$ , to that of  $\gamma_{ab} = \bar{g}_{ab}(0)$ , denoted  ${}^0\bar{\nabla}_a$ . The relation is given by an object  $Q^a_{bc} = Q^a_{cb}$  defined by

$$Q^a_{bc} = g^{ad}Q_{dbc} = \frac{1}{2}g^{ad}({}^0\bar{\nabla}_c g_{db} - {}^0\bar{\nabla}_d g_{bc} + {}^0\bar{\nabla}_b g_{cd}), \quad (87)$$

(see Wald (1984) equation (D.1)), with the property that<sup>34</sup>

$${}^\epsilon\nabla_a A^b_c = {}^0\bar{\nabla}_a A^b_c + Q^b_{ad}A^d_c - Q^d_{ac}A^b_d. \quad (88)$$

It is convenient to write  $Q^a_{bc}$  as the sum of two parts:

$$Q^a_{bc}(\epsilon) = \bar{Q}^a_{bc}(\epsilon) + \tilde{Q}^a_{bc}(\epsilon). \quad (89)$$

First, the transformation from  ${}^\epsilon\nabla_a$  to  ${}^\epsilon\bar{\nabla}_a$ , which is associated with the conformal transformation  $g_{ab}(\epsilon) = a^2\bar{g}_{ab}(\epsilon)$ , is described by

$$\bar{Q}^a_{bc}(\epsilon) = 2\delta^a_{(b}r_{c)} - \bar{g}^{ad}(\epsilon)\bar{g}_{bc}(\epsilon)r_d, \quad (90)$$

where<sup>35</sup>

$$r_a := {}^0\bar{\nabla}_a(\ln a) \quad (91)$$

(see Wald (1984), equation (D.3)). It follows that  ${}^0\bar{\nabla}_a r_b = {}^0\bar{\nabla}_b r_a$ . Second, the transformation from  ${}^\epsilon\bar{\nabla}_a$  to  ${}^0\bar{\nabla}_a$ , the covariant derivatives associated with  $\bar{g}_{ab}(\epsilon)$  and  $\bar{g}_{ab}(0)$ , respectively, is described by

$$\tilde{Q}^a_{bc}(\epsilon) = \frac{1}{2}\bar{g}^{ad}(\epsilon)({}^0\bar{\nabla}_c \bar{g}_{db}(\epsilon) - {}^0\bar{\nabla}_d \bar{g}_{bc}(\epsilon) + {}^0\bar{\nabla}_b \bar{g}_{cd}(\epsilon)). \quad (92)$$

It follows from  ${}^0\bar{\nabla}_a \gamma_{bc} = 0$  that

$$\tilde{Q}^a_{bc}(0) = 0. \quad (93)$$

To calculate  $R^{ab}_{cd}(\epsilon)$  we first perform the conformal transformation from  $g_{ab}$  to  $\bar{g}_{ab}$ , which yields

$$a^2 R^{ab}_{cd}(\epsilon) = \bar{R}^{ab}_{cd}(\epsilon) + 4\delta^{[a}_{[c}\bar{U}^{b]}_{d]}(\epsilon), \quad (94)$$

<sup>33</sup>We use the sign convention of Wald (1984) for defining the Riemann tensor.

<sup>34</sup>This example establishes the pattern for a general tensor.

<sup>35</sup>Note that we always use the vector  $r_a$  in covariant form, since  $r_a$  is independent of  $\epsilon$ , whereas  $r^a = g^{ab}(\epsilon)r_b$  is not.

where

$$\bar{U}^b_d(\epsilon) = - \left[ \bar{g}^{be} ({}^\epsilon \bar{\nabla}_d - r_d) + \frac{1}{2} \delta^b_d \bar{g}^{ef} r_f \right] r_e, \quad (95)$$

and  $\bar{R}^{ab}_{cd}(\epsilon)$  is the curvature tensor of the metric  $\bar{g}_{ab}(\epsilon)$  (see Wald (1984), equation (D.7)). Second, by performing the transition from  ${}^\epsilon \bar{\nabla}_a$  to  ${}^0 \bar{\nabla}_a$  we obtain

$$\bar{R}^{ab}_{cd}(\epsilon) = \bar{g}^{be} \bar{R}^a_{ecd}(\epsilon) = \bar{g}^{be} \left( {}^0 \bar{R}^a_{ecd} + 2 {}^0 \bar{\nabla}_{[c} \tilde{Q}^a_{d]e} + 2 \tilde{Q}^a_{f[c} \tilde{Q}^f_{d]e} \right), \quad (96)$$

where  ${}^0 \bar{R}^a_{bcd}$  is the curvature tensor of the metric  $\gamma_{ab}$  (see Wald (1984), equation (D.7)). The term  $2 \bar{g}^{be} {}^0 \bar{\nabla}_{[c} \tilde{Q}^a_{d]e}$  in (96) can be written as<sup>36</sup>

$$\begin{aligned} 2 \bar{g}^{be} {}^0 \bar{\nabla}_{[c} \tilde{Q}^a_{d]e} &= 2 \bar{g}^{be} ({}^0 \bar{\nabla}_{[c} \bar{g}^{af}) \tilde{Q}_{|f|d]e} \\ &+ \bar{g}^{be} \bar{g}^{af} ({}^0 \bar{\nabla}_{[c} {}^0 \bar{\nabla}_{|e|} \bar{g}_{d]f} - {}^0 \bar{\nabla}_{[c} {}^0 \bar{\nabla}_{|f|} \bar{g}_{d]e}) - \gamma_{ef} \bar{g}^{e(b} {}^0 \bar{R}^{a)f}_{cd}, \end{aligned} \quad (97)$$

which we use to rearrange (96), in conjunction with the relation  ${}^0 \bar{\nabla}_c \bar{g}^{ab} = -2 \tilde{Q}^{(ab)}_c$ . In summary,  $R^{ab}_{cd}(\epsilon)$  is given by equation (94) with

$$\bar{R}^{ab}_{cd}(\epsilon) = -2 \bar{g}^{e[a} \bar{g}^{b]f} {}^0 \bar{\nabla}_{[c} {}^0 \bar{\nabla}_{|e|} \bar{g}_{d]f} - \gamma_{ef} \bar{g}^{e[a} {}^0 \bar{R}^{b]f}_{cd} - 2 \tilde{Q}^f{}^{[a}{}_{[c} \tilde{Q}^b]{}_{d]f}, \quad (98a)$$

$$\bar{U}^b_d(\epsilon) = - \left[ \bar{g}^{be} ({}^0 \bar{\nabla}_d - r_d) + \frac{1}{2} \delta^b_d \bar{g}^{ef} r_f - \bar{g}^{bf} \tilde{Q}^e_{df} \right] r_e, \quad (98b)$$

where we have used  ${}^\epsilon \bar{\nabla}_a r_b = {}^0 \bar{\nabla}_a r_b - \tilde{Q}^c_{ab} r_c$  in obtaining (98b) from (95).

### Calculation of ${}^{(1)}R^{ab}_{cd}$

We now calculate the perturbation  ${}^{(1)}R^{ab}_{cd}$  of the Riemann tensor, defined via equation (3), expressing it in terms of the covariant derivative  ${}^0 \bar{\nabla}_a$  associated with  $\gamma_{ab}$  and the metric perturbation  $f_{ab} = {}^{(1)}\bar{g}_{ab}$  (see (13)). We note that

$${}^{(1)}\bar{g}^{ab} = -f^{ab}, \quad (99)$$

where the indices on  $f^{ab}$  are raised using  $\gamma^{ab}$ . It follows from (3), (92) (94) and (98), in conjunction with (93) and (99), that<sup>37</sup>

$$a^2 {}^{(1)}R^{ab}_{cd} = {}^{(1)}\bar{R}^{ab}_{cd} + 4 \delta^{[a}{}_{[c} {}^{(1)}\bar{U}^{b]}{}_{d]}, \quad (100a)$$

where

$${}^{(1)}\bar{R}^{ab}_{cd} = -2 {}^0 \bar{\nabla}_{[c} {}^0 \bar{\nabla}^{[a} f_{d]b]} + f_e^{[a} {}^0 \bar{R}^{b]e}_{cd}, \quad (100b)$$

$${}^{(1)}\bar{U}^a_b = \left[ f^{ac} ({}^0 \bar{\nabla}_b - r_b) + \frac{1}{2} \delta^a_b f^{cd} r_d + \gamma^{ad} {}^{(1)}\tilde{Q}^c_{bd} \right] r_c, \quad (100c)$$

$${}^{(1)}\tilde{Q}_{abc} = \frac{1}{2} ({}^0 \bar{\nabla}_c f_{ab} - {}^0 \bar{\nabla}_a f_{bc} + {}^0 \bar{\nabla}_b f_{ca}). \quad (100d)$$

Introducing local coordinates  $x^\mu = (\eta, x^i)$  as in section 2.1 leads to

$$r_\alpha = \mathcal{H} \delta^0_\alpha, \quad {}^0 \bar{\nabla}_0 = \partial_\eta, \quad {}^0 \bar{\nabla}_i = \mathbf{D}_i. \quad (101)$$

<sup>36</sup>Note that  ${}^0 \bar{R}^{ab}_{cd} = \gamma^{be} {}^0 \bar{R}^a_{ecd}$ .

<sup>37</sup>Note that  $R^{ab}_{cd}(\epsilon)$  depends on  $\epsilon$  through  $\bar{g}_{ab}(\epsilon)$ ,  $\bar{g}^{ab}(\epsilon)$  and  $\tilde{Q}^c_{ab}(\epsilon)$ .

In addition we note that the quantity  ${}^0\bar{R}^a{}_{bcd}$ , the curvature tensor of the metric  $\gamma_{ab}$ , is zero if one index is temporal, while if all indices are spatial

$${}^0\bar{R}^{ij}{}_{km} = 2K\delta_{[k}^{[i}\delta_{m]}^j], \quad (102)$$

where the constant  $K$  describes the curvature of the maximally symmetric three-space. Equation (100), in conjunction with (101) and (102), yields the following expressions:

$$a^{2(1)}R^{0j}{}_{0m} = \frac{1}{2}[\mathbf{D}^j\mathbf{D}_m + (\mathcal{H}' - \mathcal{H}^2)\delta_m^j]f_{00} + (\partial_\eta + \mathcal{H})Y_m^j, \quad (103a)$$

$$a^{2(1)}R^{0j}{}_{km} = 2\mathbf{D}_{[k}Y_{m]}^j, \quad (103b)$$

$$a^{2(1)}R^{ij}{}_{km} = -2(\mathbf{D}_{[k}\mathbf{D}^{[i} + K\delta_{[k}^{[i}f_{m]}^{j]}) + 4\mathcal{H}\delta_{[k}^{[i}Y_{m]}^{j]}, \quad (103c)$$

where<sup>38</sup>

$$Y_{ij} = \frac{1}{2}\gamma_{ij}\mathcal{H}f_{00} - \mathbf{D}_{(i}f_{j)0} + \frac{1}{2}\partial_\eta f_{ij}. \quad (103d)$$

### Calculation of the Riemann gauge invariants

We now apply the Replacement Principle to (103), which entails performing the following replacements:

$$f_{ab} \rightarrow \mathbf{f}_{ab}[X], \quad Y_{ij} \rightarrow \mathbf{Y}_{ij}[X], \quad a^{2(1)}R^{ab}{}_{cd} \rightarrow \mathbf{R}^{ab}{}_{cd}[X], \quad (104)$$

where the gauge invariants are defined by equation (5). All components of the Riemann tensor can be obtained from the ‘curvature spanning set’ ( $R^{0i}{}_{0j}$ ,  $R^{0i}{}_{jk}$ ,  $R^{im}{}_{jm}$ ) or, alternatively, their spatial traces and their trace-free parts:

$$(R^{0m}{}_{0m}, R^{0m}{}_{jm}, R^{km}{}_{km}), \quad (\hat{R}^{0i}{}_{0j}, \hat{R}^{0i}{}_{jk}, \hat{R}^{im}{}_{jm}), \quad (105)$$

where

$$\hat{R}^{0i}{}_{0j} = R^{0i}{}_{0j} - \frac{1}{3}\delta_j^i R^{0m}{}_{0m}, \quad \hat{R}^{im}{}_{jm} = R^{im}{}_{jm} - \frac{1}{3}\delta_j^i R^{km}{}_{km}, \quad (106a)$$

$$\hat{R}^{0i}{}_{jk} = R^{0i}{}_{jk} - \delta_{[k}^i R^{0m}{}_{j]m}. \quad (106b)$$

Our motivation for choosing these particular components as the spanning set is that the first set of terms in (105) are invariant under spatial gauge transformations, while the hatted quantities are fully gauge-invariant, as follows from (4).

We denote the gauge invariants associated with the spanning set (105) by

$$(\mathbf{R}^{0m}{}_{0m}[X], \mathbf{R}^{0m}{}_{jm}[X], \mathbf{R}^{km}{}_{km}[X]), \quad (\hat{\mathbf{R}}^{0i}{}_{0j}, \hat{\mathbf{R}}^{0i}{}_{jk}, \hat{\mathbf{R}}^{im}{}_{jm}), \quad (107)$$

and refer to them as the *Riemann gauge invariants*. As indicated by the notation (*i.e.* no dependence on the gauge field  $X$ ) the hatted quantities are intrinsic gauge invariants. We now substitute the expressions<sup>39</sup> for  $\mathbf{f}_{ab}[X]$  given by (24) into the

<sup>38</sup>Note that  $\tilde{Q}^0{}_{ij} = -\mathbf{D}_{(i}f_{j)0} + \frac{1}{2}f'_{ij}$ .

<sup>39</sup>In using these expressions we are making the choice for  $X_i$  given in equation (23). Choosing  $X_i$  in this way simplifies the calculation but not the final form of the Riemann gauge invariants, since, as mentioned earlier, the spanning set is invariant under spatial gauge transformations.

bold-face version of (103), and calculate the gauge invariants (107). It is convenient to split  $\mathbf{Y}_{ij}$  into a trace and a trace-free part:

$$\hat{\mathbf{Y}}_{ij} = \mathbf{Y}_{ij} - \frac{1}{3}\gamma_{ij}\mathbf{Y}, \quad \mathbf{Y} = \mathbf{Y}^i_i, \quad (108)$$

and to use the trace-free second derivative operator  $\mathbf{D}_{ij}$  defined in (49f). We obtain<sup>40</sup>

$$\mathbf{R}^{0m}_{0m}[X] = -[\mathbf{D}^2 + 3(\mathcal{H}' - \mathcal{H}^2)]\Phi[X] + (\partial_\eta + \mathcal{H})\mathbf{Y}[X], \quad (109a)$$

$$\hat{\mathbf{R}}^{0i}_{0j} = -\mathbf{D}_j^i\Phi[X] + (\partial_\eta + \mathcal{H})\hat{\mathbf{Y}}^i_j[X], \quad (109b)$$

$$\mathbf{R}^{km}_{km}[X] = 4[(\mathbf{D}^2 + 3K)\Psi[X] + \mathcal{H}\mathbf{Y}[X]], \quad (109c)$$

$$\hat{\mathbf{R}}^{im}_{jm} = \mathbf{D}_j^i\Psi[X] + \mathcal{H}\hat{\mathbf{Y}}^i_j[X] - (\mathbf{D}^2 - 2K)\mathbf{C}^i_j, \quad (109d)$$

$$\mathbf{R}^{0m}_{jm}[X] = \frac{2}{3}\mathbf{D}_j\mathbf{Y}[X] - \mathbf{D}_m\hat{\mathbf{Y}}^m_j[X], \quad (109e)$$

$$\hat{\mathbf{R}}^{0i}_{jk} = 2\mathbf{D}_{[j}\hat{\mathbf{Y}}^i_{k]}[X] + \mathbf{D}_m\hat{\mathbf{Y}}^m_{[j}[X]\delta^i_{k]}, \quad (109f)$$

where

$$\mathbf{Y}[X] = -3(\partial_\eta\Psi[X] + \mathcal{H}\Phi[X]) - \mathbf{D}^2\mathbf{B}[X], \quad (109g)$$

$$\hat{\mathbf{Y}}^i_j[X] = -\mathbf{D}_{ij}\mathbf{B}[X] - \mathbf{D}_{(i}\mathbf{B}_{j)} + \partial_\eta\mathbf{C}_{ij}. \quad (109h)$$

These equations constitute one of the main results of this paper. They express the Riemann gauge invariants (107) in terms of the metric gauge invariants (24). They depend only on the choice of the temporal gauge field  $X^0$ , as can be seen from (24d).

### Calculation of the Einstein gauge invariants

The Einstein tensor and the Weyl conformal curvature tensor are defined in terms of the Riemann tensor according to

$$G^a_b := R^a_b - \frac{1}{2}\delta^a_b R, \quad \text{where} \quad R^a_b := R^{ac}_{bc}, \quad R := R^a_a, \quad (110a)$$

$$C^{ab}_{cd} := R^{ab}_{cd} - 2\delta^{[a}_{[c}R^{b]}_{d]} + \frac{1}{3}\delta^{[a}_{[c}\delta^{b]}_{d]}R. \quad (110b)$$

The curvature spanning set (105) can be replaced with the following spatially irreducible components of the Einstein tensor and the Weyl tensor:<sup>41</sup>

$$(G^0_0, G^m_m, G^0_i, \hat{G}^i_j), \quad (C^{0i}_{0j}, C^{0i}_{jk}), \quad (111)$$

where

$$\hat{G}^i_j := G^i_j - \frac{1}{3}\delta^i_j G^m_m. \quad (112)$$

It follows from (110) that

$$G^0_0 = -\frac{1}{2}R^{km}_{km}, \quad G^m_m = -\frac{1}{2}(R^{km}_{km} + 4R^{0m}_{0m}), \quad (113a)$$

$$G^0_i = R^{0m}_{im}, \quad \hat{G}^i_j = \hat{R}^{0i}_{0j} + \hat{R}^{im}_{jm}, \quad (113b)$$

$$C^{0i}_{0j} = \frac{1}{2}(\hat{R}^{0i}_{0j} - \hat{R}^{im}_{jm}), \quad C^{0i}_{jk} = \hat{R}^{0i}_{jk}. \quad (113c)$$

<sup>40</sup>Use the identities (125c), (125d) and (125h).

<sup>41</sup>Note that  $C^{ij}_{km} = -4C^{0[i}_{0[k}\delta^{j]}_{m]}$  in an orthonormal frame.

The Einstein gauge invariants, as defined by equations (38), (39a) and (39b) with  $A$  replaced by  $G$ , can be expressed in terms of the curvature spanning set (105) by using the bold-face version of (113). This yields

$$\hat{\mathbf{G}}_j^i := \hat{\mathbf{G}}_j^i[X] = \hat{\mathbf{R}}_{0j}^{0i} + \hat{\mathbf{R}}_{jm}^{im}, \quad (114a)$$

$$\mathbf{G}_i := -(\mathbf{D}_i \mathbf{G}_0^0[X] + 3\mathcal{H} \mathbf{G}_i^0[X]) = \frac{1}{2} \mathbf{D}_i \mathbf{R}^{km}_{km}[X] - 3\mathcal{H} \mathbf{R}^{0m}_{im}[X], \quad (114b)$$

$$\mathbf{G} := \mathcal{C}_G^2 \mathbf{G}_0^0[X] + \frac{1}{3} \mathbf{G}_m^m[X] = -\frac{1}{6} ((1 + 3\mathcal{C}_G^2) \mathbf{R}^{km}_{km}[X] + 4\mathbf{R}^{0m}_{0m}[X]). \quad (114c)$$

We find that it is simplest to express the Einstein gauge invariants (114) in terms of the *uniform curvature* metric gauge invariants  $\mathbf{A}$  and  $\mathbf{B}$  defined by (30). We accomplish this directly by choosing  $X = X_c$  in (109), and noting that by (26) we have  $\Psi[X_c] = 0$ . After simplifying using the identities (125e) and (125f) we obtain<sup>42</sup>

$$\hat{\mathbf{G}}_{ij} = \mathbf{D}_{ij} \mathbb{G} - \mathbf{D}_{(i} (\partial_\eta + 2\mathcal{H}) \mathbf{B}_{j)} + (\partial_\eta^2 + 2\mathcal{H} \partial_\eta + 2K - \mathbf{D}^2) \mathbf{C}_{ij}, \quad (115a)$$

$$\mathbf{G}_i = 2\mathcal{H} \mathbf{D}_i (\mathbf{D}^2 + 3K) \mathbf{B} + \frac{3}{2} \mathcal{H} (\mathbf{D}^2 + 2K) \mathbf{B}_i, \quad (115b)$$

$$\mathbf{G} = 2\mathcal{H} [(\partial_\eta + \mathcal{B}\mathcal{H}) \mathbf{A} + \mathcal{C}_G^2 \mathbf{D}^2 \mathbf{B}] - \frac{2}{3} \mathbf{D}^2 \mathbb{G}, \quad (115c)$$

where we have introduced the notation

$$\mathbb{G} := -[\mathbf{A} + (\partial_\eta + 2\mathcal{H}) \mathbf{B}], \quad \mathcal{B} := \frac{2\mathcal{H}'}{\mathcal{H}^2} + 1 + 3\mathcal{C}_G^2. \quad (116)$$

We also need

$$\mathbf{G}_j^0[X] = \mathbf{R}^{0m}_{jm}[X]. \quad (117)$$

We choose  $X = X_p$  in this equation, and using (109) in conjunction with the identity (125f) we obtain

$$\mathbf{G}_j^0[X_p] = -2\mathbf{D}_j (\partial_\eta \Psi + \mathcal{H} \Phi) + \frac{1}{2} (\mathbf{D}^2 + 2K) \mathbf{B}_j. \quad (118)$$

We now use (31) to express the right side of this equation in terms of  $\mathbf{A}$  and  $\mathbf{B}$ , which yields

$$\mathbf{G}_i^0[X_p] = -2\mathbf{D}_i (\mathcal{H} \mathbf{A} + (\mathcal{A}_G - K) \mathbf{B}) + \frac{1}{2} (\mathbf{D}^2 + 2K) \mathbf{B}_i. \quad (119)$$

## The Weyl tensor

The perturbation of the Weyl tensor is automatically gauge-invariant on account of the Stewart-Walker lemma (Stewart and Walker (1974)) since the Weyl tensor is zero in the background. We thus use bold-face notation for its components. From (113c) we obtain

$$\mathbf{C}^{0i}_{0j} = a^{2(1)} C^{0i}_{0j} = \frac{1}{2} (\hat{\mathbf{R}}_{0j}^{0i} - \hat{\mathbf{R}}_{jm}^{im}), \quad \mathbf{C}^{0i}_{jk} = a^{2(1)} C^{0i}_{jk} = \hat{\mathbf{R}}_{jk}^{0i}. \quad (120)$$

The Weyl tensor has a simpler form if we use Poisson gauge invariants and hence we choose  $X = X_p$  in (109). Noting that  $\mathbf{B}[X_p] = 0$  leads to

$$\mathbf{C}^{0i}_{0j} = -\frac{1}{2} [\mathbf{D}_j^i (\Psi + \Phi) + \partial_\eta \mathbf{B}_j^i - (\partial_\eta^2 + \mathbf{D}^2 - 2K) \mathbf{C}_j^i], \quad (121a)$$

$$\mathbf{C}^{0i}_{jk} = -2\mathbf{D}_{[j} (\mathbf{B}_{k]}^i - \partial_\eta \mathbf{C}_{k]}^i) - \mathbf{D}_m \mathbf{B}_{[j}^m \delta_{k]}^i, \quad \mathbf{B}_{ij} := \mathbf{D}_{[i} \mathbf{B}_{j]}. \quad (121b)$$

<sup>42</sup>Here for convenience we use  $\hat{\mathbf{G}}_{ij} = \gamma_{ik} \hat{\mathbf{G}}^k_j$ .

## B.1 Uniqueness of the decomposition into modes

**Proposition:** If the inverses of the operators  $\mathbf{D}^2$ ,  $\mathbf{D}^2 + 2K$  and  $\mathbf{D}^2 + 3K$  exist, then the equation

$$B_i = \mathbf{D}_i B + \tilde{B}_i, \quad \text{with} \quad \mathbf{D}^i \tilde{B}_i = 0, \quad (122)$$

determines  $B$  and  $\tilde{B}_i$  uniquely in terms of  $B_i$ , and the equation

$$C_{ij} = \mathbf{D}_{ij} C + \mathbf{D}_{(i} C_{j)} + \tilde{C}_{ij}, \quad (123)$$

with

$$\mathbf{D}^i C_i = 0, \quad \tilde{C}_{ij} = \tilde{C}_{ji}, \quad \tilde{C}_i^i = 0, \quad \mathbf{D}^i \tilde{C}_{ij} = 0,$$

determines  $C$ ,  $C_i$  and  $\tilde{C}_{ij}$  uniquely in terms of  $C_{ij}$ . In particular, if  $B_i = 0$  then  $B = 0$ ,  $\tilde{B}_i = 0$ , and if  $C_{ij} = 0$  then  $C = 0$ ,  $C_i = 0$ ,  $\tilde{C}_{ij} = 0$ .

*Proof.* Apply  $\mathbf{D}^i$  to (122) obtaining  $\mathbf{D}^i B_i = \mathbf{D}^2 B$ . Using the inverse operator of  $\mathbf{D}^2$  this equation determines  $B$ , and then (122) determines  $\tilde{B}_i$  uniquely in terms of  $B_i$ . Next, apply  $\mathbf{D}^{ij}$  and  $\mathbf{D}^i$  to (123), obtaining

$$\mathbf{D}^{ij} C_{ij} = \frac{2}{3} \mathbf{D}^2 (\mathbf{D}^2 + 3K) C, \quad \mathbf{D}^i C_{ij} = \frac{2}{3} \mathbf{D}_j (\mathbf{D}^2 + 3K) C + (\mathbf{D}^2 + 2K) C_j. \quad (124)$$

By using the inverse operators these equations, in conjunction with (123), successively determine  $C$ ,  $C_i$  and  $\tilde{C}_{ij}$  uniquely in terms of  $C_{ij}$ .  $\square$

## B.2 Identities

In obtaining our results we found the following identities useful:

$$\mathbf{D}_{[i} \mathbf{D}_{j]} A^k = K \delta_{[i}^k A_{j]}, \quad (125a)$$

$$\mathbf{D}_{[k} \mathbf{D}_{m]} A^{ij} = 2K \delta_{[k}^{(i} A_{m]}^{j)}, \quad (125b)$$

$$4(\mathbf{D}_{[k} \mathbf{D}^{[i} + K \delta_{[k}^{[i} \delta_{m]}^{m]}] A = (\mathbf{D}_k^i + \frac{4}{3} (\mathbf{D}^2 + 3K) \delta_k^i) A, \quad (125c)$$

$$4(\mathbf{D}_{[k} \mathbf{D}^{[i} + K \delta_{[k}^{[i} C_{j]}^{j]}] = (\mathbf{D}^2 - 2K) C_k^i, \quad (125d)$$

$$\mathbf{D}_j \mathbf{D}_i^j A = \frac{2}{3} \mathbf{D}_i (\mathbf{D}^2 + 3K) A, \quad (125e)$$

$$\mathbf{D}^i \mathbf{D}_{(i} A_{j)} = \frac{1}{2} (\mathbf{D}^2 + 2K) A_j, \quad (125f)$$

$$\mathbf{D}_i \mathbf{D}^2 A^i = (\mathbf{D}^2 + 2K) \mathbf{D}_i A^i, \quad (125g)$$

$$\delta_{[i}^{[i} A_{m]}^{j]} = \frac{1}{4} (A_m^j + \delta_m^j A), \quad (125h)$$

where  $A_{ij} = A_{ji}$ ,  $C_{ij} = C_{ji}$ ,  $C_i^i = 0$  and  $\mathbf{D}_i C_j^i = 0$ .

## B.3 Kinematic quantities

The kinematic quantities associated with a timelike congruence  $u^a$  are defined by the following decomposition into irreducible parts:

$$\nabla_a u_b = -u_a \dot{u}_b + H(g_{ab} + u_a u_b) + \sigma_{ab} + \omega_{ab}. \quad (126)$$

A routine calculation starting with equations (61)-(63) and (88) applied to  $u_a$  yields the following non-zero components:

$$a^{(1)}H = \left[ \frac{1}{3} \mathbf{D}^2(v - \chi) - (\partial_\eta \psi + \mathcal{H}\varphi) \right], \quad (127a)$$

$$\dot{\mathbf{u}}_i := {}^{(1)}\dot{u}_i = \mathbf{D}_i(\varphi + (\partial_\eta + \mathcal{H})v) + (\partial_\eta + \mathcal{H})\tilde{\mathbf{v}}_i, \quad (127b)$$

$$\boldsymbol{\sigma}_j^i := a^{(1)}\sigma_j^i = \mathbf{D}_j^i(v - \chi) + \gamma^{ik}\mathbf{D}_{(k}(\tilde{\mathbf{v}}_j) - \mathbf{B}_j) + \partial_\eta \mathbf{C}_j^i, \quad (127c)$$

$$\boldsymbol{\omega}_j^i := a^{(1)}\omega_j^i = \gamma^{ik}\mathbf{D}_{[k}\tilde{\mathbf{v}}_j], \quad (127d)$$

with the bold-face quantities being gauge-invariant on account of the Stewart-Walker lemma.

## C Scalar field

A minimally coupled scalar field  $\phi$  is described by a stress-energy tensor of the form

$$T_b^a = \nabla^a \phi \nabla_b \phi - \left[ \frac{1}{2} \nabla^c \phi \nabla_c \phi + U(\phi) \right] \delta_b^a, \quad (128)$$

with the associated Klein-Gordon equation  $\nabla^c \nabla_c \phi - U_{,\phi} = 0$ , where the potential  $U(\phi)$  has to be specified. This stress-energy tensor is of the form (59) with

$$\rho + p = -\nabla^a \phi \nabla_a \phi, \quad \rho - p = 2U(\phi), \quad \pi_{ab} = 0. \quad (129)$$

When evaluated on the RW background, equation (129) leads to

$$a^2({}^{(0)}\rho + {}^{(0)}p) = ({}^{(0)}\phi')^2, \quad ({}^{(0)}\rho - {}^{(0)}p) = 2U({}^{(0)}\phi). \quad (130)$$

On using (130) to calculate  ${}^{(0)}\rho'$ , the conservation equation (42) leads to

$${}^{(0)}\phi'' + 2\mathcal{H}({}^{(0)}\phi') + a^2 U_{,\phi} = 0, \quad (131)$$

which is the Klein-Gordon equation in the RW background. Further, by means of (41), (42), (130) and (131) we obtain

$$\mathcal{A}_T = \frac{1}{2}({}^{(0)}\phi')^2, \quad \mathcal{C}_T^2 = 1 + \frac{2a^2 U_{,\phi}}{3\mathcal{H}({}^{(0)}\phi')} = -\frac{1}{3} \left( 1 + \frac{2({}^{(0)}\phi'')}{\mathcal{H}({}^{(0)}\phi')} \right). \quad (132)$$

Viewing  $T_b^a$  and  $\phi$  as functions of the perturbation parameter  $\epsilon$ , we can use (128), in conjunction with (3), to calculate  ${}^{(1)}T_b^a$ , obtaining

$${}^{(1)}\hat{T}_j^i = 0, \quad a^2 {}^{(1)}T_i^0 = -{}^{(0)}\phi' \mathbf{D}_i({}^{(1)}\phi), \quad ({}^{(1)}T_0^0 + \frac{1}{3}({}^{(1)}T_i^i) = -2U_{,\phi}({}^{(1)}\phi). \quad (133)$$

It follows using (37) with  $A$  replaced by  $T$  and (132), that the matter gauge invariants assume the form

$$\hat{\mathbf{T}}_j^i = 0, \quad \mathbf{T}_i^0[X] = -{}^{(0)}\phi' \mathbf{D}_i \phi[X], \quad \mathbf{T}_0^0[X] + \frac{1}{3}\mathbf{T}_i^i[X] = -2a^2 U_{,\phi} \phi[X], \quad (134)$$

where  $\phi[X]$  is the gauge invariant associated with  ${}^{(1)}\phi$  by X-replacement, given by

$$\phi[X] = ({}^{(1)}\phi) - ({}^{(0)}\phi') X^0. \quad (135)$$

Equations (134) and (49) immediately lead to the expressions for the matter gauge invariants (80b) and (80c), including

$$V[X] = -\frac{1}{2}{}^{(0)}\phi' \phi[X]. \quad (136)$$

Equation (134), in conjunction with (132) and (136), yields

$$\mathbf{T}^0_0[X] + \frac{1}{3}\mathbf{T}^i_i[X] = -6(1 - \mathcal{C}_T^2)\mathcal{H}V[X]. \quad (137)$$

We now substitute (137) into the expression for  $\Gamma$  given by (45c) and (49c) to obtain<sup>43</sup>

$$\Gamma = (1 - \mathcal{C}_T^2)(-\mathbf{T}^0_0[X] - 6\mathcal{H}V[X]). \quad (138)$$

which on comparison with (67) leads to equation (80a).

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<sup>43</sup>Write the expression for  $\Gamma$  in the form  $\Gamma = -(1 - \mathcal{C}_T^2)\mathbf{T}^0_0 + (\mathbf{T}^0_0 + \frac{1}{3}\mathbf{T}^i_i)$ .

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# COSMOLOGICAL PERTURBATION THEORY REVISITED

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## Abstract

Increasingly accurate observations are driving theoretical cosmology toward the use of more sophisticated descriptions of matter and the study of nonlinear perturbations of FL cosmologies, whose governing equations are notoriously complicated. Our goal in this paper is to formulate the governing equations for linear perturbation theory in a particularly simple and concise form in order to facilitate the extension to nonlinear perturbations. Our approach has several novel features. We show that the use of so-called *intrinsic gauge invariants* has two advantages. It naturally leads to: (i) a physically motivated choice of a gauge invariant associated with the matter density, and (ii) two distinct and complementary ways of formulating the evolution equations for scalar perturbations, associated with the work of Bardeen and of Kodama and Sasaki. In the first case the perturbed Einstein tensor gives rise to a *second order* (in time) linear differential operator, and in the second case to a pair of coupled *first order* (in time) linear differential operators. These operators are of fundamental importance in cosmological perturbation theory, since they provide the leading order terms in the governing equations for nonlinear perturbations.

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# 1 Introduction

The analysis of linear perturbations of Friedmann-Lemaitre (FL) cosmologies was initiated by Lifshitz (1946) in a paper of far-reaching importance. Working in the so-called synchronous gauge, this paper showed that an arbitrary linear perturbation can be written as the sum of three modes, a scalar mode that describes perturbations in the matter density, a vector mode that describes vorticity and a tensor mode that describes gravitational waves. For many years, however, the theory was plagued by gauge problems, *i.e.* by the fact that the behaviour of the scalar mode depends significantly on the choice of gauge. A major step in alleviating this difficulty was taken by Bardeen (1980), who reformulated the linearized Einstein field equations in terms of a set of *gauge-invariant* variables, as an alternative to the traditional use of the synchronous gauge. Central to Bardeen's paper are two gauge-invariant equations that govern the behaviour of scalar perturbations. The first of these governs the evolution in time of a gauge-invariant gravitational (*i.e.* metric) potential and the second determines a gauge-invariant perturbation of the matter density in terms of the spatial Laplacian of the gravitational potential. Since this potential continues to play a central role in the study of scalar perturbations, it seems appropriate to refer to it as the *Bardeen potential*. Bardeen's paper makes clear, however, that there is no unique way of constructing gauge-invariant variables.

From our perspective, one drawback of Bardeen's paper is that he performs a harmonic decomposition of the variables *ab initio*, with the result that the mathematical structure of the governing equations is somewhat obscured. In a subsequent paper, Brandenberger, Khan and Press (1983) address this deficiency by giving a new derivation of Bardeen's gauge-invariant equations. They do not perform a harmonic decomposition, with the result that their evolution equation is a partial differential equation rather than an ordinary differential equation as in Bardeen's paper. However, unlike Bardeen they restrict consideration to a spatially flat Robertson-Walker (RW) background.<sup>1</sup>

In subsequent developments the status of the Bardeen potential was further enhanced by the appearance of the major review paper by Mukhanov *et al* (1992), which contains a simplified derivation of the Bardeen potential and the evolution equation for scalar perturbations, without performing a harmonic decomposition. However, the treatment in Mukhanov *et al* (1992) is less general than that of Bardeen (1980) and Brandenberger *et al* (1983) in two respects. First, they assume the anisotropic stresses are zero, and second, they make a specific choice of gauge invariants *a priori*, namely those associated with the so-called longitudinal gauge.

Currently, increasingly accurate observations are driving theoretical cosmology towards more sophisticated models of matter and the study of possible nonlinear deviations from FL cosmology. Motivated by this state of affairs, our long term goal is to provide a general but concise description of nonlinear perturbations of FL cosmologies that will reveal the mathematical structure of the governing equations and enable one to make the transition between different gauge-invariant formulations,

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<sup>1</sup>In our nomenclature an FL cosmology is a RW geometry that satisfies Einstein's field equations.

thereby simplifying and relating the different approaches<sup>2</sup> that have been used to date. In pursuing this objective we have found it necessary to revisit linear perturbation theory, even though it is by now a mature discipline. Our intent in the present paper is to formulate the governing equations for the linear theory in a particularly simple and concise form in order to facilitate the extension to nonlinear perturbations.

Based on earlier work by Bruni *et al* (1997) on gauge-invariant higher order perturbation theory, Nakamura (2003) introduced a geometrical method for constructing gauge invariants for linear and nonlinear (second order) perturbations which he later applied to derive the governing equations (see Nakamura (2006) and Nakamura (2007)). In the present paper we use Nakamura's method for constructing gauge invariants, but we complement it with the observation that gauge invariants are of two distinct types: *intrinsic gauge invariants*, i.e., gauge invariants that can be constructed from a given tensor alone, and *hybrid gauge invariants*, i.e. gauge invariants that are constructed from more than one tensor.

In Nakamura's approach, the linear perturbation of any tensor is written as the sum of a gauge-invariant quantity and a gauge-variant quantity, which is the Lie derivative of the zero order tensor with respect to a suitably chosen vector field  $X$ . A choice of  $X$  yields a set of gauge-invariant variables that are associated with a specific fully fixed gauge. We will show that for the metric tensor there exist two natural complementary choices of  $X$  that yield intrinsic metric gauge invariants. One choice, used in all of Nakamura's papers, leads to the two gauge-invariant metric potentials of Bardeen (1980), which are associated with the so-called *Poisson gauge*.<sup>3</sup> The other choice leads to the two gauge-invariant metric potentials of Kodama and Sasaki (1984), which are associated with the so-called *uniform curvature gauge*.<sup>4</sup> We will show that these two preferred choices lead to two distinct ways in which to present the linearized Einstein field equations: with the Bardeen choice the evolution of linear scalar perturbations is governed by a *second order* (in time) linear partial differential operator, while with the Kodama-Sasaki choice the evolution is governed by two coupled temporal *first order* linear operators.

The plan of the paper is as follows. In Section 2 we discuss the geometrical construction of gauge-invariants: we focus on the metric tensor and, with the Einstein tensor and the stress-energy tensor in mind, on mixed rank two tensors. In Section 3 we use intrinsic gauge invariants to derive the general governing equations for linear perturbations in two gauge-invariant forms associated with the Poisson and the uniform curvature gauges. The required expressions for the Einstein gauge invariants are derived efficiently in Appendix B. One of the ingredients in our derivation is the so-called Replacement Principle, which is formulated in Appendix A. Another ingredient is a general formula that expresses the Riemann gauge invariants in terms of the metric gauge invariants. In Section 4 we give an interpretation of the intrinsic matter gauge invariants and specialize our equations to the cases of a perfect fluid and a scalar field. Section 5 contains a brief discussion of future developments.

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<sup>2</sup>See, for example, Noh and Hwang (2004), Nakamura (2007) and Malik (2007).

<sup>3</sup>The Poisson gauge, which was introduced by Bertschinger (1996) (see his equation (4.46)), is a generalization of the longitudinal gauge, which only applies to scalar perturbations.

<sup>4</sup>See, for example, Malik and Wands (2009), page 20, and other references given there.

## 2 Geometrical definition of gauge invariants

### 2.1 General formulation

In cosmological perturbation theory we consider a 1-parameter family of metrics<sup>5</sup>  $g_{ab}(\epsilon)$ , where  $g_{ab}(0)$ , the unperturbed metric, is a RW metric, and  $\epsilon$  is referred to as the *perturbation parameter*. It is natural to assign physical dimension *length* to the scale factor  $a$  of the RW metric and  $(length)^2$  to  $g_{ab}$ . Hence the conformal transformation

$$g_{ab}(\epsilon) = a^2 \bar{g}_{ab}(\epsilon) \quad (1)$$

yields a dimensionless conformal metric  $\bar{g}_{ab}(\epsilon)$ .

The Riemann tensor associated with the metric  $g_{ab}(\epsilon)$  is a function of  $\epsilon$ , denoted  $R^{ab}_{cd}(\epsilon)$ , as is the Einstein tensor,  $G^a_b(\epsilon)$ . The stress-energy tensor of the matter distribution is also assumed to be a function of  $\epsilon$ , denoted  $T^a_b(\epsilon)$ . We include all these possibilities by considering a 1-parameter family of tensor fields  $A(\epsilon)$ , which we assume can be expanded in powers of  $\epsilon$ , *i.e.* as a Taylor series:

$$A(\epsilon) = {}^{(0)}A + \epsilon {}^{(1)}A + \frac{1}{2}\epsilon^2 {}^{(2)}A + \dots \quad (2)$$

The coefficients are given by<sup>6</sup>

$${}^{(0)}A = A(0), \quad {}^{(1)}A = \left. \frac{\partial A}{\partial \epsilon} \right|_{\epsilon=0}, \quad {}^{(2)}A = \left. \frac{\partial^2 A}{\partial \epsilon^2} \right|_{\epsilon=0}, \quad \dots, \quad (3)$$

where  ${}^{(0)}A$  is called the *unperturbed value*,  ${}^{(1)}A$  is called the *first order (linear) perturbation* and  ${}^{(2)}A$  is called the *second order perturbation* of  $A(\epsilon)$ .

Each of the geometric objects we work with has a well-defined physical dimension. We will use the scale-factor  $a$  of the unperturbed RW metric to define dimensionless quantities, as in (1), and to write the equations in dimensionless form. We thus consider a family of tensor fields  $A(\epsilon)$  such that  $a^n A(\epsilon)$  is dimensionless for some integer  $n$ .

The primary difficulty in cosmological perturbation theory is that the perturbations of a tensor field  $A(\epsilon)$  depend on the choice of gauge (*i.e.* on the choice of coordinates), and hence cannot be directly related to observations. The first goal is thus to formulate the theory in terms of *gauge-invariant* quantities, *i.e.* to replace the gauge-variant perturbations  ${}^{(1)}A, {}^{(2)}A, \dots$  of  $A(\epsilon)$  by gauge-invariant quantities. In this paper we restrict our attention to first order, *i.e.* linear, perturbations, but with a view to subsequently working with higher order perturbations, we use a method pioneered by Nakamura (2003).

Given a family of tensor fields  $A(\epsilon)$  such that  $a^n A(\epsilon)$  is dimensionless, the change induced in the first order perturbation  ${}^{(1)}A$  by a gauge transformation generated by a dimensionless vector field  $\xi(\epsilon)$  with expansion

$$\xi^a(\epsilon) = \epsilon {}^{(1)}\xi^a + \dots, \quad (4)$$

<sup>5</sup>We use Latin letters  $a, b, \dots, f$  to denote abstract spacetime indices.

<sup>6</sup>The notation  $A(\epsilon)$  should be viewed as shorthand for  $A(x, \epsilon)$ , indicating that the tensor fields are functions of the spacetime coordinates, which necessitates the use of partial differentiation with respect to  $\epsilon$ .

can be expressed using the Lie derivative  $\mathcal{L}$ :

$$\Delta^{(1)}A = \mathcal{L}_{(1)\xi}^{(0)}A, \quad (5)$$

(see, for example, Bruni *et al* (1997), equation (1.2)). We now introduce an as yet arbitrary dimensionless vector field  $X$  which we use to define the dimensionless object

$${}^{(1)}\mathbf{A}[X] := a^n ({}^{(1)}A - \mathcal{L}_X^{(0)}A). \quad (6)$$

It follows from (5) and (6) that

$$\Delta^{(1)}\mathbf{A}[X] = a^n (\mathcal{L}_{(1)\xi}^{(0)}A - \mathcal{L}_{\Delta X}^{(0)}A) = a^n \mathcal{L}_{(1)\xi - \Delta X}^{(0)}A. \quad (7)$$

The key step is to choose an  $X$  that satisfies

$$\Delta X^a = (1)\xi^a, \quad (8)$$

under a gauge transformation. With this choice, (7) implies that  $\Delta^{(1)}\mathbf{A}[X] = 0$ , *i.e.*,  ${}^{(1)}\mathbf{A}[X]$  is gauge-invariant. We say that  ${}^{(1)}\mathbf{A}[X]$  is the *gauge invariant associated with  ${}^{(1)}A$  by  $X$ -compensation*. Equations (5), (6) and (8) are central to Nakamura's method for constructing gauge invariants associated with the first order perturbation of a tensor  $A$  (see Nakamura (2007), equations (2.19), (2.23) and (2.26)). In what follows we will drop the superscript  $(1)$  on  $\mathbf{A}$  and  $\xi$  for convenience since in this paper we are dealing only with first order perturbations.

The above vector field  $X$ , which we shall refer to as the *gauge field*, requires comment. Unlike the geometric and matter tensor fields such as  $g_{ab}(\epsilon)$  and  $T^a_b(\epsilon)$  it is not the perturbation of a corresponding quantity on the background spacetime. Instead it should be viewed as a vector field on the background spacetime that is constructed from the linear perturbations of the geometric and matter tensors in such a way that (8) holds. We will construct specific examples of  $X$  in section 2.2.

Before continuing we briefly digress to point out that gauge invariants  $\mathbf{A}$  associated with a tensor  $A$  are of two distinct types. If  $\mathbf{A}$  is constructed solely from components of  ${}^{(1)}A$  and  ${}^{(0)}A$  then we call  $\mathbf{A}$  an *intrinsic* gauge invariant, while if  $\mathbf{A}$  also depends on the components of another perturbed tensor, then we call  $\mathbf{A}$  a *hybrid* gauge invariant. In particular if the gauge field  $X$  is formed solely from components of  ${}^{(1)}A$  and  ${}^{(0)}A$ , then  $\mathbf{A}[X]$  is an intrinsic gauge invariant; otherwise,  $\mathbf{A}[X]$  is a hybrid gauge invariant.

In the following sections we will calculate the quantities in equations (5) and (6) for various geometric objects  $A$ . To do this it is necessary to use the well known formulae for the Lie derivative. The formula for a tensor of type (1,1), which we now give, establishes the pattern:

$$\mathcal{L}_\xi A^a_b = A^a_{b,c}\xi^c + \xi^c_{,b}A^a_c - \xi^a_{,c}A^c_b, \quad (9)$$

where  $\cdot$  denotes partial differentiation. In a formula such as (9) one can replace the partial derivatives by covariant derivatives. For our purposes it is convenient to use the covariant derivative  ${}^0\bar{\nabla}_a$  associated with the *unperturbed conformal metric*  $\bar{g}_{ab}(0)$ :

$$\mathcal{L}_\xi A^a_b = ({}^0\bar{\nabla}_c A^a_b)\xi^c + ({}^0\bar{\nabla}_b \xi^c)A^a_c - ({}^0\bar{\nabla}_c \xi^a)A^c_b. \quad (10)$$

We also need to work in a coordinate frame so that we can calculate time and spatial components separately. We thus introduce local coordinates<sup>7</sup>  $x^\mu = (\eta, x^i)$ , with  $\eta$  being the usual conformal time coordinate for the RW metric  $g_{ab}(0)$ , and such that the unperturbed conformal metric  $\gamma_{ab} := \bar{g}_{ab}(0)$  has components

$$\gamma_{00} = -1, \quad \gamma_{0i} = 0, \quad \gamma_{ij}, \quad (11)$$

where  $\gamma_{ij}$  is the metric of a spatial geometry of constant curvature. The spacetime covariant derivative  ${}^0\nabla_a$  determines a temporal derivative  ${}^0\nabla_0 A = \partial_\eta A$ , where  $\partial_\eta$  denotes partial differentiation with respect to  $\eta$ , and a spatial covariant derivative  ${}^0\nabla_i$  that is associated with the spatial metric  $\gamma_{ij}$ . We introduce the notation

$$\mathbf{D}_i A := {}^0\nabla_i A \quad (12)$$

The derivative operators  $\partial_\eta$  and  $\mathbf{D}_i$  will be used throughout this paper once local coordinates have been introduced. The scale factor  $a$  determines the *dimensionless Hubble scalar*  $\mathcal{H}$  according to

$$\mathcal{H} = \frac{a'}{a} = aH, \quad (13)$$

where  $H$  is the true Hubble scalar. Here and elsewhere in this paper  $'$  denotes the derivative with respect to  $\eta$  of a function that depends only on  $\eta$ .

## 2.2 Metric gauge invariants

We expand  $\bar{g}_{ab}(\epsilon)$ , defined by equation (1), in powers of  $\epsilon$ :

$$\bar{g}_{ab}(\epsilon) = {}^{(0)}\bar{g}_{ab} + \epsilon {}^{(1)}\bar{g}_{ab} + \dots,$$

and label the unperturbed metric and (linear) metric perturbation according to

$$\gamma_{ab} := {}^{(0)}\bar{g}_{ab} = \bar{g}_{ab}(0), \quad f_{ab} := {}^{(1)}\bar{g}_{ab} = \frac{\partial \bar{g}_{ab}}{\partial \epsilon}(0), \quad (14)$$

which is consistent with (3). Applying the general transformation law (5) to the metric tensor  $g_{ab}(\epsilon) = a^2 \bar{g}_{ab}(\epsilon)$  we obtain

$$\Delta^{(1)}g_{ab} = \mathcal{L}_{(1)\xi} {}^{(0)}g_{ab}, \quad \text{or, equivalently,} \quad \Delta f_{ab} = a^{-2} \mathcal{L}_\xi (a^2 \gamma_{ab}), \quad (15)$$

in terms of the notation (14). The gauge invariant  $\mathbf{f}_{ab}[X]$  associated with the metric perturbation  $f_{ab}$  by  $X$ -compensation, given by (6), assumes the form

$$\mathbf{f}_{ab}[X] = f_{ab} - a^{-2} \mathcal{L}_X (a^2 \gamma_{ab}). \quad (16)$$

Introducing local coordinates and using (9) and (10) adapted to a (0,2) tensor, equations (15) and (16) lead to

$$\Delta f_{00} = -2(\partial_\eta + \mathcal{H})\xi^0, \quad \mathbf{f}_{00}[X] = f_{00} + 2(\partial_\eta + \mathcal{H})X^0, \quad (17a)$$

$$\Delta f_{0i} = -\mathbf{D}_i \xi^0 + \partial_\eta \xi_i, \quad \mathbf{f}_{0i}[X] = f_{0i} + \mathbf{D}_i X^0 - \partial_\eta X_i, \quad (17b)$$

$$\Delta f_{ij} = 2\mathcal{H} \xi^0 \gamma_{ij} + 2\mathbf{D}_{(i} \xi_{j)}, \quad \mathbf{f}_{ij}[X] = f_{ij} - 2\mathcal{H} X^0 \gamma_{ij} - 2\mathbf{D}_{(i} X_{j)}. \quad (17c)$$

---

<sup>7</sup>We use Greek letters to denote spacetime coordinate indices on the few occasions that they occur, and we use Latin letters  $i, j, k, m$  to denote spatial coordinate indices, which are lowered and raised using  $\gamma_{ij}$  and its inverse  $\gamma^{ij}$ , respectively.

In order to construct a gauge field  $X$  that satisfies (8), *using only the metric*, we need to decompose the metric perturbation  $f_{ab}$  into scalar, vector and tensor modes.<sup>8</sup> We introduce the notation<sup>9</sup>

$$f_{00} = -2\varphi, \quad (18a)$$

$$f_{0i} = \mathbf{D}_i B + B_i, \quad (18b)$$

$$f_{ij} = -2\psi\gamma_{ij} + 2\mathbf{D}_i \mathbf{D}_j C + 2\mathbf{D}_{(i} C_{j)} + 2C_{ij}, \quad (18c)$$

where the vectors  $B_i$  and  $C_i$  and the tensor  $C_{ij}$  satisfy

$$\mathbf{D}^i B_i = 0, \quad \mathbf{D}^i C_i = 0, \quad C^i_i = 0, \quad \mathbf{D}^i C_{ij} = 0.$$

The vector  $\xi$  is also decomposed into a scalar mode and a vector mode with components

$$\xi^0, \quad \xi^i = \mathbf{D}^i \xi + \tilde{\xi}^i. \quad (19)$$

It follows from (17), (18) and (19) that

$$\Delta\varphi = (\partial_\eta + \mathcal{H})\xi^0, \quad \Delta B = \xi^0 + \partial_\eta \xi, \quad \Delta C = \xi, \quad \Delta\psi = -\mathcal{H}\xi^0, \quad (20a)$$

$$\Delta B_i = \partial_\eta \tilde{\xi}_i, \quad \Delta C_i = \tilde{\xi}_i, \quad (20b)$$

$$\Delta C_{ij} = 0. \quad (20c)$$

We can draw two immediate conclusions. First, it follows from (20b) and (20c) that  $B_i - C'_i$  and  $C_{ij}$  are gauge invariants. We introduce the following bold-face notation:

$$\mathbf{B}_i := B_i - \partial_\eta C_i, \quad \mathbf{C}_{ij} := C_{ij}. \quad (21)$$

Second, by inspection of (19), (20a) and (20b) we obtain

$$\Delta(\mathbf{D}_i C + C_i) = \xi_i, \quad \Delta\chi = \Delta\left(\frac{\psi}{\mathcal{H}}\right) = -\xi^0, \quad (22)$$

where we have introduced the notation

$$\chi := B - \partial_\eta C. \quad (23)$$

We are now in a position to satisfy the requirement (8). Firstly, referring to (22), we can satisfy the *spatial part*  $\Delta X^i = \xi^i$  of the requirement by choosing

$$X_i = \mathbf{D}_i C + C_i, \quad (24)$$

---

<sup>8</sup>In order to guarantee that the functions  $B, B_i, C, C_i$  and  $C_{ij}$  in (18) are uniquely determined by  $f_{0i}$  and  $f_{ij}$  we need to assume that the inverses of  $\mathbf{D}^2$ ,  $\mathbf{D}^2 + 2K$  and  $\mathbf{D}^2 + 3K$  exist. See the proposition in Appendix B.1. See also Nakamura (2007), following equation (4.15), for a helpful discussion of this matter.

<sup>9</sup>We are denoting the scalar mode functions by  $\varphi, B, C$  and  $\psi$ , in agreement with Mukhanov *et al* (1992) (see equation (2.10), but note the different signature) and Malik and Wands (2009) (see equations (2.7)-(2.12)), with the difference that we use  $C$  instead of  $E$ . Bardeen (1980) used the notation  $A, -B, H_T$  and  $-H_L + \frac{1}{3}\mathbf{D}^2 H_T$  for these functions, the choice of the fourth one being motivated by harmonic decomposition. Bardeen's notation has been used by subsequent authors, *e.g.* Kodama and Sasaki (1984) and Durrer (1994), although the latter author replaced  $-B$  by  $B$ .

which we will take to be our default choice for  $X_i$ . With this choice, the expressions (17) for the components of the gauge invariant  $\mathbf{f}_{ab}[X]$ , when combined with (18), assume the form

$$\mathbf{f}_{00}[X] = -2\Phi[X], \quad (25a)$$

$$\mathbf{f}_{0i}[X] = \mathbf{D}_i\mathbf{B}[X] + \mathbf{B}_i, \quad (25b)$$

$$\mathbf{f}_{ij}[X] = -2\Psi[X]\gamma_{ij} + 2\mathbf{C}_{ij}. \quad (25c)$$

where

$$\Phi[X] := \varphi - (\partial_\eta + \mathcal{H})X^0, \quad \Psi[X] := \psi + \mathcal{H}X^0, \quad \mathbf{B}[X] := \chi + X^0, \quad (25d)$$

and  $\mathbf{B}_i$ ,  $\mathbf{C}_{ij}$  and  $\chi$  are given by (21) and (23), respectively.

Secondly, referring to (22), we can satisfy the *timelike part*  $\Delta X^0 = \xi^0$  of the requirement (8) in two obvious ways, by choosing

$$X^0 = X_{\text{P}}^0 := -\chi, \quad \text{or} \quad X^0 = X_{\text{C}}^0 := -\frac{\psi}{\mathcal{H}}, \quad (26)$$

which leads to the metric gauge invariants associated with the *Poisson gauge*, or the *uniform curvature gauge*, respectively. On substituting these choices into (25d) we obtain the conditions

$$\mathbf{B}[X_{\text{P}}] = 0 \quad \text{and} \quad \Psi[X_{\text{C}}] = 0, \quad (27)$$

which characterize these two gauge choices.

### The Poisson gauge invariants

On substituting the first of equations (26) into (25) we obtain

$$\mathbf{f}_{00}[X_{\text{P}}] := -2\Phi, \quad \mathbf{f}_{0i}[X_{\text{P}}] := \mathbf{B}_i, \quad \mathbf{f}_{ij}[X_{\text{P}}] := -2\Psi\gamma_{ij} + 2\mathbf{C}_{ij}, \quad (28)$$

where

$$\Phi := \Phi[X_{\text{P}}] = \varphi + (\partial_\eta + \mathcal{H})\chi, \quad \Psi := \Psi[X_{\text{P}}] = \psi - \mathcal{H}\chi. \quad (29)$$

Here  $\Phi$  and  $\Psi$  are the scalar metric gauge invariants associated with the Poisson gauge,<sup>10</sup> and  $\Psi$  is the Bardeen potential.

### The uniform curvature gauge invariants

On substituting the second of equations (26) into (25) we obtain

$$\mathbf{f}_{00}[X_{\text{C}}] = -2\mathbf{A}, \quad \mathbf{f}_{0i}[X_{\text{C}}] = \mathbf{D}_i\mathbf{B} + \mathbf{B}_i, \quad \mathbf{f}_{ij}[X_{\text{C}}] = 2\mathbf{C}_{ij}, \quad (30)$$

where

$$\mathbf{A} := \Phi[X_{\text{C}}] = \varphi + (\partial_\eta + \mathcal{H})\frac{\psi}{\mathcal{H}}, \quad \mathbf{B} := \mathbf{B}[X_{\text{C}}] = \chi - \frac{\psi}{\mathcal{H}}. \quad (31)$$

Here  $\mathbf{A}$  and  $\mathbf{B}$  are the scalar metric gauge invariants associated with the uniform curvature gauge,<sup>11</sup> introduced by Kodama and Sasaki (1984).<sup>12</sup>

<sup>10</sup>The gauge-fixing conditions for the Poisson gauge are  $B = C = 0, C_i = 0$  in (18).

<sup>11</sup>The gauge-fixing conditions for the uniform curvature gauge are  $\psi = C = 0, C_i = 0$  in (18).

<sup>12</sup>See equations (3.4) and (3.5), noting that  $H_L + n^{-1}H_T \equiv -\psi$  and  $B - k^{-1}H_T' \equiv \chi$ .

In concluding this section we note that the gauge fields  $X$  used to construct the above gauge invariants have the same spatial components  $X^i$  given by (24) in both cases, leading to (25), with the vector and tensor modes described by the gauge invariants  $\mathbf{B}_i$  and  $\mathbf{C}_{ij}$ , respectively. The difference lies in the scalar metric gauge invariants which are related according to<sup>13</sup>

$$\mathbf{A} = \Phi + (\partial_\eta + \mathcal{H}) \frac{\Psi}{\mathcal{H}}, \quad \mathbf{B} = -\frac{\Psi}{\mathcal{H}}, \quad (32)$$

as follows from (29) and (31). In both cases the gauge invariants are *intrinsic* since the gauge field  $X$  depends only on the metric.

A reader of this paper should be aware of the lack of agreement in the literature on labelling the scalar metric gauge invariants associated with the Poisson gauge. Our choice of  $(\Phi, \Psi)$  in (29) is the one initiated by Mukhanov *et al* (1992), and subsequently used by Nakamura (see, for example, Nakamura (2006)) and Malik and Wands (2009). On the other hand Durrer (2008) and Liddle and Lyth (2000) reverse the roles and use  $(\Psi, \Phi)$ , while Kodama and Sasaki (1984) use  $(\Psi, -\Phi)$ . Bardeen's original notation is  $(\Phi_A, -\Phi_H)$ .

### 2.3 Gauge invariants for mixed rank 2 tensors

In this subsection we consider a rank two tensor  $A^a_b$ , such that  $A_{ab}$  is symmetric and  $a^2 A^a_b$  is dimensionless. We expand  $A^a_b$  in a Taylor series in  $\epsilon$  as in (2), and assume that  ${}^{(0)}A^a_b$  obeys the background symmetries, which means it is spatially homogeneous and isotropic:

$$\mathbf{D}_i {}^{(0)}A^\alpha_\beta = 0, \quad {}^{(0)}A^0_i = {}^{(0)}A^i_0 = 0, \quad {}^{(0)}A^i_j = \frac{1}{3} \delta^i_j {}^{(0)}A^k_k. \quad (33)$$

We introduce the notation

$$\mathcal{A}_A := \frac{1}{2} a^2 (-{}^{(0)}A^0_0 + \frac{1}{3} {}^{(0)}A^k_k), \quad \mathcal{C}_A^2 := -\frac{({}^{(0)}A^k_k)'}{3({}^{(0)}A^0_0)'}, \quad (34)$$

where as before  $'$  denotes differentiation with respect to  $\eta$ . We further assume that  $A^a_b$  satisfies the conservation law  $\nabla_a A^a_b = 0$ . It follows that in the background

$$a^2 ({}^{(0)}A^0_0)' = 3a^2 \mathcal{H} (-{}^{(0)}A^0_0 + \frac{1}{3} {}^{(0)}A^k_k) = 6\mathcal{H}\mathcal{A}_A, \quad (35)$$

which, in conjunction with (34), implies that

$$\mathcal{A}'_A = -(1 + 3\mathcal{C}_A^2) \mathcal{H}\mathcal{A}_A. \quad (36)$$

We can now calculate the gauge invariants  $\mathbf{A}^a_b[X]$  associated with  ${}^{(1)}A^a_b$  by  $X$ -compensation, as defined by equation (6) with  $n = 2$ . It is convenient to decompose  ${}^{(1)}A^i_j$  into its trace  ${}^{(1)}A^k_k$  and tracefree part defined by

$${}^{(1)}\hat{A}^i_j := {}^{(1)}A^i_j - \frac{1}{3} {}^{(1)}A^k_k \delta^i_j. \quad (37)$$

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<sup>13</sup>These relation have recently been given by Christopherson *et al* (2011). See their equations (4.22) and (4.23).

A straightforward calculation using (6), (9), (10) and (33) leads to<sup>14</sup>

$$\mathbf{A}_0^0[X] = a^2 ({}^1A_0^0 - 6\mathcal{H}\mathcal{A}_A X^0) \quad (38a)$$

$$\mathbf{A}_i^0[X] = a^2 ({}^1A_i^0 + 2\mathcal{A}_A \mathbf{D}_i X^0), \quad (38b)$$

$$\mathbf{A}_k^k[X] = a^2 ({}^1A_k^k + 18\mathcal{H}\mathcal{A}_A \mathcal{C}_A^2 X^0), \quad (38c)$$

$$\hat{\mathbf{A}}_j^i[X] = a^2 ({}^1\hat{A}_j^i). \quad (38d)$$

In deriving these equations we have used (34) and (35) to express  $({}^0A_0^0)$ ,  $({}^0A_k^k)$  and their derivatives in terms of  $\mathcal{A}_A$  and  $\mathcal{C}_A^2$ .

Equation (38d) implies that  $\hat{\mathbf{A}}_j^i[X]$  is an *intrinsic* gauge invariant since it is constructed solely from the components of  $({}^1A_b^a)$ . We denote this quantity by

$$\hat{\mathbf{A}}_j^i := \hat{\mathbf{A}}_j^i[X] = a^2 ({}^1\hat{A}_j^i). \quad (39)$$

One can form two additional intrinsic gauge invariants by taking suitable combinations of  $\mathbf{A}_0^0[X]$ ,  $\mathbf{A}_i^0[X]$  and  $\mathbf{A}_k^k[X]$ . Indeed it follows from (38) that

$$\mathbf{A} := \mathcal{C}_A^2 \mathbf{A}_0^0[X] + \frac{1}{3} \mathbf{A}_k^k[X] = a^2 (\mathcal{C}_A^2 ({}^1A_0^0) + \frac{1}{3} ({}^1A_k^k)), \quad (40a)$$

$$\mathbf{A}_i := -(\mathbf{D}_i \mathbf{A}_0^0[X] + 3\mathcal{H} \mathbf{A}_i^0[X]) = -a^2 (\mathbf{D}_i ({}^1A_0^0) + 3\mathcal{H} ({}^1A_i^0)), \quad (40b)$$

which implies that  $\mathbf{A}$  and  $\mathbf{A}_i$  are *intrinsic gauge-invariants*.

In summary, the tensor  $A_b^a$  can be described by the three intrinsic gauge invariants  $\hat{\mathbf{A}}_j^i$ ,  $\mathbf{A}$ , and  $\mathbf{A}_i$ , given by (39), (40a) and (40b), and one hybrid gauge invariant  $\mathbf{A}_i^0[X]$ , given by (38b). In section 3.1 we will use these objects, constructed in terms of the Einstein tensor and the stress-energy tensor, to give a concise derivation of the governing equations in gauge-invariant form for linear perturbations of FL.

## 3 Linearized governing equations

### 3.1 General formulation

In this section we work with the linear perturbations of the Einstein tensor and the stress-energy tensor, denoted by  $({}^1G_b^a)$  and  $({}^1T_b^a)$ , and defined via equation (3). The corresponding unperturbed quantities are labelled by a superscript  $({}^0)$ .

We begin by imposing the background Einstein equations  $({}^0G_b^a) = ({}^0T_b^a)$ . The non-zero components are given by<sup>15</sup>

$$a^2 ({}^0G_0^0) = -3(\mathcal{H}^2 + K) = -a^2 ({}^0\rho) = a^2 ({}^0T_0^0), \quad (41a)$$

$$a^2 ({}^0G_j^i) = -(2\mathcal{H}' + \mathcal{H}^2 + K)\delta_j^i = a^2 ({}^0p)\delta_j^i = a^2 ({}^0T_j^i), \quad (41b)$$

where  $\mathcal{H}$  is given by (13). It follows from (41), (34) and (35), with  $A$  replaced by  $G$  and  $T$ , respectively, that

$$\mathcal{A}_G = -\mathcal{H}' + \mathcal{H}^2 + K, \quad \mathcal{A}_T = \frac{1}{2} a^2 ({}^0\rho + ({}^0p)), \quad (42a)$$

$$\mathcal{A}'_G = -(1 + 3\mathcal{C}_G^2)\mathcal{H}\mathcal{A}_G, \quad \mathcal{C}_T^2 = \frac{({}^0p)'}{({}^0\rho)'}. \quad (42b)$$

<sup>14</sup>We do not include the  $({}^1A_b^a)$  components since they can be expressed in terms of the other components and the metric perturbation, due to the assumed symmetry.

<sup>15</sup>See, for example, Mukhanov *et al* (1992), equation (4.2), noting the difference in signature.

The conservation law (35), with  $A$  replaced by  $T$ , gives

$$a^2({}^{(0)}\rho)' = -6\mathcal{H}\mathcal{A}_T = -3\mathcal{H}a^2({}^{(0)}\rho + {}^{(0)}p). \quad (43)$$

The background Einstein equations imply that  $\mathcal{A}_G = \mathcal{A}_T$  and  $\mathcal{C}_G^2 = \mathcal{C}_T^2$ . We denote the common values by  $\mathcal{A}$  and  $\mathcal{C}^2$ :

$$\mathcal{A} = \mathcal{A}_G = \mathcal{A}_T, \quad \mathcal{C}^2 = \mathcal{C}_G^2 = \mathcal{C}_T^2. \quad (44)$$

The linearized Einstein field equations are given by

$${}^{(1)}G_b^a = {}^{(1)}T_b^a. \quad (45)$$

In simplifying the linearized field equations we will make use of the intrinsic gauge invariants associated with the Einstein tensor and with the stress-energy tensor, which are given, in analogy with (39), (40a) and (40b), by

$$\hat{\mathbf{G}}_j^i = a^2 {}^{(1)}\hat{G}_j^i, \quad \hat{\mathbf{T}}_j^i = a^2 {}^{(1)}\hat{T}_j^i \quad (46a)$$

$$\mathbf{G}_i = -a^2 (\mathbf{D}_i {}^{(1)}G_0^0 + 3\mathcal{H} {}^{(1)}G_i^0), \quad \mathbf{T}_i = -a^2 (\mathbf{D}_i {}^{(1)}T_0^0 + 3\mathcal{H} {}^{(1)}T_i^0), \quad (46b)$$

$$\mathbf{G} = a^2 (\mathcal{C}_G^2 {}^{(1)}G_0^0 + \frac{1}{3} {}^{(1)}G_k^k), \quad \mathbf{T} = a^2 (\mathcal{C}_T^2 {}^{(1)}T_0^0 + \frac{1}{3} {}^{(1)}T_k^k), \quad (46c)$$

where

$${}^{(1)}\hat{G}_j^i = {}^{(1)}G_j^i - \frac{1}{3}\delta_j^i {}^{(1)}G_k^k, \quad {}^{(1)}\hat{T}_j^i = {}^{(1)}T_j^i - \frac{1}{3}\delta_j^i {}^{(1)}T_k^k. \quad (47)$$

We also need the hybrid gauge invariants  $\mathbf{G}_i^0[X]$  and  $\mathbf{T}_i^0[X]$ , which are given by (38b) with  $A$  replaced by  $G$  and  $T$ :

$$\mathbf{G}_i^0[X] = a^2 {}^{(1)}G_i^0 + 2\mathcal{A}_G \mathbf{D}_i X^0, \quad \mathbf{T}_i^0[X] = a^2 {}^{(1)}T_i^0 + 2\mathcal{A}_T \mathbf{D}_i X^0. \quad (48)$$

Since the gauge invariants (46) and (48) are linear in  ${}^{(1)}G_b^a$  and  ${}^{(1)}T_b^a$  with coefficients depending on  ${}^{(0)}G_b^a$  and  ${}^{(0)}T_b^a$ , respectively, it follows that the linearized Einstein field equations immediately imply the following relations:

$$\hat{\mathbf{G}}_j^i - \hat{\mathbf{T}}_j^i = 0, \quad \mathbf{G}_i - \mathbf{T}_i = 0, \quad \mathbf{G} - \mathbf{T} = 0, \quad (49a)$$

$$\mathbf{G}_i^0[X] - \mathbf{T}_i^0[X] = 0. \quad (49b)$$

Expressions for the Einstein gauge invariants  $\hat{\mathbf{G}}_j^i$ ,  $\mathbf{G}_i$ ,  $\mathbf{G}$  and  $\mathbf{G}_i^0[X]$  in terms of the metric gauge invariants, decomposed into scalar, vector, and tensor modes, are given in equations (116) and (120) in Appendix B. To proceed we likewise decompose the matter gauge invariants  $\hat{\mathbf{T}}_j^i$ ,  $\mathbf{T}_i$ ,  $\mathbf{T}$  and  $\mathbf{T}_i^0[X]$  into scalar, vector, and tensor modes and label them as follows:<sup>16</sup>

$$\hat{\mathbf{T}}_j^i = \mathbf{D}_j^i \Pi + 2\gamma^{ik} \mathbf{D}_{(k} \Pi_{j)} + \Pi_j^i, \quad (50a)$$

$$\mathbf{T}_i = \mathbf{D}_i \Delta + \Delta_i, \quad (50b)$$

$$\mathbf{T} = \Gamma, \quad (50c)$$

$$\mathbf{T}_i^0[X] = 2(\mathbf{D}_i V[X] + V_i), \quad (50d)$$

<sup>16</sup>In subsection 4.1 we comment on the choice of the symbols  $\Pi$ ,  $\Gamma$ ,  $\Delta$  and  $V$ .

where

$$\mathbf{D}^i \Pi_i = 0, \quad \Pi_k^k = 0, \quad \mathbf{D}_i \Pi_j^i = 0, \quad \mathbf{D}^i \Delta_i = 0, \quad \mathbf{D}^i V_i = 0, \quad (50e)$$

and

$$\mathbf{D}_{ij} := \mathbf{D}_{(i} \mathbf{D}_{j)} - \frac{1}{3} \gamma_{ij} \mathbf{D}^2, \quad \mathbf{D}^2 := \mathbf{D}^i \mathbf{D}_i. \quad (50f)$$

We stress that in making this decomposition we are not making any assumptions about the physical nature of the stress-energy tensor. By inspecting (116), (120) and (50) one concludes that equations (49) decompose into a scalar mode, a vector mode and a tensor mode, which we label as follows:

$$\begin{aligned} \mathbf{D}_{ij} \mathbb{A} + \mathbf{D}_{(i} \mathbb{A}_{j)} + \mathbb{A}_{ij} &= 0, \\ \mathbf{D}_i \mathbb{B} + \mathbb{B}_i &= 0, \\ \mathbb{C} &= 0, \\ \mathbf{D}_i \mathbb{E}[X] + \mathbb{E}_i &= 0. \end{aligned}$$

Since we are assuming that the inverses of the operators  $\mathbf{D}^2, \mathbf{D}^2 + 2K$  and  $\mathbf{D}^2 + 3K$  exist we can use the proposition in Appendix B.1 to write the linearized field equations concisely as

$$\textit{Scalar mode: } \mathbb{A} = 0, \quad \mathbb{B} = 0, \quad \mathbb{C} = 0, \quad \mathbb{E}[X] = 0. \quad (51a)$$

$$\textit{Vector mode: } \mathbb{A}_i = 0, \quad \mathbb{B}_i = 0, \quad \mathbb{E}_i = 0. \quad (51b)$$

$$\textit{Tensor mode: } \mathbb{A}_{ij} = 0. \quad (51c)$$

## 3.2 Scalar mode

In this subsection we give the governing equations (51a) for the scalar mode, first expressing them in terms of the uniform curvature gauge invariants  $\mathbf{A} = \Phi[X_C]$  and  $\mathbf{B} = \mathbf{B}[X_C]$  (see (31)). The scalars  $\mathbb{A}, \mathbb{B}$  and  $\mathbb{C}$  in (51a) are obtained without any calculation by taking the differences of equations (116) and (50) and reading off the scalar part. The scalar  $\mathbb{E}[X]$  is obtained in a similar manner from (120) and (50d) with  $X = X_P$ . The resulting equations are<sup>17</sup>

$$(\partial_\eta + 2\mathcal{H}) \mathbf{B} + \mathbf{A} = -\Pi \quad (52a)$$

$$\mathcal{H} [(\partial_\eta + \mathcal{B}\mathcal{H}) \mathbf{A} + \mathcal{C}^2 \mathbf{D}^2 \mathbf{B}] = \frac{1}{2} \Gamma + \frac{1}{3} \mathbf{D}^2 \Pi, \quad (52b)$$

$$\mathcal{H} (\mathbf{D}^2 + 3K) \mathbf{B} = -\frac{1}{2} \Delta, \quad (52c)$$

$$\mathcal{H} \mathbf{A} + (\mathcal{A} - K) \mathbf{B} = -V, \quad (52d)$$

where

$$\mathcal{B} = \frac{2\mathcal{H}'}{\mathcal{H}^2} + 1 + 3\mathcal{C}^2, \quad (53)$$

(see equation (117) in Appendix B), and  $V = V[X_P]$ . We shall refer to these equations as the *uniform curvature form* of the governing equations for the scalar mode.

<sup>17</sup>In deriving (52b) we use (52a) to replace  $(\partial_\eta + 2\mathcal{H}) \mathbf{B} + \mathbf{A}$  by  $-\Pi$ .

We now give the governing equations in terms for the Poisson gauge invariants  $\Psi$  and  $\Phi$ . We eliminate  $\mathbf{A}$  in (52b) using (52a) and in (52d) using (32), and eliminate  $\mathbf{B}$  using  $\mathcal{H}\mathbf{B} = -\Psi$ . The resulting equations are

$$\Psi - \Phi = \Pi, \quad (54a)$$

$$(\mathcal{L} - \mathcal{C}^2 \mathbf{D}^2) \Psi = \frac{1}{2} \Gamma + \left( \frac{1}{3} \mathbf{D}^2 + \mathcal{H}(\partial_\eta + \mathcal{B}\mathcal{H}) \right) \Pi, \quad (54b)$$

$$(\mathbf{D}^2 + 3K) \Psi = \frac{1}{2} \Delta, \quad (54c)$$

$$\partial_\eta \Psi + \mathcal{H}\Phi = -V, \quad (54d)$$

where the differential operator  $\mathcal{L}$  is defined by

$$\mathcal{L}(\bullet) := \mathcal{H}(\partial_\eta + \mathcal{B}\mathcal{H})(\partial_\eta + 2\mathcal{H}) \left( \frac{\bullet}{\mathcal{H}} \right). \quad (55)$$

Expanding the brackets yields<sup>18</sup>

$$\mathcal{L} = \partial_\eta^2 + 3(1 + \mathcal{C}^2) \mathcal{H} \partial_\eta + \mathcal{H}^2 \mathcal{B} - (1 + 3\mathcal{C}^2) K. \quad (56)$$

We shall refer to the above field equations as the *Poisson form* of the governing equations for the scalar mode.

Equations (52) and (54) constitute one of the main results of this paper. Either system of equations determine the behaviour of linear scalar perturbations of an FL cosmology with arbitrary stress-energy content whose scalar mode is described by the gauge invariants  $\Gamma, \Pi, \Delta$  and  $V$ . The structure of these two systems of equations differs in one significant way. In the system (52) the time dependence is governed by two *first order* differential operators, while in the system (54) the time dependence is governed by a *second order* linear differential operator. Note that all the coefficients in equations (52) and equations (54) are functions of  $\mathcal{H}, \mathcal{H}', \mathcal{H}''$  and  $K$ , *i.e.* they depend on the background geometry, which is determined by the the assumed stress-energy content and the field equations in the background FL model.

To the best of our knowledge equations (52) have not been given in the literature, although if one performs a harmonic decomposition one obtains a system of first order ordinary differential equations closely related to that given by Kodama and Sasaki (1984) (see Chapter 2, equations (4.6a-d)). Likewise, the governing equations in Poisson form have not appeared in the literature in the above fully general form, although the operator  $\mathcal{L}$ , given by (56), has a lengthy history as we now describe. This operator is ubiquitous<sup>19</sup> in the theory of scalar perturbations of FL universes, but is never given in the above purely geometric form, with its coefficients depending only on the background RW geometry. The form in the literature that is closest to the above is that given by Mukhanov *et al* (1992), equation (5.22), who replace  $\mathcal{C}^2$  by the matter quantity  $c_s^2$ , using (44) and (79). Nakamura (2007) gives the same expression (see his equation (5.30)). A more common form of  $\mathcal{L}$  in the literature has

<sup>18</sup>Referring to (42) express  $\mathcal{H}'$  in terms of  $\mathcal{A}_G$  and then use the equation for  $\mathcal{A}'_G$ .

<sup>19</sup>The operator  $\mathcal{L}$  appears in papers that use the Bardeen-Mukhanov gauge-invariant potentials, or that use the so-called longitudinal gauge, or that use the 1 + 3 gauge-invariant approach to perturbations of FL.

$\mathcal{B}$ , defined in equation (117), expressed in terms of the background matter variables<sup>20</sup> using the background field equations:

$$\mathcal{H}^2\mathcal{B} = (c_s^2 - w)\rho a^2 + (1 + c_s^2)\Lambda a^2 - (1 + 3c_s^2)K. \quad (57)$$

The earliest occurrence of which we are aware is Harrison (1967), equation (182), followed by Bardeen (1980), equation (5.30), after making the suitable changes of notation and setting  $\Lambda = 0$ . See also Ellis, Hwang and Bruni (1989), equation (31) and Hwang and Vishniac (1990), equation (105).<sup>21</sup>

### 3.3 Vector and tensor modes

First, we give the governing equations (51b) for the *vector mode*. The vectors  $\mathbb{A}_i$  and  $\mathbb{B}_i$  in (51b) are obtained without any calculation by taking the differences of equations (116) and (50) and reading off the vector part. The vector  $\mathbb{E}_i$  is obtained in a similar manner from (120) and (50d). The resulting equations are

$$(\partial_\eta + 2\mathcal{H})\mathbf{B}_i = -2\Pi_i, \quad (58a)$$

$$(\mathbf{D}^2 + 2K)\mathbf{B}_i = 4V_i, \quad (58b)$$

as well as the relation  $\Delta_i = 6\mathcal{H}V_i$ , which is satisfied identically (see equation (68)). If  $\Pi_i$  is specified and can be regarded as a source term, the evolution equation (58a) is a first order linear ordinary differential equation that determines  $\mathbf{B}_i$ , which in turn determines  $V_i$  by differentiation using (58b).

Second, we give the governing equations (51c) for the *tensor mode*. The tensor  $\mathbb{A}_{ij}$  in (51c) is obtained without any calculation by taking the differences of equations (116) and (50) and reading off the tensor part, leading to

$$(\partial_\eta^2 + 2\mathcal{H}\partial_\eta + 2K - \mathbf{D}^2)\mathbf{C}_{ij} = \Pi_{ij}. \quad (59)$$

If  $\Pi_{ij}$  is specified and can be regarded as source term, this is a second order linear partial differential equation that determines  $\mathbf{C}_{ij}$ .

## 4 Interpretations and examples

### 4.1 Interpretation of the matter gauge invariants

In this section we give the physical interpretation of the gauge invariants  $\Pi, \Gamma, \Delta$  and  $V[X]$  associated with the scalar mode of the stress-energy tensor.

We begin with the decomposition of a stress-energy tensor with respect to a unit timelike vector field  $u^a$ , which is given by

$$T^a_b = (\rho + p)u^a u_b + p\delta^a_b + (q^a u_b + u^a q_b) + \pi^a_b, \quad (60)$$

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<sup>20</sup>We are here assuming as matter content a barotropic perfect fluid with linear equation of state, and a cosmological constant.

<sup>21</sup>In these two references, the evolution equation in question arises in the 1 + 3 gauge-invariant approach to perturbations of FL, and the unknown is a vector quantity that is related to the scalar  $\Psi$ .

where

$$u^a q_b = 0, \quad \pi^a_a = 0, \quad u_a \pi^a_b = 0. \quad (61)$$

We choose  $u^a$  to be the timelike eigenvector of  $T^a_b$ , which implies  $q^a = 0$ , *i.e.* we are using the so-called energy frame (see for example, Bruni *et al* (1992), page 37).

Assuming that the unperturbed stress-energy tensor  ${}^{(0)}T^a_b$  has the isotropy and homogeneity properties of the RW geometry, the expansion (2) to linear order for  $\rho, p, u_a$  and  $\pi^a_b$  has the form:<sup>22</sup>

$$\rho = {}^{(0)}\rho + \epsilon {}^{(1)}\rho, \quad p = {}^{(0)}p + \epsilon {}^{(1)}p, \quad (62a)$$

$$\pi^0_0 = 0 = \pi^0_i, \quad \pi^i_j = 0 + \epsilon {}^{(1)}\pi^i_j, \quad (62b)$$

$$u_0 = -a(1 + \epsilon \varphi), \quad u_i = a(0 + \epsilon v_i). \quad (62c)$$

Decomposing  $v_i$  into a scalar and vector mode yields

$$v_i = \mathbf{D}_i v + \tilde{\mathbf{v}}_i, \quad \mathbf{D}^i \tilde{\mathbf{v}}_i = 0. \quad (63)$$

We use boldface in writing  $\tilde{\mathbf{v}}_i$  in view of the fact that this quantity is a dimensionless gauge invariant, as can be verified by applying (5) to  $u_a$ .

For ease of comparison with other work, we note that the expansion of  $u^a = g^{ab}u_b$  to linear order, expressed in terms of  $v, \tilde{\mathbf{v}}^i$  and the linearly perturbed metric, is given by

$$u^0 = a^{-1}(1 - \epsilon \varphi), \quad u^i = a^{-1} [0 + \epsilon (\mathbf{D}^i(v - B) + (\tilde{\mathbf{v}}^i - B^i))]. \quad (64)$$

We digress briefly to mention that our expansion of the four-velocity differs from the usual approach in the literature in that we use the *covariant* vector  $u_a$  to define the perturbed three-velocity instead of the contravariant vector  $u^a$ , since we find that this leads to a number of simplifications. For example, Malik and Wands (2009) (see equation (4.4)) have

$$u^i = a^{-1} [0 + \epsilon (\mathbf{D}^i v_{MW} + \tilde{v}^i_{MW})],$$

so that

$$v_{MW} = v - B, \quad \tilde{v}^i_{MW} = \tilde{\mathbf{v}}^i - B^i.$$

From (60) and (62), and making use of (3), we obtain the following expressions for the components of the linear perturbation of the stress-energy tensor:

$${}^{(1)}T^0_0 = -{}^{(1)}\rho, \quad {}^{(1)}T^k_k = 3 {}^{(1)}p, \quad {}^{(1)}T^0_i = ({}^{(0)}\rho + {}^{(0)}p)v_i, \quad {}^{(1)}\hat{T}^i_j = {}^{(1)}\pi^i_j. \quad (65)$$

It follows from (46), (50) and (65), in conjunction with (42) and (43), that the matter gauge invariants are determined by

$$a^2 {}^{(1)}\pi^i_j = \mathbf{D}^i_j \Pi + 2\gamma^{ik} \mathbf{D}_{(k} \Pi_{j)} + \Pi^i_j, \quad (66a)$$

$$\Gamma = a^2 (-\mathcal{C}_T^2 {}^{(1)}\rho + {}^{(1)}p), \quad (66b)$$

$$\Delta = a^2 ({}^{(1)}\rho + ({}^{(0)}\rho)' v), \quad (66c)$$

$$V[X] = \mathcal{A}_T(v + X^0), \quad V_i = \mathcal{A}_T \tilde{\mathbf{v}}_i. \quad (66d)$$

---

<sup>22</sup>The form of  $u_0$  is determined by the requirement that  $u^a$  is a unit vector. Recall that  $\varphi$  is one of the metric potentials in (18).

Before continuing we derive an additional relation. It follows from (40b) with  $A$  replaced by  $T$  that

$$\mathbf{T}_i = -\mathbf{D}_i \mathbf{T}_0^0[X] - 3\mathcal{H}\mathbf{T}_i^0[X]. \quad (67)$$

On substituting from (50b) and (50d) into this equation, we conclude that

$$\Delta = -\mathbf{T}_0^0[X] - 6\mathcal{H}V[X], \quad \Delta_i = -6\mathcal{H}V_i. \quad (68)$$

We can now give the physical interpretation of the matter gauge invariants. First, the gauge invariants  $\Pi, \Pi_i$  and  $\Pi_{ij}$  represent the anisotropic stresses. The interpretation of  $\Gamma$  is given in the context of a perfect fluid in the next section. Next, the gauge invariants  $V = V[X_P]$  and  $V_i$  play a role in determining the shear and vorticity of  $u_a$ . The relevant formulae are given in (128) in Appendix B.3. In particular,  $V[X_P]$  determines the scalar mode of the shear according to

$$\mathbf{D}_i^j \boldsymbol{\sigma}^i_j = \frac{2}{3} \mathcal{A}_T^{-1} \mathbf{D}^2 (\mathbf{D}^2 + 3K) V[X_P], \quad (69)$$

as follows from (128) in conjunction with (66d) with  $X = X_P$  and the identity (126e). We will hence use  $V := V[X_P]$  as our standard choice for the gauge invariant  $V[X]$ . However, since the choice  $V[X_C]$  is also of interest we note that

$$V[X_C] - V[X_P] = \mathcal{A}_T \mathbf{B}, \quad (70)$$

as follows from (66d), (26) and (31).

Finally, in order to interpret  $\Delta$  we need to make a small digression. For any scalar field  $A$  with the property that  $a^n A$  is dimensionless we can define a dimensionless gauge invariant  $\mathbf{A}[X]$  according to<sup>23</sup>

$$\mathbf{A}[X] = a^n \left( {}^{(1)}A - ({}^{(0)}A)' X^0 \right). \quad (71)$$

For the matter density  $\rho$  we denote the gauge invariant by  $\Delta[X]$ :

$$\Delta[X] = a^2 \left( {}^{(1)}\rho - ({}^{(0)}\rho)' X^0 \right). \quad (72)$$

On choosing  $X = X_v$  with  $X_v^0 := -v$  it follows from (66c) that  $\Delta = \Delta[X_v]$ . By comparing (72) with equation (3.13) in Bardeen (1980),<sup>24</sup> we conclude that  $\Delta[X_v]$ , and hence  $\Delta$ , equals the well-known Bardeen gauge-invariant density perturbation  $\epsilon_m$ , up to a factor of  $a^2 ({}^{(0)}\rho)$ . The specific relation is

$$\Delta = (a^2 ({}^{(0)}\rho)) \epsilon_m. \quad (73)$$

We note that the choice  $X_v^0 = -v$ , in conjunction with our default choice (24) for the spatial components of  $X$ , is associated with the so-called *total matter gauge* (see, for example, Malik and Wands (2009), pages 23-24). Thus  $\Delta$  is the density perturbation in the total matter gauge. In addition it turns out that  $\Delta$  is closely related to the 1 + 3 gauge-invariant approach to perturbations of FL, pioneered by

<sup>23</sup>This is equation (6) specialized to the case of a scalar field.

<sup>24</sup>One has to take into account differences in notation, the conservation equation (43), and the fact that Bardeen has performed a harmonic decomposition.

Ellis and collaborators (see Ellis *et al* (1989)), in which the spatial gradient of the matter density orthogonal to  $u^a$  plays a key role. To elucidate the relation we define the dimensionless spatial density gradient<sup>25</sup>

$$\mathcal{D}_a = a^2 h_a^b \nabla_b \rho, \quad h_a^b = \delta_a^b + u_a u^b. \quad (74)$$

A straight-forward calculation shows that to linear order

$$\mathcal{D}_0 = 0, \quad \mathcal{D}_i = \mathbf{D}_i \Delta - 6\mathcal{H}V_i, \quad (75)$$

which shows that  $\Delta$  equals the scalar mode of the spatial density gradient. In addition it follows from (50b) and (68) that  $\mathbf{T}_i = \mathcal{D}_i$ , giving a physical interpretation of the intrinsic gauge-invariant  $\mathbf{T}_i$ .

To end this section we comment on our choice of notation. In using the symbols  $\Pi, \Gamma, \Delta$  and  $V$  for the matter gauge invariants we are following Kodama and Sasaki (1984) with the difference that we scale the variables as follows:

$$\Pi = a^2 p \Pi_{KS}, \quad \Gamma = a^2 p \Gamma_{KS}, \quad \Delta = a^2 \rho \Delta_{KS}, \quad V = \mathcal{A}_T V_{KS}, \quad (76)$$

where  $p$  and  $\rho$  refer to the background. These scalings simplify the equations considerably.

## 4.2 Perfect fluid

For a perfect fluid the matter gauge invariants are restricted according to

$$\Pi = 0, \quad \Pi_i = 0, \quad \Pi_j^i = 0. \quad (77)$$

In addition it follows from (42b) and (66b) that

$$\Gamma = 0 \quad \text{if and only if} \quad p = p(\rho), \quad (78)$$

*i.e.* if and only if the equation of state is barotropic. In this case it is customary to introduce the notation

$$c_s^2 := \mathcal{C}_T^2, \quad w := \frac{{}^{(0)}p}{{}^{(0)}\rho}, \quad (79)$$

where  $c_s^2 = w$  if  $w$  is constant, as follows from (42b).

On account of (77) the governing equations in the Poisson form (54) for scalar perturbations imply that  $\Psi - \Phi = 0$ , which reduces the governing equations for the scalar mode in the perfect fluid case to

$$(\mathcal{L} - \mathcal{C}^2 \mathbf{D}^2) \Psi = \frac{1}{2} \Gamma, \quad (80a)$$

$$(\mathbf{D}^2 + 3K) \Psi = \frac{1}{2} \Delta, \quad (80b)$$

$$\Psi' + \mathcal{H} \Psi = -V, \quad (80c)$$

where  $\mathcal{L}$  is given by (56) with  $\mathcal{C}^2 = c_s^2$  and  $\mathcal{B}$  is expressed in terms of the background matter variables according to (57).

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<sup>25</sup>Our  $\mathcal{D}_a$  differs from that in Bruni, Dunsby and Ellis (1992) by a factor of  $\rho a^2$  (see their equation (24)).

### 4.3 Scalar field

For a minimally coupled scalar field we show in Appendix C that the matter gauge invariants are given by

$$\Gamma = (1 - \mathcal{C}_T^2)\Delta, \quad (81a)$$

$$V[X] = -\frac{1}{2}{}^{(0)}\phi' \phi[X], \quad V_i = 0, \quad (81b)$$

$$\Pi = 0, \quad \Pi_i = 0, \quad \Pi_j^i = 0, \quad (81c)$$

where  $\phi[X]$  is the gauge invariant associated with  ${}^{(1)}\phi$  by X-replacement, given by<sup>26</sup>

$$\phi[X] = {}^{(1)}\phi - {}^{(0)}\phi' X^0. \quad (82)$$

The governing equations (54) in Poisson form imply that  $\Psi - \Phi = 0$ , and then reduce to

$$(\mathcal{L} - \mathcal{C}^2 \mathbf{D}^2)\Psi = \frac{1}{2}(1 - \mathcal{C}^2)\Delta, \quad (83a)$$

$$(\mathbf{D}^2 + 3K)\Psi = \frac{1}{2}\Delta, \quad (83b)$$

$$\Psi' + \mathcal{H}\Psi = \frac{1}{2}{}^{(0)}\phi' \phi_P, \quad (83c)$$

where  $\phi_P := \phi[X_P]$ . By combining (83a) and (83b) we obtain an evolution equation for  $\Psi$  without a source term:

$$(\mathcal{L} - 3(1 - \mathcal{C}^2)K - \mathbf{D}^2)\Psi = 0, \quad (84)$$

where  $\mathcal{L}$  is given by (56). Having solved this equation one can calculate  $\phi_P$  and  $\Delta$  from (83). If one expresses  $\mathcal{C}^2$  in  $\mathcal{L}$  in terms of the unperturbed scalar field and its derivatives (see (133)) and sets  $K = 0$ , equation (84) coincides with equation (6.48) in Mukhanov *et al* (1992). For the generalization to arbitrary  $K$ , see Nakamura (2007), equation (5.39).<sup>27</sup>

One can also use the governing equations (52) in uniform curvature form, obtaining equations equivalent to those derived by Malik (2007) (see equations (2.20)-(2.23), noting that he is considering multiple scalar fields).

## 5 Discussion

We have given a systematic account of the gauge-invariant quantities that are associated with a linearly perturbed RW geometry and stress-energy tensor, emphasizing the role of intrinsic gauge invariants. First, we have shown that there are two distinct choices of intrinsic gauge invariants for the perturbed metric, which are the gauge invariants associated with the Poisson gauge and the uniform curvature gauge, through the work of Bardeen (1980) and Kodama and Sasaki (1984), respectively. Second, we have introduced intrinsic gauge invariants for the Einstein tensor and the stress-energy tensor, which we used to derive a particularly simple and concise form

<sup>26</sup>This is a special case of equation (71).

<sup>27</sup>We note a minor typo: a factor of 2 multiplying  $\partial_\eta^2$  should be deleted.

of the governing equations for linear perturbations of FL models. The specific form of the governing equations for the scalar mode depends on the choice of intrinsic gauge invariants for the perturbed metric. The Kodama-Sasaki choice leads to a coupled system of two *first order* (in time) linear differential operators that govern the evolution of the uniform curvature metric gauge invariants (see equations (52)). On going over to the Poisson picture, the product of these two operators<sup>28</sup> yields the well-known *second order* linear differential operator  $\mathcal{L}$  that governs the evolution of the Bardeen potential (see equation (56)), thereby providing the link between the two forms of the governing equations. A common feature of both systems is the appearance of the physically motivated gauge-invariant density perturbation  $\Delta$  that is one of the intrinsic gauge invariants associated with the stress-energy tensor (see equations (52c) and (54c)).

The mathematical structure of the governing equations for linear perturbations that we have elucidated here has in fact a much wider significance. Indeed, as one might expect on the basis of elementary perturbation theory, the governing equations for second order (nonlinear) perturbations have precisely the same form, apart from the inclusion of a source term that depends quadratically on the linear metric perturbation.<sup>29</sup> As an illustration of this we give the form of the equations that govern second order scalar perturbations using the metric gauge invariants associated with the Poisson gauge:

$${}^{(2)}\Psi - {}^{(2)}\Phi = {}^{(2)}\Pi + S_{aniso}({}^{(1)}\mathbf{f}), \quad (85a)$$

$$(\mathcal{L} - \mathcal{C}_G^2 \mathbf{D}^2) {}^{(2)}\Psi = \frac{1}{2} {}^{(2)}\Gamma + \left(\frac{1}{3} \mathbf{D}^2 + \mathcal{H}(\partial_\eta + \mathcal{B}\mathcal{H})\right) {}^{(2)}\Pi + S_{evol}({}^{(1)}\mathbf{f}), \quad (85b)$$

$$(\mathbf{D}^2 + 3K) {}^{(2)}\Psi = \frac{1}{2} {}^{(2)}\Delta + S_{matter}({}^{(1)}\mathbf{f}), \quad (85c)$$

$$\partial_\eta {}^{(2)}\Psi + \mathcal{H} {}^{(2)}\Phi = -{}^{(2)}V + S_{velocity}({}^{(1)}\mathbf{f}), \quad (85d)$$

where  $S_\bullet({}^{(1)}\mathbf{f})$  is a *source term* that depends quadratically on the first order gauge-invariant metric perturbation  ${}^{(1)}\mathbf{f}_{ab} \equiv \mathbf{f}_{ab}$  in equation (28). The key point is that, apart from the source terms, equations (85) have the same form as equations (54), with the variables  ${}^{(2)}\Psi$  and  ${}^{(2)}\Phi$  being the metric gauge invariants at second order determined by the Nakamura procedure. The second order matter terms  ${}^{(2)}\Pi$ ,  ${}^{(2)}\Gamma$ ,  ${}^{(2)}\Delta$  and  ${}^{(2)}V$  are defined in analogy with the first order terms  $\Pi$ ,  $\Gamma$ ,  $\Delta$  and  $V$  after expanding the stress-energy tensor  $T^a_b$  to second order in powers of  $\epsilon$ . All the complications lie in the source terms, whose explicit form has to be found by calculating the Riemann tensor to second order. In order to solve the above second order equations the source terms, which include scalar, vector and tensor modes, first have to be obtained by solving the governing equations for the scalar, vector and tensor linear perturbations. In a subsequent article we will derive both the above Poisson form and the corresponding uniform curvature form of the governing equations for second order perturbations, relating our formulation to other recent work.

<sup>28</sup>This factorization property (55) of  $\mathcal{L}_s$  appears to be new, and enables one to simplify a number of results relating to  $\mathcal{L}_s$ . We will discuss these elsewhere.

<sup>29</sup>This behaviour has been noted in general terms by Nakamura (2006), equations (38)-(39).

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## A The Replacement Principle

The expression for the perturbation of the Riemann tensor given in equation (104) in Appendix B, can be written symbolically in the form:

$$a^{2(1)}R^{ab}{}_{cd} = \mathbf{L}^{ab}{}_{cd}(f), \quad (86)$$

where  $\mathbf{L}^{ab}{}_{cd}$  is a linear operator and  $f$  is shorthand for  $f_{ab}$ . The Replacement Principle for the Riemann curvature states that the gauge invariants associated with  ${}^{(1)}R^{ab}{}_{cd}$  and with  $f_{ab}$  by  $X$ -compensation are related by *the same* linear operator:

$$\mathbf{R}^{ab}{}_{cd}[X] = \mathbf{L}^{ab}{}_{cd}(\mathbf{f}[X]), \quad (87)$$

where  $\mathbf{f}[X]$  is shorthand for  $\mathbf{f}_{ab}[X]$ .

This result is adapted from more general results given by Nakamura (2005) (see in particular, his equations (3.12), (3.15) and (3.23)). Similar results hold for the Einstein and Weyl tensors. Use of the Replacement Principle in Appendix B makes the transition from gauge-variant to gauge-invariant equations particularly easy and transparent.

## B Derivation of the curvature formula

In this appendix we derive expressions for the Einstein gauge invariants, namely, the three intrinsic gauge invariants  $\hat{\mathbf{G}}_j^i$ ,  $\mathbf{G}_i$  and  $\mathbf{G}$ , and the single hybrid gauge invariant  ${}^{(1)}\mathbf{G}_i^0[X]$ , defined by equations (46) and (48). Our strategy incorporates the following ideas:

- i) *Conformal structure.* We adapt to the conformal structure of the background geometry, determined by the scale factor  $a$  of the RW metric, from the outset. In particular we create dimensionless quantities by multiplying with appropriate powers of  $a$ , which simplifies the equations considerably.
- ii) *Index conventions.* We represent tensors of even rank, apart from the metric tensor, with equal numbers of covariant and contravariant indices. This make contractions trivial to perform and ensures that the components of the tensor have the same physical dimension as the associated contracted scalar.
- iii) *Timing of specialization.* We defer performing the decomposition into scalar, vector and tensor modes as long as possible, and do not make harmonic decompositions. This strategy helps to reveal structure in the equations and serves to reduce the amount of calculation.

**Calculation of  $R^{ab}_{cd}(\epsilon)$** 

We begin by deriving an exact expression for the Riemann tensor<sup>30</sup>  $R^{ab}_{cd}(\epsilon)$  of the metric  $g_{ab}(\epsilon)$  in terms of the covariant derivative of the conformal background metric  $\gamma_{ab}$ . We thus relate the covariant derivative of  $g_{ab}(\epsilon)$ , denoted  ${}^\epsilon\nabla_a$ , to that of  $\gamma_{ab} = \bar{g}_{ab}(0)$ , denoted  ${}^0\bar{\nabla}_a$ . The relation is given by an object  $Q^a_{bc} = Q^a_{cb}$  defined by

$$Q^a_{bc} = g^{ad}Q_{dbc} = \frac{1}{2}g^{ad} \left( {}^0\bar{\nabla}_c g_{db} - {}^0\bar{\nabla}_d g_{bc} + {}^0\bar{\nabla}_b g_{cd} \right), \quad (88)$$

(see Wald (1984) equation (D.1)), with the property that<sup>31</sup>

$${}^\epsilon\nabla_a A^b_c = {}^0\bar{\nabla}_a A^b_c + Q^b_{ad}A^d_c - Q^d_{ac}A^b_d. \quad (89)$$

It is convenient to write  $Q^a_{bc}$  as the sum of two parts:

$$Q^a_{bc}(\epsilon) = \bar{Q}^a_{bc}(\epsilon) + \tilde{Q}^a_{bc}(\epsilon). \quad (90)$$

First, the transformation from  ${}^\epsilon\nabla_a$  to  ${}^\epsilon\bar{\nabla}_a$ , which is associated with the conformal transformation  $g_{ab}(\epsilon) = a^2\bar{g}_{ab}(\epsilon)$ , is described by

$$\bar{Q}^a_{bc}(\epsilon) = 2\delta^a_{(b}r_{c)} - \bar{g}^{ad}(\epsilon)\bar{g}_{bc}(\epsilon)r_d, \quad (91)$$

where<sup>32</sup>

$$r_a := {}^0\bar{\nabla}_a(\ln a) \quad (92)$$

(see Wald (1984), equation (D.3)). It follows that  ${}^0\bar{\nabla}_a r_b = {}^0\bar{\nabla}_b r_a$ . Second, the transformation from  ${}^\epsilon\nabla_a$  to  ${}^0\bar{\nabla}_a$ , the covariant derivatives associated with  $\bar{g}_{ab}(\epsilon)$  and  $\bar{g}_{ab}(0)$ , respectively, is described by

$$\tilde{Q}^a_{bc}(\epsilon) = \frac{1}{2}\bar{g}^{ad}(\epsilon) \left( {}^0\bar{\nabla}_c \bar{g}_{db}(\epsilon) - {}^0\bar{\nabla}_d \bar{g}_{bc}(\epsilon) + {}^0\bar{\nabla}_b \bar{g}_{cd}(\epsilon) \right). \quad (93)$$

It follows from  ${}^0\bar{\nabla}_a \gamma_{bc} = 0$  that

$$\tilde{Q}^a_{bc}(0) = 0. \quad (94)$$

To calculate  $R^{ab}_{cd}(\epsilon)$  we first perform the conformal transformation from  $g_{ab}$  to  $\bar{g}_{ab}$ , which yields

$$a^2 R^{ab}_{cd}(\epsilon) = \bar{R}^{ab}_{cd}(\epsilon) + 4\delta^{[a}_{[c}\bar{U}^{b]}_{d]}(\epsilon), \quad (95)$$

where

$$\bar{U}^b_d(\epsilon) = - \left[ \bar{g}^{be} ({}^\epsilon\bar{\nabla}_d - r_d) + \frac{1}{2}\delta^b_d \bar{g}^{ef} r_f \right] r_e, \quad (96)$$

and  $\bar{R}^{ab}_{cd}(\epsilon)$  is the curvature tensor of the metric  $\bar{g}_{ab}(\epsilon)$ , (see Wald (1984), equation (D.7)). Second, by performing the transition from  ${}^\epsilon\nabla_a$  to  ${}^0\bar{\nabla}_a$  we obtain

$$\bar{R}^{ab}_{cd}(\epsilon) = \bar{g}^{be} \bar{R}^a_{ecd}(\epsilon) = \bar{g}^{be} \left( {}^0\bar{R}^a_{ecd} + 2{}^0\bar{\nabla}_{[c}\tilde{Q}^a_{d]e} + 2\tilde{Q}^a_{f[c}\tilde{Q}^f_{d]e} \right), \quad (97)$$

<sup>30</sup>We use the sign convention of Wald (1984) for defining the Riemann tensor.

<sup>31</sup>This example establishes the pattern for a general tensor.

<sup>32</sup>Note that we always use the vector  $r_a$  in covariant form, since  $r_a$  is independent of  $\epsilon$ , whereas  $r^a = g^{ab}(\epsilon)r_b$  is not.

where  ${}^0\bar{R}^a{}_{bcd}$  is the curvature tensor of the metric  $\gamma_{ab}$  (see Wald (1984), equation (D.7)). The term  $2\bar{g}^{be} {}^0\bar{\nabla}_{[c}\tilde{Q}^a{}_{d]e}$  in (97) can be written as<sup>33</sup>

$$2\bar{g}^{be} {}^0\bar{\nabla}_{[c}\tilde{Q}^a{}_{d]e} = 2\bar{g}^{be} ({}^0\bar{\nabla}_{[c}\bar{g}^{af}) \tilde{Q}_{|f|d]e} + \bar{g}^{be} \bar{g}^{af} ({}^0\bar{\nabla}_{[c} {}^0\bar{\nabla}_{|e|}\bar{g}_{d]f} - {}^0\bar{\nabla}_{[c} {}^0\bar{\nabla}_{|f|}\bar{g}_{d]e}) - \gamma_{ef}\bar{g}^{e(b} {}^0\bar{R}^{a)f}{}_{cd}, \quad (98)$$

which we use to rearrange (97), in conjunction with the relation  ${}^0\bar{\nabla}_c\bar{g}^{ab} = -2\tilde{Q}^{(ab)}{}_c$ . In summary,  $R^{ab}{}_{cd}(\epsilon)$  is given by equation (95) with

$$\bar{R}^{ab}{}_{cd}(\epsilon) = -2\bar{g}^{e[a}\bar{g}^{b]f} {}^0\bar{\nabla}_{[c} {}^0\bar{\nabla}_{|e|}\bar{g}_{d]f} - \gamma_{ef}\bar{g}^{e[a} {}^0\bar{R}^{b]f}{}_{cd} - 2\tilde{Q}^f{}_{[c} \tilde{Q}_{|f|}{}^{b]}{}_{d]}, \quad (99a)$$

$$\bar{U}^b{}_d(\epsilon) = -\left[\bar{g}^{be} ({}^0\bar{\nabla}_d - r_d) + \frac{1}{2}\delta^b{}_d \bar{g}^{ef} r_f - \bar{g}^{bf} \tilde{Q}^e{}_{df}\right] r_e, \quad (99b)$$

where we have used  ${}^\epsilon\bar{\nabla}_a r_b = {}^0\bar{\nabla}_a r_b - \tilde{Q}^c{}_{ab} r_c$  in obtaining (99b) from (96).

### Calculation of ${}^{(1)}R^{ab}{}_{cd}$

We now calculate the perturbation  ${}^{(1)}R^{ab}{}_{cd}$  of the Riemann tensor, defined via equation (3), expressing it in terms of the covariant derivative  ${}^0\bar{\nabla}_a$  associated with  $\gamma_{ab}$  and the metric perturbation  $f_{ab} = {}^{(1)}\bar{g}_{ab}$  (see (14)). We note that

$${}^{(1)}\bar{g}^{ab} = -f^{ab}, \quad (100)$$

where the indices on  $f^{ab}$  are raised using  $\gamma^{ab}$ . It follows from (3), (93) (95) and (99), in conjunction with (94) and (100), that<sup>34</sup>

$$a^2{}^{(1)}R^{ab}{}_{cd} = {}^{(1)}\bar{R}^{ab}{}_{cd} + 4\delta^{[a}{}_{[c} {}^{(1)}\bar{U}^{b]}{}_{d]}, \quad (101a)$$

where

$${}^{(1)}\bar{R}^{ab}{}_{cd} = -2 {}^0\bar{\nabla}_{[c} {}^0\bar{\nabla}^{[a} f_{d]}{}^{b]} + f_e^{[a} {}^0\bar{R}^{b]e}{}_{cd}, \quad (101b)$$

$${}^{(1)}\bar{U}^a{}_b = \left[f^{ac} ({}^0\bar{\nabla}_b - r_b) + \frac{1}{2}\delta^a{}_b f^{cd} r_d + \gamma^{ad} {}^{(1)}\tilde{Q}^c{}_{bd}\right] r_c, \quad (101c)$$

$${}^{(1)}\tilde{Q}^c{}_{abc} = \frac{1}{2} ({}^0\bar{\nabla}_c f_{ab} - {}^0\bar{\nabla}_a f_{bc} + {}^0\bar{\nabla}_b f_{ca}). \quad (101d)$$

Introducing local coordinates  $x^\mu = (\eta, x^i)$  as in section 2.1 leads to

$$r_\alpha = \mathcal{H} \delta^0{}_\alpha, \quad {}^0\bar{\nabla}_0 = \partial_\eta, \quad {}^0\bar{\nabla}_i = \mathbf{D}_i. \quad (102)$$

In addition we note that the quantity  ${}^0\bar{R}^a{}_{bcd}$ , the curvature tensor of the metric  $\gamma_{ab}$ , is zero if one index is temporal, while if all indices are spatial

$${}^0\bar{R}^{ij}{}_{km} = 2K \delta^{[i}{}_{[k} \delta^{j]}{}_{m]}, \quad (103)$$

where the constant  $K$  describes the curvature of the maximally symmetric three-space. Equation (101), in conjunction with (102) and (103), yields the following expressions:

$$a^2{}^{(1)}R^{0j}{}_{0m} = \frac{1}{2}[\mathbf{D}^j \mathbf{D}_m + (\mathcal{H}' - \mathcal{H}^2)\delta^j{}_m]f_{00} + (\partial_\eta + \mathcal{H})Y^j{}_m, \quad (104a)$$

$$a^2{}^{(1)}R^{0j}{}_{km} = 2\mathbf{D}_{[k} Y^j{}_{m]}, \quad (104b)$$

$$a^2{}^{(1)}R^{ij}{}_{km} = -2(\mathbf{D}_{[k} \mathbf{D}^{[i} + K \delta_{[k}^{[i} \delta^{j]}{}_{m]})f_m{}^{j]} + 4\mathcal{H}\delta_{[k}^{[i} Y_m{}^{j]}], \quad (104c)$$

<sup>33</sup>Note that  ${}^0\bar{R}^{ab}{}_{cd} = \gamma^{be} {}^0\bar{R}^a{}_{ecd}$ .

<sup>34</sup>Note that  $R^{ab}{}_{cd}(\epsilon)$  depends on  $\epsilon$  through  $\bar{g}_{ab}(\epsilon)$ ,  $\bar{g}^{ab}(\epsilon)$  and  $\tilde{Q}^c{}_{ab}(\epsilon)$ .

where<sup>35</sup>

$$Y_{ij} = \frac{1}{2}\gamma_{ij}\mathcal{H}f_{00} - \mathbf{D}_{(i}f_{j)0} + \frac{1}{2}\partial_\eta f_{ij}. \quad (104d)$$

### Calculation of the Riemann gauge invariants

We now apply the Replacement Principle to (104), which entails performing the following replacements:

$$f_{ab} \rightarrow \mathbf{f}_{ab}[X], \quad a^{2(1)}R^{ab}_{cd} \rightarrow \mathbf{R}^{ab}_{cd}[X], \quad Y_{ij} \rightarrow \mathbf{Y}_{ij}[X], \quad (105)$$

where the gauge invariants are defined by equation (6). All components of the Riemann tensor can be obtained from the ‘curvature spanning set’ ( $R^{0i}_{0j}$ ,  $R^{0i}_{jk}$ ,  $R^{im}_{jm}$ ) or, alternatively, their spatial traces and their trace-free parts:

$$(R^{0m}_{0m}, R^{0m}_{jm}, R^{km}_{km}), \quad (\hat{R}^{0i}_{0j}, \hat{R}^{0i}_{jk}, \hat{R}^{im}_{jm}), \quad (106)$$

where

$$\hat{R}^{0i}_{0j} = R^{0i}_{0j} - \frac{1}{3}\delta_j^i R^{0m}_{0m}, \quad \hat{R}^{im}_{jm} = R^{im}_{jm} - \frac{1}{3}\delta_j^i R^{km}_{km}, \quad (107a)$$

$$\hat{R}^{0i}_{jk} = R^{0i}_{jk} - \delta_{[k}^i R^{0m}_{j]m}. \quad (107b)$$

Our motivation for choosing these particular components as the spanning set is that the first set of terms in (106) are invariant under spatial gauge transformations, while the hatted quantities are fully gauge-invariant, as follows from (5).

We denote the gauge invariants associated with spanning set (106) by

$$(\mathbf{R}^{0m}_{0m}[X], \mathbf{R}^{0m}_{jm}[X], \mathbf{R}^{km}_{km}[X]), \quad (\hat{\mathbf{R}}^{0i}_{0j}, \hat{\mathbf{R}}^{0i}_{jk}, \hat{\mathbf{R}}^{im}_{jm}), \quad (108)$$

and refer to them as as the *Riemann gauge invariants*. As indicated by the notation (*i.e.* no dependence on the gauge field  $X$ ) the hatted quantities are intrinsic gauge invariants. We now substitute the expressions<sup>36</sup> for  $\mathbf{f}_{ab}[X]$  given by (25) into the bold-face version of (104), and calculate the gauge invariants (108). It is convenient to split  $\mathbf{Y}_{ij}$  into a trace and a trace-free part:

$$\hat{\mathbf{Y}}_{ij} = \mathbf{Y}_{ij} - \frac{1}{3}\gamma_{ij}\mathbf{Y}, \quad \mathbf{Y} = \mathbf{Y}^i_i, \quad (109)$$

and to use the trace-free second derivative operator  $\mathbf{D}_{ij}$  defined in (50f). We obtain<sup>37</sup>

$$\mathbf{R}^{0m}_{0m}[X] = -[\mathbf{D}^2 + 3(\mathcal{H}' - \mathcal{H}^2)]\Phi[X] + (\partial_\eta + \mathcal{H})\mathbf{Y}[X], \quad (110a)$$

$$\hat{\mathbf{R}}^{0i}_{0j} = -\mathbf{D}_j^i\Phi[X] + (\partial_\eta + \mathcal{H})\hat{\mathbf{Y}}^i_j[X], \quad (110b)$$

$$\mathbf{R}^{km}_{km}[X] = 4[(\mathbf{D}^2 + 3K)\Psi[X] + \mathcal{H}\mathbf{Y}[X]], \quad (110c)$$

$$\hat{\mathbf{R}}^{im}_{jm} = \mathbf{D}_j^i\Psi[X] + \mathcal{H}\hat{\mathbf{Y}}^i_j[X] - (\mathbf{D}^2 - 2K)\mathbf{C}^i_j, \quad (110d)$$

$$\mathbf{R}^{0m}_{jm}[X] = \frac{2}{3}\mathbf{D}_j\mathbf{Y}[X] - \mathbf{D}_m\hat{\mathbf{Y}}^m_j[X], \quad (110e)$$

$$\hat{\mathbf{R}}^{0i}_{jk} = 2\mathbf{D}_{[j}\hat{\mathbf{Y}}^i_{k]}[X] + \mathbf{D}_m\hat{\mathbf{Y}}^m_{[j}[X]\delta^i_{k]}, \quad (110f)$$

<sup>35</sup>Note that  $\tilde{Q}^{0i}_{ij} = -\mathbf{D}_{(i}f_{j)0} + \frac{1}{2}f'_{ij}$ .

<sup>36</sup>In using these expressions we are making the choice for  $X_i$  given in equation (24). Choosing  $X_i$  in this way simplifies the calculation but not the final form of the Riemann gauge invariants, since, as mentioned earlier, the spanning set is invariant under spatial gauge transformations.

<sup>37</sup>Use the identities (126c), (126d) and (126h).

where

$$\mathbf{Y}[X] = -3(\partial_\eta \Psi[X] + \mathcal{H}\Phi[X]) - \mathbf{D}^2 \mathbf{B}[X], \quad (110g)$$

$$\hat{\mathbf{Y}}_{ij}[X] = -\mathbf{D}_{ij} \mathbf{B}[X] - \mathbf{D}_{(i} \mathbf{B}_{j)} + \partial_\eta \mathbf{C}_{ij}. \quad (110h)$$

These equations constitute one of the main results of this paper. They express the Riemann gauge invariants (108) in terms of the metric gauge invariants (25). They depend only on the choice of the temporal gauge field  $X^0$ , as can be seen from (25d).

### Calculation of the Einstein gauge invariants

The Einstein tensor and the Weyl conformal curvature tensor are defined in terms of the Riemann tensor according to

$$G^a_b := R^a_b - \frac{1}{2} \delta^a_b R, \quad \text{where} \quad R^a_b := R^{ac}_{bc}, \quad R := R^a_b, \quad (111a)$$

$$C^{ab}_{cd} := R^{ab}_{cd} - 2 \delta^{[a}_{[c} R^{b]}_{d]} + \frac{1}{3} \delta^{[a}_{[c} \delta^{b]}_{d]} R. \quad (111b)$$

The curvature spanning set (106) can be replaced with the following spatially irreducible components of the Einstein tensor and the Weyl tensor:<sup>38</sup>

$$(G^0_0, G^m_m, G^0_i, \hat{G}^i_j), \quad (C^{0i}_{0j}, C^{0i}_{jk}), \quad (112)$$

where

$$\hat{G}^i_j := G^i_j - \frac{1}{3} \delta^i_j G^m_m. \quad (113)$$

It follows from (111) that

$$G^0_0 = -\frac{1}{2} R^{km}_{km}, \quad G^m_m = -\frac{1}{2} (R^{km}_{km} + 4R^{0m}_{0m}), \quad (114a)$$

$$G^0_i = R^{0m}_{im}, \quad \hat{G}^i_j = \hat{R}^{0i}_{0j} + \hat{R}^{im}_{jm}, \quad (114b)$$

$$C^{0i}_{0j} = \frac{1}{2} (\hat{R}^{0i}_{0j} - \hat{R}^{im}_{jm}), \quad C^{0i}_{jk} = \hat{R}^{0i}_{jk}. \quad (114c)$$

The Einstein gauge invariants, as defined by equations (39), (40a) and (40b) with  $A$  replaced by  $G$ , can be expressed in terms of the curvature spanning set (106) by using the bold-face version of (114). This yields

$$\hat{\mathbf{G}}^i_j := \hat{\mathbf{G}}^i_j[X] = \hat{\mathbf{R}}^{0i}_{0j} + \hat{\mathbf{R}}^{im}_{jm}, \quad (115a)$$

$$\mathbf{G}_i := -(\mathbf{D}_i \mathbf{G}^0_0[X] + 3\mathcal{H} \mathbf{G}^0_i[X]) = \frac{1}{2} \mathbf{D}_i \mathbf{R}^{km}_{km}[X] - 3\mathcal{H} \mathbf{R}^{0m}_{im}[X], \quad (115b)$$

$$\mathbf{G} := \mathcal{C}_G^2 \mathbf{G}^0_0[X] + \frac{1}{3} \mathbf{G}^m_m[X] = -\frac{1}{6} ((1 + 3\mathcal{C}_G^2) \mathbf{R}^{km}_{km}[X] + 4\mathbf{R}^{0m}_{0m}[X]). \quad (115c)$$

We find that it is simplest to express the Einstein gauge invariants (115) in terms of the *uniform curvature* metric gauge invariants  $\mathbf{A}$  and  $\mathbf{B}$  defined by (31). We accomplish this directly by choosing  $X = X_C$  in (110), and noting that by (27) we have  $\Psi[X_C] = 0$ . After simplifying using the identities (126e) and (126f) we obtain<sup>39</sup>

$$\hat{\mathbf{G}}_{ij} = \mathbf{D}_{ij} \mathbb{G} - \mathbf{D}_{(i} (\partial_\eta + 2\mathcal{H}) \mathbf{B}_{j)} + (\partial_\eta^2 + 2\mathcal{H} \partial_\eta + 2K - \mathbf{D}^2) \mathbf{C}_{ij}, \quad (116a)$$

$$\mathbf{G}_i = 2\mathcal{H} \mathbf{D}_i (\mathbf{D}^2 + 3K) \mathbf{B} + \frac{3}{2} \mathcal{H} (\mathbf{D}^2 + 2K) \mathbf{B}_i, \quad (116b)$$

$$\mathbf{G} = 2\mathcal{H} [(\partial_\eta + \mathcal{B}\mathcal{H}) \mathbf{A} + \mathcal{C}_G^2 \mathbf{D}^2 \mathbf{B}] - \frac{2}{3} \mathbf{D}^2 \mathbb{G}, \quad (116c)$$

<sup>38</sup>Note that  $C^{ij}_{km} = -4C^{0[i}_{0[k} \delta^{j]}_{m]}$  in an orthonormal frame.

<sup>39</sup>Here for convenience we use  $\hat{\mathbf{G}}_{ij} = \gamma_{ik} \hat{\mathbf{G}}^k_j$ .

where we have introduced the notation

$$\mathbb{G} := -[\mathbf{A} + (\partial_\eta + 2\mathcal{H})\mathbf{B}], \quad \mathcal{B} := \frac{2\mathcal{H}'}{\mathcal{H}^2} + 1 + 3\mathcal{C}_G^2. \quad (117)$$

We also need

$$\mathbf{G}_j^0[X] = \mathbf{R}^{0m}_{jm}[X]. \quad (118)$$

We choose  $X = X_P$  in this equation, and using (110) in conjunction with the identity (126f) we obtain

$$\mathbf{G}_j^0[X_P] = -2\mathbf{D}_j(\partial_\eta\Psi + \mathcal{H}\Phi) + \frac{1}{2}(\mathbf{D}^2 + 2K)\mathbf{B}_j. \quad (119)$$

We now use (32) to express the right side of this equation in terms of  $\mathbf{A}$  and  $\mathbf{B}$ , which yields

$$\mathbf{G}_i^0[X_P] = -2\mathbf{D}_i(\mathcal{H}\mathbf{A} + (\mathcal{A}_G - K)\mathbf{B}) + \frac{1}{2}(\mathbf{D}^2 + 2K)\mathbf{B}_i. \quad (120)$$

### The Weyl tensor

The perturbation of the Weyl tensor is automatically gauge-invariant on account of the Stewart-Walker lemma (Stewart and Walker (1974)) since the Weyl tensor is zero in the background. We thus use bold-face notation for its components. From (114c) we obtain

$$\mathbf{C}^{0i}_{0j} = a^{2(1)}C^{0i}_{0j} = \frac{1}{2}(\hat{\mathbf{R}}^{0i}_{0j} - \hat{\mathbf{R}}^{im}_{jm}), \quad \mathbf{C}^{0i}_{jk} = a^{2(1)}C^{0i}_{jk} = \hat{\mathbf{R}}^{0i}_{jk}. \quad (121)$$

The Weyl tensor has a simpler form if we use Poisson gauge invariants and hence we choose  $X = X_P$  in (110). Noting that  $\mathbf{B}[X_P] = 0$  leads to

$$\mathbf{C}^{0i}_{0j} = -\frac{1}{2}[\mathbf{D}^i_j(\Psi + \Phi) + \partial_\eta\mathbf{B}^i_j - (\partial_\eta^2 + \mathbf{D}^2 - 2K)\mathbf{C}^i_j], \quad (122a)$$

$$\mathbf{C}^{0i}_{jk} = -2\mathbf{D}_{[j}(\mathbf{B}^i_{k]} - \partial_\eta\mathbf{C}^i_{k]}) - \mathbf{D}_m\mathbf{B}^m_{[j}\delta^i_{k]}, \quad \mathbf{B}_{ij} := \mathbf{D}_{[i}\mathbf{B}_{j]}. \quad (122b)$$

## B.1 Uniqueness of the decomposition into modes

**Proposition:** If the inverses of the operators  $\mathbf{D}^2$ ,  $\mathbf{D}^2 + 2K$  and  $\mathbf{D}^2 + 3K$  exist, then the equation

$$B_i = \mathbf{D}_i B + \tilde{B}_i, \quad \text{with} \quad \mathbf{D}^i \tilde{B}_i = 0, \quad (123)$$

determines  $B$  and  $\tilde{B}_i$  uniquely in terms of  $B_i$ , and the equation

$$C_{ij} = \mathbf{D}_{ij}C + \mathbf{D}_{(i}C_{j)} + \tilde{C}_{ij}, \quad (124)$$

with

$$\mathbf{D}^i C_i = 0, \quad \tilde{C}_{ij} = \tilde{C}_{ji}, \quad \tilde{C}^i_i = 0, \quad \mathbf{D}^i \tilde{C}_{ij} = 0,$$

determines  $C$ ,  $C_i$  and  $\tilde{C}_{ij}$  uniquely in terms of  $C_{ij}$ . In particular, if  $B_i = 0$  then  $B = 0$ ,  $\tilde{B}_i = 0$ , and if  $C_{ij} = 0$  then  $C = 0$ ,  $C_i = 0$ ,  $\tilde{C}_{ij} = 0$ .

*Proof.* Apply  $\mathbf{D}^i$  to (123) obtaining  $\mathbf{D}^i B_i = \mathbf{D}^2 B$ . Using the inverse operator of  $\mathbf{D}^2$  this equation determines  $B$ , and then (123) determines  $\tilde{B}_i$  uniquely in terms of  $B_i$ . Next, apply  $\mathbf{D}^{ij}$  and  $\mathbf{D}^i$  to (124), obtaining

$$\mathbf{D}^{ij}C_{ij} = \frac{2}{3}\mathbf{D}^2(\mathbf{D}^2 + 3K)C, \quad \mathbf{D}^i C_{ij} = \frac{2}{3}\mathbf{D}_j(\mathbf{D}^2 + 3K)C + (\mathbf{D}^2 + 2K)C_j. \quad (125)$$

By using the inverse operators these equations, in conjunction with (124), successively determine  $C$ ,  $C_i$  and  $\tilde{C}_{ij}$  uniquely in terms of  $C_{ij}$ .  $\square$

## B.2 Identities

In obtaining our results we found the following identities useful:

$$\mathbf{D}_{[i}\mathbf{D}_{j]}A^k = K\delta_{[i}^k A_{j]}, \quad (126a)$$

$$\mathbf{D}_{[k}\mathbf{D}_{m]}A^{ij} = 2K\delta_{[k}^{(i} A_{m]}^{j)}, \quad (126b)$$

$$4(\mathbf{D}_{[k}\mathbf{D}^{[i} + K\delta_{[k}^{[i}]\delta_{m]}^m]A = (\mathbf{D}_k^i + \frac{4}{3}(\mathbf{D}^2 + 3K)\delta_k^i)A, \quad (126c)$$

$$4(\mathbf{D}_{[k}\mathbf{D}^{[i} + K\delta_{[k}^{[i}])C_{j]}^j] = (\mathbf{D}^2 - 2K)C_k^i, \quad (126d)$$

$$\mathbf{D}_j\mathbf{D}_i^j A = \frac{2}{3}\mathbf{D}_i(\mathbf{D}^2 + 3K)A, \quad (126e)$$

$$\mathbf{D}^i\mathbf{D}_{(i}A_{j)} = \frac{1}{2}(\mathbf{D}^2 + 2K)A_j, \quad (126f)$$

$$\mathbf{D}_i\mathbf{D}^2 A^i = (\mathbf{D}^2 + 2K)\mathbf{D}_i A^i, \quad (126g)$$

$$\delta_{[i}^{[i} A_{m]}^{j]} = \frac{1}{4}(A_m^j + \delta_m^j A), \quad (126h)$$

where  $A_{ij} = A_{ji}$ ,  $C_{ij} = C_{ji}$ ,  $C_i^i = 0$  and  $\mathbf{D}_i C_j^i = 0$ .

## B.3 Kinematic quantities

The kinematic quantities associated with a timelike congruence  $u^a$  are defined by the following decomposition into irreducible parts:

$$\nabla_a u_b = -u_a \dot{u}_b + H(g_{ab} + u_a u_b) + \sigma_{ab} + \omega_{ab}. \quad (127)$$

A routine calculation starting with equations (62)-(64) and (89) applied to  $u_a$  yields the following non-zero components:

$$a^{(1)}H = \left[\frac{1}{3}\mathbf{D}^2(v - \chi) - (\partial_\eta \psi + \mathcal{H}\varphi)\right], \quad (128a)$$

$$\dot{\mathbf{u}}_i := {}^{(1)}\dot{u}_i = \mathbf{D}_i(\varphi + (\partial_\eta + \mathcal{H})v) + (\partial_\eta + \mathcal{H})\tilde{\mathbf{v}}_i, \quad (128b)$$

$$\boldsymbol{\sigma}_j^i := a^{(1)}\sigma_j^i = \mathbf{D}_j^i(v - \chi) + \gamma^{ik}\mathbf{D}_{(k}(\tilde{\mathbf{v}}_j) - \mathbf{B}_j) + \partial_\eta \mathbf{C}_j^i, \quad (128c)$$

$$\boldsymbol{\omega}_j^i := a^{(1)}\omega_j^i = \gamma^{ik}\mathbf{D}_{[k}\tilde{\mathbf{v}}_j], \quad (128d)$$

with the bold-face quantities being gauge-invariant on account of the Stewart-Walker lemma.

## C Scalar field

A minimally coupled scalar field  $\phi$  is described by a stress-energy tensor of the form

$$T_b^a = \nabla^a \phi \nabla_b \phi - \left[\frac{1}{2}\nabla^c \phi \nabla_c \phi + U(\phi)\right] \delta_b^a, \quad (129)$$

with the associated Klein-Gordon equation  $\nabla^c \nabla_c \phi - U_{,\phi} = 0$ , where the potential  $U(\phi)$  has to be specified. This stress-energy tensor is of the form (60) with

$$\rho + p = -\nabla^a \phi \nabla_a \phi, \quad \rho - p = 2U(\phi), \quad \pi_{ab} = 0. \quad (130)$$

When evaluated on the RW background equation (130) leads to

$$a^2({}^{(0)}\rho + {}^{(0)}p) = ({}^{(0)}\phi')^2, \quad ({}^{(0)}\rho - {}^{(0)}p) = 2U({}^{(0)}\phi). \quad (131)$$

On using (131) to calculate  ${}^{(0)}\rho'$  the conservation equation (43) leads to

$${}^{(0)}\phi'' + 2\mathcal{H}{}^{(0)}\phi' + a^2U_{,\phi} = 0, \quad (132)$$

which is the Klein-Gordon equation in the RW background. Further, by means of (42), (43), (131) and (132) we obtain

$$\mathcal{A}_T = \frac{1}{2}({}^{(0)}\phi')^2, \quad \mathcal{C}_T^2 = 1 + \frac{2a^2U_{,\phi}}{3\mathcal{H}{}^{(0)}\phi'} = -\frac{1}{3} \left( 1 + \frac{2{}^{(0)}\phi''}{\mathcal{H}{}^{(0)}\phi'} \right). \quad (133)$$

Viewing  $T^a_b$  and  $\phi$  as functions of the perturbation parameter  $\epsilon$ , we can use (129), in conjunction with (3), to calculate  ${}^{(1)}T^a_b$ , obtaining

$${}^{(1)}\hat{T}^i_j = 0, \quad a^2{}^{(1)}T^0_i = -{}^{(0)}\phi' \mathbf{D}_i{}^{(1)}\phi, \quad {}^{(1)}T^0_0 + \frac{1}{3}{}^{(1)}T^i_i = -2U_{,\phi}{}^{(1)}\phi. \quad (134)$$

It follows using (38) with  $A$  replaced by  $T$  and (133), that the matter gauge invariants assume the form

$$\hat{\mathbf{T}}^i_j = 0, \quad \mathbf{T}^0_i[X] = -{}^{(0)}\phi' \mathbf{D}_i\phi[X], \quad \mathbf{T}^0_0[X] + \frac{1}{3}\mathbf{T}^i_i[X] = -2a^2U_{,\phi}\phi[X], \quad (135)$$

where  $\phi[X]$  is the gauge invariant associated with  ${}^{(1)}\phi$  by X-replacement, given by

$$\phi[X] = {}^{(1)}\phi - {}^{(0)}\phi' X^0. \quad (136)$$

Equations (135) and (50) immediately lead to the expressions for the matter gauge invariants (81b) and (81c), including

$$V[X] = -\frac{1}{2}{}^{(0)}\phi' \phi[X]. \quad (137)$$

Equation (135), in conjunction with (133) and (137), yields

$$\mathbf{T}^0_0[X] + \frac{1}{3}\mathbf{T}^i_i[X] = -6(1 - \mathcal{C}_T^2)\mathcal{H}V[X]. \quad (138)$$

We now substitute (138) into the expression for  $\Gamma$  given by (46c) and (50c) to obtain<sup>40</sup>

$$\Gamma = (1 - \mathcal{C}_T^2)(-\mathbf{T}^0_0[X] - 6\mathcal{H}V[X]). \quad (139)$$

which on comparison with (68) leads to equation (81a).

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<sup>40</sup>Write the expression for  $\Gamma$  in the form  $\Gamma = -(1 - \mathcal{C}_T^2)\mathbf{T}^0_0 + (\mathbf{T}^0_0 + \frac{1}{3}\mathbf{T}^i_i)$ .

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# COSMOLOGICAL PERTURBATION THEORY REVISITED

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## Abstract

Increasingly accurate observations are driving theoretical cosmology toward the use of more sophisticated descriptions of matter and the study of nonlinear perturbations of Friedmann-Lemaitre cosmologies, whose governing equations are notoriously complicated. Our goal in this paper is to formulate the governing equations for linear perturbation theory in a particularly simple and concise form in order to facilitate the extension to nonlinear perturbations. Our approach has several novel features. We show that the use of so-called *intrinsic gauge invariants* has two advantages. It naturally leads to: (i) a physically motivated choice of a gauge invariant associated with the matter density, and (ii) two distinct and complementary ways of formulating the evolution equations for scalar perturbations, associated with the work of Bardeen and of Kodama and Sasaki. In the first case the perturbed Einstein tensor gives rise to a second order (in time) linear differential operator, and in the second case to a pair of coupled first order (in time) linear differential operators. These operators are of fundamental importance in cosmological perturbation theory, since they provide the leading order terms in the governing equations for nonlinear perturbations.

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# 1 Introduction

The analysis of linear perturbations of Friedmann-Lemaitre (FL) cosmologies was initiated by Lifshitz (1946) in a paper of far-reaching importance. Working in the so-called synchronous gauge, this paper showed that an arbitrary linear perturbation can be written as the sum of three modes, a scalar mode that describes perturbations in the matter density, a vector mode that describes vorticity and a tensor mode that describes gravitational waves. For many years, however, the theory was plagued by gauge problems, *i.e.* by the fact that the behaviour of the scalar mode depends significantly on the choice of gauge. A major step in alleviating this difficulty was taken by Bardeen (1980), who reformulated the linearized Einstein field equations in terms of a set of *gauge-invariant* variables, as an alternative to the traditional use of the synchronous gauge. Central to Bardeen's paper are two gauge-invariant equations that govern the behaviour of scalar perturbations. The first of these governs the evolution in time of a gauge-invariant gravitational (*i.e.* metric) potential and the second determines a gauge-invariant perturbation of the matter density in terms of the spatial Laplacian of the gravitational potential. Since this potential continues to play a central role in the study of scalar perturbations, it seems appropriate to refer to it as the *Bardeen potential*. Bardeen's paper makes clear, however, that there is no unique way of constructing gauge-invariant variables.

From our perspective, one drawback of Bardeen's paper is that he performs a harmonic decomposition of the variables *ab initio*, with the result that the mathematical structure of the governing equations is somewhat obscured. In a subsequent paper, Brandenberger, Khan and Press (1983) address this deficiency by giving a new derivation of Bardeen's gauge-invariant equations. They do not perform a harmonic decomposition, with the result that their evolution equation is a partial differential equation rather than an ordinary differential equation as in Bardeen's paper. However, unlike Bardeen they restrict consideration to a spatially flat Robertson-Walker (RW) background.<sup>1</sup>

In subsequent developments the status of the Bardeen potential was further enhanced by the appearance of the major review paper by Mukhanov *et al* (1992), which contains a simplified derivation of the Bardeen potential and the evolution equation for scalar perturbations, without performing a harmonic decomposition. However, the treatment in Mukhanov *et al* (1992) is less general than that of Bardeen (1980) and Brandenberger *et al* (1983) in two respects. First, they assume the anisotropic stresses are zero, and second, they make a specific choice of gauge invariants *a priori*, namely those associated with the so-called longitudinal gauge.

Currently, increasingly accurate observations are driving theoretical cosmology towards more sophisticated models of matter and the study of possible nonlinear deviations from FL cosmology. Motivated by this state of affairs, our long term goal is to provide a general but concise description of nonlinear perturbations of FL cosmologies that will reveal the mathematical structure of the governing equations and enable one to make the transition between different gauge-invariant formula-

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<sup>1</sup>We follow the nomenclature of Wainwright and Ellis (1997) where an FL cosmology is a RW geometry that satisfies Einstein's field equations.

tions, thereby simplifying and relating the different approaches that have been used to date.<sup>2</sup> In pursuing this objective we have found it necessary to revisit linear perturbation theory, even though it is by now a mature discipline.<sup>3</sup> Our intent in the present paper is to formulate the governing equations for the linear theory in a particularly simple and concise form in order to facilitate the extension to nonlinear perturbations.

Based on earlier work by Bruni *et al* (1997) on gauge-invariant higher order perturbation theory, Nakamura (2003) introduced a geometrical method for constructing gauge invariants for linear and nonlinear (second order) perturbations which he later applied to derive the governing equations (see Nakamura (2006) and Nakamura (2007)). In the present paper we use a dimensionless version of Nakamura's method for constructing gauge invariants, but we complement it with the observation that gauge invariants are of two distinct types: *intrinsic gauge invariants*, i.e., gauge invariants that can be constructed from a given tensor alone, and *hybrid gauge invariants*, i.e. gauge invariants that are constructed from more than one tensor.

In Nakamura's approach, the linear perturbation of any tensor is written as the sum of a gauge-invariant quantity and a gauge-variant quantity, which is the Lie derivative of the zero order tensor with respect to a suitably chosen vector field  $X$ . A choice of  $X$  yields a set of gauge-invariant variables that are associated with a specific fully fixed gauge. We will show that for the metric tensor there exist two natural complementary choices of  $X$  that yield intrinsic metric gauge invariants. One choice, used in all of Nakamura's papers, leads to the two gauge-invariant metric potentials of Bardeen (1980), which are associated with the so-called *Poisson gauge*.<sup>4</sup> The other choice leads to the two gauge-invariant metric potentials of Kodama and Sasaki (1984), which are associated with the so-called *uniform curvature gauge*.<sup>5</sup> We will show that these two preferred choices lead to two distinct ways in which to present the linearized Einstein field equations: with the Bardeen choice the evolution of linear scalar perturbations is governed by a *second order* (in time) linear partial differential operator, while with the Kodama-Sasaki choice the evolution is governed by two coupled temporal *first order* linear operators.

The plan of the paper is as follows. In Section 2 we discuss the geometrical construction of gauge-invariants: we focus on the metric tensor and, with the Einstein tensor and the stress-energy tensor in mind, on mixed rank two tensors. In Section 3 we use intrinsic gauge invariants to derive the general governing equations for linear perturbations in two gauge-invariant forms associated with the Poisson and the uniform curvature gauges. The required expressions for the Einstein gauge invariants are derived efficiently in Appendix B, where we also give a general concise formula that expresses the Riemann gauge invariants in terms of the metric gauge invariants. One of the ingredients in our derivation is the so-called Replacement Principle, which is formulated in Appendix A. In Section 4 we give an interpre-

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<sup>2</sup>See, for example, Noh and Hwang (2004), Nakamura (2007) and Malik (2007).

<sup>3</sup>For some recent reviews and books, see, for example, Mukhanov (2005), Tsagas *et al* (2008), Weinberg (2008), Durrer (2008), Malik and Wands (2009) and Lyth and Liddle (2009).

<sup>4</sup>The Poisson gauge, which was introduced by Bertschinger (1996) (see his equation (4.46)), is a generalization of the longitudinal gauge, which only applies to scalar perturbations.

<sup>5</sup>See, for example, Malik and Wands (2009), page 20, and other references given there.

tation of the intrinsic matter gauge invariants and specialize our equations to the cases of a perfect fluid and a scalar field. Section 5 contains a brief discussion of future developments.

## 2 Geometrical definition of gauge invariants

### 2.1 General formulation

Following standard cosmological perturbation theory (see for example, Chapter 7.5 in Wald (1984)), we consider a 1-parameter family of spacetimes  $g_{ab}(\epsilon)$ , where  $g_{ab}(0)$ , the unperturbed metric, is a RW metric, and  $\epsilon$  is referred to as the *perturbation parameter*.<sup>6</sup> We assign physical dimension *length* to the scale factor  $a$  of the RW metric and  $(\text{length})^2$  to  $g_{ab}(\epsilon)$ . Then the conformal transformation

$$g_{ab}(\epsilon) = a^2 \bar{g}_{ab}(\epsilon), \quad (1)$$

yields a dimensionless metric  $\bar{g}_{ab}(\epsilon)$ . Our reason for making this choice<sup>7</sup> concerning the allocation of physical dimensions is that it enables one to create dimensionless quantities by multiplying by the appropriate power of  $a$ , leading to simple perturbation equations that do not contain  $a$  explicitly. We refer to Appendix B, where this process is applied.

The Riemann tensor associated with the metric  $g_{ab}(\epsilon)$  is a function of  $\epsilon$ , denoted  $R^{ab}_{cd}(\epsilon)$ , as is the Einstein tensor,  $G^a_b(\epsilon)$ . The stress-energy tensor of the matter distribution is also assumed to be a function of  $\epsilon$ , denoted  $T^a_b(\epsilon)$ . We include all these possibilities by considering a 1-parameter family of tensor fields  $A(\epsilon)$ , which we assume can be expanded in powers of  $\epsilon$ , *i.e.* as a Taylor series:

$$A(\epsilon) = {}^{(0)}A + \epsilon {}^{(1)}A + \frac{1}{2}\epsilon^2 {}^{(2)}A + \dots \quad (2)$$

The coefficients are given by<sup>8</sup>

$${}^{(0)}A = A(0), \quad {}^{(1)}A = \left. \frac{\partial A}{\partial \epsilon} \right|_{\epsilon=0}, \quad {}^{(2)}A = \left. \frac{\partial^2 A}{\partial \epsilon^2} \right|_{\epsilon=0}, \quad \dots, \quad (3)$$

where  ${}^{(0)}A$  is called the *unperturbed value*,  ${}^{(1)}A$  is called the *first order (linear) perturbation* and  ${}^{(2)}A$  is called the *second order perturbation* of  $A(\epsilon)$ .

The primary difficulty in cosmological perturbation theory is that the perturbations of a tensor field  $A(\epsilon)$  depend on the choice of gauge, and hence cannot be directly related to observations. It is therefore desirable to formulate the theory in

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<sup>6</sup>We use Latin letters  $a, b, \dots, f$  to denote abstract spacetime indices.

<sup>7</sup>An alternative choice in cosmology is to make  $a$  dimensionless and let the spacetime coordinates of  $\bar{g}_{ab}(0)$  have dimension *length* (see, for example, Malik and Wands (2009), page 48). This choice is unsuitable for our purposes since it does not lead naturally to perturbative equations involving dimensionless quantities. For discussions about dimensions and their uses, see for example, Eardley (1974), Martin-Garcia and Gundlach (2002), Wiesenfeld (2001), and Heinzle *et al* (2003).

<sup>8</sup>The notation  $A(\epsilon)$  should be viewed as shorthand for  $A(x, \epsilon)$ , indicating that the tensor fields are functions of the spacetime coordinates, which necessitates the use of partial differentiation with respect to  $\epsilon$ .

terms of *gauge-invariant* quantities, *i.e.* to replace the gauge-variant perturbations  ${}^{(1)}A, {}^{(2)}A, \dots$  of  $A(\epsilon)$  by gauge-invariant quantities. In this paper we restrict our attention to first order, *i.e.* linear, perturbations, but with a view to subsequently working with higher order perturbations we use a method pioneered by Nakamura (2003), and adapt it so as to create quantities that are gauge-invariant *and* dimensionless.

A linear gauge transformation is represented in coordinates by the equation

$$\tilde{x}^a = x^a + \epsilon \xi^a + \dots, \quad (4)$$

where  $\xi^a$  is an arbitrary dimensionless vector field on the background. Given a family of tensor fields  $A(\epsilon)$  the change induced in the first order perturbation  ${}^{(1)}A$  by a gauge transformation is determined by

$$\Delta^{(1)}A = \mathcal{L}_\xi^{(0)}A, \quad (5)$$

where  $\mathcal{L}_\xi$  denotes the Lie derivative with respect to  $\xi^a$  and  $\Delta^{(1)}A := {}^{(1)}\tilde{A} - {}^{(1)}A$  (see, for example, Bruni *et al* (1997), equations (1.1) and (1.2)). We now introduce an as yet arbitrary dimensionless vector field  $X$  on the background which we use to define the dimensionless object<sup>9</sup>

$${}^{(1)}\mathbf{A}[X] := a^n ({}^{(1)}A - \mathcal{L}_X^{(0)}A), \quad (6)$$

where we assume that  $A(\epsilon)$  is such that  $a^n A(\epsilon)$  is dimensionless. It follows from (5) and (6) that

$$\Delta^{(1)}\mathbf{A}[X] = a^n (\mathcal{L}_\xi^{(0)}A - \mathcal{L}_{\Delta X}^{(0)}A) = a^n \mathcal{L}_{\xi - \Delta X}^{(0)}A. \quad (7)$$

The key step is to choose an  $X$  that satisfies

$$\Delta X^a = \xi^a, \quad (8)$$

under a gauge transformation. With this choice, (7) implies that  $\Delta^{(1)}\mathbf{A}[X] = 0$ , *i.e.*,  ${}^{(1)}\mathbf{A}[X]$  is gauge-invariant. We say that  ${}^{(1)}\mathbf{A}[X]$  is the *gauge invariant associated with*  ${}^{(1)}A$  *by*  $X$ -*compensation*. Equations (5), (6) and (8) are central to our version of Nakamura's method for constructing gauge invariants associated with the first order perturbation of a tensor  $A$  (see Nakamura (2007), equations (2.19), (2.23) and (2.26)). In what follows we will drop the superscript  ${}^{(1)}$  on  $\mathbf{A}$  for convenience since in this paper we are dealing only with first order perturbations.

The above 'gauge compensating vector field'  $X$ , which for brevity we shall refer to as the *gauge field*, requires comment. Unlike the geometric and matter tensor fields such as  $g_{ab}(\epsilon)$  and  $T^a_b(\epsilon)$  it is not the perturbation of a corresponding quantity on the background spacetime. Instead it should be viewed as a vector field on the background spacetime that is constructed from the linear perturbations of the geometric and matter tensors in such a way that (8) holds. We will construct specific examples of  $X$  in section 2.2. We note that in choosing the gauge field  $X$  we are

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<sup>9</sup>When we consider second order perturbations in a subsequent paper, we will denote  $\xi^a$  and  $X^a$  by  ${}^{(1)}\xi^a$  and  ${}^{(1)}X^a$ , and introduce a second pair of vector fields denoted  ${}^{(2)}\xi^a$  and  ${}^{(2)}X^a$ .

essentially fixing the gauge (*i.e.* making a choice of gauge), which is accomplished in the traditional approach by making a choice of the vector field  $\xi$  that determines the gauge transformation.<sup>10</sup> One advantage of using the gauge field  $X$  is that one immediately obtains a geometric connection between the gauge invariants associated with different choices of gauge. This matter is discussed in more detail in Ugglá and Wainwright (2011).

Before continuing we briefly digress to point out that gauge invariants associated with a tensor  $A$  are of two distinct types: those that are solely constructed from components of  $(1)A$  and  $(0)A$  are called *intrinsic* gauge invariants, while those that depend on the components of another perturbed tensor are called *hybrid* gauge invariants. In particular if the gauge field  $X$  is formed solely from components of  $(1)A$  and  $(0)A$ , then  $\mathbf{A}[X]$  is an intrinsic gauge invariant, otherwise  $\mathbf{A}[X]$  is a hybrid gauge invariant.

In the following sections we will calculate the quantities in equations (5) and (6) for various geometric objects  $A$ . To do this it is necessary to use the well known formulae for the Lie derivative. The formula for a tensor of type (1, 1), which we now give, establishes the pattern:

$$\mathcal{L}_\xi A^a_b = A^a_{b,c}\xi^c + \xi^c_{,b}A^a_c - \xi^a_{,c}A^c_b, \quad (9)$$

where  $\cdot$  denotes partial differentiation. In a formula such as (9) one can replace the partial derivatives by covariant derivatives. For our purposes it is convenient to use the covariant derivative  ${}^0\bar{\nabla}_a$  associated with the *unperturbed conformal metric*  $\bar{g}_{ab}(0)$ :

$$\mathcal{L}_\xi A^a_b = ({}^0\bar{\nabla}_c A^a_b)\xi^c + ({}^0\bar{\nabla}_b \xi^c)A^a_c - ({}^0\bar{\nabla}_c \xi^a)A^c_b. \quad (10)$$

We also need to work in a coordinate frame so that we can calculate time and spatial components separately. We thus introduce local coordinates<sup>11</sup>  $x^\mu = (\eta, x^i)$ , with  $\eta$  being the usual conformal time coordinate<sup>12</sup> for the RW metric  $g_{ab}(0)$ , and such that the unperturbed conformal metric  $\gamma_{ab} := \bar{g}_{ab}(0)$  has components

$$\gamma_{00} = -1, \quad \gamma_{0i} = 0, \quad \gamma_{ij}, \quad (11)$$

where  $\gamma_{ij}$  is the metric of a spatial geometry of constant curvature. The curvature index of the RW metric, denoted  $K$ , determines the sign of the curvature of the spatial geometry, and if non-zero can be scaled to be +1 or -1 (see, for example, Plebanski and Krasinski (2006), page 261).

The spacetime covariant derivative  ${}^0\bar{\nabla}_a$  determines a temporal derivative  ${}^0\bar{\nabla}_0 A = \partial_\eta A$ , where  $\partial_\eta$  denotes partial differentiation with respect to  $\eta$ , and a spatial covariant derivative  ${}^0\bar{\nabla}_i$  that is associated with the spatial metric  $\gamma_{ij}$ . We introduce the notation

$$\mathbf{D}_i A := {}^0\bar{\nabla}_i A. \quad (12)$$

<sup>10</sup>See, for example, Malik and Wands (2009); equations (6.17), (7.3) and (7.4) provide an example in connection with the metric tensor.

<sup>11</sup>We use Greek letters to denote spacetime coordinate indices on the few occasions that they occur, and we use Latin letters  $i, j, k, m$  to denote spatial coordinate indices, which are lowered and raised using  $\gamma_{ij}$  and its inverse  $\gamma^{ij}$ , respectively.

<sup>12</sup>Since we assigned  $a$  to have physical dimension *length*, the conformal time  $\eta$  and the conformal spatial line-element  $\gamma_{ij}dx^i dx^j$  are dimensionless. We choose the  $x^i$  to be dimensionless, which implies that the  $\gamma_{ij}$  are also dimensionless.

The derivative operators  $\partial_\eta$  and  $\mathbf{D}_i$  will be used throughout this paper once local coordinates have been introduced. However, for simplicity we shall denote the derivative of a function  $f(\eta)$  that depends only on  $\eta$  by  $f'(\eta)$ .

With our present allocation of dimensions, the scalar  $\mathcal{H}$  defined by

$$\mathcal{H} := \frac{a'}{a} = aH, \quad (13)$$

where  $H$  is the Hubble scalar,<sup>13</sup> is dimensionless. We shall refer to it as the *dimensionless Hubble scalar*. The use of this scalar, e.g. by Mukhanov *et al* (1992) (see page 218), is essential in eliminating  $a$  from the perturbation equations.

## 2.2 Metric gauge invariants

We expand  $\bar{g}_{ab}(\epsilon)$ , defined by equation (1), in powers of  $\epsilon$ :

$$\bar{g}_{ab}(\epsilon) = {}^{(0)}\bar{g}_{ab} + \epsilon {}^{(1)}\bar{g}_{ab} + \dots,$$

and label the unperturbed metric and (linear) metric perturbation according to

$$\gamma_{ab} := {}^{(0)}\bar{g}_{ab} = \bar{g}_{ab}(0), \quad f_{ab} := {}^{(1)}\bar{g}_{ab} = \frac{\partial \bar{g}_{ab}}{\partial \epsilon}(0), \quad (14)$$

which is consistent with (3). Applying the general transformation law (5) to the metric tensor  $g_{ab}(\epsilon) = a^2 \bar{g}_{ab}(\epsilon)$  we obtain

$$\Delta^{(1)}g_{ab} = \mathcal{L}_\xi^{(0)}g_{ab}, \quad \text{or, equivalently,} \quad \Delta f_{ab} = a^{-2} \mathcal{L}_\xi(a^2 \gamma_{ab}), \quad (15)$$

in terms of the notation (14). The gauge invariant  $\mathbf{f}_{ab}[X]$  associated with the metric perturbation  $f_{ab}$  by  $X$ -compensation, given by (6), assumes the form

$$\mathbf{f}_{ab}[X] = f_{ab} - a^{-2} \mathcal{L}_X(a^2 \gamma_{ab}). \quad (16)$$

Introducing local coordinates and using (9) and (10) adapted to a (0,2) tensor, equations (15) and (16) lead to

$$\Delta f_{00} = -2(\partial_\eta + \mathcal{H})\xi^0, \quad \mathbf{f}_{00}[X] = f_{00} + 2(\partial_\eta + \mathcal{H})X^0, \quad (17a)$$

$$\Delta f_{0i} = -\mathbf{D}_i \xi^0 + \partial_\eta \xi_i, \quad \mathbf{f}_{0i}[X] = f_{0i} + \mathbf{D}_i X^0 - \partial_\eta X_i, \quad (17b)$$

$$\Delta f_{ij} = 2\mathcal{H} \xi^0 \gamma_{ij} + 2\mathbf{D}_{(i} \xi_{j)}, \quad \mathbf{f}_{ij}[X] = f_{ij} - 2\mathcal{H} X^0 \gamma_{ij} - 2\mathbf{D}_{(i} X_{j)}. \quad (17c)$$

In order to construct a gauge field  $X$  that satisfies (8), *using only the metric*, we need to decompose the metric perturbation  $f_{ab}$  into scalar, vector and tensor

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<sup>13</sup>Recall that  $H := \frac{1}{a} \frac{da}{dt}$ , where  $t$  is cosmic time, and that  $\frac{dt}{d\eta} = a$ .

modes.<sup>14</sup> We introduce the notation<sup>15</sup>

$$f_{00} = -2\varphi, \quad (18a)$$

$$f_{0i} = \mathbf{D}_i B + B_i, \quad (18b)$$

$$f_{ij} = -2\psi\gamma_{ij} + 2\mathbf{D}_i \mathbf{D}_j C + 2\mathbf{D}_{(i} C_{j)} + 2C_{ij}, \quad (18c)$$

where the vectors  $B_i$  and  $C_i$  and the tensor  $C_{ij}$  satisfy

$$\mathbf{D}^i B_i = 0, \quad \mathbf{D}^i C_i = 0, \quad C_i^i = 0, \quad \mathbf{D}^i C_{ij} = 0.$$

The vector  $\xi$  is also decomposed into a scalar mode and a vector mode with components

$$\xi^0, \quad \xi^i = \mathbf{D}^i \xi + \tilde{\xi}^i. \quad (19)$$

It follows from (17), (18) and (19) that

$$\Delta\varphi = (\partial_\eta + \mathcal{H})\xi^0, \quad \Delta B = -\xi^0 + \partial_\eta \xi, \quad \Delta C = \xi, \quad \Delta\psi = -\mathcal{H}\xi^0, \quad (20a)$$

$$\Delta B_i = \partial_\eta \tilde{\xi}_i, \quad \Delta C_i = \tilde{\xi}_i, \quad (20b)$$

$$\Delta C_{ij} = 0. \quad (20c)$$

We can draw two immediate conclusions. First, it follows from (20b) and (20c) that  $B_i - C'_i$  and  $C_{ij}$  are gauge invariants. We introduce the following bold-face notation:

$$\mathbf{B}_i := B_i - \partial_\eta C_i, \quad \mathbf{C}_{ij} := C_{ij}. \quad (21)$$

Second, by inspection of (19), (20a) and (20b) we obtain

$$\Delta(\mathbf{D}_i C + C_i) = \xi_i, \quad \Delta\chi = \Delta\left(\frac{\psi}{\mathcal{H}}\right) = -\xi^0, \quad (22)$$

where we have introduced the notation

$$\chi := B - \partial_\eta C. \quad (23)$$

We are now in a position to satisfy the requirement (8). Firstly, referring to (22), we can satisfy the *spatial part*  $\Delta X^i = \xi^i$  of the requirement by choosing

$$X_i = \mathbf{D}_i C + C_i, \quad (24)$$

---

<sup>14</sup>In order to guarantee that the functions  $B, B_i, C, C_i$  and  $C_{ij}$  in (18) are uniquely determined by  $f_{0i}$  and  $f_{ij}$  we need to assume that the inverses of  $\mathbf{D}^2$ ,  $\mathbf{D}^2 + 2K$  and  $\mathbf{D}^2 + 3K$  exist. See the proposition in Appendix B.1. See also Nakamura (2007), following equation (4.15), for a helpful discussion of this matter.

<sup>15</sup>We are denoting the scalar mode functions by  $\varphi, B, C$  and  $\psi$ , in agreement with Mukhanov *et al* (1992) (see equation (2.10), but note the different signature) and Malik and Wands (2009) (see equations (2.7)-(2.12)), with the difference that we use  $C$  instead of  $E$ . Bardeen (1980) used the notation  $A, -B, H_T$  and  $-H_L + \frac{1}{3}\mathbf{D}^2 H_T$  for these functions, the choice of the fourth one being motivated by harmonic decomposition. Bardeen's notation has been used by subsequent authors, for example, Kodama and Sasaki (1984) and Durrer (1994), although the latter author replaced  $-B$  by  $B$ .

which we will take to be our default choice for  $X_i$ . With this choice, the expressions (17) for the components of the gauge invariant  $\mathbf{f}_{ab}[X]$ , when combined with (18), assume the form

$$\mathbf{f}_{00}[X] = -2\Phi[X], \quad (25a)$$

$$\mathbf{f}_{0i}[X] = \mathbf{D}_i\mathbf{B}[X] + \mathbf{B}_i, \quad (25b)$$

$$\mathbf{f}_{ij}[X] = -2\Psi[X]\gamma_{ij} + 2\mathbf{C}_{ij}. \quad (25c)$$

where

$$\Phi[X] := \varphi - (\partial_\eta + \mathcal{H})X^0, \quad \Psi[X] := \psi + \mathcal{H}X^0, \quad \mathbf{B}[X] := \chi + X^0, \quad (25d)$$

and  $\mathbf{B}_i$ ,  $\mathbf{C}_{ij}$  and  $\chi$  are given by (21) and (23), respectively.

Secondly, referring to (22), we can satisfy the *timelike part*  $\Delta X^0 = \xi^0$  of the requirement (8) in two obvious ways, by choosing

$$X^0 = X_p^0 := -\chi, \quad \text{or} \quad X^0 = X_c^0 := -\frac{\psi}{\mathcal{H}}, \quad (26)$$

which leads to the metric gauge invariants associated with the *Poisson gauge*, or the *uniform curvature gauge*, respectively. On substituting these choices into (25d) we obtain the conditions

$$\mathbf{B}[X_p] = 0 \quad \text{and} \quad \Psi[X_c] = 0, \quad (27)$$

which characterize these two gauge choices.

### The Poisson gauge invariants

On substituting the first of equations (26) into (25) we obtain

$$\mathbf{f}_{00}[X_p] := -2\Phi, \quad \mathbf{f}_{0i}[X_p] := \mathbf{B}_i, \quad \mathbf{f}_{ij}[X_p] := -2\Psi\gamma_{ij} + 2\mathbf{C}_{ij}, \quad (28)$$

where

$$\Phi := \Phi[X_p] = \varphi + (\partial_\eta + \mathcal{H})\chi, \quad \Psi := \Psi[X_p] = \psi - \mathcal{H}\chi. \quad (29)$$

Here  $\Phi$  and  $\Psi$  are the scalar metric gauge invariants associated with the Poisson gauge,<sup>16</sup> and  $\Psi$  is the Bardeen potential.

### The uniform curvature gauge invariants

On substituting the second of equations (26) into (25) we obtain

$$\mathbf{f}_{00}[X_c] = -2\mathbf{A}, \quad \mathbf{f}_{0i}[X_c] = \mathbf{D}_i\mathbf{B} + \mathbf{B}_i, \quad \mathbf{f}_{ij}[X_c] = 2\mathbf{C}_{ij}, \quad (30)$$

where

$$\mathbf{A} := \Phi[X_c] = \varphi + (\partial_\eta + \mathcal{H})\frac{\psi}{\mathcal{H}}, \quad \mathbf{B} := \mathbf{B}[X_c] = \chi - \frac{\psi}{\mathcal{H}}. \quad (31)$$

Here  $\mathbf{A}$  and  $\mathbf{B}$  are the scalar metric gauge invariants associated with the uniform curvature gauge,<sup>17</sup> introduced by Kodama and Sasaki (1984).<sup>18</sup>

<sup>16</sup>The gauge-fixing conditions for the Poisson gauge are  $B = C = 0, C_i = 0$  in (18).

<sup>17</sup>The gauge-fixing conditions for the uniform curvature gauge are  $\psi = C = 0, C_i = 0$  in (18).

<sup>18</sup>See equations (3.4) and (3.5), noting that  $H_L + n^{-1}H_T \equiv -\psi$  and  $B - k^{-1}H'_T \equiv \chi$ .

In concluding this section we note that the gauge fields  $X$  used to construct the above gauge invariants have the same spatial components  $X^i$  given by (24) in both cases, leading to (25), with the vector and tensor modes described by the gauge invariants  $\mathbf{B}_i$  and  $\mathbf{C}_{ij}$ , respectively. The difference lies in the scalar metric gauge invariants which are related according to<sup>19</sup>

$$\mathbf{A} = \Phi + (\partial_\eta + \mathcal{H}) \frac{\Psi}{\mathcal{H}}, \quad \mathbf{B} = -\frac{\Psi}{\mathcal{H}}, \quad (32)$$

as follows from (29) and (31). In both cases the gauge invariants are *intrinsic* since the gauge field  $X$  depends only on the metric.

A reader of this paper should be aware of the lack of agreement in the literature on labelling the scalar metric gauge invariants associated with the Poisson gauge. Our choice of  $(\Phi, \Psi)$  in (29) is the one initiated by Mukhanov *et al* (1992), and subsequently used by Nakamura (see, for example, Nakamura (2006)) and Malik and Wands (2009). On the other hand Durrer (2008) and Liddle and Lyth (2000) reverse the roles and use  $(\Psi, \Phi)$ , while Kodama and Sasaki (1984) use  $(\Psi, -\Phi)$ . Bardeen's original notation is  $(\Phi_A, -\Phi_H)$ .

### 2.3 Gauge invariants for mixed rank 2 tensors

In this subsection we consider a rank two tensor  $A^a_b$ , such that  $A_{ab}$  is symmetric and  $a^2 A^a_b$  is dimensionless. We expand  $A^a_b$  in a Taylor series in  $\epsilon$  as in (2), and assume that  ${}^{(0)}A^a_b$  obeys the background symmetries, which means it is spatially homogeneous and isotropic:

$$\mathbf{D}_i {}^{(0)}A^\alpha_\beta = 0, \quad {}^{(0)}A^0_i = {}^{(0)}A^i_0 = 0, \quad {}^{(0)}A^i_j = \frac{1}{3} \delta^i_j {}^{(0)}A^k_k. \quad (33)$$

We introduce the notation

$$\mathcal{A}_A := a^2 (-{}^{(0)}A^0_0 + \frac{1}{3} {}^{(0)}A^k_k), \quad \mathcal{C}_A^2 := -\frac{({}^{(0)}A^k_k)'}{3({}^{(0)}A^0_0)'}, \quad (34)$$

where as before  $'$  denotes differentiation with respect to  $\eta$ . We further assume that  $A^a_b$  satisfies the conservation law  $\nabla_a A^a_b = 0$ . It follows that in the background

$$a^2 ({}^{(0)}A^0_0)' = 3a^2 \mathcal{H} (-{}^{(0)}A^0_0 + \frac{1}{3} {}^{(0)}A^k_k) = 3\mathcal{H} \mathcal{A}_A, \quad (35)$$

which, in conjunction with (34), implies that

$$\mathcal{A}'_A = -(1 + 3\mathcal{C}_A^2) \mathcal{H} \mathcal{A}_A. \quad (36)$$

We can now calculate the gauge invariants  $\mathbf{A}^a_b[X]$  associated with  ${}^{(1)}A^a_b$  by  $X$ -compensation, as defined by equation (6) with  $n = 2$ . It is convenient to decompose  ${}^{(1)}A^i_j$  into its trace  ${}^{(1)}A^k_k$  and tracefree part defined by

$${}^{(1)}\hat{A}^i_j := {}^{(1)}A^i_j - \frac{1}{3} {}^{(1)}A^k_k \delta^i_j. \quad (37)$$

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<sup>19</sup>These relation have recently been given by Christopherson *et al* (2011). See their equations (4.22) and (4.23).

A straightforward calculation using (6), (9), (10) and (33) leads to<sup>20</sup>

$$\mathbf{A}_0^0[X] = a^2 {}^{(1)}A_0^0 - 3\mathcal{H}\mathcal{A}_A X^0 \quad (38a)$$

$$\mathbf{A}_i^0[X] = a^2 {}^{(1)}A_i^0 + \mathcal{A}_A \mathbf{D}_i X^0, \quad (38b)$$

$$\mathbf{A}_k^k[X] = a^2 {}^{(1)}A_k^k + 9\mathcal{H}\mathcal{A}_A \mathcal{C}_A^2 X^0, \quad (38c)$$

$$\hat{\mathbf{A}}_j^i[X] = a^2 {}^{(1)}\hat{A}_j^i. \quad (38d)$$

In deriving these equations we have used (34) and (35) to express  ${}^{(0)}A_0^0$ ,  ${}^{(0)}A_k^k$  and their derivatives in terms of  $\mathcal{A}_A$  and  $\mathcal{C}_A^2$ .

Equation (38d) implies that  $\hat{\mathbf{A}}_j^i[X]$  is an *intrinsic* gauge invariant since it is constructed solely from the components of  ${}^{(1)}A_b^a$ . We denote this quantity by

$$\hat{\mathbf{A}}_j^i := \hat{\mathbf{A}}_j^i[X] = a^2 {}^{(1)}\hat{A}_j^i. \quad (39)$$

One can form two additional intrinsic gauge invariants by taking suitable combinations of  $\mathbf{A}_0^0[X]$ ,  $\mathbf{A}_i^0[X]$  and  $\mathbf{A}_k^k[X]$ . Indeed it follows from (38) that

$$\mathbf{A} := \mathcal{C}_A^2 \mathbf{A}_0^0[X] + \frac{1}{3} \mathbf{A}_k^k[X] = a^2 (\mathcal{C}_A^2 {}^{(1)}A_0^0 + \frac{1}{3} {}^{(1)}A_k^k), \quad (40a)$$

$$\mathbf{A}_i := -(\mathbf{D}_i \mathbf{A}_0^0[X] + 3\mathcal{H} \mathbf{A}_i^0[X]) = -a^2 (\mathbf{D}_i {}^{(1)}A_0^0 + 3\mathcal{H} {}^{(1)}A_i^0), \quad (40b)$$

which implies that  $\mathbf{A}$  and  $\mathbf{A}_i$  are *intrinsic gauge-invariants*.

In summary, the tensor  $A_b^a$  can be described by the three intrinsic gauge invariants  $\hat{\mathbf{A}}_j^i$ ,  $\mathbf{A}$ , and  $\mathbf{A}_i$ , given by (39), (40a) and (40b), and one hybrid gauge invariant  $\mathbf{A}_i^0[X]$ , given by (38b). In section 3.1 we will use these objects, constructed in terms of the Einstein tensor and the stress-energy tensor, to give a concise derivation of the governing equations in gauge-invariant form for linear perturbations of FL.

## 3 Linearized governing equations

### 3.1 General formulation

In this section we work with the linear perturbations of the Einstein tensor and the stress-energy tensor, denoted by  ${}^{(1)}G_b^a$  and  ${}^{(1)}T_b^a$ , and defined via equation (3). The corresponding unperturbed quantities are labelled by a superscript  ${}^{(0)}$ .

We begin by imposing the background Einstein equations  ${}^{(0)}G_b^a = {}^{(0)}T_b^a$ . The non-zero components are given by<sup>21</sup>

$$a^2 {}^{(0)}G_0^0 = -3(\mathcal{H}^2 + K) = -a^2 {}^{(0)}\rho = a^2 {}^{(0)}T_0^0, \quad (41a)$$

$$a^2 {}^{(0)}G_j^i = -(2\mathcal{H}' + \mathcal{H}^2 + K)\delta_j^i = a^2 {}^{(0)}p \delta_j^i = a^2 {}^{(0)}T_j^i, \quad (41b)$$

where  $\mathcal{H}$  is given by (13) and  $K$  is the curvature index. It follows from (41), (34) and (35), with  $A$  replaced by  $G$  and  $T$ , respectively, that

$$\mathcal{A}_G = 2(-\mathcal{H}' + \mathcal{H}^2 + K), \quad \mathcal{A}_T = a^2 ({}^{(0)}\rho + {}^{(0)}p), \quad (42a)$$

$$\mathcal{A}'_G = -(1 + 3\mathcal{C}_G^2)\mathcal{H}\mathcal{A}_G, \quad \mathcal{C}_T^2 = \frac{{}^{(0)}p'}{{}^{(0)}\rho'}. \quad (42b)$$

<sup>20</sup>We do not include the  ${}^{(1)}A_0^0$  components since they can be expressed in terms of the other components and the metric perturbation, due to the assumed symmetry.

<sup>21</sup>See, for example, Mukhanov *et al* (1992), equation (4.2), noting the difference in signature.

The conservation law (35), with  $A$  replaced by  $T$ , gives

$$a^2({}^{(0)}\rho)' = -3\mathcal{H}\mathcal{A}_T = -3\mathcal{H}a^2({}^{(0)}\rho + {}^{(0)}p). \quad (43)$$

The background Einstein equations imply that  $\mathcal{A}_G = \mathcal{A}_T$  and  $\mathcal{C}_G^2 = \mathcal{C}_T^2$ . We denote the common values by  $\mathcal{A}$  and  $\mathcal{C}^2$ :

$$\mathcal{A} = \mathcal{A}_G = \mathcal{A}_T, \quad \mathcal{C}^2 = \mathcal{C}_G^2 = \mathcal{C}_T^2. \quad (44)$$

The linearized Einstein field equations are given by

$${}^{(1)}G_b^a = {}^{(1)}T_b^a. \quad (45)$$

In simplifying the linearized field equations we will make use of the intrinsic gauge invariants associated with the Einstein tensor and with the stress-energy tensor, which are given, in analogy with (39), (40a) and (40b), by

$$\hat{\mathbf{G}}_j^i = a^2 {}^{(1)}\hat{G}_j^i, \quad \hat{\mathbf{T}}_j^i = a^2 {}^{(1)}\hat{T}_j^i \quad (46a)$$

$$\mathbf{G}_i = -a^2 (\mathbf{D}_i {}^{(1)}G_0^0 + 3\mathcal{H} {}^{(1)}G_i^0), \quad \mathbf{T}_i = -a^2 (\mathbf{D}_i {}^{(1)}T_0^0 + 3\mathcal{H} {}^{(1)}T_i^0), \quad (46b)$$

$$\mathbf{G} = a^2 (\mathcal{C}_G^2 {}^{(1)}G_0^0 + \frac{1}{3} {}^{(1)}G_k^k), \quad \mathbf{T} = a^2 (\mathcal{C}_T^2 {}^{(1)}T_0^0 + \frac{1}{3} {}^{(1)}T_k^k), \quad (46c)$$

where

$${}^{(1)}\hat{G}_j^i = {}^{(1)}G_j^i - \frac{1}{3}\delta_j^i {}^{(1)}G_k^k, \quad {}^{(1)}\hat{T}_j^i = {}^{(1)}T_j^i - \frac{1}{3}\delta_j^i {}^{(1)}T_k^k. \quad (47)$$

We also need the hybrid gauge invariants  $\mathbf{G}_i^0[X]$  and  $\mathbf{T}_i^0[X]$ , which are given by (38b) with  $A$  replaced by  $G$  and  $T$ :

$$\mathbf{G}_i^0[X] = a^2 {}^{(1)}G_i^0 + \mathcal{A}_G \mathbf{D}_i X^0, \quad \mathbf{T}_i^0[X] = a^2 {}^{(1)}T_i^0 + \mathcal{A}_T \mathbf{D}_i X^0. \quad (48)$$

Since the gauge invariants (46) and (48) are linear in  ${}^{(1)}G_b^a$  and  ${}^{(1)}T_b^a$  with coefficients depending on  ${}^{(0)}G_b^a$  and  ${}^{(0)}T_b^a$ , respectively, it follows that the linearized Einstein field equations immediately imply the following relations:

$$\hat{\mathbf{G}}_j^i - \hat{\mathbf{T}}_j^i = 0, \quad \mathbf{G}_i - \mathbf{T}_i = 0, \quad \mathbf{G} - \mathbf{T} = 0, \quad (49a)$$

$$\mathbf{G}_i^0[X] - \mathbf{T}_i^0[X] = 0. \quad (49b)$$

Expressions for the Einstein gauge invariants  $\hat{\mathbf{G}}_j^i$ ,  $\mathbf{G}_i$ ,  $\mathbf{G}$  and  $\mathbf{G}_i^0[X]$  in terms of the metric gauge invariants, decomposed into scalar, vector, and tensor modes, are given in equations (116) and (120) in Appendix B. To proceed we likewise decompose the matter gauge invariants  $\hat{\mathbf{T}}_j^i$ ,  $\mathbf{T}_i$ ,  $\mathbf{T}$  and  $\mathbf{T}_i^0[X]$  into scalar, vector, and tensor modes and label them as follows:<sup>22</sup>

$$\hat{\mathbf{T}}_j^i = \mathbf{D}_j^i \Pi + 2\gamma^{ik} \mathbf{D}_{(k} \Pi_{j)} + \Pi_j^i, \quad (50a)$$

$$\mathbf{T}_i = \mathbf{D}_i \Delta + \Delta_i, \quad (50b)$$

$$\mathbf{T} = \Gamma, \quad (50c)$$

$$\mathbf{T}_i^0[X] = \mathbf{D}_i V[X] + V_i, \quad (50d)$$

---

<sup>22</sup>In subsection 4.1 we comment on the choice of the symbols  $\Pi$ ,  $\Gamma$ ,  $\Delta$  and  $V$ .

where

$$\mathbf{D}^i \Pi_i = 0, \quad \Pi^k_k = 0, \quad \mathbf{D}_i \Pi_j^i = 0, \quad \mathbf{D}^i \Delta_i = 0, \quad \mathbf{D}^i V_i = 0, \quad (50e)$$

and

$$\mathbf{D}_{ij} := \mathbf{D}_{(i} \mathbf{D}_{j)} - \frac{1}{3} \gamma_{ij} \mathbf{D}^2, \quad \mathbf{D}^2 := \mathbf{D}^i \mathbf{D}_i. \quad (50f)$$

We stress that in making this decomposition we are not making any assumptions about the physical nature of the stress-energy tensor. By inspecting (116), (120) and (50) one concludes that equations (49) decompose into a scalar mode, a vector mode and a tensor mode, which we label as follows:

$$\begin{aligned} \mathbf{D}_{ij} \mathbb{A} + \mathbf{D}_{(i} \mathbb{A}_{j)} + \mathbb{A}_{ij} &= 0, \\ \mathbf{D}_i \mathbb{B} + \mathbb{B}_i &= 0, \\ \mathbb{C} &= 0, \\ \mathbf{D}_i \mathbb{E}[X] + \mathbb{E}_i &= 0. \end{aligned}$$

Since we are assuming that the inverses of the operators  $\mathbf{D}^2, \mathbf{D}^2 + 2K$  and  $\mathbf{D}^2 + 3K$  exist we can use the proposition in Appendix B.1 to write the linearized field equations concisely as

$$\textit{Scalar mode: } \mathbb{A} = 0, \quad \mathbb{B} = 0, \quad \mathbb{C} = 0, \quad \mathbb{E}[X] = 0. \quad (51a)$$

$$\textit{Vector mode: } \mathbb{A}_i = 0, \quad \mathbb{B}_i = 0, \quad \mathbb{E}_i = 0. \quad (51b)$$

$$\textit{Tensor mode: } \mathbb{A}_{ij} = 0. \quad (51c)$$

## 3.2 Scalar mode

In this subsection we give the governing equations (51a) for the scalar mode, first expressing them in terms of the uniform curvature gauge invariants  $\mathbf{A} = \Phi[X_c]$  and  $\mathbf{B} = \mathbf{B}[X_c]$  (see (31)). The scalars  $\mathbb{A}, \mathbb{B}$  and  $\mathbb{C}$  in (51a) are obtained without any calculation by taking the differences of equations (116) and (50) and reading off the scalar part. The scalar  $\mathbb{E}[X]$  is obtained in a similar manner from (120) and (50d) with  $X = X_p$ . The resulting equations are<sup>23</sup>

$$(\partial_\eta + 2\mathcal{H}) \mathbf{B} + \mathbf{A} = -\Pi \quad (52a)$$

$$\mathcal{H} [(\partial_\eta + \mathcal{B}\mathcal{H}) \mathbf{A} + \mathcal{C}_G^2 \mathbf{D}^2 \mathbf{B}] = \frac{1}{2} \Gamma + \frac{1}{3} \mathbf{D}^2 \Pi, \quad (52b)$$

$$\mathcal{H} (\mathbf{D}^2 + 3K) \mathbf{B} = -\frac{1}{2} \Delta, \quad (52c)$$

$$\mathcal{H} \mathbf{A} + (\frac{1}{2} \mathcal{A}_G - K) \mathbf{B} = -\frac{1}{2} V, \quad (52d)$$

where

$$\mathcal{B} = \frac{2\mathcal{H}'}{\mathcal{H}^2} + 1 + 3\mathcal{C}_G^2, \quad (53)$$

(see equation (117) in Appendix B), and  $V = V[X_p]$ . We shall refer to these equations as the *uniform curvature form* of the governing equations for the scalar mode.

<sup>23</sup>In deriving (52b) we use (52a) to replace  $(\partial_\eta + 2\mathcal{H}) \mathbf{B} + \mathbf{A}$  by  $-\Pi$ .

We now give the governing equations in terms for the Poisson gauge invariants  $\Psi$  and  $\Phi$ . We eliminate  $\mathbf{A}$  in (52b) using (52a) and in (52d) using (32), and eliminate  $\mathbf{B}$  using  $\mathcal{H}\mathbf{B} = -\Psi$ . The resulting equations are

$$\Psi - \Phi = \Pi, \quad (54a)$$

$$(\mathcal{L} - \mathcal{C}_G^2 \mathbf{D}^2) \Psi = \frac{1}{2} \Gamma + \left( \frac{1}{3} \mathbf{D}^2 + \mathcal{H}(\partial_\eta + \mathcal{B}\mathcal{H}) \right) \Pi, \quad (54b)$$

$$(\mathbf{D}^2 + 3K) \Psi = \frac{1}{2} \Delta, \quad (54c)$$

$$\partial_\eta \Psi + \mathcal{H}\Phi = -\frac{1}{2} V, \quad (54d)$$

where the differential operator  $\mathcal{L}$  is defined by

$$\mathcal{L}(\bullet) := \mathcal{H}(\partial_\eta + \mathcal{B}\mathcal{H})(\partial_\eta + 2\mathcal{H}) \left( \frac{\bullet}{\mathcal{H}} \right), \quad (55)$$

and  $\mathcal{B}$  is given by (53). Expanding the brackets yields<sup>24</sup>

$$\mathcal{L} = \partial_\eta^2 + 3(1 + \mathcal{C}_G^2) \mathcal{H} \partial_\eta + \mathcal{H}^2 \mathcal{B} - (1 + 3\mathcal{C}_G^2) K. \quad (56)$$

We shall refer to the above equations as the *Poisson form* of the governing equations for the scalar mode, and to the evolution equation (54b) as the *Bardeen equation*.

Equations (52) and (54), which are linked by the factorization property (55), constitute one of the main results of this paper. Either system of equations determine the behaviour of linear scalar perturbations of an FL cosmology with arbitrary stress-energy content whose scalar mode is described by the gauge invariants  $\Gamma$ ,  $\Pi$ ,  $\Delta$  and  $V$ . The structure of these two systems of equations differs in a significant way. In the system (52) the time dependence is governed by two *first order* differential operators  $\partial_\eta + \mathcal{B}\mathcal{H}$  and  $\partial_\eta + 2\mathcal{H}$ , while in the system (54) the time dependence is governed by the *second order* linear differential operator  $\mathcal{L}$ . A key point is that *the coefficients in these operators depend only on the background RW geometry*, and this dependence manifests itself through the appearance of  $\mathcal{H}$ ,  $\mathcal{H}'$ ,  $\mathcal{H}''$  and  $K$ . This property is significant since it means that these operators will have the same form irrespective of the nature of the source in the FL background model, *e.g.* whether it is a perfect fluid with  $p = p(\rho)$ , or a scalar field with potential  $V(\phi)$ . What will differ, however, is the functional dependence of  $\mathcal{H}(\eta)$ , which is determined by solving the Einstein equations in the background RW geometry, and hence depends on the source. Furthermore these differential operators will also appear in the linearized field equations in any geometrical theory of gravity, whose field equations depend in some way on the Einstein tensor.

To the best of our knowledge equations (52) have not been given in the literature, although if one performs a harmonic decomposition one obtains a system of first order ordinary differential equations closely related to that given by Kodama and Sasaki (1984) (see Chapter 2, equations (4.6a-d)). Likewise, the governing equations in Poisson form (54) have not appeared in the literature in the above fully general form. The use of the Poisson gauge invariants was initiated by Bardeen (1980), and the evolution equation (54b) for  $\Psi$  is now commonly used, although it is written in a variety of different forms, as a partial or ordinary differential equation with the

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<sup>24</sup>Referring to (42) to express  $\mathcal{H}'$  in terms of  $\mathcal{A}_G$  and then use the equation for  $\mathcal{A}'_G$ .

coefficients usually expressed in terms of the matter variables of the background FL model. In contrast we have written the Bardeen equation in a fully general form in terms of the purely geometric differential operator  $\mathcal{L}$ , which is defined by the factorization property (55). We can relate our form of the equation to the literature by expanding  $\mathcal{L}$  as in (56) and expressing the coefficients in terms of the matter variables. If the matter content is a barotropic perfect fluid and a cosmological constant and one imposes the background Einstein field equations then the geometric coefficients  $\mathcal{C}_G^2$  and  $\mathcal{B}$  can be written as

$$\mathcal{C}_G^2 = c_s^2, \quad \mathcal{H}^2 \mathcal{B} = (c_s^2 - w)\rho a^2 + (1 + c_s^2)\Lambda a^2 - (1 + 3c_s^2)K, \quad (57)$$

using (41), (44) and (79). The form in the literature that is closest to the purely geometric form (56) is that given by Mukhanov *et al* (1992), equation (5.22), who replace  $\mathcal{C}_G^2$  by the matter quantity  $c_s^2$  as in (57) but retain  $\mathcal{H}$  and  $\mathcal{H}'$ . Nakamura (2007) gives the same expression (see his equation (5.30)). A more common form in the literature has  $\mathcal{B}$ , in addition to  $\mathcal{C}_G^2$ , expressed in terms of the background matter variables as in (57). The earliest occurrence of which we are aware is Harrison (1967), equation (182), followed by Bardeen (1980), equation (5.30), after making the appropriate changes of notation and setting  $\Lambda = 0$ . See also Ellis, Hwang and Bruni (1989), equation (31) and Hwang and Vishniac (1990), equation (105).<sup>25</sup>

### 3.3 Vector and tensor modes

First, we give the governing equations (51b) for the *vector mode*. The vectors  $\mathbb{A}_i$  and  $\mathbb{B}_i$  in (51b) are obtained without any calculation by taking the differences of equations (116) and (50) and reading off the vector part. The vector  $\mathbb{E}_i$  is obtained in a similar manner from (120) and (50d). The resulting equations are

$$(\partial_\eta + 2\mathcal{H})\mathbf{B}_i = -2\Pi_i, \quad (58a)$$

$$(\mathbf{D}^2 + 2K)\mathbf{B}_i = 2V_i, \quad (58b)$$

as well as the relation  $\Delta_i = 3\mathcal{H}V_i$ , which is satisfied identically (see equation (68)). If  $\Pi_i$  is specified and can be regarded as a source term, the evolution equation (58a) is a first order linear ordinary differential equation that determines  $\mathbf{B}_i$ , which in turn determines  $V_i$  by differentiation using (58b).

Second, we give the governing equations (51c) for the *tensor mode*. The tensor  $\mathbb{A}_{ij}$  in (51c) is obtained without any calculation by taking the differences of equations (116) and (50) and reading off the tensor part, leading to

$$(\partial_\eta^2 + 2\mathcal{H}\partial_\eta + 2K - \mathbf{D}^2)\mathbf{C}_{ij} = \Pi_{ij}. \quad (59)$$

If  $\Pi_{ij}$  is specified and can be regarded as source term, this is a second order linear partial differential equation that determines  $\mathbf{C}_{ij}$ .

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<sup>25</sup>In these two references, the evolution equation in question arises in the 1 + 3 gauge-invariant approach to perturbations of FL, and the unknown is a vector quantity that is related to the scalar  $\Psi$ .

## 4 Interpretations and examples

### 4.1 Interpretation of the matter gauge invariants

In this section we give the physical interpretation of the gauge invariants  $\Pi, \Gamma, \Delta$  and  $V[X]$  associated with the scalar mode of the stress-energy tensor.

We begin with the decomposition of a stress-energy tensor with respect to a unit timelike vector field  $u^a$ , which is given by

$$T^a_b = (\rho + p)u^a u_b + p\delta^a_b + (q^a u_b + u^a q_b) + \pi^a_b, \quad (60)$$

where

$$u^a q_b = 0, \quad \pi^a_a = 0, \quad u_a \pi^a_b = 0. \quad (61)$$

We choose  $u^a$  to be the timelike eigenvector of  $T^a_b$ , which implies  $q^a = 0$ , *i.e.* we are using the so-called energy frame (see for example, Bruni *et al* (1992), page 37).

Assuming that the unperturbed stress-energy tensor  ${}^{(0)}T^a_b$  has the isotropy and homogeneity properties of the RW geometry, the expansion (2) to linear order for  $\rho, p, u_a$  and  $\pi^a_b$  has the form:<sup>26</sup>

$$\rho = {}^{(0)}\rho + \epsilon {}^{(1)}\rho, \quad p = {}^{(0)}p + \epsilon {}^{(1)}p, \quad (62a)$$

$$\pi^0_0 = 0 = \pi^0_i, \quad \pi^i_j = 0 + \epsilon {}^{(1)}\pi^i_j, \quad (62b)$$

$$u_0 = -a(1 + \epsilon\varphi), \quad u_i = a(0 + \epsilon v_i). \quad (62c)$$

Decomposing  $v_i$  into a scalar and vector mode yields

$$v_i = \mathbf{D}_i v + \tilde{\mathbf{v}}_i, \quad \mathbf{D}^i \tilde{\mathbf{v}}_i = 0. \quad (63)$$

We use boldface in writing  $\tilde{\mathbf{v}}_i$  in view of the fact that this quantity is a dimensionless gauge invariant, as can be verified by applying (5) to  $u_a$ .

For ease of comparison with other work, we note that the expansion of  $u^a = g^{ab}u_b$  to linear order, expressed in terms of  $v, \tilde{\mathbf{v}}^i$  and the linearly perturbed metric, is given by

$$u^0 = a^{-1}(1 - \epsilon\varphi), \quad u^i = a^{-1} [0 + \epsilon (\mathbf{D}^i(v - B) + (\tilde{\mathbf{v}}^i - B^i))]. \quad (64)$$

We digress briefly to mention that our expansion of the four-velocity differs from the usual approach in the literature in that we use the *covariant* vector  $u_a$  to define the perturbed three-velocity instead of the *contravariant* vector  $u^a$ , since we find that this leads to a number of simplifications.<sup>27</sup> For example, Malik and Wands (2009) (see equation (4.4)) have

$$u^i = a^{-1}[0 + \epsilon (\mathbf{D}^i v_{MW} + \tilde{v}^i_{MW})],$$

so that

$$v_{MW} = v - B, \quad \tilde{v}^i_{MW} = \tilde{\mathbf{v}}^i - B^i.$$

<sup>26</sup>The form of  $u_0$  is determined by the requirement that  $u^a$  is a unit vector. Recall that  $\varphi$  is one of the metric potentials in (18).

<sup>27</sup>The source of these simplifications is the fact that  $u_i$  is invariant under purely spatial gauge transformations while  $u^i$  is not.

From (60) and (62), and making use of (3), we obtain the following expressions for the components of the linear perturbation of the stress-energy tensor:

$${}^{(1)}T_0^0 = -{}^{(1)}\rho, \quad {}^{(1)}T_k^k = 3{}^{(1)}p, \quad {}^{(1)}T_i^0 = ({}^{(0)}\rho + {}^{(0)}p)v_i, \quad {}^{(1)}\hat{T}_j^i = {}^{(1)}\pi_j^i. \quad (65)$$

It follows from (46), (50) and (65), in conjunction with (42) and (43), that the matter gauge invariants are determined by

$$a^2{}^{(1)}\pi_j^i = \mathbf{D}_j^i \Pi + 2\gamma^{ik} \mathbf{D}_{(k} \Pi_{j)} + \Pi_j^i, \quad (66a)$$

$$\Gamma = a^2(-\mathcal{C}_T^2 {}^{(1)}\rho + {}^{(1)}p), \quad (66b)$$

$$\Delta = a^2 ({}^{(1)}\rho + ({}^{(0)}\rho)' v), \quad (66c)$$

$$V[X] = \mathcal{A}_T(v + X^0), \quad V_i = \mathcal{A}_T \tilde{\mathbf{v}}_i. \quad (66d)$$

Before continuing we derive an additional relation. It follows from (40b) with  $A$  replaced by  $T$  that

$$\mathbf{T}_i = -\mathbf{D}_i \mathbf{T}_0^0[X] - 3\mathcal{H} \mathbf{T}_i^0[X]. \quad (67)$$

On substituting from (50b) and (50d) into this equation, we conclude that

$$\Delta = -\mathbf{T}_0^0[X] - 3\mathcal{H}V[X], \quad \Delta_i = -3\mathcal{H}V_i. \quad (68)$$

We can now give the physical interpretation of the matter gauge invariants. First, the gauge invariants  $\Pi$ ,  $\Pi_i$  and  $\Pi_{ij}$  represent the anisotropic stresses. The interpretation of  $\Gamma$  is given in the context of a perfect fluid in the next section. Next, the gauge invariants  $V = V[X_p]$  and  $V_i$  play a role in determining the shear and vorticity of  $u_a$ . The relevant formulae are given in (128) in Appendix B.3. In particular,  $V[X_p]$  determines the scalar mode of the shear according to

$$\mathbf{D}_i^j \boldsymbol{\sigma}_j^i = \frac{2}{3} \mathcal{A}_T^{-1} \mathbf{D}^2 (\mathbf{D}^2 + 3K) V[X_p], \quad (69)$$

as follows from (128) in conjunction with (66d) with  $X = X_p$  and the identity (126e). We will hence use  $V := V[X_p]$  as our standard choice for the gauge invariant  $V[X]$ . However, since the choice  $V[X_c]$  is also of interest we note that

$$V[X_c] - V[X_p] = \mathcal{A}_T \mathbf{B}, \quad (70)$$

as follows from (66d), (26) and (31).

Finally, in order to interpret  $\Delta$  we need to make a small digression. For any scalar field  $A$  with the property that  $a^n A$  is dimensionless we can define a dimensionless gauge invariant  $\mathbf{A}[X]$  according to<sup>28</sup>

$$\mathbf{A}[X] = a^n ({}^{(1)}A - ({}^{(0)}A)' X^0). \quad (71)$$

For the matter density  $\rho$  we denote the gauge invariant by  $\boldsymbol{\rho}[X]$ :

$$\boldsymbol{\rho}[X] = a^2 ({}^{(1)}\rho - ({}^{(0)}\rho)' X^0). \quad (72)$$

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<sup>28</sup>This is equation (6) specialized to the case of a scalar field.

On choosing  $X = X_v$  with  $X_v^0 := -v$  it follows from (66c) that  $\Delta = \rho[X_v]$ . By comparing (72) with equation (3.13) in Bardeen (1980),<sup>29</sup> we conclude that  $\rho[X_v]$ , and hence  $\Delta$ , equals the well-known Bardeen gauge-invariant density perturbation  $\epsilon_m$ , up to a factor of  $a^2{}^{(0)}\rho$ . The specific relation is

$$\Delta = (a^2{}^{(0)}\rho)\epsilon_m. \quad (73)$$

We note that the choice  $X_v^0 = -v$ , in conjunction with our default choice (24) for the spatial components of  $X$ , is associated with the so-called *total matter gauge* (see, for example, Malik and Wands (2009), pages 23-24). Thus  $\Delta$  is the density perturbation in the total matter gauge. In addition it turns out that  $\Delta$  is closely related to the 1 + 3 gauge-invariant approach to perturbations of FL, pioneered by Ellis and collaborators (see for example, Ellis and Bruni (1989), Ellis *et al* (1989)), in which the spatial gradient of the matter density orthogonal to  $u^a$  plays a key role. To elucidate the relation we define the dimensionless spatial density gradient<sup>30</sup>

$$\mathcal{D}_a(\epsilon) = a^2 h_a^b(\epsilon) \epsilon \nabla_b \rho(\epsilon), \quad h_a^b(\epsilon) = \delta_a^b + u_a(\epsilon) u^b(\epsilon). \quad (74)$$

A straight-forward calculation shows that  $\mathcal{D}_a(0) = 0$  and that to linear order

$${}^{(1)}\mathcal{D}_0 = 0, \quad {}^{(1)}\mathcal{D}_i = \mathbf{D}_i \Delta - 3\mathcal{H}V_i, \quad (75)$$

from which we conclude that  $\Delta$  equals the the scalar mode of the linear perturbation of the spatial density gradient.<sup>31</sup> In addition it follows from (50b) and (68) that  ${}^{(1)}\mathcal{D}_i = \mathbf{T}_i$ , giving a physical interpretation of the intrinsic gauge-invariant  $\mathbf{T}_i$ .

To end this section we comment on our choice of notation. In using the symbols  $\Pi, \Gamma, \Delta$  and  $V$  for the matter gauge invariants we are following Kodama and Sasaki (1984) with the difference that we scale the variables as follows:

$$\Pi = a^2 p \Pi_{KS}, \quad \Gamma = a^2 p \Gamma_{KS}, \quad \Delta = a^2 \rho \Delta_{KS}, \quad V = \mathcal{A}_T V_{KS}, \quad (76)$$

where  $p$  and  $\rho$  refer to the background. Our choice of scalings simplify the equations considerably.

## 4.2 Perfect fluid

For a perfect fluid the matter gauge invariants are restricted according to

$$\Pi = 0, \quad \Pi_i = 0, \quad \Pi_j^i = 0. \quad (77)$$

In addition it follows from (42b) and (66b) that

$$\Gamma = 0 \quad \text{if and only if} \quad p = p(\rho), \quad (78)$$

<sup>29</sup>One has to take into account differences in notation, the conservation equation (43), and the fact that Bardeen has performed a harmonic decomposition.

<sup>30</sup>Our  $\mathcal{D}_a$  differs from that in Bruni, Dunsby and Ellis (1992) by a factor of  $\rho a^2$  (see their equation (24)).

<sup>31</sup>Note that  $\epsilon \nabla_a \rho(\epsilon) = {}^0 \bar{\nabla}_a \rho(\epsilon)$ .

*i.e.* if and only if the equation of state is barotropic. In this case it is customary to introduce the notation

$$c_s^2 := \mathcal{C}_T^2, \quad w := \frac{{}^{(0)}p}{{}^{(0)}\rho}, \quad (79)$$

where  $c_s^2 = w$  if  $w$  is constant, as follows from (42b).

On account of (77) the governing equations in the Poisson form (54) for scalar perturbations imply that  $\Psi - \Phi = 0$ , which (in conjunction with the background field equations) reduces the governing equations for the scalar mode in the perfect fluid case to

$$(\mathcal{L} - c_s^2 \mathbf{D}^2)\Psi = \frac{1}{2}\Gamma, \quad (80a)$$

$$(\mathbf{D}^2 + 3K)\Psi = \frac{1}{2}\Delta, \quad (80b)$$

$$\Psi' + \mathcal{H}\Psi = -\frac{1}{2}V, \quad (80c)$$

where  $\mathcal{L}$  is given by (56) with  $\mathcal{C}_G^2 = \mathcal{C}_T^2 = c_s^2$  and  $\mathcal{B}$  is expressed in terms of the background matter variables according to (57).

### 4.3 Scalar field

For a minimally coupled scalar field we show in Appendix C that the matter gauge invariants are given by

$$\Gamma = (1 - \mathcal{C}_T^2)\Delta, \quad (81a)$$

$$V[X] = -{}^{(0)}\phi' \phi[X], \quad V_i = 0, \quad (81b)$$

$$\Pi = 0, \quad \Pi_i = 0, \quad \Pi_j^i = 0, \quad (81c)$$

where  $\phi[X]$  is the gauge invariant associated with  ${}^{(1)}\phi$  by X-replacement, given by<sup>32</sup>

$$\phi[X] = {}^{(1)}\phi - {}^{(0)}\phi' X^0. \quad (82)$$

Note that  $\mathcal{A}_T$  and  $\mathcal{C}_T^2$  are given by (133). The governing equations (54) in Poisson form imply that  $\Psi - \Phi = 0$ , and then reduce to

$$(\mathcal{L} - \mathcal{C}^2 \mathbf{D}^2)\Psi = \frac{1}{2}(1 - \mathcal{C}^2)\Delta, \quad (83a)$$

$$(\mathbf{D}^2 + 3K)\Psi = \frac{1}{2}\Delta, \quad (83b)$$

$$\Psi' + \mathcal{H}\Psi = \frac{1}{2}{}^{(0)}\phi' \phi_p, \quad (83c)$$

where  $\phi_p := \phi[X_p]$ , and where we have used  $\mathcal{C}_G^2 = \mathcal{C}_T^2 = \mathcal{C}^2$ . By combining (83a) and (83b) we obtain an evolution equation for  $\Psi$  without a source term:

$$(\mathcal{L} - 3(1 - \mathcal{C}^2)K - \mathbf{D}^2)\Psi = 0, \quad (84)$$

where  $\mathcal{L}$  is given by (56). Having solved this equation one can calculate  $\phi_p$  and  $\Delta$  from (83). If one expresses  $\mathcal{C}^2$  in  $\mathcal{L}$  in terms of the unperturbed scalar field and its derivatives (see (133)) and sets  $K = 0$ , equation (84) coincides with equation (6.48)

<sup>32</sup>This is a special case of equation (71).

in Mukhanov *et al* (1992). For the generalization to arbitrary  $K$ , see Nakamura (2007), equation (5.39).<sup>33</sup>

One can also use the governing equations (52) in uniform curvature form, obtaining equations equivalent to those derived by Malik (2007) (see equations (2.20)-(2.23), noting that he is considering multiple scalar fields).

## 5 Discussion

We have given a systematic account of the gauge-invariant quantities that are associated with a linearly perturbed RW geometry and stress-energy tensor, emphasizing the role of intrinsic dimensionless gauge invariants. First, we have shown that there are two distinct choices of dimensionless intrinsic gauge invariants for the perturbed metric, which are the gauge invariants associated with the Poisson gauge and the uniform curvature gauge, through the work of Bardeen (1980) and Kodama and Sasaki (1984), respectively. Second, we have introduced dimensionless intrinsic gauge invariants for the Einstein tensor and the stress-energy tensor, which we used to derive a particularly simple and concise form of the governing equations for linear perturbations of FL models. The specific form of the governing equations for the scalar mode depends on the choice of intrinsic gauge invariants for the perturbed metric. The Kodama-Sasaki choice leads to a coupled system of two *first order* (in time) linear differential operators that govern the evolution of the uniform curvature metric gauge invariants (see equations (52)). On going over to the Poisson picture, the product of these two operators yields the *second order* linear differential operator  $\mathcal{L}$  that governs the evolution of the Bardeen potential (see equation (56)), thereby providing a link between the two forms of the governing equations. A common feature of both systems is the appearance of the physically motivated gauge-invariant density perturbation  $\Delta$  that is one of the intrinsic gauge invariants associated with the stress-energy tensor (see equations (52c) and (54c)).

The mathematical structure of the governing equations for linear perturbations that we have elucidated here has in fact a much wider significance. Indeed, as one might expect on the basis of elementary perturbation theory, the governing equations for second order (nonlinear) perturbations have precisely the same form, apart from the inclusion of a source term that depends quadratically on the linear metric perturbation.<sup>34</sup> As an illustration of this we give the form of the equations that govern second order scalar perturbations using the metric gauge invariants associated with the Poisson gauge:

$${}^{(2)}\Psi - {}^{(2)}\Phi = {}^{(2)}\Pi + S_{aniso}({}^{(1)}\mathbf{f}), \quad (85a)$$

$$(\mathcal{L} - \mathcal{C}_G^2 \mathbf{D}^2) {}^{(2)}\Psi = \frac{1}{2} {}^{(2)}\Gamma + \left(\frac{1}{3} \mathbf{D}^2 + \mathcal{H}(\partial_\eta + \mathcal{B}\mathcal{H})\right) {}^{(2)}\Pi + S_{evol}({}^{(1)}\mathbf{f}), \quad (85b)$$

$$(\mathbf{D}^2 + 3K) {}^{(2)}\Psi = \frac{1}{2} {}^{(2)}\Delta + S_{matter}({}^{(1)}\mathbf{f}), \quad (85c)$$

$$\partial_\eta {}^{(2)}\Psi + \mathcal{H} {}^{(2)}\Phi = -\frac{1}{2} {}^{(2)}V + S_{velocity}({}^{(1)}\mathbf{f}), \quad (85d)$$

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<sup>33</sup>We note a minor typo: a factor of 2 multiplying  $\partial_\eta^2$  should be deleted.

<sup>34</sup>This behaviour has been noted in general terms by Nakamura (2006), equations (38)-(39).

where  $S_{\bullet}({}^{(1)}\mathbf{f})$  is a *source term* that depends quadratically on the first order gauge-invariant metric perturbation  ${}^{(1)}\mathbf{f}_{ab} \equiv \mathbf{f}_{ab}$  in equation (28). The key point is that, apart from the source terms, equations (85) have the same form as equations (54), with the variables  ${}^{(2)}\Psi$  and  ${}^{(2)}\Phi$  being the metric gauge invariants at second order determined by the Nakamura procedure. The second order matter terms  ${}^{(2)}\Pi$ ,  ${}^{(2)}\Gamma$ ,  ${}^{(2)}\Delta$  and  ${}^{(2)}V$  are defined in analogy with the first order terms  $\Pi$ ,  $\Gamma$ ,  $\Delta$  and  $V$  after expanding the stress-energy tensor  $T^a_b$  to second order in powers of  $\epsilon$ . All the complications lie in the source terms, whose explicit form has to be found by calculating the Riemann tensor to second order. In order to solve the above second order equations the source terms, which include scalar, vector and tensor modes, first have to be obtained by solving the governing equations for the scalar, vector and tensor linear perturbations. In a subsequent paper we will derive both the above Poisson form and the corresponding uniform curvature form of the governing equations for second order perturbations, relating our formulation to other recent work.

In this paper we have focussed exclusively on using the linearized Einstein field equations to describe the dynamics of scalar perturbations. There are, however, two alternatives to the direct use of the linearized Einstein equations. First, one can use the linearized conservation equations for the stress-energy tensor, and second, one can use the 1 + 3 gauge-invariant formalism,<sup>35</sup> in which the evolution equations are obtained from the Ricci identities. An advantage of using the first approach independently of the Einstein equations is that the results are applicable to theories of gravity other than general relativity. An advantage of the second approach is that one initially derives exact nonlinear evolution equations, which are then subsequently linearized. Both of these approaches lead to a system of first order partial differential equations that describe the evolution of scalar perturbations. An additional aspect of the dynamics of scalar perturbations that we have likewise not touched on in this paper is that under certain conditions (*i.e.* in the long wavelength regime) the governing equations admit so-called conserved quantities, *i.e.* quantities that remain approximately constant during a restricted epoch. These quantities, which are related to both the linearized Einstein equations and the linearized conservation equations, have been found to be useful in analyzing the dynamics of scalar perturbations during inflation. We refer to Ugglå and Wainwright (2011), where we discuss the above aspects of the dynamics of scalar perturbations within the framework of the present paper.

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<sup>35</sup>See Bruni *et al* (1992) for a comprehensive treatment.

## A The Replacement Principle

The expression for the perturbation of the Riemann tensor given in equation (104) in Appendix B, can be written symbolically in the form:

$$a^2 {}^{(1)}R^{ab}{}_{cd} = \mathbf{L}^{ab}{}_{cd}(f), \quad (86)$$

where  $\mathbf{L}^{ab}{}_{cd}$  is a linear operator and  $f$  is shorthand for  $f_{ab}$ . The Replacement Principle for the Riemann curvature states that the gauge invariants associated with  ${}^{(1)}R^{ab}{}_{cd}$  and with  $f_{ab}$  by  $X$ -compensation are related by *the same* linear operator:

$$\mathbf{R}^{ab}{}_{cd}[X] = \mathbf{L}^{ab}{}_{cd}(\mathbf{f}[X]), \quad (87)$$

where  $\mathbf{f}[X]$  is shorthand for  $\mathbf{f}_{ab}[X]$ .

This result is adapted from more general results given by Nakamura (2005) (see in particular, his equations (3.12), (3.15) and (3.23)). Similar results hold for the Einstein and Weyl tensors. Use of the Replacement Principle in Appendix B makes the transition from gauge-variant to gauge-invariant equations particularly easy and transparent.

## B Derivation of the curvature formula

In this appendix we derive expressions for the Einstein gauge invariants, namely, the three intrinsic gauge invariants  $\hat{\mathbf{G}}_j^i$ ,  $\mathbf{G}_i$  and  $\mathbf{G}$ , and the single hybrid gauge invariant  ${}^{(1)}\mathbf{G}_i^0[X]$ , defined by equations (46) and (48). Our strategy incorporates the following ideas:

- i) *Conformal structure.* We adapt to the conformal structure of the background geometry, determined by the scale factor  $a$  of the RW metric, from the outset. In particular we create dimensionless quantities by multiplying with appropriate powers of  $a$ , which simplifies the equations considerably.
- ii) *Index conventions.* We represent tensors of even rank, apart from the metric tensor, with equal numbers of covariant and contravariant indices. This makes contractions trivial to perform and ensures that the components of the tensor have the same physical dimension as the associated contracted scalar.
- iii) *Timing of specialization.* We defer performing the decomposition into scalar, vector and tensor modes as long as possible, and do not make harmonic decompositions. This strategy helps to reveal structure in the equations and serves to reduce the amount of calculation.

### Calculation of $R^{ab}{}_{cd}(\epsilon)$

We begin by deriving an exact expression for the Riemann tensor<sup>36</sup>  $R^{ab}{}_{cd}(\epsilon)$  of the metric  $g_{ab}(\epsilon)$  in terms of the covariant derivative of the conformal background metric

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<sup>36</sup>We use the sign convention of Wald (1984) for defining the Riemann tensor.

$\gamma_{ab}$ . We thus relate the covariant derivative of  $g_{ab}(\epsilon)$ , denoted  ${}^\epsilon\nabla_a$ , to that of  $\gamma_{ab} = \bar{g}_{ab}(0)$ , denoted  ${}^0\nabla_a$ . The relation is given by an object  $Q^a{}_{bc} = Q^a{}_{cb}$  defined by

$$Q^a{}_{bc} = g^{ad}Q_{dbc} = \frac{1}{2}g^{ad}({}^0\nabla_c g_{db} - {}^0\nabla_d g_{bc} + {}^0\nabla_b g_{cd}), \quad (88)$$

(see Wald (1984) equation (D.1)), with the property that<sup>37</sup>

$${}^\epsilon\nabla_a A^b{}_c = {}^0\nabla_a A^b{}_c + Q^b{}_{ad}A^d{}_c - Q^d{}_{ac}A^b{}_d. \quad (89)$$

It is convenient to write  $Q^a{}_{bc}$  as the sum of two parts:

$$Q^a{}_{bc}(\epsilon) = \bar{Q}^a{}_{bc}(\epsilon) + \tilde{Q}^a{}_{bc}(\epsilon). \quad (90)$$

First, the transformation from  ${}^\epsilon\nabla_a$  to  ${}^\epsilon\bar{\nabla}_a$ , which is associated with the conformal transformation  $g_{ab}(\epsilon) = a^2\bar{g}_{ab}(\epsilon)$ , is described by

$$\bar{Q}^a{}_{bc}(\epsilon) = 2\delta^a{}_{(b}r_{c)} - \bar{g}^{ad}(\epsilon)\bar{g}_{bc}(\epsilon)r_d, \quad (91)$$

where<sup>38</sup>

$$r_a := {}^0\nabla_a(\ln a) \quad (92)$$

(see Wald (1984), equation (D.3)). It follows that  ${}^0\nabla_a r_b = {}^0\nabla_b r_a$ . Second, the transformation from  ${}^\epsilon\nabla_a$  to  ${}^0\nabla_a$ , the covariant derivatives associated with  $\bar{g}_{ab}(\epsilon)$  and  $\bar{g}_{ab}(0)$ , respectively, is described by

$$\tilde{Q}^a{}_{bc}(\epsilon) = \frac{1}{2}\bar{g}^{ad}(\epsilon)({}^0\nabla_c \bar{g}_{db}(\epsilon) - {}^0\nabla_d \bar{g}_{bc}(\epsilon) + {}^0\nabla_b \bar{g}_{cd}(\epsilon)). \quad (93)$$

It follows from  ${}^0\nabla_a \gamma_{bc} = 0$  that

$$\tilde{Q}^a{}_{bc}(0) = 0. \quad (94)$$

To calculate  $R^{ab}{}_{cd}(\epsilon)$  we first perform the conformal transformation from  $g_{ab}$  to  $\bar{g}_{ab}$ , which yields

$$a^2 R^{ab}{}_{cd}(\epsilon) = \bar{R}^{ab}{}_{cd}(\epsilon) + 4\delta^{[a}{}_{[c}\bar{U}^{b]}{}_{d]}(\epsilon), \quad (95)$$

where

$$\bar{U}^b{}_d(\epsilon) = -[\bar{g}^{be}({}^\epsilon\nabla_d - r_d) + \frac{1}{2}\delta^b{}_d \bar{g}^{ef} r_f] r_e, \quad (96)$$

and  $\bar{R}^{ab}{}_{cd}(\epsilon)$  is the curvature tensor of the metric  $\bar{g}_{ab}(\epsilon)$  (see Wald (1984), equation (D.7)). Second, by performing the transition from  ${}^\epsilon\nabla_a$  to  ${}^0\nabla_a$  we obtain

$$\bar{R}^{ab}{}_{cd}(\epsilon) = \bar{g}^{be}\bar{R}^a{}_{ecd}(\epsilon) = \bar{g}^{be}\left({}^0\bar{R}^a{}_{ecd} + 2{}^0\nabla_{[c}\tilde{Q}^a{}_{d]e} + 2\tilde{Q}^a{}_{f[c}\tilde{Q}^f{}_{d]e}\right), \quad (97)$$

where  ${}^0\bar{R}^a{}_{bcd}$  is the curvature tensor of the metric  $\gamma_{ab}$  (see Wald (1984), equation (D.7)). The term  $2\bar{g}^{be}{}^0\nabla_{[c}\tilde{Q}^a{}_{d]e}$  in (97) can be written as<sup>39</sup>

$$2\bar{g}^{be}{}^0\nabla_{[c}\tilde{Q}^a{}_{d]e} = 2\bar{g}^{be}({}^0\nabla_{[c}\bar{g}^{af]})\tilde{Q}^a{}_{|f|d]e} + \bar{g}^{be}\bar{g}^{af}({}^0\nabla_{[c}{}^0\nabla_{|e|}\bar{g}_{d]f} - {}^0\nabla_{[c}{}^0\nabla_{|f|}\bar{g}_{d]e}) - \gamma_{ef}\bar{g}^{(b}{}^0\bar{R}^{a)f}{}_{cd}, \quad (98)$$

<sup>37</sup>This example establishes the pattern for a general tensor.

<sup>38</sup>Note that we always use the vector  $r_a$  in covariant form, since  $r_a$  is independent of  $\epsilon$ , whereas  $r^a = g^{ab}(\epsilon)r_b$  is not.

<sup>39</sup>Note that  ${}^0\bar{R}^{ab}{}_{cd} = \gamma^{be}{}^0\bar{R}^a{}_{ecd}$ .

which we use to rearrange (97), in conjunction with the relation  ${}^0\bar{\nabla}_c \bar{g}^{ab} = -2\tilde{Q}^{(ab)}{}_c$ . In summary,  $R^{ab}{}_{cd}(\epsilon)$  is given by equation (95) with

$$\bar{R}^{ab}{}_{cd}(\epsilon) = -2\bar{g}^{e[a}\bar{g}^{b]f} {}^0\bar{\nabla}_{[c} {}^0\bar{\nabla}_{|e|} \bar{g}_{d]f} - \gamma_{ef}\bar{g}^{e[a} {}^0\bar{R}^{b]f}{}_{cd} - 2\tilde{Q}^{f[a}{}_{[c} \tilde{Q}_{|f|}{}^{b]}{}_{d]}, \quad (99a)$$

$$\bar{U}^b{}_d(\epsilon) = -\left[\bar{g}^{be} ({}^0\bar{\nabla}_d - r_d) + \frac{1}{2}\delta^b{}_d \bar{g}^{ef} r_f - \bar{g}^{bf} \tilde{Q}^e{}_{df}\right] r_e, \quad (99b)$$

where we have used  ${}^\epsilon\bar{\nabla}_a r_b = {}^0\bar{\nabla}_a r_b - \tilde{Q}^c{}_{ab} r_c$  in obtaining (99b) from (96).

### Calculation of ${}^{(1)}R^{ab}{}_{cd}$

We now calculate the perturbation  ${}^{(1)}R^{ab}{}_{cd}$  of the Riemann tensor, defined via equation (3), expressing it in terms of the covariant derivative  ${}^0\bar{\nabla}_a$  associated with  $\gamma_{ab}$  and the metric perturbation  $f_{ab} = {}^{(1)}\bar{g}_{ab}$  (see (14)). We note that

$${}^{(1)}\bar{g}^{ab} = -f^{ab}, \quad (100)$$

where the indices on  $f^{ab}$  are raised using  $\gamma^{ab}$ . It follows from (3), (93) (95) and (99), in conjunction with (94) and (100), that<sup>40</sup>

$$a^2{}^{(1)}R^{ab}{}_{cd} = {}^{(1)}\bar{R}^{ab}{}_{cd} + 4\delta^{[a}{}_{[c} {}^{(1)}\bar{U}^{b]}{}_d], \quad (101a)$$

where

$${}^{(1)}\bar{R}^{ab}{}_{cd} = -2{}^0\bar{\nabla}_{[c} {}^0\bar{\nabla}^{[a} f_{d]}{}^{b]} + f_e^{[a} {}^0\bar{R}^{b]e}{}_{cd}, \quad (101b)$$

$${}^{(1)}\bar{U}^a{}_b = \left[f^{ac} ({}^0\bar{\nabla}_b - r_b) + \frac{1}{2}\delta^a{}_b f^{cd} r_d + \gamma^{ad} {}^{(1)}\tilde{Q}^c{}_{bd}\right] r_c, \quad (101c)$$

$${}^{(1)}\tilde{Q}_{abc} = \frac{1}{2} ({}^0\bar{\nabla}_c f_{ab} - {}^0\bar{\nabla}_a f_{bc} + {}^0\bar{\nabla}_b f_{ca}). \quad (101d)$$

Introducing local coordinates  $x^\mu = (\eta, x^i)$  as in section 2.1 leads to

$$r_\alpha = \mathcal{H} \delta^0{}_\alpha, \quad {}^0\bar{\nabla}_0 = \partial_\eta, \quad {}^0\bar{\nabla}_i = \mathbf{D}_i. \quad (102)$$

In addition we note that the quantity  ${}^0\bar{R}^a{}_{bcd}$ , the curvature tensor of the metric  $\gamma_{ab}$ , is zero if one index is temporal, while if all indices are spatial

$${}^0\bar{R}^{ij}{}_{km} = 2K \delta^{[i}{}_{[k} \delta^{j]}{}_{m]}, \quad (103)$$

where the constant  $K$  describes the curvature of the maximally symmetric three-space. Equation (101), in conjunction with (102) and (103), yields the following expressions:

$$a^2{}^{(1)}R^{0j}{}_{0m} = \frac{1}{2}[\mathbf{D}^j \mathbf{D}_m + (\mathcal{H}' - \mathcal{H}^2)\delta^j{}_m]f_{00} + (\partial_\eta + \mathcal{H})Y^j{}_m, \quad (104a)$$

$$a^2{}^{(1)}R^{0j}{}_{km} = 2\mathbf{D}_{[k} Y^j{}_{m]}, \quad (104b)$$

$$a^2{}^{(1)}R^{ij}{}_{km} = -2(\mathbf{D}_{[k} \mathbf{D}^{[i} + K\delta_{[k}^{[i} \delta^{j]}{}_{m]})f_m{}^{j]} + 4\mathcal{H}\delta_{[k}^{[i} Y_m{}^{j]}], \quad (104c)$$

where<sup>41</sup>

$$Y_{ij} = \frac{1}{2}\gamma_{ij}\mathcal{H}f_{00} - \mathbf{D}_{(i}f_{j)0} + \frac{1}{2}\partial_\eta f_{ij}. \quad (104d)$$

<sup>40</sup>Note that  $R^{ab}{}_{cd}(\epsilon)$  depends on  $\epsilon$  through  $\bar{g}_{ab}(\epsilon)$ ,  $\bar{g}^{ab}(\epsilon)$  and  $\tilde{Q}^c{}_{ab}(\epsilon)$ .

<sup>41</sup>Note that  $\tilde{Q}^0{}_{ij} = -\mathbf{D}_{(i}f_{j)0} + \frac{1}{2}f'_{ij}$ .

### Calculation of the Riemann gauge invariants

We now apply the Replacement Principle to (104), which entails performing the following replacements:

$$f_{ab} \rightarrow \mathbf{f}_{ab}[X], \quad Y_{ij} \rightarrow \mathbf{Y}_{ij}[X], \quad a^{2(1)}R^{ab}_{cd} \rightarrow \mathbf{R}^{ab}_{cd}[X], \quad (105)$$

where the gauge invariants are defined by equation (6). All components of the Riemann tensor can be obtained from the ‘curvature spanning set’ ( $R^{0i}_{0j}$ ,  $R^{0i}_{jk}$ ,  $R^{im}_{jm}$ ) or, alternatively, their spatial traces and their trace-free parts:

$$(R^{0m}_{0m}, R^{0m}_{jm}, R^{km}_{km}), \quad (\hat{R}^{0i}_{0j}, \hat{R}^{0i}_{jk}, \hat{R}^{im}_{jm}), \quad (106)$$

where

$$\hat{R}^{0i}_{0j} = R^{0i}_{0j} - \frac{1}{3}\delta^i_j R^{0m}_{0m}, \quad \hat{R}^{im}_{jm} = R^{im}_{jm} - \frac{1}{3}\delta^i_j R^{km}_{km}, \quad (107a)$$

$$\hat{R}^{0i}_{jk} = R^{0i}_{jk} - \delta^i_{[k} R^{0m}_{j]m}. \quad (107b)$$

Our motivation for choosing these particular components as the spanning set is that the first set of terms in (106) are invariant under spatial gauge transformations, while the hatted quantities are fully gauge-invariant, as follows from (5).

We denote the gauge invariants associated with the spanning set (106) by

$$(\mathbf{R}^{0m}_{0m}[X], \mathbf{R}^{0m}_{jm}[X], \mathbf{R}^{km}_{km}[X]), \quad (\hat{\mathbf{R}}^{0i}_{0j}, \hat{\mathbf{R}}^{0i}_{jk}, \hat{\mathbf{R}}^{im}_{jm}), \quad (108)$$

and refer to them as the *Riemann gauge invariants*. As indicated by the notation (*i.e.* no dependence on the gauge field  $X$ ) the hatted quantities are intrinsic gauge invariants. We now substitute the expressions<sup>42</sup> for  $\mathbf{f}_{ab}[X]$  given by (25) into the bold-face version of (104), and calculate the gauge invariants (108). It is convenient to split  $\mathbf{Y}_{ij}$  into a trace and a trace-free part:

$$\hat{\mathbf{Y}}_{ij} = \mathbf{Y}_{ij} - \frac{1}{3}\gamma_{ij}\mathbf{Y}, \quad \mathbf{Y} = \mathbf{Y}^i_i, \quad (109)$$

and to use the trace-free second derivative operator  $\mathbf{D}_{ij}$  defined in (50f). We obtain<sup>43</sup>

$$\mathbf{R}^{0m}_{0m}[X] = -[\mathbf{D}^2 + 3(\mathcal{H}' - \mathcal{H}^2)]\Phi[X] + (\partial_\eta + \mathcal{H})\mathbf{Y}[X], \quad (110a)$$

$$\hat{\mathbf{R}}^{0i}_{0j} = -\mathbf{D}^i_j\Phi[X] + (\partial_\eta + \mathcal{H})\hat{\mathbf{Y}}^i_j[X], \quad (110b)$$

$$\mathbf{R}^{km}_{km}[X] = 4[(\mathbf{D}^2 + 3K)\Psi[X] + \mathcal{H}\mathbf{Y}[X]], \quad (110c)$$

$$\hat{\mathbf{R}}^{im}_{jm} = \mathbf{D}^i_j\Psi[X] + \mathcal{H}\hat{\mathbf{Y}}^i_j[X] - (\mathbf{D}^2 - 2K)\mathbf{C}^i_j, \quad (110d)$$

$$\mathbf{R}^{0m}_{jm}[X] = \frac{2}{3}\mathbf{D}_j\mathbf{Y}[X] - \mathbf{D}_m\hat{\mathbf{Y}}^m_j[X], \quad (110e)$$

$$\hat{\mathbf{R}}^{0i}_{jk} = 2\mathbf{D}_{[j}\hat{\mathbf{Y}}^i_{k]}[X] + \mathbf{D}_m\hat{\mathbf{Y}}^m_{[j}[X]\delta^i_{k]}, \quad (110f)$$

<sup>42</sup>In using these expressions we are making the choice for  $X_i$  given in equation (24). Choosing  $X_i$  in this way simplifies the calculation but not the final form of the Riemann gauge invariants, since, as mentioned earlier, the spanning set is invariant under spatial gauge transformations.

<sup>43</sup>Use the identities (126c), (126d) and (126h).

where

$$\mathbf{Y}[X] = -3(\partial_\eta \Psi[X] + \mathcal{H}\Phi[X]) - \mathbf{D}^2 \mathbf{B}[X], \quad (110g)$$

$$\hat{\mathbf{Y}}_{ij}[X] = -\mathbf{D}_{ij} \mathbf{B}[X] - \mathbf{D}_{(i} \mathbf{B}_{j)} + \partial_\eta \mathbf{C}_{ij}. \quad (110h)$$

These equations constitute one of the main results of this paper. They express the Riemann gauge invariants (108) in terms of the metric gauge invariants (25). They depend only on the choice of the temporal gauge field  $X^0$ , as can be seen from (25d).

### Calculation of the Einstein gauge invariants

The Einstein tensor and the Weyl conformal curvature tensor are defined in terms of the Riemann tensor according to

$$G^a_b := R^a_b - \frac{1}{2} \delta^a_b R, \quad \text{where} \quad R^a_b := R^{ac}_{bc}, \quad R := R^a_b, \quad (111a)$$

$$C^{ab}_{cd} := R^{ab}_{cd} - 2 \delta^{[a}_{[c} R^{b]}_{d]} + \frac{1}{3} \delta^{[a}_{[c} \delta^{b]}_{d]} R. \quad (111b)$$

The curvature spanning set (106) can be replaced with the following spatially irreducible components of the Einstein tensor and the Weyl tensor:<sup>44</sup>

$$(G^0_0, G^m_m, G^0_i, \hat{G}^i_j), \quad (C^{0i}_{0j}, C^{0i}_{jk}), \quad (112)$$

where

$$\hat{G}^i_j := G^i_j - \frac{1}{3} \delta^i_j G^m_m. \quad (113)$$

It follows from (111) that

$$G^0_0 = -\frac{1}{2} R^{km}_{km}, \quad G^m_m = -\frac{1}{2} (R^{km}_{km} + 4R^{0m}_{0m}), \quad (114a)$$

$$G^0_i = R^{0m}_{im}, \quad \hat{G}^i_j = \hat{R}^{0i}_{0j} + \hat{R}^{im}_{jm}, \quad (114b)$$

$$C^{0i}_{0j} = \frac{1}{2} (\hat{R}^{0i}_{0j} - \hat{R}^{im}_{jm}), \quad C^{0i}_{jk} = \hat{R}^{0i}_{jk}. \quad (114c)$$

The Einstein gauge invariants, as defined by equations (39), (40a) and (40b) with  $A$  replaced by  $G$ , can be expressed in terms of the curvature spanning set (106) by using the bold-face version of (114). This yields

$$\hat{\mathbf{G}}^i_j := \hat{\mathbf{G}}^i_j[X] = \hat{\mathbf{R}}^{0i}_{0j} + \hat{\mathbf{R}}^{im}_{jm}, \quad (115a)$$

$$\mathbf{G}_i := -(\mathbf{D}_i \mathbf{G}^0_0[X] + 3\mathcal{H} \mathbf{G}^0_i[X]) = \frac{1}{2} \mathbf{D}_i \mathbf{R}^{km}_{km}[X] - 3\mathcal{H} \mathbf{R}^{0m}_{im}[X], \quad (115b)$$

$$\mathbf{G} := \mathcal{C}_G^2 \mathbf{G}^0_0[X] + \frac{1}{3} \mathbf{G}^m_m[X] = -\frac{1}{6} ((1 + 3\mathcal{C}_G^2) \mathbf{R}^{km}_{km}[X] + 4\mathbf{R}^{0m}_{0m}[X]). \quad (115c)$$

We find that it is simplest to express the Einstein gauge invariants (115) in terms of the *uniform curvature* metric gauge invariants  $\mathbf{A}$  and  $\mathbf{B}$  defined by (31). We accomplish this directly by choosing  $X = X_c$  in (110), and noting that by (27) we have  $\Psi[X_c] = 0$ . After simplifying using the identities (126e) and (126f) we obtain<sup>45</sup>

$$\hat{\mathbf{G}}_{ij} = \mathbf{D}_{ij} \mathbb{G} - \mathbf{D}_{(i} (\partial_\eta + 2\mathcal{H}) \mathbf{B}_{j)} + (\partial_\eta^2 + 2\mathcal{H} \partial_\eta + 2K - \mathbf{D}^2) \mathbf{C}_{ij}, \quad (116a)$$

$$\mathbf{G}_i = 2\mathcal{H} \mathbf{D}_i (\mathbf{D}^2 + 3K) \mathbf{B} + \frac{3}{2} \mathcal{H} (\mathbf{D}^2 + 2K) \mathbf{B}_i, \quad (116b)$$

$$\mathbf{G} = 2\mathcal{H} [(\partial_\eta + \mathcal{B}\mathcal{H}) \mathbf{A} + \mathcal{C}_G^2 \mathbf{D}^2 \mathbf{B}] - \frac{2}{3} \mathbf{D}^2 \mathbb{G}, \quad (116c)$$

<sup>44</sup>Note that  $C^{ij}_{km} = -4C^{0[i}_{0[k} \delta^{j]}_{m]}$  in an orthonormal frame.

<sup>45</sup>Here for convenience we use  $\hat{\mathbf{G}}_{ij} = \gamma_{ik} \hat{\mathbf{G}}^k_j$ .

where we have introduced the notation

$$\mathbb{G} := -[\mathbf{A} + (\partial_\eta + 2\mathcal{H})\mathbf{B}], \quad \mathcal{B} := \frac{2\mathcal{H}'}{\mathcal{H}^2} + 1 + 3\mathcal{C}_G^2. \quad (117)$$

We also need

$$\mathbf{G}_j^0[X] = \mathbf{R}^{0m}_{jm}[X]. \quad (118)$$

We choose  $X = X_p$  in this equation, and using (110) in conjunction with the identity (126f) we obtain

$$\mathbf{G}_j^0[X_p] = -2\mathbf{D}_j(\partial_\eta\Psi + \mathcal{H}\Phi) + \frac{1}{2}(\mathbf{D}^2 + 2K)\mathbf{B}_j. \quad (119)$$

We now use (32) to express the right side of this equation in terms of  $\mathbf{A}$  and  $\mathbf{B}$ , which yields

$$\mathbf{G}_i^0[X_p] = -2\mathbf{D}_i(\mathcal{H}\mathbf{A} + (\frac{1}{2}\mathcal{A}_G - K)\mathbf{B}) + \frac{1}{2}(\mathbf{D}^2 + 2K)\mathbf{B}_i. \quad (120)$$

### The Weyl tensor

The perturbation of the Weyl tensor is automatically gauge-invariant on account of the Stewart-Walker lemma (Stewart and Walker (1974)) since the Weyl tensor is zero in the background. We thus use bold-face notation for its components. From (114c) we obtain

$$\mathbf{C}^{0i}_{0j} = a^{2(1)}C^{0i}_{0j} = \frac{1}{2}(\hat{\mathbf{R}}^{0i}_{0j} - \hat{\mathbf{R}}^{im}_{jm}), \quad \mathbf{C}^{0i}_{jk} = a^{2(1)}C^{0i}_{jk} = \hat{\mathbf{R}}^{0i}_{jk}. \quad (121)$$

The Weyl tensor has a simpler form if we use Poisson gauge invariants and hence we choose  $X = X_p$  in (110). Noting that  $\mathbf{B}[X_p] = 0$  leads to

$$\mathbf{C}^{0i}_{0j} = -\frac{1}{2}[\mathbf{D}^i_j(\Psi + \Phi) + \partial_\eta\mathbf{B}^i_j - (\partial_\eta^2 + \mathbf{D}^2 - 2K)\mathbf{C}^i_j], \quad (122a)$$

$$\mathbf{C}^{0i}_{jk} = -2\mathbf{D}_{[j}(\mathbf{B}^i_{k]} - \partial_\eta\mathbf{C}^i_{k]}) - \mathbf{D}_m\mathbf{B}^m_{[j}\delta^i_{k]}, \quad \mathbf{B}_{ij} := \mathbf{D}_{[i}\mathbf{B}_{j]}. \quad (122b)$$

## B.1 Uniqueness of the decomposition into modes

**Proposition:** If the inverses of the operators  $\mathbf{D}^2$ ,  $\mathbf{D}^2 + 2K$  and  $\mathbf{D}^2 + 3K$  exist, then the equation

$$B_i = \mathbf{D}_i B + \tilde{B}_i, \quad \text{with} \quad \mathbf{D}^i \tilde{B}_i = 0, \quad (123)$$

determines  $B$  and  $\tilde{B}_i$  uniquely in terms of  $B_i$ , and the equation

$$C_{ij} = \mathbf{D}_{ij}C + \mathbf{D}_{(i}C_{j)} + \tilde{C}_{ij}, \quad (124)$$

with

$$\mathbf{D}^i C_i = 0, \quad \tilde{C}_{ij} = \tilde{C}_{ji}, \quad \tilde{C}^i_i = 0, \quad \mathbf{D}^i \tilde{C}_{ij} = 0,$$

determines  $C$ ,  $C_i$  and  $\tilde{C}_{ij}$  uniquely in terms of  $C_{ij}$ . In particular, if  $B_i = 0$  then  $B = 0$ ,  $\tilde{B}_i = 0$ , and if  $C_{ij} = 0$  then  $C = 0$ ,  $C_i = 0$ ,  $\tilde{C}_{ij} = 0$ .

*Proof.* Apply  $\mathbf{D}^i$  to (123) obtaining  $\mathbf{D}^i B_i = \mathbf{D}^2 B$ . Using the inverse operator of  $\mathbf{D}^2$  this equation determines  $B$ , and then (123) determines  $\tilde{B}_i$  uniquely in terms of  $B_i$ . Next, apply  $\mathbf{D}^{ij}$  and  $\mathbf{D}^i$  to (124), obtaining

$$\mathbf{D}^{ij}C_{ij} = \frac{2}{3}\mathbf{D}^2(\mathbf{D}^2 + 3K)C, \quad \mathbf{D}^i C_{ij} = \frac{2}{3}\mathbf{D}_j(\mathbf{D}^2 + 3K)C + (\mathbf{D}^2 + 2K)C_j. \quad (125)$$

By using the inverse operators these equations, in conjunction with (124), successively determine  $C$ ,  $C_i$  and  $\tilde{C}_{ij}$  uniquely in terms of  $C_{ij}$ .  $\square$

## B.2 Identities

In obtaining our results we found the following identities useful:

$$\mathbf{D}_{[i}\mathbf{D}_{j]}A^k = K\delta_{[i}^k A_{j]}, \quad (126a)$$

$$\mathbf{D}_{[k}\mathbf{D}_{m]}A^{ij} = 2K\delta_{[k}^{(i} A_{m]}^{j)}, \quad (126b)$$

$$4(\mathbf{D}_{[k}\mathbf{D}^{[i} + K\delta_{[k}^{[i}]\delta_{m]}^m]A = (\mathbf{D}_k^i + \frac{4}{3}(\mathbf{D}^2 + 3K)\delta_k^i)A, \quad (126c)$$

$$4(\mathbf{D}_{[k}\mathbf{D}^{[i} + K\delta_{[k}^{[i}])C_{j]}^j] = (\mathbf{D}^2 - 2K)C_k^i, \quad (126d)$$

$$\mathbf{D}_j\mathbf{D}_i^j A = \frac{2}{3}\mathbf{D}_i(\mathbf{D}^2 + 3K)A, \quad (126e)$$

$$\mathbf{D}^i\mathbf{D}_{(i}A_{j)} = \frac{1}{2}(\mathbf{D}^2 + 2K)A_j, \quad (126f)$$

$$\mathbf{D}_i\mathbf{D}^2 A^i = (\mathbf{D}^2 + 2K)\mathbf{D}_i A^i, \quad (126g)$$

$$\delta_{[i}^{[i} A_{m]}^{j]} = \frac{1}{4}(A_m^j + \delta_m^j A), \quad (126h)$$

where  $A_{ij} = A_{ji}$ ,  $C_{ij} = C_{ji}$ ,  $C_i^i = 0$  and  $\mathbf{D}_i C_j^i = 0$ .

## B.3 Kinematic quantities

The kinematic quantities associated with a timelike congruence  $u^a$  are defined by the following decomposition into irreducible parts:

$$\nabla_a u_b = -u_a \dot{u}_b + H(g_{ab} + u_a u_b) + \sigma_{ab} + \omega_{ab}. \quad (127)$$

A routine calculation starting with equations (62)-(64) and (89) applied to  $u_a$  yields the following non-zero components:

$$a^{(1)}H = \left[\frac{1}{3}\mathbf{D}^2(v - \chi) - (\partial_\eta \psi + \mathcal{H}\varphi)\right], \quad (128a)$$

$$\dot{\mathbf{u}}_i := {}^{(1)}\dot{u}_i = \mathbf{D}_i(\varphi + (\partial_\eta + \mathcal{H})v) + (\partial_\eta + \mathcal{H})\tilde{\mathbf{v}}_i, \quad (128b)$$

$$\boldsymbol{\sigma}_j^i := a^{(1)}\sigma_j^i = \mathbf{D}_j^i(v - \chi) + \gamma^{ik}\mathbf{D}_{(k}(\tilde{\mathbf{v}}_j) - \mathbf{B}_j) + \partial_\eta \mathbf{C}_j^i, \quad (128c)$$

$$\boldsymbol{\omega}_j^i := a^{(1)}\omega_j^i = \gamma^{ik}\mathbf{D}_{[k}\tilde{\mathbf{v}}_j], \quad (128d)$$

with the bold-face quantities being gauge-invariant on account of the Stewart-Walker lemma.

## C Scalar field

A minimally coupled scalar field  $\phi$  is described by a stress-energy tensor of the form

$$T_b^a = \nabla^a \phi \nabla_b \phi - \left[\frac{1}{2}\nabla^c \phi \nabla_c \phi + U(\phi)\right] \delta_b^a, \quad (129)$$

with the associated Klein-Gordon equation  $\nabla^c \nabla_c \phi - U_{,\phi} = 0$ , where the potential  $U(\phi)$  has to be specified. This stress-energy tensor is of the form (60) with

$$\rho + p = -\nabla^a \phi \nabla_a \phi, \quad \rho - p = 2U(\phi), \quad \pi_{ab} = 0. \quad (130)$$

When evaluated on the RW background, equation (130) leads to

$$a^2({}^{(0)}\rho + {}^{(0)}p) = ({}^{(0)}\phi')^2, \quad ({}^{(0)}\rho - {}^{(0)}p) = 2U({}^{(0)}\phi). \quad (131)$$

On using (131) to calculate  ${}^{(0)}\rho'$ , the conservation equation (43) leads to

$${}^{(0)}\phi'' + 2\mathcal{H}{}^{(0)}\phi' + a^2 U_{,\phi} = 0, \quad (132)$$

which is the Klein-Gordon equation in the RW background. Further, by means of (42), (43), (131) and (132) we obtain

$$\mathcal{A}_T = ({}^{(0)}\phi')^2, \quad \mathcal{C}_T^2 = 1 + \frac{2a^2 U_{,\phi}}{3\mathcal{H}{}^{(0)}\phi'} = -\frac{1}{3} \left( 1 + \frac{2{}^{(0)}\phi''}{\mathcal{H}{}^{(0)}\phi'} \right). \quad (133)$$

Viewing  $T^a_b$  and  $\phi$  as functions of the perturbation parameter  $\epsilon$ , we can use (129), in conjunction with (3), to calculate  ${}^{(1)}T^a_b$ , obtaining

$${}^{(1)}\hat{T}^i_j = 0, \quad a^2 {}^{(1)}T^0_i = -{}^{(0)}\phi' \mathbf{D}_i {}^{(1)}\phi, \quad {}^{(1)}T^0_0 + \frac{1}{3} {}^{(1)}T^i_i = -2U_{,\phi} {}^{(1)}\phi. \quad (134)$$

It follows using (38) with  $A$  replaced by  $T$  and (133), that the matter gauge invariants assume the form

$$\hat{\mathbf{T}}^i_j = 0, \quad \mathbf{T}^0_i[X] = -{}^{(0)}\phi' \mathbf{D}_i \phi[X], \quad \mathbf{T}^0_0[X] + \frac{1}{3} \mathbf{T}^i_i[X] = -2a^2 U_{,\phi} \phi[X], \quad (135)$$

where  $\phi[X]$  is the gauge invariant associated with  ${}^{(1)}\phi$  by X-replacement, given by

$$\phi[X] = {}^{(1)}\phi - {}^{(0)}\phi' X^0. \quad (136)$$

Equations (135) and (50) immediately lead to the expressions for the matter gauge invariants (81b) and (81c), including

$$V[X] = -{}^{(0)}\phi' \phi[X]. \quad (137)$$

Equation (135), in conjunction with (133) and (137), yields

$$\mathbf{T}^0_0[X] + \frac{1}{3} \mathbf{T}^i_i[X] = -3(1 - \mathcal{C}_T^2) \mathcal{H} V[X]. \quad (138)$$

We now substitute (138) into the expression for  $\Gamma$  given by (46c) and (50c) to obtain<sup>46</sup>

$$\Gamma = (1 - \mathcal{C}_T^2)(-\mathbf{T}^0_0[X] - 3\mathcal{H}V[X]). \quad (139)$$

which on comparison with (68) leads to equation (81a).

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<sup>46</sup>Write the expression for  $\Gamma$  in the form  $\Gamma = -(1 - \mathcal{C}_T^2)\mathbf{T}^0_0 + (\mathbf{T}^0_0 + \frac{1}{3}\mathbf{T}^i_i)$ .

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