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APPROXIMATION OF STOCHASTIC PROCESSES WITH CONTINUOUS PETRI NETS AND CLASSIFICATION METHODS

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ABSTRACT: *Reliability analysis is often based on stochastic discrete event models like stochastic Petri nets. For complex dynamical systems with numerous components, analytical expressions of the steady state are tedious to work out because of the combinatory explosion with discrete models. For this reason, fluidification is an interesting alternative to estimate the asymptotic behavior of stochastic processes with continuous Petri nets. Unfortunately, the asymptotic mean marking of stochastic and continuous Petri nets are mainly often different. This paper combines a geometric approach that leads to a homothetic approximation of the stochastic steady state in sub regions of the marking space with a classifier that selects the sub region of interest and maps the parameters of the stochastic model with the ones of the fluid model.*

KEYWORDS: *Reliability analysis, SPN, continuous Petri nets, fluidification, steady state, geometric approach*

1 INTRODUCTION

Reliability analysis is a major challenge to improve the safety of industrial processes. For complex dynamical systems with numerous interdependent components, such studies are mainly based on stochastic discrete event models like Markov models (Rausand, M. and A. Hoyland, 2004) or stochastic Petri nets (SPNs) (Molloy, M.K., 1982). Such models are mathematically well founded and lead either to analytical results or numerical simulations. But in case of large systems, the combinatory explosion limits their use. In particular when high availability constraints are considered (high availability implies no human intervention to restore operation and refers to availability at less equal to 99%), the number of states becomes rapidly huge and state enumeration is no longer computable.

In such cases, fluidification can be discussed as a relaxation method (Recalde L. and M. Silva, 2002), (Recalde L. and M. Silva, 2004). The main idea of Petri nets fluidification is to replace a discrete Petri net (PN) by a continuous one. An open issue is that numerous structural and behavioral properties are not preserved with standard fluidification (standard means that both models have the same structure, parameters and initial state) (Julvez J. et *al*., 2005). The standard fluidification of SPNs leads to continuous models so that the steady states of SPNs and contPNs do not coincide in many cases, particularly for non-ordinary PNs or non join-free PNs. Approximations provided by the steady state of contPNs are "acceptable" only if the net is heavy loaded and the marking vector does not leave the neighborhood of initial marking (Vazquez R. et *al*., 2008). Markovian and Hybrid Markovian Continuous Petri Nets have been introduced to relax these conditions (Vazquez R. and M. Silva, 2009) but in the former works the continuous models are no longer deterministic. In (Lefebvre D. et *al*., 2009), (Lefebvre D. and E. Leclercq, 2011), piecewise constant timed continuous PNs have been proposed that are suitable to compute the SPNs steady state in non critical regions (non critical means that each join is driven by a different place). Finally, a homothetic approach has been developed to provide an approximation of the SPNs steady state in critical region (Lefebvre D et *al*., 2010), (Lefebvre D., 2011). The limitation of the previous works is that they are not constructive and provide only a global understanding of the SPNs and contPNs behaviors.

This paper continues the investigation of this problem and proposes a parameter classification approach that is suitable to compute the transitions maximal firing speeds that lead to a good approximation of SPNs steady states. For this purpose, the state space is divided into polyhedral cells, classifying the asymptotic mean markings of SPNs. A mapping is defined between these cells, the firing rates of the stochastic model and the parameters of the continuous model. This mapping is based on a homothetic approximation of the stochastic steady state previously developed by the author (Lefebvre D. et *al*., 2010), (Lefebvre D., 2011) and the classifier is trained with a set of arbitrary firing parameters. After training, it can be used in order to obtain proper parameters for contPNs that approximate the steady state of SPNs with arbitrary firing parameters.

2 PROBLEM STATEMENT

2.1 Petri nets and reduced marking space

A Petri net (PN) is defined as $<\boldsymbol{P}, \boldsymbol{T}, W_{PR}, W_{PO}>$ where $\boldsymbol{P} = \{P_i\}$ is a set of n places and $\boldsymbol{T} = \{T_j\}$ is a set of q transitions, $W = W_{PO} - W_{PR} \in (\mathbf{Z})^{n \times q}$ is the incidence matrix, $M(t)$ is the PN marking vector and M_I the PN initial marking (David *et al.*, 1992) (David R., and H. Alla, 1992). $X(t)$ stands for the PN transition throughputs. Depending on the incidence matrix, PNs may have P-semiflows. A P-semiflows $y \in (\mathbf{Z}^+)^n$ is a non-zero solution of equation $y^T.W = 0$. Let define $Y = \{y_1, ..., y_{hp}\}$ as a basis of W^T kernel, composed of hp minimal P-semiflows. For simplicity, the basis Y will be represented as a matrix $Y \in (\mathbf{Z}^+)^{n \text{ x } hp}$ that satisfies (1):

$$Y^T.M(t) = Y^T.M_I = C, \ t \geq 0 . \quad (1)$$

According to Y, let us define a regular permutation matrix D, such that $(Y_1 \mid Y_2) = Y^T.D^{-1}$ with $Y_1 \in (\mathbf{Z}^+)^{hp \text{ x } (n-hp)}$ and $Y_2 \in (\mathbf{Z}^+)^{hp \text{ x } hp}$ of full rank hp. The permutation matrix D may be written as $D = (D^T_1 \mid D^T_2)^T$ with $D_1 \in \{0, 1\}^{(n-hp) \text{ x } n}$ and $D_2 \in \{0, 1\}^{hp \text{ x } n}$ and similarly the matrix D^{-1} satisfies $D^{-1} = (D'_1 \mid D'_2)$ with $D'_1 \in \{0, 1\}^{n \text{ x } (n-hp)}$ and $D'_2 \in \{0, 1\}^{n \text{ x } hp}$. Let us define the driven marking vector $M_2 = D_2.M \in (\mathbf{R}^+)^{hp}$ that may be recovered with respect to Y ($\boldsymbol{P_2} \subset \boldsymbol{P}$ is a places subset of dimension hp) and the reduced marking vector $M_1 = D_1.M \in (\mathbf{R}^+)^{n-hp}$ as the marking of the places that do not belong to $\boldsymbol{P_2}$ (these places form a subset $\boldsymbol{P_1} \subset \boldsymbol{P}$ of dimension $n - hp$).
It is possible to work out M_2 from the reduced marking vector M_1 and as a consequence to write the full marking vector M according to M_1. The equation (1) can be rewritten as $Y^T.D^{-1}.D.M(t) = Y_1.M_1(t) + Y_2.M_2(t) = C$ that leads to (2):

$$M_2(t) = (Y_2)^{-1}.(C - Y_1.M_1(t)) . \quad (2)$$

Then, $M(t) = D'_1.M_1(t) + D'_2.M_2(t)$ and (3) holds:

$$M(t) = F.M_1(t) + G.C . \quad (3)$$

with :

$$F = D'_1 - D'_2.(Y_2)^{-1}.Y_1$$
$$G = D'_2.(Y2)^{-1} . \quad (4)$$

2.2 Stochastic Petri nets

A stochastic Petri net (SPN) is a timed PN whose transitions firing periods are characterized a firing rate vector $\mu = (\mu_j) \in (\mathbf{R}^+)^q$ (Ajmone M. and G. Chiola, 1987.), (Molloy M. K., 1982). The marking vectors of a marked SPN at time t will be referred as $M_s(t, M_I)$. The SPNs considered in this paper are bounded, reinitialisable, with infinite server semantic, race policy and resampling memory. As a consequence, the considered SPNs have a reachability graph with a finite number N of states and their marking process is mapped into a Markov model with state space isomorphic to the reachability graph (Bobbio, A. et *al.*, 1998). The Markov model has an asymptotic state propability vector $\Pi_{ss} = (\pi_{ss\,k}) \in [0, 1]^{1 \text{ x } N}$ and the asymptotic mean marking $M_{mms} = (m_{mms\ i}) \in (\boldsymbol{R}^+)^n$ of SPNs depends on Π_{ss}:

$$m_{mms\,i} = \sum_{k=1,...,N} m_{ki}.\pi_{ss\,k}, i = 1,..,n . \quad (5)$$

2.3 An introductive example

The system in figure 1 models a simple manufacturing system. The final product is composed of two different parts, A and B, that are processed in machines M1 and M2 (represented by transitions T_1 and T_2), and stored in buffers P_4 and P_6, respectively. Then, they are assembled by M3 (i.e. transition T_3), and processed in M4 (i.e. transition T_4). Finally, M5 (i.e. transition T_5) packages them. During the processing of parts A and B, tool1 (tokens in place P_5) and tool2 (tokens in place P_7) are needed. Also tool3 (tokens in place P_3) has to be used in the three final operations. The machines M1, M2, M4 and M5 are assumed to be reliable and an active redundancy (n = 3) is considered for the assembly machine M3 that is assumed to have failure and repair rates λ = 1.5e-2 TU^{-1} and μ = 1e-1 TU^{-1}. To achieve high availability requirements 3 active redundancies are considered for M3 (place P_{10}) The productivity of the workshop is evaluated with the computation of the output flow $X(t, T5)$ with respect to the number k of pallets and tools : $M_I = (2k\ 2k\ k\ 0\ k\ 0\ k\ 0\ 0\ 3\ 0)^T$.

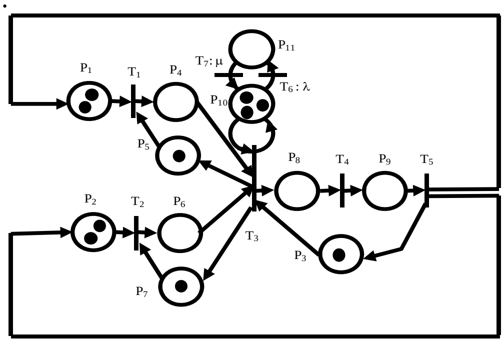


Figure 1: Assembly workshop.

The results obtained with Markov models and SPNs simulation over a time interval of D = 1000 TU are summed up in tables 1 and 2.

k	N	$x_5(t)$ Markov model	Computational effort (TU)
1	48	0.29	0.1
2	216	0.61	0.9
3	640	0.93	12
4	1500	1.25	108
…	…	…	…

Table 1: Performance evaluation with Markov models

For $k > 4$, the computational effort becomes heavy because of the large number N of states and the performance evaluation with Markov model analysis is no longer computable.

k	$x_5(t)$ SPN	Computational effort (TU)
1	0.30	0.59
2	0.61	2.0
3	0.94	4.0
4	1.24	6.8
5	1.55	11
10	2.41	30
…	…	…

Table 2: Performance evaluation with SPNs

Simulation with SPNs can be used to overcome the computational limitation with Markov model. The simulation error does not exceed 1%. One can also notice that the computational effort increases but remains acceptable up to k =10. For $k > 10$, fluidification must be introduced.

3 FLUIDIFICATION OF SPNS

3.1 Timed continuous Petri nets

Timed continuous PNs under infinite server semantic (contPNs) have been developed in order to provide continuous approximations of the discrete behaviors of timed PNs (Recalde L. et *al*., 1999), (Recalde L. and M. Silva, 2002), (Recalde L. and M. Silva. 2004). The marking of each place is a continuous non negative real valued function of time and $M(t, M_I) \in (\boldsymbol{R}^+)^n$, $t \geq 0$ is the continuous marking trajectory that starts with M_I at $t = 0$. X_{max} = diag($x_{max\,j}$) $\in (\boldsymbol{R}^+)^{qxq}$ is the diagonal matrix of maximal firing speeds $x_{max\,j}$, $j = 1,\ldots q$ and $X(t, M_I) = (x_j(t, M_I)) \in (\boldsymbol{R}^+)^q$ is the firing speeds vector at time t in free regime that depends continuously on the marking of the places. The flow through the transition T_j is defined by (6):

$$x_j(t, M_I) = x_{maxj} \,.\, enab_j(M(t), M_I)\,. \tag{6}$$

with:

$$enab_j(M) = min\,\{m_k \,/\, w^{PR}_{kj} : P_k \in {}^{\circ}T_j\}\,. \tag{7}$$

where $^{\circ}\boldsymbol{T_j}$ stands for the set of T_j upstream places. Switches occur in contPNs according to the function "min(.)" in (7). Let us define the critical place(s) for transition T_j at time t as the place(s) P_i such that i = arg-min $\{m_k(t, M_I) / w^{PR}_{kj}, P_k \in {}^{\circ}T_j\}$. For a contPN with P-semiflows represented by matrix Y, any reachable marking $M(t, M_I) \in (\boldsymbol{R}^+)^n$ satisfies $Y^T.M(t, M_I) = C$. So, linear dependencies between marking variables appear. The limit timed reachable set, $LTR(contPN, M_I) \subset (\mathbf{R}^+)^n$, is defined as the set of all reachable markings in finite or infinite time, from a given initial marking M_I and for all constant matrices $X_{max} \in (\boldsymbol{R}^+)^{q\ x\ q}$ of maximal firing speeds. In comparison with the usual untimed reachable set, $LTR(contPN, M_I)$ concerns timed nets and includes also the limit reachable markings (i.e. the asymptotic mean markings) (Lefebvre D., 2011), (Mahulea C. et *al*., 2008).

$LTR(contPN, M_I)$ can be partitioned in K reachable regions (r-regions) with $K \leq \Pi\{|^{\circ}T_j|, j = 1,...,q\}$: $LTR(contPN, M_I) = \mathbf{A_1} \cup \ldots \cup \mathbf{A_k}$. PN configurations (Mahulea C. et *al*., 2006), (Zerhouni N. and H. Alla, 1990) are used to define the r-regions. A configuration is a cover of $\boldsymbol{T}$ by its input arcs and assigns to each transition a single input place: $config(k) = \{(P_{i(k,j)}, T_j), j = 1,...q\}$, $k = 1,...,\Pi\{|^{\circ}T_j|, j = 1,...,q\}$, where $P_{i(k,j)} \in {}^{\circ}T_j$ is the single input place of transition T_j in configuration k. The r-region $\mathbf{A_k} \subset LTR(contPN, M_I)$, $k = 1,\ldots, K$ of a marked contPN, $< PN, X_{max}, M_I >$, is defined for a given configuration $config(k)$, and for all matrices $X_{max} \in \boldsymbol{R}^+)^{qxq}$ as the set of all reachable markings $M(t, M_I)$, $t \geq 0$, that satisfy (1) $Y^T.M(t, M_I) = C$, (2) $\forall\ T_j \in \boldsymbol{T}$, $P_{i(k,j)}$ is the critical place of transition T_j for marking $M(t, M_I)$.

Each r-region $\mathbf{A_k}$ is characterized by a constraint matrix $A_k = (a^k_{ij}) \in (\boldsymbol{R}^+)^{q\ x\ n}$, $k = 1,\ldots,K$, $i = 1,..., q$ and $j = 1,...,n$:

- $a^k_{ji(k,j)} = 1/w^{PR}_{i(k,j)j}$ for all $T_j \in T$,
- $a^k_{ji(k,j)}$=0 otherwise.

The constraint matrices A_k lead to a linear matrix inequality (LMI) that characterizes the r-regions:

Proposition 1 (Lefebvre D. et *al*., 2010), (Lefebvre D., 2011): Let us consider a contPN with K r-regions $\mathbf{A_k}$. Each r-region $\mathbf{A_k}$ is a polyhedral set characterized by the LMI $H_k.M \leq h_k$ with:

$$H_k = \begin{pmatrix} -I_n \\ A(k) \\ Y^T \\ -Y^T \end{pmatrix},\ h_k = \begin{pmatrix} 0 \\ 0 \\ C \\ -C \end{pmatrix},\ A(k) = \begin{pmatrix} A_k - A_1 \\ .... \\ A_k - A_{k-1} \\ .... \\ A_k - A_{k+1} \\ A_k - A_K \end{pmatrix} \tag{8}$$

and I_n is the identity matrix of size n.

3.2 Fluidication of discrete model

The main idea of PNs fluidification is to replace a discrete PN by a continuous one with same structure, initial marking and parameter (i.e. standard fluidification). The origin of the approach is that continuous models have been intensively used from the 90th to approximate the behavior of deterministic discrete event systems, in particular for control issues (Recalde L. et *al*., 1999), (Recalde L. and M. Silva, 2002), (Recalde L. and M. Silva, 2004). The advantage of fluidification is that the enumeration of discrete states is no longer required with continuous models and that standard tools exist for such model. Unfortunately, numerous structural and behavioral properties are not preserved with fluidification. In particular, the throughput of a contPN is mainly not identical to the throughput of a discrete PN. The example of figure 1 is considered again and simulated as a contPN. Standard fluidification is used and the results are reported in table 3.

k	$x_5(t)$ contPN	Computational effort (TU)
1	0.33	0.25
2	0.66	0.22
3	1	0.22
4	1.33	0.20
5	1.66	0.22
10	2.61	0.23
20	2.61	0.25
100	2.61	0.22
200	2.61	0.23

Table 3: Performance evaluation with contPNs

One can notice that the computation effort does not depend on the marking magnitude. So, fluidification can be used for rapid performance evaluation. But simulation with contPNs leads to biased results even if the magnitude of errors decreases as k increases. For this reason, alternative fluidification methods are introduced and discussed in the next section.

4 HOMOTHETIC ESTIMATION OF SPNS ASYMPTOTIC MEAN MARKINGS

In our preceding works, standard fluidification (models with same structure, initial state and $x_{maxj} = \mu_j$ $j = 1,...,q$) has been discussed to approximate the steady states of SPNs (Lefebvre D. et *al*., 2009), (Lefebvre D. and E. Leclercq, 2011), (Lefebvre D et *al*., 2010), (Lefebvre D., 2011). We have proposed a geometric approach to compute contPNs with modified maximal firing speeds and initial markings that estimate the asymptotic stochastic mean marking in non critical and critical regions (Lefebvre D. et *al*., 2010), (Lefebvre D., 2011). The main results are summed up in the next sections.

4.1 SPNs and contPNs local equivalence in long runs

Proposition 2 provides sufficient conditions such that $M(t, M_I)$ reaches M_{mms} when $M(t, M_I)$ stays in a single non critical region $\mathbf{A_k}$:

Proposition 2: (Lefebvre D. et *al*., 2010) Let us consider SPN(W_{PR}, W_{PO}, μ, M_I) with initial marking $M_I \in \boldsymbol{A}_\mathbf{k}$, asymptotic mean marking $M_{mms} \in \boldsymbol{A}_\mathbf{k}$ and $\boldsymbol{A}_\mathbf{k}$ is non critical. ContPN(W_{PR}, W_{PO}, X_{max}, M_I) with same structure and initial marking has a marking vector $M(t, M_I)$ that tends asymptotically to M_{mms} if there exist X_{max} such that $M(t, M_I)$ satisfies LMI $H_k.M \leq h_k$ for all $t \geq 0$ and (9) holds:

$$D_1.W.X_{max}.X_{cr} = 0 \,. \qquad (9)$$

with $X_{cr} = A_k.(F.D_1.M_{mms} + G.C) \in (\boldsymbol{R}^+)^q$.

Critical regions are not concerned by proposition 2 because the set of solutions for equation (9) is mainly often empty in such regions.

4.2 SPNs and contPNs global equivalence in long runs

When the asymptotic mean marking M_{mms} and the initial marking M_I are in different regions, a corrected contPN is defined with same structure but partial homothetic initial marking and modified transitions maximal firing speeds so that the continuous marking vector will converge partially to M_{mms}. The considered problem is to reach M_{mms} when $M_{mms} \in \mathbf{A_i}$ ($\mathbf{A_i}$ may be a critical region) and $M_I \in \mathbf{A_k}$ ($\mathbf{A_k}$ is a non critical region) with $\mathbf{A_i} \neq \mathbf{A_k}$. The proposition 3 provides conditions in reduced marking space to work out admissible homothetic transformations of ratio α such that $(\alpha.(M_{mms1})^T\ (M'_{mms2})^T)^T \in \mathbf{A_k}$, with $M_{mms1} = D_1.M_{mms}$ and $M'_{mms2} = (Y_2)^{-1}.(C - \alpha.Y_1.M_{mms1})$.

Proposition 3: (Lefebvre D. et *al*., 2011) A partial homothetic transformation of ratio α exists such that $(\alpha.(M_{mms1})^T\ (M'_{mms2})^T)^T \in \mathbf{A_k}$ with $M'_{mms2} = (Y_2)^{-1}.(C - \alpha.Y_1.M_{mms1})$ if α satisfies (10):

$$\left(\begin{pmatrix} -I_{n-hp} \\ A(k) \\ Y^T \\ -Y^T \end{pmatrix} .F.M_{mms1}\right).\alpha \leq \begin{pmatrix} G \\ -A(k).G \\ I_{nh} - Y^T.G \\ Y^T.G - I_{hp} \end{pmatrix}.C \qquad (10)$$

with matrices F and G defined by equation (4) and $A(k)$ is defined by equation (8).

A set of modified constant firing speeds is worked out with proposition 4.

Proposition 4: (Lefebvre D., 2011), Consider SPN(W_{PR}, W_{PO}, μ, M_I) with $M_I \in \boldsymbol{A}_\mathbf{k}$, asymptotic mean marking $M_{mms} \in \boldsymbol{A}_\mathbf{i}$ with $\mathbf{A_i} \neq \mathbf{A_k}$. ContPN($W_{PR}$, W_{PO}, X_{max}, M_I) with same structure and initial marking has a marking $M(t, M_I)$ that tends asymptotically to M_{mmc} such that $M_{mmc1} = \alpha.M_{mms1}$ if there exists α that satisfies proposition 3 and X_{max} such that $M(t, M_I)$ satisfies LMI $H_k.M \leq h_k$ for all $t \geq 0$ and equation (11) holds:

$$D_1.W.X_{max}.Xcr(\alpha) = 0.. \quad (11)$$

with :

$X_{cr}(\alpha) = A_k.(\alpha.F.D_1.M_{mms} + G.C) \in (\mathbf{R}^+)^q$.

Finally, proposition 5 uses the scaling properties of contPNs to provide sufficient conditions with respect to X_{max} so that $M(t, M_I)$ converges partially to the asymptotic mean marking M_{mms}.

Proposition 5: (Lefebvre D., 2011) Consider SPN(W_{PR}, W_{PO}, μ, M_I) with $M_I \in \mathbf{A_k}$, asymptotic mean marking $M_{mms} \in \mathbf{A_i}$ with $\mathbf{A_i} \neq \mathbf{A_k}$. ContPN($W_{PR}$, W_{PO}, X_{max}, M_I/α) with same structure and homothetic initial marking M_I/α has a marking $M(t, M_I/\alpha)$ that tends asymptotically to M_{mmc} such that $M_{mmc1} = M_{mms1}$ if there exists α that satisfies proposition 3 and X_{max} such that $M(t, M_I/\alpha)$ satisfies LMI $H_k.M \leq h_k$ for all $t \geq 0$ and equations (12) holds with:

$$X_{cr}(\alpha) = A_k.(F.D_1.M_{mms} + (1/\alpha).G.C). \quad (12)$$

The proposition 5 leads to the following algorithm that transforms a considered SPN into a fluid model contPN that converges partially to M_{mms}:

1. Work out permutation matrix D according to thecritical regions and P-semiflows,
2. Select a parameter α so that the partial homothetic transformation of M_{mms} and M_I are in same non critical region (proposition 3),
3. Work out the modified constant firing speeds that drive $M(t, M_I/\alpha)$ to the steady state M_{mmc} with $D_1.M_{mmc} = M_{mms1}$ (proposition 5),
4. Recover the full asymptotic stochastic mean marking M_{mms} with $M_{mms} = F.\ M_{mms1} + G.\ Y^T.M_I$.

5 APPROXIMATION BY MEANS OF CLASSIFICATION

The partial homothetic estimation described in section III provides a global understanding of SPN steady states distribution. The main drawback of the proposed method is that it is not constructive. In this section, the geometric approach is combined with a classifier to provide an acceptable approximation of the SPN mean markings directly from the firing rates of SPN transitions. The geometric approach is used to map the firing rates of SPNs with the modified maximal firing speeds and homothetic ratio of corrected contPNs. The mapping is computed from a training set of arbitrary firing rate vectors SET_μ that are supposed to cover the domain of SPN parameters. This domain is defined a priori according to the specifications of the system under consideration. If the firing parameters satisfy $\mu_j \in [0, \mu_{MAX}]$, $j = 1,\ldots, q$, a simple way to obtain SET_μ is to mesh the domain $[0, \mu_{MAX}]^q$ with a regular grid. For each vector $\mu \in SET_\mu$, the modified maximal firing speed vector $X_{max}(\mu)$ and homothetic ratio $\alpha(\mu)$ are worked out so that the reduced marking of corrected contPN(W_{PR}, W_{PO}, $X_{max}(\mu)$, $M_I/\alpha(\mu)$) tends to $M_{mms1}(\mu)$. Let us define SET_{Xmax} and SET_α as the sets of maximal firing speeds and homothetic ratio obtained for all $\mu \in SET_\mu$. The resulting sets of asymptotic mean markings for SPNs and corrected contPNs worked out according to the algorithm in section III are identical and named SET_{Mmm}.

The limit timed reachable set is then divided into N polyhedral cells. For each cell $CELL_{Mmm}(k) \subset LTR(contPN, M_I)$, $k = 1,\ldots,N$, a polyhedral region $CELL_\mu(k) \subset [0, \mu_{MAX}]^q$ is worked out in order to be an upper bound of the domain of firing rates that result in asymptotic mean markings $M_{mms} \in CELL_{Mmm}(k)$ (figure 3). The gravity center of $CELL_\mu(k)$ is defined as $C\mu(k)$ and the perimeter of $CELL_\mu(k)$ are obtained according to 1-norm. $CM_{mm}(k)$ is the asymptotic stochastic mean marking corresponding to $C\mu(k)$ and $X_{max}(k)$ and $\alpha(k)$ are the maximal firing rates and homothetic ratio that lead to the same asymptotic marking $CM_{mm}(k)$ with corrected contPN.
Considering finally, any new firing rate vector μ, the mapping consists to find the cell $CELL_\mu(k^*)$ whose center $C\mu(k^*)$ is the nearest from μ according to Euclidean distance (the cells in domain $[0, \mu_{MAX}]^q$ are not disjoint and a single firing rate μ may belong to several cells):

$$k^* = argmin\{||C\ \mu(k) - \mu\ ||:k = 1,...,N\}. \quad (13)$$

The modified contPN parameters are approximated by $X_{max}(k^*)$ and $\alpha(k^*)$ (figure 4) and the approximation error is defined as:

$$E(M_{mms}, M_{mmc}) = ((M_{mms} - CM_{mm}(k^*))^T.(M_{mms} - CM_{mm}(k^*)))^{1/2} \quad (14)$$

6 EXAMPLE

PN1 described in figure 2 is has 2 P-semiflows: $y_{11} = (1\ 1\ 1\ 0)^T$, $y_{12} = (1\ 0\ 4\ 1)^T$, and one can define $Y = (y_{11}\ y_{12}) \in (\mathbf{Z}^+)^{5 \times 2}$ and $C = (17\ 19)^T$. Four regions $\mathbf{A_1}$ to $\mathbf{A_4}$ exist in reachable marking space of PN1. The regions are defined by the constraint matrices A_1 to A_4:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

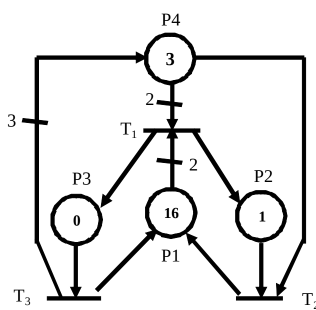


Figure 2: Examples PN1 with $M_I = (16\ 1\ 0\ 3)^T$

ContPN1 has a single critical region $\mathbf{A_1}$ (rank $A_1 = 2$) and the set of critical places for $\mathbf{A_1}$ is $\boldsymbol{P_{cr}}(\mathbf{A_1}) = \{P_3, P_4\}$. The P-semiflows Y and $\boldsymbol{P_{cr}}(\mathbf{A_1})$ are used to define the subsets of places $\boldsymbol{P_1} = \boldsymbol{P_{cr}}(\mathbf{A_1})$ and $\boldsymbol{P_2} = \{P_1, P_2\}$. The marking M depends only on the reduced marking vector $M_1 = (m_3, m_4)^T$ and the matrices F and G are given by (15):

$$F = \begin{pmatrix} -4 & -1 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} G = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{15}$$

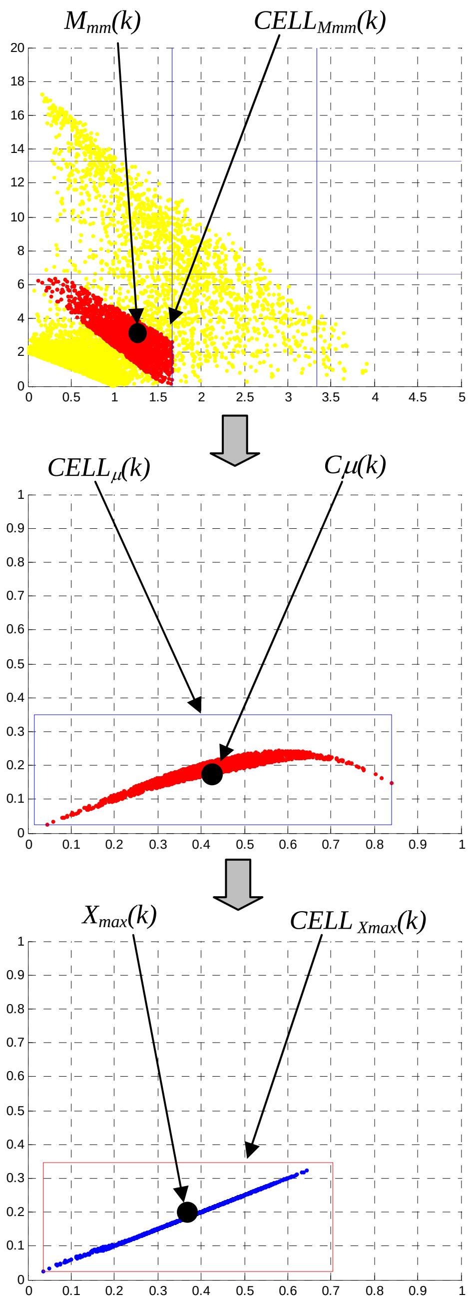


Figure 3: Classifier learning

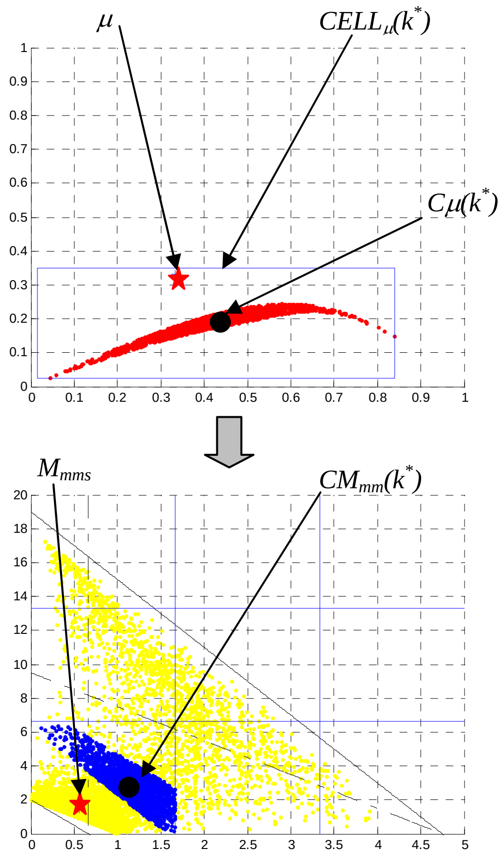


Figure 4: contPN parameters mapping

The approximation algorithm proposed in preceding section is applied to PN1. For this purpose, the asymptotic stochastic marking space is meshed with 3 x 3 x 3 x 3 = 81 cells. In order to discuss the influence of the cells number, two additive meshes with respectively 6 x 6 x 6 x 6 = 1296 cells and 12 x 12 x 12 x 12 = 20736 cells are also considered. These meshes are used to divide the domain of firing parameters into cells $CELL_\mu(k)$ with centers $C\mu(k)$ (figure 5). The centers are mapped with maximal firing speeds $X_{max}(k)$ and homothetic ratio $\alpha(k)$. Each new firing rate vector μ is then associated to the more representative cell: $CELL_\mu(k^*)$ (i.e. the cell with the nearest center $C\mu(k)$ according to Euclidean distance (14)) and the corresponding asymptotic mean marking vector is worked out according to the corrected contPN with modified firing speed $X_{max}(k^*)$ and homothetic ratio $\alpha(k^*)$.

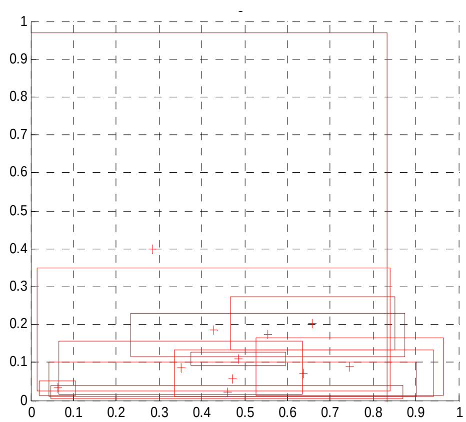
In order to evaluate the performances of the approximation a validation set of firing rate vectors is used (table 4). For each sample the approximations resulting from the three meshes are worked out and compared with standard fluidification according to distance (14). Results depicted in table 5 illustrate the performance of the proposed method that lead improved approximations in most cases. For test 4 the best approximation is provided by standard fluidification, because in few regions of the marking space standard fluidification leads to exact estimation of the asymptotic stochastic mean marking. Another expected conclusion is that the approximation error decreases in the most cases with respect to the number of cells: class 12 provides better results than class 6 and class 3 classifiers.

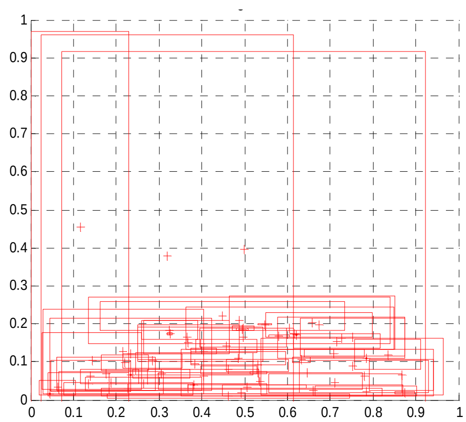
μ	μ1	μ2	μ3
Test1	0.33	0.33	0.33
Test2	0.8	0.1	0.1
Test3	0.1	0.8	0.1
Test4	0.1	0.1	0.8
Test5	0.45	0.45	0.1
Test6	0.45	0.1	0.45
Test7	0.1	0.45	0.45

Table 4: Set of validation firing rates

Class	3	6	12	standard
Test1	0.11	0.02	0.07	1.23
Test2	4.05	13.74	0.15	2.64
Test3	0.46	0.74	0.04	0.69
Test4	215.8	87.1	1.79	0.01
Test5	0.97	0.23	0.06	2.22
Test6	15.24	0.60	0.63	1.38
Test7	1.57	0.03	0.004	0.10

Table 5: Performance evaluation *(E(…))*^{2}





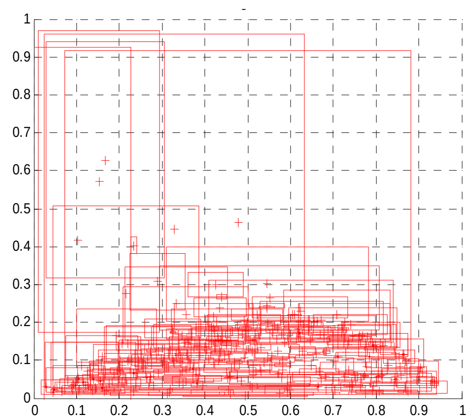


Figure 5: Different meshes for the firing rate space up) Class 3 classifier with 81 cells; middle) Class 6 classifier with 1296 cells; down) Class 12 classifier with 20736 cells

7 CONCLUSION

SPNs and contPNs are mainly often not equivalent in long run when standard fluidification is used. The combined used of corrected contPNs with modified maximal firing speeds and homothetic ratio and classification method with a partition of the reachable marking space has been developed. This method leads to better approximations of the asymptotic stochastic mean markings in comparison with standard fluidification.

Several questions will attract our interest in the next future. First at all, we will improve the estimation by using interpolation tools if the SPN firing rate vector belongs simultaneously to several cells. Computational complexity will be also investigated according to the PN structure and to the number of cells. The determination of the number of cells will be studied to upper-bound the approximation error. Finally, we noticed that the performance of the classifier is more sensitive to the dispersion of homothetic ratio than to the number of cells.

Poor results have been observed for cells that are not included in a single r-region. As a consequence, we will investigate partitions of the reachable marking space that are driven by the r-regions definition and an alternative estimation for cells with large ratio dispersion will be developed.

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