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HKT Geometry and Fake Five Dimensional Supergravity

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Abstract. Recent results on the relation between hyper-Kähler geometry with Torsion and solutions admitting Killing spinors in minimal de sitter supergravity are extended to more general supergravity models with vector multiplets.

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1. Introduction

The strong relation between complex geometry and supersymmetry has been known for some time by now. It was observed first by Zumino [1] that demanding $N = 2$ supersymmetry on a two-dimensional non-linear sigma model puts the restriction that the target space metric must be described by a Kähler manifold. Extending the supersymmetry to $N = 4$, the target manifold then becomes a hyper-Kähler manifold [2]. The Wess-Zumino-Witten couplings [3] in the non-linear sigma model can be interpreted as torsion potentials from the target space viewpoint [4, 5]. Thus, it is natural to expect that adding such couplings will lead to Kähler and hyper-Kähler torsion (HKT) target space geometries [6, 7, 8, 9, 10].

In string compactifications, demanding that the four dimensional low energy action has $N = 1$ supersymmetry forces the six dimensional compact manifold to be a Calabi-Yau 3-fold [11]. Another connection between complex geometry and supersymmetry was also revealed in the study of the moduli space metrics of supersymmetric electrically charged five-dimensional black holes which was found to be described by a HKT manifold [12, 13].

More recently, Kähler and hyper-Kähler geometry also arise in connection with the study of supersymmetric solutions in supergravity theories. For instance, the four-dimensional base spaces of time-like supersymmetric solutions of ungauged and gauged five dimensional supergravity are given, respectively, by a hyper-Kähler and Kähler manifold [14, 15, 16].

The embedding of cosmological Einstein gravity in a supergravity theory is allowed provided that the cosmological constant is either vanishing or negative. However, in the case of a positive cosmological constant, the concept of fake supergravity can be introduced as a solution generating technique. In this case, a Killing spinor equation is the analytic continuation of the equation resulting from the vanishing of the gravitini supersymmetry variation in the corresponding theory with negative cosmological constant. Recently, the programme of the classification of all solutions admitting (pseudo-)Killing spinors in de Sitter supergravity theories was initiated in [17, 18]. There it was shown that the base space of time-like solutions of five dimensional de Sitter supergravity is given by four dimensional HKT geometry. Moreover, solutions admitting null Killing vectors were later analysed in [19] where it was found that those solutions are related to a one-parameter family of Gauduchon-Tod spaces [20]. Our present work is the generalisation of the results of [17, 18] to five dimensional supergravity models with scalar fields which could be of relevance to cosmological models.

This paper is organized as follows. Section two contains a brief description of the models under study and the analysis of the fake gravitino and gaugino Killing spinor equations of the five dimensional de Sitter supergravity with vector multiplets. The general structure of the pseudo-supersymmetric solutions, admitting Killing spinors that give rise to a timelike vector field, is obtained. In section three and four we provide some

examples and in section five we give some final remarks.

2. Fake $N = 2$ supergravity and Killing Spinors

The models we will be considering in our present work are the $N = 2$, $D = 5$ gauged supergravity coupled to abelian vector multiplets [21] whose bosonic action is given by

$$S = \frac{1}{16\pi G} \int (R + 2g^2\mathcal{V}) *1 - Q_{IJ} (dX^I \wedge *dX^J + F^I \wedge *F^J) - \frac{C_{IJK}}{6} F^I \wedge F^J \wedge A^K \quad (1)$$

where I, J, K take values $1, \dots, n$ and $F^I = dA^I$ are the two-forms representing gauge field strengths (one of the gauge fields corresponds to the graviphoton). The constants C_{IJK} are symmetric in IJK , we will assume that Q_{IJ} is invertible, with inverse Q^{IJ} . The X^I are scalar fields subject to the condition

$$\frac{1}{6} C_{IJK} X^I X^J X^K = X_I X^I = 1. \quad (2)$$

The fields X^I can thus be regarded as being functions of $n - 1$ unconstrained scalars ϕ^r . We list some useful relations

$$\begin{aligned} Q_{IJ} &= \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K, \\ Q_{IJ} X^J &= \frac{3}{2} X_I, \quad Q_{IJ} dX^J = -\frac{3}{2} dX_I, \\ \mathcal{V} &= 9V_I V_J (X^I X^J - \frac{1}{2} Q^{IJ}), \end{aligned} \quad (3)$$

here V_I are constants.

Fake supergravity theory is obtained by sending g^2 to $-g^2$ in the above action. We begin the analysis of pseudo-supersymmetric de-Sitter solutions by examining the fake gravitino Killing spinor equation:

$$\left[\nabla_M - \frac{i}{8} \Gamma_M H_{N_1 N_2} \Gamma^{N_1 N_2} + \frac{3i}{4} H_M{}^N \Gamma_N - g \left(\frac{i}{2} X \Gamma_M - \frac{3}{2} A_M \right) \right] \epsilon = 0, \quad (4)$$

where we have defined

$$V_I X^I = X, \quad V_I A^I{}_M = A_M, \quad X_I F^I{}_{MN} = H_{MN}. \quad (5)$$

We shall analyse the solutions of the Killing spinor equations using spinorial geometry techniques originally developed to analyse supersymmetric solutions in ten and eleven dimensional supergravity [22, 23], and which have since been used to analyse a large variety of supersymmetric solutions in numerous theories. For de Sitter supergravity in five-dimensions, one takes the space of Dirac spinors to be the space of complexified forms on \mathbb{R}^2 , which are spanned over \mathbb{C} by $\{1, e_1, e_2, e_{12}\}$ where $e_{12} = e_1 \wedge e_2$. The action of complexified Γ -matrices on these spinors is given by

$$\begin{aligned} \Gamma_\alpha &= \sqrt{2} e_\alpha \wedge, \\ \Gamma_{\bar{\alpha}} &= \sqrt{2} i e_\alpha, \end{aligned} \quad (6)$$

for $\alpha = 1, 2$, and Γ_0 satisfies

$$\Gamma_0 1 = -i1, \quad \Gamma_0 e^{12} = -ie^{12}, \quad \Gamma_0 e^j = ie^j \quad j = 1, 2. \quad (7)$$

The spacetime metric has signature $(-, +, +, +, +)$ and is written in the following basis

$$ds^2 = -(e^0)^2 + 2\delta_{\alpha\bar{\beta}} e^\alpha e^{\bar{\beta}}. \quad (8)$$

The $Spin(4, 1)$ gauge transformations can be used to fix the Killing spinor to take the form $\epsilon = f1$. Moreover, we can set $f = 1$, using the \mathbb{R} transformation [17, 18]

$$\epsilon \rightarrow e^\lambda \epsilon, \quad V_I A^I \rightarrow V_I A^I - \frac{2}{3g} d\lambda, \quad (9)$$

which leaves the Killing spinor equation invariant. With all this information, we obtain from (4), the following conditions:

$$\begin{aligned} H_\alpha{}^\alpha + 2gX - 6gA_0 - 2\Omega_{0,\alpha}{}^\alpha &= 0, \\ H_{0\alpha} - \Omega_{0,0\alpha} &= 0, \\ \left(\Omega_{0,\bar{\alpha}\bar{\beta}} - \frac{1}{2} H_{\bar{\alpha}\bar{\beta}} \right) \epsilon^{\bar{\alpha}\bar{\beta}} &= 0, \\ \frac{1}{2} \Omega_{\beta,\alpha}{}^\alpha + \frac{3}{4} H_{0\beta} + \frac{3g}{2} A_\beta &= 0, \\ \Omega_{\alpha,0\bar{\beta}} + \frac{1}{2} H_{\mu}{}^\mu \delta_{\alpha\bar{\beta}} - \frac{3}{2} H_{\alpha\bar{\beta}} + gX \delta_{\alpha\bar{\beta}} &= 0, \\ \Omega_{\beta,\bar{\mu}\bar{\nu}} \epsilon^{\bar{\mu}\bar{\nu}} + H^{0\mu} \epsilon_{\alpha\mu} &= 0, \\ \Omega_{\bar{\alpha},0\bar{\beta}} - \frac{1}{2} H_{\bar{\alpha}\bar{\beta}} &= 0, \\ \Omega_{\bar{\beta},\mu}{}^\mu + \frac{1}{2} H_{0\bar{\beta}} + 3gA_{\bar{\beta}} &= 0, \\ \Omega_{\bar{\beta},\bar{\mu}\bar{\nu}} \epsilon^{\bar{\mu}\bar{\nu}} &= 0. \end{aligned} \quad (10)$$

The above equations then imply:

$$\begin{aligned} A_0 &= \frac{X}{3}, \\ A_\alpha &= -\frac{1}{3g} \Omega_{0,0\alpha}, \\ H_{0\alpha} &= \Omega_{0,0\alpha}, \\ H_{\alpha\bar{\beta}} &= \frac{2}{3} (\Omega_{\alpha,0\bar{\beta}} + \Omega_{\mu,0}{}^\mu \delta_{\alpha\bar{\beta}} + 3gX \delta_{\alpha\bar{\beta}}), \\ H_{\bar{\alpha}\bar{\beta}} &= 2\Omega_{\bar{\alpha},0\bar{\beta}}, \end{aligned} \quad (11)$$

together with the purely geometric conditions

$$\begin{aligned}
 \Omega_{[0,\alpha]\beta} &= 0, \\
 \Omega_{0,\mu}{}^\mu - \Omega_{\mu,0}{}^\mu - 2gX &= 0, \\
 \Omega_{(\alpha,|0|\bar{\beta})} &= -gX\delta_{\alpha\bar{\beta}}, \tag{12}
 \end{aligned}$$

and

$$\begin{aligned}
 \Omega_{\alpha,\mu\nu} &= 0, \\
 \Omega_{\alpha,\beta}{}^\beta + \frac{1}{2}\Omega_{0,0\alpha} &= 0, \\
 \Omega_{\alpha,\bar{\mu}\bar{\nu}} - \frac{1}{2}\delta_{\alpha\bar{\mu}}\Omega_{0,0\bar{\nu}} + \frac{1}{2}\delta_{\alpha\bar{\nu}}\Omega_{0,0\bar{\mu}} &= 0. \tag{13}
 \end{aligned}$$

The gaugino Killing spinor equation is given by

$$\left((F_{MN}^I - X^I H_{MN})\Gamma^{MN} - 2i\nabla_M X^I \Gamma^M - 4gV_J (X^I X^J - \frac{3}{2}Q^{IJ}) \right) \epsilon = 0 \tag{14}$$

which, on setting $\epsilon = 1$, implies

$$\begin{aligned}
 F^I{}_\alpha{}^\alpha &= X^I H_\alpha{}^\alpha, \\
 F^I{}_{0\alpha} &= X^I H_{0\alpha} - \partial_\alpha X^I, \\
 F^I{}_{\alpha\beta} &= X^I H_{\alpha\beta}, \\
 \partial_0 X^I &= 2g(X^I V_J X^J - \frac{3}{2}Q^{IJ} V_J). \tag{15}
 \end{aligned}$$

To proceed, we examine the conditions implied by (12) and (13). Define the 1-form $V = \mathbf{e}^0$, and introduce a t coordinate such that the dual vector field is $V = -\frac{\partial}{\partial t}$. We also introduce the real coordinates x^m , for $m = 1, 2, 3, 4$. The vielbein is then given by

$$\mathbf{e}^0 = dt + \omega_m dx^m, \quad \mathbf{e}^\alpha = \mathbf{e}^\alpha{}_m dx^m. \tag{16}$$

From (12), it can be easily demonstrated that

$$\begin{aligned}
 (\mathcal{L}_V \mathbf{e}^\alpha)_{\bar{\beta}} &= 0, \\
 (\mathcal{L}_V \mathbf{e}^\alpha)_\beta &= \left(\Omega_{0,\beta}{}^\alpha - \Omega_{\beta,0}{}^\alpha + \frac{1}{2}(\Omega_{0,\mu}{}^\mu + \Omega_{\mu,0}{}^\mu) \delta^\alpha{}_\beta \right) - gX\delta^\alpha{}_\beta. \tag{17}
 \end{aligned}$$

The quantity

$$\Omega_{0,\beta}{}^\alpha - \Omega_{\beta,0}{}^\alpha + \frac{1}{2}(\Omega_{0,\mu}{}^\mu + \Omega_{\mu,0}{}^\mu) \delta^\alpha{}_\beta \tag{18}$$

is antihermitian and traceless (i.e. $\in su(2)$) and as such it can be gauged away by applying a $SU(2) \subset Spin(4, 1)$ gauge transformation to the Killing spinors, which leaves 1 invariant. In this gauge,

$$\mathcal{L}_V \mathbf{e}^\alpha = -gX\mathbf{e}^\alpha. \tag{19}$$

Define $\hat{\mathbf{e}}^\alpha$ by

$$\mathbf{e}^\alpha = G\hat{\mathbf{e}}^\alpha, \quad (20)$$

with

$$\frac{\partial_t G}{G} = gX, \quad (21)$$

then

$$\mathcal{L}_V \hat{\mathbf{e}}^\alpha = 0. \quad (22)$$

In what follows we introduce the base manifold B with the t -independent metric

$$ds_B^2 = 2\delta_{\alpha\bar{\beta}}\hat{\mathbf{e}}^\alpha\hat{\mathbf{e}}^{\bar{\beta}}. \quad (23)$$

Let us denote the spin connections on the manifold B by $\hat{\Omega}$ and rewrite the conditions in (13) in terms of $\hat{\Omega}$. The third condition in (13) can be written as

$$2\hat{\Omega}_{\alpha,\bar{\mu}\bar{\nu}} - 2G^{-1}(\tilde{\partial}_{\hat{\mu}}G\delta_{\alpha\bar{\nu}} - \tilde{\partial}_{\bar{\nu}}G\delta_{\alpha\hat{\mu}}) - G^{-2}(\delta_{\alpha\bar{\mu}}\partial_t(G^2\omega_{\hat{\nu}}) - \delta_{\alpha\bar{\nu}}\partial_t(G^2\omega_{\hat{\mu}})) = 0. \quad (24)$$

Contracting with $\delta^{\alpha\bar{\nu}}$ we obtain

$$\partial_t(G^2\omega_{\hat{\mu}}) = -2G^2\hat{\Omega}_{\alpha,\bar{\mu}}{}^\alpha + (\tilde{d}G^2)_{\hat{\mu}} \quad (25)$$

which gives

$$\omega_{\hat{\mu}} = M\hat{\Omega}_{\alpha,\bar{\mu}}{}^\alpha + G^{-2}(\mathcal{Q} + \tilde{d} \int G^2)_{\hat{\mu}} \quad (26)$$

with

$$\partial_t \mathcal{Q}_{\hat{\mu}} = 0, \quad M = -\frac{2}{G^2} \int G^2. \quad (27)$$

Therefore we can write

$$\omega = M\mathcal{P} + G^{-2}(\mathcal{Q} + \tilde{d} \int G^2), \quad (28)$$

where

$$\mathcal{P} \equiv \hat{\Omega}_{\bar{\beta},\alpha}{}^{\bar{\beta}}\hat{\mathbf{e}}^\alpha + \hat{\Omega}_{\beta,\bar{\alpha}}{}^\beta\hat{\mathbf{e}}^{\bar{\alpha}} = \mathcal{P}_m dx^m, \quad (29)$$

and \mathcal{Q} are 1-forms on the base manifold B satisfying

$$\mathcal{L}_V \mathcal{Q} = 0, \quad \mathcal{L}_V \mathcal{P} = 0. \quad (30)$$

The remaining two conditions in (13) give

$$\hat{\Omega}_{\alpha,\mu\nu} = 0, \quad (31)$$

and

$$\hat{\Omega}_{\alpha,\mu}{}^\mu - \hat{\Omega}_{\bar{\mu},\alpha}{}^{\bar{\mu}} = 0. \quad (32)$$

It is convenient to define the almost hypercomplex structure

$$\begin{aligned}
 \mathbf{J}^1 &= \hat{\mathbf{e}}^1 \wedge \hat{\mathbf{e}}^2 + \hat{\mathbf{e}}^{\bar{1}} \wedge \hat{\mathbf{e}}^{\bar{2}} , \\
 \mathbf{J}^2 &= i\hat{\mathbf{e}}^1 \wedge \hat{\mathbf{e}}^{\bar{1}} + i\hat{\mathbf{e}}^2 \wedge \hat{\mathbf{e}}^{\bar{2}} , \\
 \mathbf{J}^3 &= -i\hat{\mathbf{e}}^1 \wedge \hat{\mathbf{e}}^2 + i\hat{\mathbf{e}}^{\bar{1}} \wedge \hat{\mathbf{e}}^{\bar{2}} ,
 \end{aligned} \tag{33}$$

where \mathbf{J}^i satisfy the algebra of the imaginary unit quaternions. The conditions (31) and (32) are then equivalent to

$$d\mathbf{J}^i = -2\mathcal{P} \wedge \mathbf{J}^i , \quad i = 1, 2, 3 , \tag{34}$$

where \mathcal{P} is given by (29). The condition (34) implies that the base B is hyper-Kähler with torsion (HKT) ‡, i.e.

$$\nabla^+ \mathbf{J}^i = 0 , \tag{35}$$

where the connection of the covariant derivative ∇^+ is given by

$$\Gamma^{(+)\ i}_{\ jk} = \{^i_{jk}\} + \Theta^i_{\ jk} , \tag{36}$$

and where Θ is the torsion 3-form on B given by

$$\Theta = \star_4 \mathcal{P} , \tag{37}$$

where we take the volume form on the base space B to be $-\frac{1}{2}\mathbf{J}^1 \wedge \mathbf{J}^1$, in this convention \mathbf{J}^i are anti-self-dual. Note that (34) implies that

$$d\mathcal{P} \wedge \mathbf{J}^i = 0 \tag{38}$$

for $i = 1, 2, 3$. Equivalently, the anti-self-dual projection of $d\mathcal{P}$ vanishes,

$$(d\mathcal{P})^- = 0 . \tag{39}$$

The conditions (10) give for the gauge fields

$$A = V_I A^I = -\frac{2}{3g} G^{-1} dG + X \mathbf{e}^0 + \frac{2}{3g} \mathcal{P} \tag{40}$$

and

$$H = d\mathbf{e}^0 + \Psi \tag{41}$$

where Ψ is a traceless (1,1) form on B , i.e. Ψ is a self-dual 2-form on B , with

$$\Psi = \frac{4}{3} (G^{-2} \int G^2) d\mathcal{P} - \frac{2}{3} G^{-2} (d\mathcal{Q} + 2\mathcal{Q} \wedge \mathcal{P})^+ \tag{42}$$

where here $^+$ denotes the self-dual projection onto the base space B . Next we consider the conditions obtained from (15). The first three conditions imply that

$$F^I = d(X^I \mathbf{e}^0) + \Psi^I \tag{43}$$

‡ In [24], a subclass of the pseudo-supersymmetric solutions with hyper-Kähler base space was examined. However, in general the base space is hyper-Kähler with torsion, and the classification constructed in this paper describes all pseudo-supersymmetric solutions.

where Ψ^I are closed, t -independent self-dual 2-forms on B , satisfying, as a consequence of (40),

$$V_I \Psi^I = \frac{2}{3g} d\mathcal{P} \quad (44)$$

and as a consequence of (41)

$$X_I \Psi^I = \Psi \quad (45)$$

where Ψ is given by (42).

The final condition in (15) implies that

$$X_I = 2g(G^{-2} \int G^2) V_I + G^{-2} Z_I, \quad \partial_t Z_I = 0 \quad (46)$$

where Z_I are t -independent functions on B .

On substituting (46) into (45), and making use of (44) one obtains

$$Z_I \Psi^I = -\frac{2}{3} (d\mathcal{Q} + 2\mathcal{Q} \wedge \mathcal{P})^+ . \quad (47)$$

Next, it is convenient to make a co-ordinate transformation to simplify the solution, and define

$$u = \int G^2. \quad (48)$$

The metric, gauge field strengths and scalars are then given by

$$\begin{aligned} ds^2 &= -G^{-4} (du - 2u\mathcal{P} + \mathcal{Q})^2 + G^2 ds_B^2, \\ F^I &= d\left(G^{-2} X^I (du - 2u\mathcal{P} + \mathcal{Q})\right) + \Psi^I, \\ X_I &= G^{-2} (2guV_I + Z_I). \end{aligned} \quad (49)$$

where ds_B^2 is the u -independent metric on a (strong) HKT manifold B . \mathcal{P} is a u -independent 1-form on B satisfying (34) and $d\mathcal{P}$ is a self-dual 2-form on B . \mathcal{Q} is another u -independent 1-form on B , Z_I are u -independent functions on B , and Ψ^I are closed self-dual, u -independent 2-forms on B satisfying

$$V_I \Psi^I = \frac{2}{3g} d\mathcal{P} \quad (50)$$

and

$$Z_I \Psi^I = -\frac{2}{3} (d\mathcal{Q} + 2\mathcal{Q} \wedge \mathcal{P})^+ . \quad (51)$$

It remains to consider the gauge field equations; one finds, after some computation, that

$$\hat{\nabla}^i \left(-\frac{3}{2} dZ_I + 3Z_I \mathcal{P} + 3gV_I \mathcal{Q} \right)_i + \frac{1}{8} C_{IJK} \Psi_{ij}^J \Psi^{Kij} = 0 \quad (52)$$

where $\hat{\nabla}$ denotes the Levi-Civita connection of B , and here all indices are frame indices on B . We remark that, as a consequence of the integrability conditions examined in Appendix B of [25], the Killing spinor equations, together with the gauge field

equations and Bianchi identity are sufficient to imply that the Einstein and the scalar field equations hold automatically, without any further conditions.

Note that the solution (49) together with the conditions in (50), (51) and (52) are invariant under the conformal re-scaling

$$ds_B^2 = e^{-2h} ds_B'^2 \quad (53)$$

where h is a u -independent function, together with the re-definitions

$$u = e^{2h} u', \quad \mathcal{P} = \mathcal{P}' + dh, \quad \mathcal{Q} = e^{2h} \mathcal{Q}', \quad Z_I = e^{2h} Z_I', \quad G = e^h G'. \quad (54)$$

A HKT manifold is called *strong* HKT if the associated torsion Θ is closed, or equivalently

$$d \star_4 \mathcal{P} = 0 \quad (55)$$

where \star_4 denotes the Hodge dual on B . For the solutions under consideration here, by making an appropriate conformal transformation as described above, one can without loss of generality take B to be a strong HKT manifold.

In order to recover the solutions for the minimal theory determined in [17, 18], one sets

$$C_{111} = \frac{2}{\sqrt{3}}, \quad X^1 = \sqrt{3}, \quad X_1 = \frac{1}{\sqrt{3}} \quad (56)$$

and hence we set

$$V_1 = \frac{1}{\sqrt{3}}, \quad Z_1 = 0. \quad (57)$$

In addition, one has

$$G = e^{gt}, \quad g = -\frac{\chi}{2\sqrt{3}}, \quad \Psi^1 = -\frac{4}{\chi} d\mathcal{P}, \quad F^1 = 2F \quad (58)$$

where F is the Maxwell field strength of the minimal theory.

It is also useful to consider a co-ordinate transformation of the form

$$u' = u - \Theta \quad (59)$$

where the function Θ does not depend on u , and set

$$\mathcal{Q}' = \mathcal{Q} - 2\Theta\mathcal{P} + d\Theta, \quad Z_I' = Z_I + 2g\Theta V_I. \quad (60)$$

Under these transformations, the solution given in (49), (50), (51), together with the gauge equations (52) are invariant. It is clear that one can always choose the function Θ such that

$$d \star_4 \mathcal{Q}' = 0 \quad (61)$$

and one can therefore work in a gauge for which *both* \mathcal{P} and \mathcal{Q} are co-closed. It should however be noted that the gauge in which \mathcal{Q} is co-closed is not the same gauge in which the solutions to the minimal theory are constructed as described in [17, 18]; this is because the minimal theory gauge has $Z_1 = 0$ and G is a function only of t . One cannot in general make a gauge transformation of the form described above and keep $Z_1' = 0$ as well. In what follows it will be most convenient to work with the gauge choice for which

$$d \star_4 \mathcal{P} = d \star_4 \mathcal{Q} = 0. \quad (62)$$

3. Solutions with a tri-holomorphic isometry

It is straightforward to analyse the case when the base manifold B is strong HKT and admits a tri-holomorphic isometry, which we denote by $\frac{\partial}{\partial x^5}$, and we take this isometry to be a symmetry of the full solution. Such base spaces have been classified in [20, 26, 27], and the metric on B is given by

$$ds_B^2 = W^{-1}(dx^5 + \varphi)^2 + W ds_E^2 \quad (63)$$

where E is a constrained 3-dimensional Einstein Weyl geometry, consisting of a x^5 -independent 3-metric γ_{ij} , a x^5 -independent 1-form α on E , and an x^5 -independent scalar α_0 on E , satisfying

$$\star_E d\alpha = -d\alpha_0 - \alpha_0 \alpha, \quad d \star_E \alpha = 0 \quad (64)$$

where \star_E denotes the Hodge dual on E , and the Ricci tensor of E satisfies

$${}^{(E)}R_{ij} + \nabla_{(i}\alpha_{j)} + \alpha_i\alpha_j = \gamma_{ij}\left(\frac{1}{2}\alpha_0^2 + \alpha^\ell\alpha_\ell\right) \quad (65)$$

where here ∇ denotes the Levi-Civita connection of E , and φ is a x^5 independent 1-form on E satisfying

$$\star_E d\varphi = dW + W\alpha \quad (66)$$

and the function W does not depend on x^5 . The volume form of B , ϵ_B , and the volume form of E , $d\text{vol}_E$ are related by

$$\epsilon_B = W(dx^5 + \varphi) \wedge d\text{vol}_E. \quad (67)$$

The torsion of B is determined by \mathcal{P} , with

$$\mathcal{P} = -\frac{\alpha_0}{2W}(dx^5 + \varphi) - \frac{1}{2}\alpha \quad (68)$$

which is co-closed as a consequence of the previous conditions. We further remark that the functions W , α_0 satisfy

$$(\Delta_E + \alpha^i \nabla_i)W = (\Delta_E + \alpha^i \nabla_i)\alpha_0 = 0 \quad (69)$$

where Δ_E is the Laplacian on E .

To proceed further with the analysis, note that self-duality of Ψ^I , together with the requirement that $d\Psi^I = 0$, are equivalent to

$$\Psi^I = -\frac{1}{2}(dx^5 + \varphi) \wedge d(W^{-1}K^I) - \frac{1}{2}W \star_E d(W^{-1}K^I) \quad (70)$$

where K^I are x^5 -independent functions on E satisfying

$$(\Delta_E + \alpha^i \nabla_i)K^I = 0. \quad (71)$$

The condition (50) constrains the K^I via

$$V_I K^I = \frac{1}{3g}\alpha_0 + kW \quad (72)$$

for constant k . Next, it is straightforward to solve the gauge equation (52) to find

$$Z_I = \frac{1}{24}C_{IJK}W^{-1}K^J K^K + L_I \quad (73)$$

where L_I are x^5 -independent functions on E satisfying

$$(\Delta_E + \alpha^i \nabla_i)L_I = 0. \quad (74)$$

Finally we solve for the 1-form \mathcal{Q} . We decompose this 1-form as

$$\mathcal{Q} = Q_5(dx^5 + \chi) + \tilde{Q} \quad (75)$$

where the function Q_5 does not depend on x^5 , and \tilde{Q} is a x^5 -independent 1-form on E . The condition $d \star_B \mathcal{Q} = 0$ implies that

$$d \star_E \tilde{Q} = 0 \quad (76)$$

and the condition (51), after some manipulation, implies that

$$Q_5 = -\frac{1}{48}W^{-2}C_{IJK}K^I K^J K^K - \frac{3}{4}W^{-1}L_I K^I + M \quad (77)$$

where M is a x^5 -independent function on E satisfying

$$(\Delta_E + \alpha^i \nabla_i)M = 0 \quad (78)$$

and \tilde{Q} also must satisfy

$$d\tilde{Q} + \alpha \wedge \tilde{Q} + \alpha_0 \star_E \tilde{Q} = W \star_E dM - M \star_E dW - \frac{3}{4}(K^I \star_E dL_I - L_I \star_E dK^I). \quad (79)$$

4. Solutions with a Conformally Hyper-Kähler base

Suppose that the base space B is conformally hyper-Kähler. Then \mathcal{P} is closed, and using the conformal transformation described in (53) and (54), one can without loss of generality set $\mathcal{P} = 0$, i.e. one can take B to be a hyper-Kähler manifold, which we denote by HK . We shall also work in a gauge for which \mathcal{Q} is co-closed on HK , as described previously. Hence the solution can be written as

$$\begin{aligned} ds^2 &= -G^{-4}(du + \mathcal{Q})^2 + G^2 ds_{HK}^2, \\ F^I &= d\left(G^{-2}X^I(du + \mathcal{Q})\right) + \Psi^I, \\ X_I &= G^{-2}(2guV_I + Z_I), \end{aligned} \quad (80)$$

where ds_{HK}^2 is the u -independent metric on a hyper-Kähler manifold HK , \mathcal{Q} is a u -independent 1-form on HK , Z_I are u -independent functions on HK , and Ψ^I are self-dual, u -independent 2-forms on HK satisfying

$$d\Psi^I = 0 \quad (81)$$

and

$$V_I \Psi^I = 0 \quad (82)$$

and

$$Z_I \Psi^I = -\frac{2}{3}(d\mathcal{Q})^+ \quad (83)$$

and

$$d \star_4 \mathcal{Q} = 0 \tag{84}$$

and

$$d \star_4 dZ_I + \frac{1}{6} C_{IJK} \Psi^J \wedge \Psi^K = 0 . \tag{85}$$

In order to recover the special case for which the base space is hyper-Kähler with a triholomorphic isometry, i.e. a Gibbons-Hawking manifold, for which the triholomorphic isometry is a symmetry of the full solution, one takes the analysis of the previous section and sets $E = \mathbb{R}^3$, $\alpha_0 = 0$, $\alpha = 0$, with $W = H$ where H is a harmonic function on \mathbb{R}^3 and φ is a x^5 -independent 1-form on \mathbb{R}^3 satisfying

$$d\varphi = \star_{\mathbb{R}^3} dH \tag{86}$$

and the remaining functions K^I , L_I , M which are used in the construction of the solution are also harmonic functions on \mathbb{R}^3 . We remark that one can also allow Z_I to depend linearly on x^5 by taking

$$Z_I = \frac{1}{24} C_{IJK} H^{-1} K^J K^K + L_I + c V_I x^5 \tag{87}$$

for constant c , with Ψ^I , \mathcal{Q} unchanged (and H , K^I , L_I , M still x^5 -independent). It is straightforward to show that adding such a term linear in x^5 to Z_I does not give any contribution to the LHS of the conditions (83) and (85).

The black hole solutions found in [28, 29, 30] are a special case of the solutions found here, for which all the harmonic functions depend only on r , and hence have poles only at $r = 0$.

5. Final Remarks

In this paper we have studied timelike solutions admitting Killing spinors of five dimensional de Sitter supergravity with Abelian vector multiplets. The four dimensional base space of these solutions was found to be given by a four dimensional HKT geometry. In our present work we have also described two special classes of solution. First, we considered the case when the HKT manifold admits a tri-holomorphic Killing vector field. Then we considered the case for which the HKT manifold is conformally hyper-Kähler. The conformally hyper-Kähler class of solutions include all previously constructed solutions in the literature as special cases. It would be of great interest to construct new solutions in the non-conformally hyper-Kähler case, as these might be of relevance to black hole physics and cosmology. It would also be particularly interesting to determine whether there exist regular (pseudo) supersymmetric black ring solutions in de Sitter supergravity. Finally, a continuation of our present work is to study the solutions of the null case for the theories considered here and possibly generalising these results to de Sitter supergravity in other dimensions. Work along these directions is in progress.

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