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Isocausal spacetimes may have different causal boundaries

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Abstract

We construct an example which shows that two isocausal spacetimes, in the sense introduced recently in [12], may have c-boundaries which are not equal (more precisely, not equivalent, as no bijection between the completions can preserve all the binary relations induced by causality). This example also suggests that isocausality can be useful for the understanding and computation of the c-boundary.

Keywords: causal boundary, causal mappings, isocausality, conformal structure of spacetimes, Causality.

2010 MSC: 83C75, 53C50, 83C20.

1 Introduction

The *causal boundary*, or c-boundary for short, is a well-known tool for the study of the conformal structure of a spacetime and related topics such as event horizons or singularities. The first approximation to this boundary was introduced four decades ago by Geroch, Kronheimer and Penrose (GKP) in the seminal paper [16]. Since then, a long series of redefinitions and new contributions has been carried out, and a renewed interest comes from the recent contributions by Harris [17, 18, 19] and Marolf and Ross [20, 21] (see the review in [24] for complete references). Recently, the authors have carried out an extensive revision of both, the notion of c-boundary and the tools for its computation [5, 7, 8]. So, the c-boundary can be regarded now as a useful and consistent notion, which is well related to other geometric objects. Along this paper we understand by c-boundary the last redefinition in [7]. Nevertheless, the properties to be considered here appear at a much more basic level (say, whenever Harris' universal properties for the partial boundaries are satisfied [17]). So, they are valid for any definition of the c-boundary obtained by using the basic ingredients in the seminal GKP construction and, in particular, for all the previous redefinitions of the c-boundary along the literature.

Some years ago, García-Parrado and Senovilla [12] introduced the notions of *causal mapping*, *causal relation* and *isocausality* for two spacetimes V, V' . Namely, V is *causally related* to V' , denoted $V \prec V'$, if there exists a diffeomorphism $\Phi : V \rightarrow V'$ which is a *causal mapping*, that is, such that all the future-directed causal vectors of V are mapped by the differential of Φ into future-directed causal ones of V' . Then, V is *isocausal* to V' if $V \prec V'$ and $V' \prec V$. In that article and subsequent developments [13, 14, 11], many applications and properties of such notions were carried out. Recall that isocausality is a generalization of conformal equivalence, adding more flexibility. This flexibility yields appealing properties, as the fact that any spacetime is *locally isocausal* to Lorentz-Minkowski one, even if it is not conformally flat. So, isocausality preserves some relevant global properties associated to the conformal structure, but not all of them —as stressed in [11] for the case of two levels of the causal ladder of spacetimes.

It was also suggested in [12, Sect. 6] that causal mappings could be used to obtain *causal extensions* and boundaries for spacetimes as a generalization of the (Penrose) *conformal boundary*. Concretely, a *causal extension* is an embedding of the spacetime in a larger one such that the former is isocausal to its image in the larger. Clearly, a boundary can be then naturally associated to such an extension (here, we will avoid the name *causal boundary* for this last boundary, in order to avoid confusions with the c-boundary). Recall that, in spite of its generalized usage in General Relativity, the conformal boundary has serious problems of existence and uniqueness. The problems come from the fact that, in order to find a reasonable conformal boundary, one has to find an appropriate open conformal embedding of the spacetime in some (*aphysical*) spacetime. It is not clear when such an embedding will exist and, in this case, if the properties of the corresponding boundary will be

independent of the embedding¹. The flexibility of causal mappings and isocausal properties, allows to check their existence much more easily than their conformal counterparts, even though with a cost of uniqueness.

In the present note, we explore the connections between the c-boundary and the notion of isocausality by means of a concrete example. This example firstly shows that *two isocausal spacetimes may have different c-boundaries*. That is, even though the c-boundary relies on the global conformal structure of the spacetime, it is not an object naturally invariant by isocausality. At a first glance, this property would seem a drawback for the notion of isocausality. On one hand, isocausality would be insufficient to distinguish between two spacetimes with different asymptotic causal behaviors. On the other, the boundaries obtained by using different causal extensions appear as extremely non-unique —as the causal extensions of isocausal spacetimes with different c-boundaries, may look very different. However, a deeper study suggests that, when the causal extensions are compared with the conformal ones, these properties are not a disadvantage. Notice that, essentially, the conformal boundary becomes interesting when it agrees with some intrinsic element of the spacetime, and conditions to ensure this agreement are commonly imposed (see [2, 7]). But the most important intrinsic element of the spacetime which may match with the conformal boundary is the c-boundary; so, basically, the conformal boundary becomes useful as an auxiliary tool to compute the more general c-boundary. On the contrary, the properties which remain true for all the elements of a class of isocausal spacetimes (in particular, the possible similarities of their c-boundaries or of the boundaries obtained through causal extensions), become a genuinely new type of information, which reveals new connections among non-conformally related spacetimes. In fact, a closer look at our example in this article, suggests that *causal mappings and isocausality may yield a very valuable information on the c-boundary*. Here, we explain this possibility only for our particular example, in order to provide a natural intuitive picture. By using the machinery introduced in [8], this idea will be developed technically in a further work [9].

2 The example

Typical background and terminology in Lorentzian Geometry as in [3, 22, 23] and on causal boundaries as in [3, 7, 15] will be used. From the technical viewpoint, our example will be very simple, and the c-boundary will be handled at a very elementary level. Basically, the idea to construct the c-boundary ∂V of a (strongly causal) spacetime V starts by defining its *future causal boundary* $\hat{\partial}V$ and the dual past one $\check{\partial}V$. The former is the set of all the *TIP*'s (terminal indecomposable past subsets) of V , where any TIP can be regarded as the chronological past $I^-[\gamma]$ of

¹ An example of the difficulties can be found in the recent article [4]. In order to ensure uniqueness, some technical assumptions (which involve any pair of lightlike curves) must be assumed. Remarkably, the existence of a maximal conformal extension is also ensured in [4]. However, this does not exclude the possibility that no extension exists, nor ensures a priori good properties for such an extension.

some future-directed inextendible timelike curve γ (an obvious dual definition appears for the elements of $\hat{\partial}V$, or *TIFs*). A long-standing problem for the definition of the c-boundary appears when one realizes that, eventually, some points in $\hat{\partial}V$ must be paired with some others in $\hat{\partial}V$. Even though this problem can be solved satisfactorily [7], we will not worry about it. In fact, our example will be robust, in the sense that even the partial boundary $\hat{\partial}V$ will not be preserved by isocausality. Moreover, our concrete example is bidimensional, and the TIPs can be also generated as the chronological past of (piecewise smooth) lightlike curves, instead of timelike ones —this will be straightforward here, however, one can find in [10, Proposition 2] and [7, Sect. 3.5] a precise justification. So, the picture of the c-boundary is simplified, as in dimension two the (smooth) lightlike curves must lie in two families of geodesics.

2.1 Abstract properties. Our aim is to endow the manifold $V = \mathbb{R} \times (-\infty, 0)$ with three metrics g_{cl} , g , g_{op} satisfying the following properties:

- (i) $g_{cl} \prec_0 g \prec_0 g_{op}$ where the symbol $\prec_0$ means that the future causal cones of the metric at the left-hand side are included in the ones of the metric at the right-hand one (i.e., the identity in V is a causal mapping from V endowed with the left metric to V endowed with the right one).
- (ii) g_{cl} and g_{op} are conformally related. So the future causal boundaries $\hat{\partial}_{cl}V$, $\hat{\partial}_{op}V$ for, resp., $V_{cl} := (V, g_{cl})$ and $V_{op} := (V, g_{op})$ agree and, taking into account property (i), (V, g) is isocausal to V_{cl} (and V_{op}). Moreover g_{cl} and g_{op} will be simple standard static metrics, so that its causal boundary will be easily computable.
- (iii) g presents a future causal boundary $\hat{\partial}V$ “strictly greater” than the one of g_{cl} (or g_{op}), in a precise sense explained below. Essentially, a segment of *causally but not chronologically* related points, appears for $\hat{\partial}V$ where only a point (in a timelike part of the boundary) existed for $\hat{\partial}_{cl}V$ and $\hat{\partial}_{op}V$.

Note that the non-preservation of the c-boundary by isocausality follows from these properties. So, once the metrics are achieved, we will pass to discuss the interplay between the c-boundary and isocausality.

2.2 Explicit construction. Define the metrics g_{cl} , g , g_{op} on $V = \mathbb{R} \times (-\infty, 0)$ in the following way:

$$g_{cl} = -dt^2 + dx^2, \quad g = -dt^2 + \beta(t/x)dx^2, \quad g_{op} = -dt^2 + (1/4) dx^2,$$

where $\beta : \mathbb{R} \rightarrow (0, \infty)$ is a smooth function which satisfies:

- $\beta(u) \equiv 1/4$ if $u(= t/x) \leq 1/2$, that is, $g = g_{op}$ in the region $x \leq 2t$.
- $\beta(u) \equiv 1$ if $u \geq 1$, that is, $g = g_{cl}$ in the region $t \leq x(< 0)$.
- β increases strictly from $1/4$ to 1 on the interval $1/2 \leq u \leq 1$, so that the causal cones of g_{cl} (resp. of g) are strictly contained in the ones of g (resp. of g_{op}) in the region $2x < 2t < x$.

Note that the announced property (i) becomes clear from the properties of β . About (ii), the conformal relation between g_{cl} and g_{op} is also obvious. Moreover, the future causal boundary $\hat{\partial}_{cl}V$ can be represented by two lines $\mathcal{T}, \mathcal{J}^+$ with a common endpoint i^+ , which is the TIP equal to all V (see [19, 1, 6, 8] for much more general computations, which include the c-boundary of all the standard static spacetimes). More precisely, the TIPs which constitute $\mathcal{T}$ are the chronological past of all the future-directed lightlike geodesics ρ with endpoint at $x = 0$. $\mathcal{T}$ is timelike in the sense that any two distinct TIPs $P, P' \in \mathcal{T}$ satisfy either $P \ll P'$ or $P' \ll P$, where the *extended chronological relation* $\ll$ can be defined here as: $P \ll P'$ if and only if there exists some $p' \in P'$ such that $p \ll p'$ for all $p \in P$. It is also clear that, for the (future) chronological topology on $\hat{\partial}V$ (which here reduces to the point set convergence of the corresponding TIP's as subsets of V , see [6, 7, 8]) $\mathcal{T}$ will be homeomorphic to $\mathbb{R}$. That is, in the following, $\mathcal{T}$ will be identified with $\mathbb{R} \times \{0\}$ (each $P \in \mathcal{T}$ is identified with the endpoint in $\mathbb{R} \times \{0\}$ of the lightlike geodesic whose past is equal to P), and this identification holds at the point set, chronological and topological levels. The TIPs which constitute $\mathcal{J}^+$ are the chronological pasts of all the future-directed lightlike ρ as above which goes to infinity (reaching arbitrarily large values of $-x$). We will not pay attention to this line $\mathcal{J}^+$, but we point out that it is *horismotic*. This means that any two distinct TIPs $P, P' \in \mathcal{J}^+$ are *horismotically related*, i.e. they satisfy either $P \subset P'$ or $P' \subset P$, but neither $P \ll P'$ nor $P' \ll P$.

For the property (iii), let us focus on the timelike line $\mathcal{T}$, identified with $\mathbb{R} \times \{0\}$. Our aim is to prove that, in addition to this timelike line, the future causal boundary $\hat{\partial}V$ of (V, g) contains other boundary points $P = I^-[\rho]$ such that $(0, 0)$ is the endpoint of the generating future-directed lightlike curve ρ .

Consider the lightlike vector field $X(t, x) = (\sqrt{\beta(t/x)}, 1)$ for g . All the integral curves of X can be written as $\gamma_t(s) = (r_t(s), s)$, with $s < 0$ and $r_t : (-\infty, 0) \rightarrow \mathbb{R}$ satisfying:

$$\begin{cases} \dot{r}_t(s) = \sqrt{\beta\left(\frac{r_t(s)}{s}\right)} \\ r_t(-1) = t \end{cases} \quad (2.1)$$

(see Figure 1). Note the following properties of the curves γ_t :

- (a) For $t_1 < t_2$, necessarily $r_{t_1}(s) < r_{t_2}(s)$, as $r_{t_1}(-1) = t_1 < t_2 = r_{t_2}(-1)$ and γ_{t_1} does not intersect γ_{t_2} .
- (b) $\gamma_{-1/2}(s) = (s/2, s)$ and $\gamma_{-1}(s) = (s, s)$ for all $s < 0$, and thus, any intermediate γ_t satisfies:

$$\lim_{s \rightarrow 0} \gamma_t(s) = (0, 0) \quad \forall t \in [-1, -1/2].$$

- (c) $I^-[\gamma_{t_1}] \subsetneq I^-[\gamma_{t_2}]$ for all $t_1 < t_2$.

In fact, (a) and (b) are direct consequences of the definition of γ_t . The property (c) is a consequence of (a) and the following characterization:

$$I^-[\gamma_t] = \{(t', s) : t' < r_t(s)\} \quad \forall t \in \mathbb{R}. \quad (2.2)$$

The inclusion $\supset$ follows because, for the metric g , $t' < r_t(s)$ implies $(t', s) \ll (r_t(s), s)$. For $\subset$, recall that $V \setminus \{\gamma_t(s) : s < 0\}$ has two connected components, and the right-hand side of (2.2) is equal to one of them. Any past-directed timelike curve α starting at a point p on γ_t must enter initially in this region (as any tangent vector in the past timelike cone at p , points to it). Moreover, α cannot touch γ_t at a distinct (first) point q , as α and γ_t would intersect transversally and, so, the velocity α' would point out to the future on q . As consequence α remains totally contained in that region up to the initial point p .

From the properties (b) and (c), different TIPs $P_t := I^-[\gamma_t]$, with $t \in [-1, -1/2]$, become naturally associated to the point $(0, 0)$ (which was identified with a point of $\hat{\partial}_{cl}V$). This implies the required property (iii). In fact, the description of the the boundary $\hat{\partial}V$ for g is similar to the one of $\hat{\partial}_{cl}V$. However, now in the analog to the timelike line $\mathcal{T} \subset \partial_{cl}V$, the boundary point associated to $(0, 0)$ must be replaced by all the TIPs in the *strain* $\text{Str} := \{P_t : -1 \leq t \leq -1/2\}$. So, we can regard $\mathcal{T}_{\text{Str}} = ((\mathbb{R} \setminus \{0\}) \times \{0\}) \cup \text{Str}$, as a part of $\hat{\partial}V$ (see Fig. 2). Recall that all the points in the strain are horismotically related. So, $\mathcal{T}_{\text{Str}}$ differs from $\mathcal{T}$ from the chronological viewpoint (there exists no bijection from $\mathcal{T}_{\text{Str}}$ in $\mathcal{T}$ which preserves the chronologically and horismotically related points). Nevertheless, if one replaces the whole strain by any of its elements, this bijection appears naturally. Summing up, the claimed property (iii), as well as the non-equivalence of $\hat{\partial}V$ and $\hat{\partial}_{cl}V$, are justified in a precise way.

2.3. Final discussion. We can understand the behavior of the causal boundary in the previous example as follows.

Consider two causally related spacetimes $V_1 \prec_0 V_2$ (we will write I_1^-, I_2^- instead of I^- in each spacetime). A natural map between the future boundaries $\hat{j} : \hat{\partial}V_1 \rightarrow \hat{\partial}V_2$ can be defined by taking into account that if $P \in \hat{\partial}V_1$ then $I_2^-(P) \in \hat{\partial}V_2$. In fact, if $P = I_1^-[\gamma]$ for some inextendible future-directed timelike curve γ , then γ must be timelike also for V_2 , and $I_2^-[\gamma] = I_2^-(P)$. So, we can define:

$$\hat{j}(P) := I_2^-(P), \quad \forall P \in \hat{\partial}V_1.$$

Nevertheless, $\hat{j}$ may be very bad-behaved, even if V_1 and V_2 are isocausal. Concretely, our example above shows that the map $\hat{j}_{cl} : \hat{\partial}V_{cl} \rightarrow \hat{\partial}V$ associated to $V_{cl} \prec_0 V$ cannot be continuous (nor surjective), as it induces a map $\mathcal{T} \rightarrow \mathcal{T}_{\text{Str}}$ where $\hat{j}(0, 0)$ chooses just the point $I^-(\gamma_{-1})$ of the strain. Even more, the map $\hat{j}_{op} : \hat{\partial}V \rightarrow \hat{\partial}_{op}V$ associated to $V \prec_0 V_{op}$ is continuous, but it is not injective, as all the strain is mapped into $(0, 0)$. It is worth pointing out that, in spite of these properties, the composition $\hat{j}_{op} \circ \hat{j}_{cl} : \hat{\partial}_{cl}V \rightarrow \hat{\partial}_{op}V$ is an isomorphism (a homeomorphism which also preserves the chronological relation). Our example shows that, this nice last property does not imply a straightforward good relation between $\hat{\partial}V$ and $\hat{\partial}_{cl}V$.

However, the example suggests another possibility. Assume that all the elements in the strain of $\hat{\partial}V$ were identified to a single one. Then $\hat{\partial}_{cl}V$ would be naturally embedded in this quotient space (in this particular example, they would be naturally

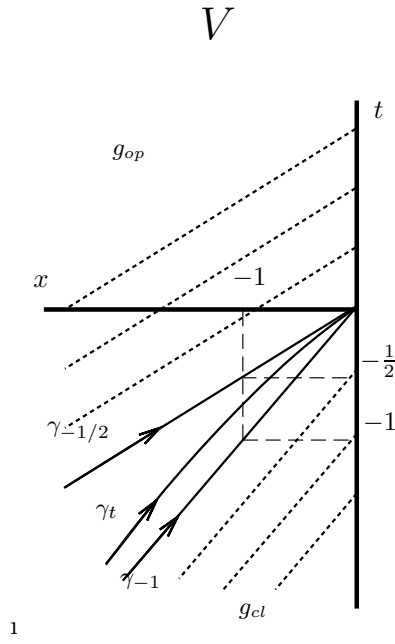


Figure 1: Computed with the metric g , the curves $\gamma_{-1/2}$, γ_t and γ_{-1} are lightlike and define different TIPs. These TIPs are naturally associated to the point $(0, 0)$. However, the point $(0, 0)$ is associated only to one TIP when the metric g_{op} or g_{cl} is considered (recall that these two metrics are conformal).

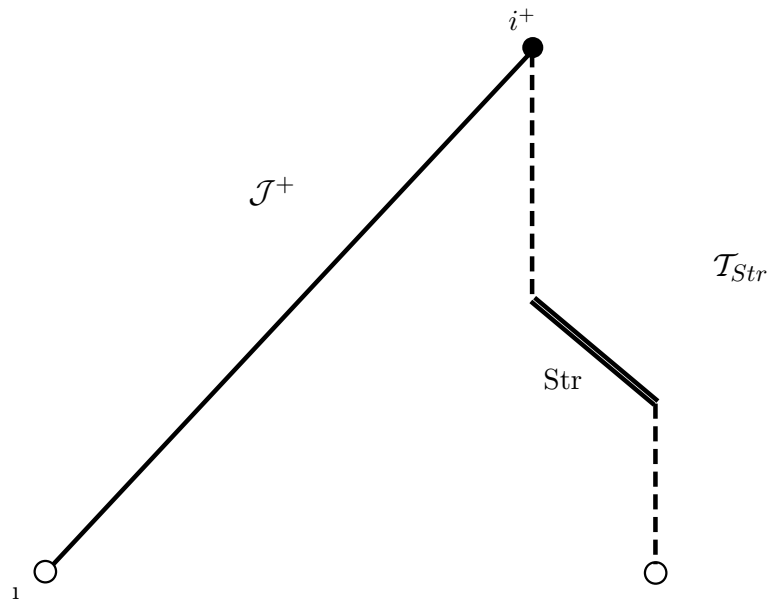


Figure 2: Structure of the future causal boundary for (V, g) . The part of the boundary $\mathcal{T}_{Str}$ is composed by two timelike lines and a lightlike one, denoted by Str in the picture, which corresponds with the *strain* $\text{Str} = \{P_t : -1 \leq t \leq -1/2\}$.

isomorphic). In this sense, the boundary $\hat{\partial}_{cl}V$ yields an important information about the boundary $\hat{\partial}V$, namely: $\hat{\partial}_{cl}V$ represents the quotient of a part of $\hat{\partial}V$ (alternatively, $\hat{\partial}V$ can be seen as an enlargement of $\hat{\partial}_{cl}V$). At what extent is this property generalizable? We will prove that it can be extended to a wide family of spacetimes which are isocausal to the standard stationary ones. However, the computation of such boundaries requires the machinery on Finsler metrics and Busemann functions developed in [8]. So, it is postponed to a forthcoming paper [9].

Appendix

Our example can be understood more clearly as the spacetime (V, g) is conformal (thus, isocausal) to the following open region of Minkowski spacetime:

$$V' = \mathbb{L}^2 \setminus (\{x \geq a\} \cup \{t + x \geq 0, 0 \leq x \leq a\}),$$

where $a = (\pi/2) - \arctan(1/2)$. A conformal map $f : (V, g) \rightarrow (V', g_0)$ is represented in Figure 3, and can be described as follows.

The spacetime (V, g) is divided in three regions: (A) the wedge (i.e., the region between γ_{-1} and $\gamma_{-1/2}$), (B) the region above the wedge (above $\gamma_{-1/2}$), and (C) the region below the wedge (below γ_{-1}). Accordingly, the spacetime (V', g_0) is also divided in three regions: (A') $\{(t, x) \in V' : -2a \leq t - x \leq 0\}$, (B') $\{(t, x) \in V' : t - x \geq 0\}$ and (C') $\{(t, x) \in V' : t - x \leq -2a\}$. Given a point p_A of the region (A), there exist two lightlike geodesics $\gamma_{p_A}, \sigma_{p_A}$ passing through it, which are integral curves of the lightlike vector fields $X(t, x) = (\sqrt{\beta(t/x)}, 1)$, $Y(t, x) = (\sqrt{\beta(t/x)}, -1)$, resp. These curves determine the parameters r_{p_A} (the natural Euclidean distance from $\sigma_{p_A} \cap \gamma_{-1/2}$ to the origin) and α_{p_A} (the Euclidean angle between the velocities of $\gamma_{-1/2}$ and γ_{p_A} at the origin), as indicated in the figure. Then, the image $f(p_A)$ is defined as the point in the region (A') which lies in the line $t - x = -2\alpha_{p_A}$ at the natural Euclidean distance r_{p_A} from $(-\alpha_{p_A}, \alpha_{p_A})$. Next, given a point p_B in region (B), it is clearly determined by the parameters t_{p_B} (where $(t_{p_B}, 0)$ is the future endpoint of the integral curve of the lightlike vector field X through p_B) and r_{p_B} (Euclidean distance to this endpoint from p_B). Then, the image $f(p_B)$ is defined as the point in region (B') determined by the analogous parameters for an integral curve of $\partial_t + \partial_x$, as indicated in the figure. Finally, for any p_C belonging to region (C) we proceed similarly to obtain parameters r_{p_C}, t_{p_C} , and define $f(p_C)$ in the region (C') of (V', g_0) as the point determined by $t_{p_C} - a$ (which selects an integral curve of $\partial_t + \partial_x$) and r_{p_C} (which selects a point in this curve).

Recall that this map f is obviously continuous and piecewise smooth. Its conformal character is ensured as it clearly maps lightlike curves in (V, g) into lightlike curves in (V', g_0) .

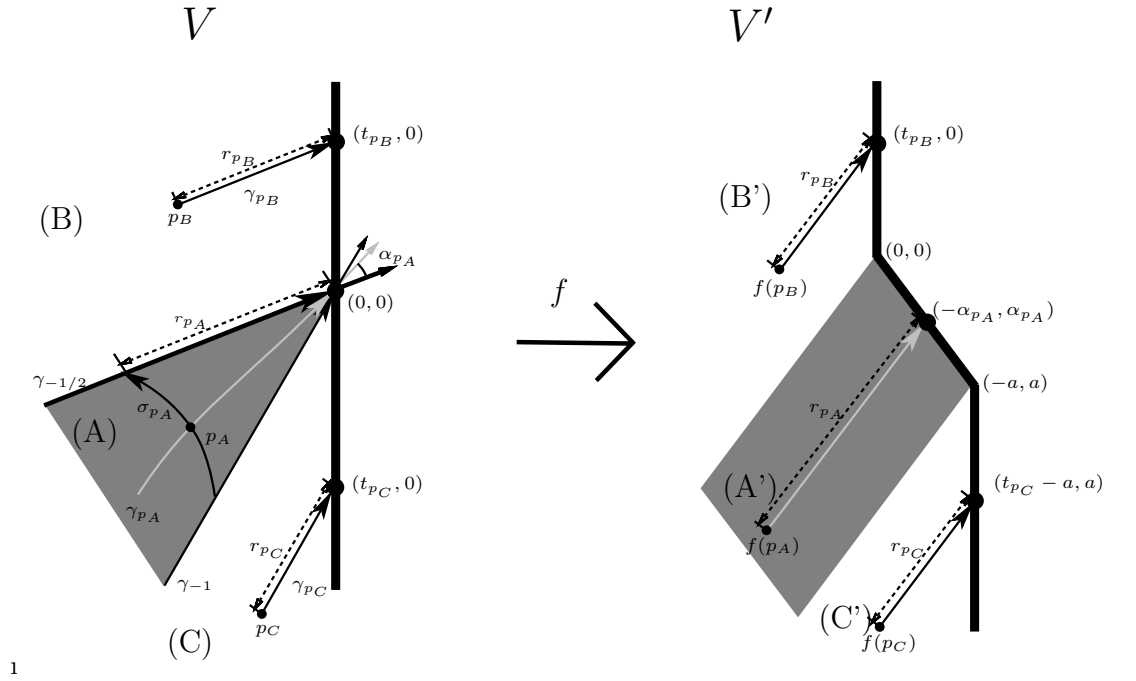


Figure 3: This figure represents a conformal map between the spacetime (V, g) (at the left) and an open region V' of Minkowski spacetime (at the right).

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