Weighted Sobolev spaces for the Laplace equation in periodic infinite strips

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Abstract

This paper establishes isomorphisms for the Laplace operator in weighted Sobolev spaces (WSS). These spaces are similar to standard Sobolev spaces $H^m(\mathbb{R}^n)$, but they are endowed with weights prescribing the functions' growth or decay at infinity. Although well established in $\mathbb{R}^n$ [1], these weighted results do not apply in the specific hypothesis of periodicity. This kind of problem appears when studying singularly perturbed domains (roughness, sieves, porous media, etc). When zooming on a single perturbation pattern, one often ends with a periodic problem set on an infinite strip. Our results set a unified framework that enable a systematic treatment of such problems. We provide existence and uniqueness of solution in our WSS. This gives a refined description of solutions behavior at infinity which is a matter of importance in the mutli-scale context. These isomorphism results hold for any weight exponent and any regularity index. We then identify these solutions with the convolution of a Green function (specific to periodical infinite strips) and the given data. This identification is valid again for any weight and any regularity index modulo some harmonic polynomials.

Keywords:
weighted Sobolev spaces, Hardy inequality, isomorphisms of the Laplace operator, periodic infinite strip, Green function, convolution

1. Introduction

In this article, we solve the Laplace equation in a 1-periodic infinite strip in two space dimensions:

$$\Delta u = f, \quad \text{in } Z :=]0,1[\times\mathbb{R}. \tag{1}$$

As the domain is infinite in the vertical direction, one introduces weighted Sobolev spaces describing the behavior at infinity of the solution. This behavior is related to weighted Sobolev properties of f.

The usual weights, when adapted to our problem, are polynomial functions at infinity and regular bounded functions in the neighborhood of the origin: they are powers of $\rho(y_2) := (1+y_2^2)^{1/2}$ and, in some critical cases, higher order derivatives are completed by logarithmic functions $(\rho(y_2)^\alpha \log^\beta(1+\rho(y_2)^2))$.

The literature on the weighted Sobolev spaces is wide [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and deals with various types of domains. To our knowledge, this type of weights has not

been applied to problem (1). The choice of the physical domain comes from periodic singular problems : in [2, 14, 3], a zoom around a domain's periodic ϵ-perturbation leads to set an obstacle of size 1 in Z and to consider a microscopic problem defined on *a boundary layer*. The behavior at infinity of this *microscopic* solution is of importance: it provides an averaged feed-back on the macroscopic scale (see [15] and references therein). This paper is a first step towards a systematic analysis of such microscopic problems. We intend to give a standard framework to skip tedious and particular proofs related to the unboundedness of Z.

We provide isomorphisms of the Laplace operator between our weighted Sobolev spaces. It is the first step among results in the spirit of [5, 1, 16]. Since error estimates for boundary layer problems [2, 3, 14] are mostly performed in the H^s framework we focus here on weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(Z)$ with $p = 2$. There are three types of tools used here: arguments specifically related to weighted Sobolev spaces [1, 17], variational techniques from the homogenization literature [3, 18, 19] and some potential theory methods [20]. A general scheme might illustrate how these ideas relate one to each other :

- a Green function G specific to the periodic infinite strip is exhibited for the Laplace operator. The convolution of f with G provides an explicit solution to (1). A particular attention is provided to give the weakest possible meaning to the latter convolution under minimal requirements on f.

- variational *inf-sup* techniques provide *a priori* estimates, in an initial range of weights, for interior and exterior problems with Dirichlet boundary conditions. Then a lift is built in order to cancel the lineic Dirac mass that's introduced by artificial interfaces. This leads to first isomorphism results.

- these arguments are then applied to weighted derivatives and give natural regularity shift results in $H^m_{\alpha,\#}(Z)$ for $m \geq 2$ (see below).

- by duality and appropriate use of generalized Poincaré estimates (leading to interactions - orthogonality or quotient spaces - with various polynomial families), one ends with generic isomorphism result that reads

  Theorem 1.1. *For any $\alpha \in \mathbb{R}$ and any $m \in \mathbb{Z}$, the mapping*

  $$\Delta : X^{m+2}_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{q(m+2,\alpha)} \mapsto X^{m}_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{q(-m,-\alpha)}$$

  is an isomorphism.

  The spaces $X^m_{\alpha,\#}(Z)$ are generalized WSS whose precise definition is given in section 6. They are introduced in order to deal with critical values of (m, α). The spaces $\mathbb{P}'^{\Delta}_{q(m,\alpha)}$ of harmonic polynomials included in $X^m_{\alpha,\#}(Z)$ are defined in section 2.

- for all values of $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}$, we identify explicit solutions obtained via convolution with solutions given in Theorem 1.1. This gives our main result:

  Theorem 1.2. *Let $m \in \mathbb{Z}$, $\alpha \in \mathbb{R}$ and $f \in X^m_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{q(-m,-\alpha)}$. Then $G * f \in X^{m+2}_{\alpha,\#}(Z)$ is the unique solution of the Laplace equation* (21) *up to a polynomial of $\mathbb{P}'^{\Delta}_{q(m+2,\alpha)}$. Moreover, we have the estimate*

  $$\|G * f\|_{X^{m+2}_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{q(m+2,\alpha)}} \leq \|f\|_{X^m_{\alpha,\#}(Z)}.$$

The paper is organized as follows. In section 2, we define the basic functional framework and some preliminary results used in this article. In section 3, we adapt weighted Poincaré estimates to our setting. Then (section 4), we introduce a mixed Fourier transform: it is a discrete Fourier transform in the horizontal direction and a continuous transform in the vertical direction. This operator allows the explicit computation of the Green function, and we derive weighted and standard estimates of the convolution with this fundamental solution. At this stage, we prove a series of isomorphisms in the non-critical case (section 5) by variational techniques. In the last section, we extend these to the critical cases. In a last step we identify any of the solutions above with the convolution between G and the data f, this proves Theorem 1.2.

2. Notation, preliminary results and functional framework

2.1. Notation and preliminaries

We denote by Z the two-dimensional infinite strip defined by

$$Z :=]0,1[\times\mathbb{R}.$$

We use bold characters for vector or matrix fields. A point in $\mathbb{R}^2$ is denoted by $\boldsymbol{y} = (y_1, y_2)$ and its distance to the origin by

$$r := |\boldsymbol{y}| = \left(y_1^2 + y_2^2\right)^{1/2}.$$

Let $\mathbb{N}$ denote the set of non-negative integers, $\mathbb{Z}$ the set of all integers and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. We denote by $[k]$ the integer part of k. For any $j \in \mathbb{Z}$, $\mathbb{P}'_j$ stands for the polynomial space of degree less than or equal to j that only depends on y_2. If j is a negative integer, we set by convention $\mathbb{P}'_j = \{0\}$. We define $\mathbb{P}'^{\Delta}_j$ the subspace of harmonic polynomials of $\mathbb{P}'_j$. The support of a function φ is denoted by $\mathrm{supp}(\varphi)$. We recall that $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R}^2)$ are spaces of $\mathcal{C}^\infty$ functions with compact support in $\mathbb{R}$ and $\mathbb{R}^2$ respectively, $\mathcal{D}'(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R}^2)$ their dual spaces, namely the spaces of distributions. We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of functions in $\mathcal{C}^\infty(\mathbb{R})$ with rapid decrease at infinity, and by $\mathcal{S}'(\mathbb{R})$ its dual, i.e. the space of tempered distributions. We recall that, for $m \in \mathbb{N}$, H^m is the classical Sobolev space and we denote by $H^m_{\#}(Z)$ the space of functions that belong to $H^m(Z)$ and that are 1-periodic in the y_1 direction. Given a Banach space B with its dual B' and a closed subspace X of B, we denote by $B'\perp X$ the subspace of B' orthogonal to X, *i.e.*:

$$B'\perp X = \{f \in B',\ \forall v \in X, \langle f, v\rangle = 0\} = (B/X)'.$$

We introduce τ_λ the operator of translation of $\lambda \in \mathbb{Z}$ in the y_1 direction. If $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}$ is a function, then we have

$$(\tau_\lambda \Phi)(\boldsymbol{y}) := \Phi(y_1 - \lambda, y_2).$$

For any $\Phi \in \mathcal{D}(\mathbb{R}^2)$, we set $\overline{\omega}\Phi := \sum_{\lambda\in\mathbb{Z}} \tau_\lambda \Phi$, which is the y_1-periodical transform of Φ. The mapping $y_1 \mapsto \overline{\omega}\Phi(\boldsymbol{y})$ belongs to $\mathcal{C}^\infty(\mathbb{R})$ and is 1-periodic. Observe that there exists a function θ satisfying

$$\theta \in \mathcal{D}(\mathbb{R}) \text{ and } \overline{\omega}\theta = 1.$$

More precisely, consider a function $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi > 0$ on the interior of its support. Then we simply set $\theta = \frac{\psi}{\overline{\omega}\psi}$. The function θ is called a periodical $\mathcal{D}(\mathbb{R})$-partition of unity.

If $T \in \mathcal{D}'(\mathbb{R}^2)$, then for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we set

$$\langle \tau_\lambda T, \varphi \rangle := \langle T, \tau_{-\lambda}\varphi \rangle, \ \ \lambda \in \mathbb{Z}.$$

Similarly, if $T \in \mathcal{D}'(\mathbb{R}^2)$ has a compact support in the y_1 direction, then the y_1-periodical transform of T, denoted by $\overline{\omega}T$, is defined by

$$\langle \overline{\omega}T, \varphi \rangle := \langle T, \overline{\omega}\varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^2).$$

These definitions are well-known, we refer for instance to [21] and [22].

Remark 2.1. Let Φ be in $\mathcal{D}(\mathbb{R}^2)$. Then we have

$$\tau_\lambda \partial_i \Phi = \partial_i(\tau_\lambda \Phi), \quad \forall \lambda \in \mathbb{Z}, \tag{2}$$

$$\overline{\omega}(\partial_i \Phi) = \partial_i(\overline{\omega}\Phi) \tag{3}$$

and

$$\|\tau_\lambda \Phi\|_{L^2(Z)} \leq \|\Phi\|_{L^2(\mathbb{R}^2)}. \tag{4}$$

As a consequence of equality (3), if $u \in \mathcal{D}'(\mathbb{R}^2)$, we also have

$$\overline{\omega}(\partial_i u) = \partial_i(\overline{\omega}u). \tag{5}$$

The next lemma is used to prove the density result of Proposition 2.2.

Lemma 2.1. *Let K be a compact of $\mathbb{R}^2$. Let u be in $H^m(\mathbb{R}^2)$ and have a compact support included in K. Then we have*

$$\|\overline{\omega}\, u\|_{H^m(Z)} \leq N(K)\|u\|_{H^m(\mathbb{R}^2)},$$

where $N(K)$ is an integer only depending on K.

Proof. Let us first notice that, since K is compact, there is a finite number of $\lambda \in \mathbb{Z}$ such that $\mathrm{supp}(\tau_\lambda u) \cap [0,1] \times \mathbb{R}$ is not an empty set. This number is bounded by a finite integer $N(K)$ that only depends on K. It follows that $\overline{\omega}\, u$ is a finite sum and

$$\|\overline{\omega}\, u\|_{L^2(Z)} \leq N(K)\|u\|_{L^2(\mathbb{R}^2)}.$$

The end of the proof then follows from (5). □

We now define the following spaces

$$\begin{aligned}
\mathcal{C}^\infty_\#(Z) &:= \{ \quad \varphi : Z \mapsto \mathbb{R},\ y_1 \mapsto \varphi(\boldsymbol{y}) \in \mathcal{C}^\infty([0,1])\ \text{1-periodic},\ y_2 \mapsto \varphi(\boldsymbol{y}) \in \mathcal{C}^\infty(\mathbb{R})\},\\
\mathcal{D}_\#(Z) &:= \{ \quad \varphi : Z \mapsto \mathbb{R},\ y_1 \mapsto \varphi(\boldsymbol{y}) \in \mathcal{C}^\infty([0,1])\ \text{1-periodic},\ y_2 \mapsto \varphi(\boldsymbol{y}) \in \mathcal{D}(\mathbb{R})\},\\
\mathcal{S}_\#(Z) &:= \{ \quad \varphi : Z \mapsto \mathbb{R},\ y_1 \mapsto \varphi(\boldsymbol{y}) \in \mathcal{C}^\infty([0,1])\ \text{1-periodic},\ y_2 \mapsto \varphi(\boldsymbol{y}) \in \mathcal{S}(\mathbb{R})\}.
\end{aligned}$$

The dual spaces of $\mathcal{D}_\#(Z)$ and $\mathcal{S}_\#(Z)$ are denoted by $\mathcal{D}'_\#(Z)$ and $\mathcal{S}'_\#(Z)$ respectively.

2.2. Weighted Sobolev spaces in an infinite strip

We introduce the weight

$$\rho \equiv \rho(y_2) := \left(1 + y_2^2\right)^{1/2}.$$

Observe that, for $\lambda \in \mathbb{N}$ and for $\gamma \in \mathbb{R}$, as $|y_2|$ tends to infinity, we have

$$\left| \frac{\partial^\lambda \rho^\gamma}{\partial y_2^\lambda} \right| \leq C \rho^{\gamma - \lambda}. \tag{6}$$

For $\alpha \in \mathbb{R}$, we define the weighted space

$$L^2_\alpha(Z) \equiv H^0_{\alpha,\#}(Z) = \{u \in \mathcal{D}'(Z),\ \rho^\alpha u \in L^2(Z)\},$$

which is a Banach space equipped with the norm

$$\|u\|_{L^2_\alpha(Z)} = \|\rho^\alpha u\|_{L^2(Z)}.$$

Proposition 2.1. *The space $\mathcal{D}(Z)$ is dense in $L^2_\alpha(Z)$.*

Proof. Let u be in $L^2_\alpha(Z)$. Then, by definition of the space $L^2_\alpha(Z)$, it follows that $\rho^\alpha u \in L^2(Z)$. Therefore there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(Z)$ such that u_n converges to $\rho^\alpha u$ in $L^2(Z)$ as $n \to \infty$. Thus, setting $v_n = \rho^{-\alpha} u_n$, we see that v_n converges to u in $L^2_\alpha(Z)$ as $n \to \infty$. □

Remark 2.2. Observe that we have the algebraic inclusion $\mathcal{D}(Z) \subset \mathcal{D}_\#(Z)$. It follows that the space $\mathcal{D}_\#(Z)$ is dense in $L^2_\alpha(Z)$.

For a non-negative integer m and a real number α, we set

$$k = k(m,\alpha) := \begin{cases} -1 & \text{if} \quad \alpha \notin \{1/2, ..., m - 1/2\} \\ m - 1/2 - \alpha & \text{if} \quad \alpha \in \{1/2, ..., m - 1/2\} \end{cases}$$

and we introduce the weighted Sobolev space

$$\begin{aligned} H^m_{\alpha,\#}(Z) := \{u \in \mathcal{D}'_\#(Z); \forall \lambda \in \mathbb{N}^2, 0 \leq |\lambda| \leq k,\ \rho^{-m+|\lambda|}(\ln(1+\rho^2))^{-1}\partial^\lambda u \in L^2_\alpha(Z), \\ k+1 \leq |\lambda| \leq m,\ \rho^{-m+|\lambda|}\partial^\lambda u \in L^2_\alpha(Z)\}, \end{aligned}$$

which is a Banach space when endowed with the norm

$$\|u\|_{H^m_{\alpha,\#}(Z)} := \left(\sum_{0\leq|\lambda|\leq k} \|\rho^{-m+|\lambda|}(\ln(1+\rho^2))^{-1}\partial^\lambda u\|^2_{L^2_\alpha(Z)} + \sum_{k+1\leq|\lambda|\leq m} \|\rho^{-m+|\lambda|}\partial^\lambda u\|^2_{L^2_\alpha(Z)} \right)^{1/2}.$$

We define the semi-norm

$$|u|_{H^m_{\alpha,\#}(Z)} := \left(\sum_{|\lambda|=m} \|\partial^\lambda u\|^2_{L^2_\alpha(Z)} \right)^{1/2}.$$

Observe that the logarithmic weight function only appears for the so-called critical cases $\alpha \in \{1/2, ..., m - 1/2\}$. Local properties of the spaces $H^m_{\alpha,\#}(Z)$ coincide with those of the

classical Sobolev space $H^m_{\#}(Z)$. When $\alpha \notin \{1/2, ..., m-1/2\}$ we have the following algebraic and topological inclusions :

$$H^m_{\alpha,\#}(Z) \subset H^{m-1}_{\alpha-1,\#}(Z) \subset ... \subset L^2_{\alpha-m}(Z). \tag{7}$$

When $\alpha \in \{1/2, ..., m-1/2\}$, the logarithmic weight function appears, so that we only have the inclusions

$$H^m_{\alpha,\#}(Z) \subset ... \subset H^{m-\alpha+1/2}_{1/2,\#}(Z). \tag{8}$$

Observe that the mapping

$$u \in H^m_{\alpha,\#}(Z) \mapsto \partial^\lambda u \in H^{m-|\lambda|}_{\alpha,\#}(Z) \tag{9}$$

is continuous for $\lambda \in \mathbb{N}^2$. Using (6), for $\alpha, \gamma \in \mathbb{R}$ such that $\alpha \notin \{1/2, ..., m-1/2\}$ and $\alpha - \gamma \notin \{1/2, ..., m-1/2\}$, the mapping

$$u \in H^m_{\alpha,\#}(Z) \mapsto \rho^\gamma u \in H^m_{\alpha-\gamma,\#}(Z) \tag{10}$$

is an isomorphism.
Let q be the greatest non-negative integer such that $y_2^q \in H^m_{\alpha,\#}(Z)$. An easy computation shows that q reads :

$$q \equiv q(m,\alpha) := \begin{cases} m - 3/2 - \alpha, & \text{if } \alpha + 1/2 \in \{i \in \mathbb{Z}; i \leq 0\}, \\ [m - 1/2 - \alpha], & \text{otherwise.} \end{cases} \tag{11}$$

For example (see also tab. 1), if $m = 0$ one has :

$$q = q(0,\alpha) := \begin{cases} -3/2 - \alpha, & \text{if } \alpha = \pm(\frac{1}{2} + i), i \in \mathbb{Z}^*, \\ [-1/2 - \alpha], & \text{otherwise.} \end{cases} \tag{12}$$

$m \setminus \alpha$	$\left[-\frac{5}{2}; -\frac{3}{2}\right[$	$\left[-\frac{3}{2}; -\frac{1}{2}\right[$	$\left[-\frac{1}{2}; \frac{1}{2}\right[$	$\left[\frac{1}{2}; \frac{3}{2}\right[$
$\mathbb{P}'_{q(0,\alpha)}$	$\mathbb{P}'_1$	$\mathbb{P}'_0$	0	0
$\mathbb{P}'_{q(1,\alpha)}$	$\mathbb{P}'_2$	$\mathbb{P}'_1$	$\mathbb{P}'_0$	0
$\mathbb{P}'_{q(2,\alpha)}$	$\mathbb{P}'_3$	$\mathbb{P}'_2$	$\mathbb{P}'_1$	$\mathbb{P}'_0$

Table 1: Polynomial spaces included in $H^m_{\alpha,\#}(Z)$ for various values of α and m

Proposition 2.2. *The space $\mathcal{D}_{\#}(Z)$ is dense in $H^m_{\alpha,\#}(Z)$.*

Proof. The idea of the proof comes from [1] and [23]. Let u be in $H^m_{\alpha,\#}(Z)$.

(i) We first approximate u by functions with compact support in the y_2 direction. Let $\Phi \in \mathcal{C}^\infty([0,\infty[)$ such that $\Phi(t) = 0$ for $0 \leq t \leq 1$, $0 \leq \Phi(t) \leq 1$, for $1 \leq t \leq 2$ and $\Phi(t) = 1$, for $t \geq 2$. For $\ell \in \mathbb{N}$, we introduce the function Φ_ℓ, defined by

$$\Phi_\ell(t) = \begin{cases} \Phi\left(\dfrac{\ell}{\ln t}\right) & \text{if } t > 1, \\ 1 & \text{otherwise.} \end{cases} \tag{13}$$

Note that we have $\Phi_\ell(t) = 1$ if $0 \le t \le e^{\ell/2}$, $0 \le \Phi_\ell(t) \le 1$ if $e^{\ell/2} \le t \le e^\ell$ and $\Phi_\ell(t) = 0$ if $t \ge e^\ell$. Moreover, for all $\ell \ge 2$, $t \in [e^{\ell/2}, e^\ell]$, $\lambda \in \mathbb{N}$, owing that $t \le \sqrt{1+t^2} \le \sqrt{2}\,t$ and $\ln t \le \ln(2+t^2) \le 3 \ln t$, we prove that (see [1], Lemma 7.1):

$$\left| \frac{d^\lambda}{dt^\lambda} \Phi\left(\frac{\ell}{\ln t}\right) \right| \le \frac{C}{(1+t^2)^{\lambda/2} \ln(2+t^2)},$$

where C is a constant independent of ℓ. We set $u_\ell(\boldsymbol{y}) = u(\boldsymbol{y})\Phi_\ell(y_2)$. Then, proceeding as in [1] (Theorem 7.2), one checks easily that u_ℓ belongs to $H^m_{\alpha,\#}(Z)$, has a compact support in the y_2 direction, and that u_ℓ converges to u in $H^m_{\alpha,\#}(Z)$ as ℓ tends to ∞. Thus, the functions of $H^m_{\alpha,\#}(Z)$ with compact support in the y_2 direction are dense in $H^m_{\alpha,\#}(Z)$ and, we may assume that u has a compact support in the y_2 direction.

(ii) Let θ be a periodical $\mathcal{D}(\mathbb{R})$-partition of unity and let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence such that $\alpha_j \in \mathcal{D}(\mathbb{R}^2)$, $\alpha_j \ge 0$, $\int_{\mathbb{R}^2} \alpha_j(\boldsymbol{x})d\boldsymbol{x} = 1$ and the support of α_j is included in the closed ball of radius $r_j > 0$ and centered at $(0,0)$ where $r_j \to 0$ as $j \to \infty$. It is well known that as $j \to \infty$, α_j converges in the distributional sense to the Dirac distribution. We set $w(\boldsymbol{y}) = \theta(y_1)u(\boldsymbol{y})$. Then w belongs to $H^m(\mathbb{R}^2)$ and has a compact support. Moreover, since $\overline{\omega}\theta = 1$, we have $\overline{\omega}w = \overline{\omega}(\theta u) = (\overline{\omega}\theta)u = u$. We define $\varphi_j = w * \alpha_j$. Then φ_j belongs to $\mathcal{D}(\mathbb{R}^2)$ and converges to w in $H^m(\mathbb{R}^2)$ as j tends to ∞. Let $\psi_j = \overline{\omega}\varphi_j$, then ψ_j belongs to $\mathcal{D}_\#(Z)$ and thanks to Lemma 2.1, ψ_j converges to $\overline{\omega}\, w = u$ in $H^m_{\alpha,\#}(Z)$ as j tends to ∞.

□

The above proposition implies that the dual space of $H^m_{\alpha,\#}(Z)$ denoted by $H^{-m}_{-\alpha,\#}(Z)$ is a subspace of $\mathcal{D}'_\#(Z)$.

3. Weighted Poincaré estimates

Let R be a positive real number. For $\beta \ne -1$, one uses the standard Hardy estimates:

$$\int_R^\infty |f(r)|^2 r^\beta dr \le \left(\frac{2}{\beta+1}\right)^2 \int_R^\infty |f'(r)|^2 r^{\beta+2} dr, \tag{14}$$

while for the specific case when $\beta = -1$, one switches to

$$\int_R^\infty \frac{|f(r)|^2}{(\ln r)^2\, r} dr \le \left(\frac{4}{3}\right)^2 \int_R^\infty |f'(r)|^2\, r dr.$$

We now introduce the truncated domain

$$Z_R :=]0,1[\times(]-\infty, -R[\cup]R, +\infty[).$$

Using the above Hardy inequalities in the y_2 direction, we can easily prove the following lemma.

Lemma 3.1. *Let α, $R > 1$ be real numbers and let $m \ge 1$ be an integer. Then there exists a constant C^m_α such that*

$$\forall \varphi \in \mathcal{D}_\#(Z_R), \quad \|\varphi\|_{H^m_{\alpha,\#}(Z_R)} \le C^m_\alpha \|\partial_2 \varphi\|_{H^{m-1}_{\alpha,\#}(Z_R)}. \tag{15}$$

As a consequence, we also have

$$\forall \varphi \in \mathcal{D}_\#(Z_R), \quad \|\varphi\|_{H^m_{\alpha,\#}(Z_R)} \le C^m_\alpha |\varphi|_{H^m_{\alpha,\#}(Z_R)}. \tag{16}$$

For the particular case when $\alpha \in \{1/2, ..., m-1/2\}$, (16) cannot hold without introducing logarithmic weights in the definition of the space $H^m_{\alpha,\#}(Z)$.
Proceeding as in [1] (Theorem 8.3), we have the Poincaré-type inequalities :

Theorem 3.1. *Let α be a real number and $m \geq 1$ an integer. Let $j := \min(q(m,\alpha), m-1)$ where $q(m,\alpha)$ is defined by (11). Then there exists a constant $C > 0$, such that for any $u \in H^m_{\alpha,\#}(Z)$, we have*

$$\inf_{\lambda \in \mathbb{P}'_j} \|u + \lambda\|_{H^m_{\alpha,\#}(Z)} \leq C|u|_{H^m_{\alpha,\#}(Z)}. \tag{17}$$

In other words, the semi-norm $|.|_{H^m_{\alpha,\#}(Z)}$ defines on $H^m_{\alpha,\#}(Z)/\mathbb{P}'_j$ a norm which is equivalent to the quotient norm.

4. The mixed Fourier transform (MFT) and the Green function

The purpose of this section is twofold:

(i) we look for the fundamental solution for the Laplace equation in Z,
(ii) we estimate the convolution with this solution in weighted Sobolev spaces.

To achieve this goal, we define the MFT and an adequate functional setting.

4.1. Rapidly decreasing functions

Let set $\Gamma := \mathbb{Z} \times \mathbb{R}$ and let us write the locally convex linear topological spaces :

$$\tilde{\mathcal{S}}(\Gamma) := \Big\{ \tilde{\varphi} : \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \ \text{ s.t. } \forall k \in \mathbb{Z} \quad \tilde{\varphi}(k,\cdot) \in \mathcal{S}(\mathbb{R}) \\ \text{and } \sup_{k \in \mathbb{Z}, l \in \mathbb{R}} \left| k^\alpha l^\beta \partial^\gamma_{l^\gamma} \tilde{\varphi}(k,l) \right| < \infty, \, \forall (\alpha,\beta,\gamma) \in \mathbb{N}^3 \Big\}.$$

The space $\tilde{\mathcal{S}}(\Gamma)$ is endowed with the semi-norms

$$|\tilde{\varphi}|_{\alpha,\beta,\gamma} := \sup_{k \in \mathbb{Z}, l \in \mathbb{R}} \left| k^{\alpha'} l^{\beta'} \partial^{\gamma'}_{l^{\gamma'}} \tilde{\varphi} \right|, \quad \forall \alpha' \leq \alpha, \, \beta' \leq \beta, \, \gamma' \leq \gamma.$$

We define also

$$l^2(\mathbb{Z}; L^2(\mathbb{R})) := \left\{ u \in \tilde{\mathcal{S}}'(\Gamma) \text{ s.t. } \sum_{k \in \mathbb{Z}} |u(k,\cdot)|^2_{L^2(\mathbb{R})} < \infty \right\}.$$

Proposition 4.1. *$\tilde{\mathcal{S}}(\Gamma)$ is dense in $l^2(\mathbb{Z}; L^2(\mathbb{R}))$.*

Proof. As $u \in l^2(\mathbb{Z}; L^2(\mathbb{R}))$

$$\forall \epsilon > 0 \quad \exists k_0 > 0 \text{ s.t. } \sum_{|k| > k_0} \|u(k,\cdot)\|^2_{L^2(\mathbb{R})} < \frac{\epsilon}{2}.$$

We set

$$u^\delta := \begin{cases} u^\delta(k,\cdot) & \text{if } |k| < k_0, \\ 0 & \text{otherwise }, \end{cases}$$

where $u^\delta(k,\cdot)$ is a standard smooth approximation of $u(k,\cdot)$ in $\mathcal{D}(\mathbb{R})$. Then, we choose $\delta(k_0)$ s.t.

$$\forall k \text{ s.t. } |k| \leq k_0 \quad \left\| u(k,\cdot) - u^\delta(k,\cdot) \right\|^2_{L^2(\mathbb{R})} < \frac{\epsilon}{2k_0},$$

which finally gives

$$\left\| u - u^\delta \right\|^2_{l^2(\mathbb{Z};L^2(\mathbb{R}))} \leq \sum_{|k|\leq k_0} \left\| u(k,\cdot) - u^\delta(k,\cdot) \right\|^2_{L^2(\mathbb{R})} + \sum_{|k|>k_0} \|u(k,\cdot)\|^2_{L^2(\mathbb{R})} < \epsilon$$

which proves the claim, since u^δ trivially belongs to $\tilde{\mathcal{S}}(\Gamma)$. □

Definition 4.1. *We define the MFT operator $\mathcal{F} : \mathcal{S}_\#(Z) \to \tilde{\mathcal{S}}(\Gamma)$ as*

$$\mathcal{F}(\varphi)(k,l) := \int_Z \varphi(\boldsymbol{y}) e^{-i(\tilde{k}y_1 + \tilde{l}y_2)} d\boldsymbol{y}, \quad \forall (k,l) \in \Gamma,$$

where $\tilde{k} := 2\pi k$, and the same holds for $\tilde{l}$.

We list below some basic tools concerning the MFT needed in the remaining of the paper. They can be proved following classical arguments ([24], [25], [26]).

Proposition 4.2. (i) *The operator $\mathcal{F} : \mathcal{S}_\#(Z) \to \tilde{\mathcal{S}}(\Gamma)$ is an isomorphism and the inverse operator is explicit:*

$$\mathcal{F}^{-1}(\tilde{\varphi})(\boldsymbol{y}) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \tilde{\varphi}(k,l) e^{i(\tilde{k}y_1 + \tilde{l}y_2)} dl, \quad \forall \tilde{\varphi} \in \tilde{\mathcal{S}}(\Gamma), \quad \forall \boldsymbol{y} \in Z.$$

(ii) *One has in a classical way, for any $(f,g) \in \mathcal{S}_\#(Z) \times \mathcal{S}_\#(Z)$, that*

$$\int_Z f \cdot \overline{g}\, d\boldsymbol{y} = \sum_k \int_{\mathbb{R}} \mathcal{F}(f)(k,l) \cdot \overline{\mathcal{F}(g)}(k,l) dl, \text{ and } \mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g), \quad \forall (k,l) \in \Gamma.$$

4.2. Tempered distributions and MFT

In a quite natural manner we extend the concepts introduced above to tempered distributions.

Definition 4.2. (i) *A linear form $\tilde{T}$ acting on $\tilde{\mathcal{S}}(\Gamma)$ is s.t. $\exists (\alpha,\beta,\gamma) \in \mathbb{N}^3$*

$$\left| \tilde{T}(\tilde{\varphi}) \right| \leq C |\tilde{\varphi}|_{\alpha,\beta,\gamma}, \quad \forall \tilde{\varphi} \in \tilde{\mathcal{S}}(\Gamma).$$

(ii) *The MFT applied to a tempered distribution $T \in \mathcal{S}'_\#(Z)$ is defined as*

$$< \mathcal{F}(T), \tilde{\varphi} >_{\tilde{\mathcal{S}}',\tilde{\mathcal{S}}} = \mathcal{F}(T)(\tilde{\varphi}) := T(\mathcal{F}^T(\tilde{\varphi})) =< T, \mathcal{F}^T(\tilde{\varphi}) >_{\mathcal{S}'_\#\times\mathcal{S}_\#}, \quad \forall \tilde{\varphi} \in \tilde{\mathcal{S}}(\Gamma),$$

where

$$\mathcal{F}^T(\tilde{\varphi}) := \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \tilde{\varphi}(k,l) e^{-i(\tilde{k}y_1 + \tilde{l}y_2)} dl, \quad \forall \tilde{\varphi} \in \tilde{\mathcal{S}}(\Gamma), \quad \forall \boldsymbol{y} \in Z.$$

In the same way, we define the reciprocal operator denoted $\mathcal{F}^{-1}$ which associates to each distribution $\tilde{T} \in \tilde{\mathcal{S}}'(\Gamma)$ a distribution $\mathcal{F}^{-1}(\tilde{T})$ s.t.

$$< \mathcal{F}^{-1}(\tilde{T}), \varphi >_{\mathcal{S}'\times\mathcal{S}} :=< \tilde{T}, \breve{\mathcal{F}}(\varphi) >_{\tilde{\mathcal{S}}'\times\tilde{\mathcal{S}}}$$

where $\breve{\tilde{\varphi}}(k,l) = \tilde{\varphi}(-k,-l)$.

We sum up properties extending classical results of Fourier analysis to our MFT.

Proposition 4.3. (i) *If $T \in \mathcal{S}'_{\#}(Z)$ then $\mathcal{F}(T) \in \tilde{\mathcal{S}}'(\Gamma)$. Similarly, if $\tilde{T} \in \tilde{\mathcal{S}}'(\Gamma)$ then $\mathcal{F}^{-1}(\tilde{T}) \in \mathcal{S}'_{\#}(Z)$.*

(ii) *The Dirac measure belongs to $\mathcal{S}'_{\#}(Z)$ and :*

$$\mathcal{F}(\delta_0) = 1, \quad \forall k \in \mathbb{Z}, \quad a.e.\ \ell \in \mathbb{R}.$$

(iii) *the MFT acts on the derivatives in a polynomial fashion:*

$$\forall T \in \mathcal{S}'_{\#}(Z), \quad \mathcal{F}(\partial^\alpha T) = i^{|\alpha|}(\tilde{k}^\alpha + \tilde{l}^\alpha)\mathcal{F}(T),$$

for any multi-index $\alpha \in \mathbb{N}^2$.

(iv) *Plancherel's Theorem : if $f \in L^2(Z)$ then the Fourier transform $\mathcal{F}(T_f)$ is defined by a function $\mathcal{F}(f) \in l^2(\mathbb{Z}, L^2(\mathbb{R}))$ i.e.*

$$\mathcal{F}(T_f) = T_{\mathcal{F}(f)} \quad \forall f \in L^2(Z) \text{ and } \|\mathcal{F}(f)\|_{l^2(\mathbb{Z};L^2(\mathbb{R}))} = \|f\|_{L^2(Z)}.$$

4.3. The Green function for periodic strips

We aim at solving the fundamental equation:

$$-\Delta G = \delta_0 \quad \text{in } Z, \tag{18}$$

stated here in the sense of tempered distributions. Then G should satisfy, in $\tilde{\mathcal{S}}(\Gamma)'$:

$$(\tilde{k}^2 + \tilde{l}^2)\mathcal{F}(G) = 1.$$

Proposition 4.4. *The Green function G solving* (18) *is a tempered distribution and it reads*

$$G(\boldsymbol{y}) = \frac{1}{2}\left\{\sum_{k\in\mathbb{Z}^*} \frac{e^{-|\tilde{k}||y_2|+i\tilde{k}y_1}}{|\tilde{k}|} - |y_2|\right\} = G_1(\boldsymbol{y}) + G_2(\boldsymbol{y}), \quad \forall \boldsymbol{y} \in Z,$$

where

$$G_1(\boldsymbol{y}) := \frac{1}{2}\sum_{k\in\mathbb{Z}^*} \frac{e^{-|\tilde{k}||y_2|+i\tilde{k}y_1}}{|\tilde{k}|} \quad \text{and} \quad G_2(\boldsymbol{y}) := -\frac{1}{2}|y_2|.$$

Moreover, one has $G_1 \in L^1(Z) \cap L^2(Z)$ and thus $G \in L^2_{\text{loc}}(Z)$. The Green function can be written in a more compact expression :

$$G(\boldsymbol{y}) = -\frac{1}{4\pi}\ln(2\,(\cosh(2\pi y_2) - \cos(2\pi y_1))).$$

Notice that the Green function is odd with respect to $\boldsymbol{y}$.

Proof. We define:

$$G = \lim_{N\to\infty} G_N := \lim_{N\to\infty} \sum_{|k|<N} J_k e^{i\tilde{k}y_1}, \text{ with } J_k := \int_{\mathbb{R}} \frac{e^{i\tilde{l}y_2}}{\tilde{k}^2+\tilde{l}^2} dl = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i\tilde{l}|y_2|}}{\tilde{k}^2+\tilde{l}^2} dl.$$

The absolute value in the last right hand side is added as follows:

$$\int_{\mathbb{R}} \frac{e^{i\tilde{l}y_2}}{\tilde{k}^2+\tilde{l}^2} dl = \int_0^\infty \frac{e^{i\tilde{l}y_2} + e^{-i\tilde{l}y_2}}{\tilde{k}^2+\tilde{l}^2} dl = 2\int_0^\infty \frac{\cos(\tilde{l}y_2)}{\tilde{k}^2+\tilde{l}^2} dl = 2\int_0^\infty \frac{\cos(\tilde{l}|y_2|)}{\tilde{k}^2+\tilde{l}^2} dl.$$

Proceeding as in example 1 p. 58 [26], one has, extending the integral to the complex plane, that

$$J_k \equiv \lim_{R\to\infty} \oint_{C_R} \frac{e^{i\tilde{z}|y_2|}}{\tilde{k}^2+\tilde{z}^2} dz = \lim_{R\to\infty} \oint_{C_R} \frac{e^{i\tilde{z}|y_2|}}{(\tilde{z}+i|\tilde{k}|)(\tilde{z}-i|\tilde{k}|)} dz$$
$$= \lim_{R\to\infty} \frac{1}{2\pi} \oint_{C_R} \frac{\mathfrak{f}(z)}{z-i|k|} dz = i\mathfrak{f}(i|k|) = \frac{e^{-|\tilde{k}||y_2|}}{2|\tilde{k}|},$$

where $C_R := \{z \in \mathbb{C};\ |z| = R \text{ and } \mathfrak{Im}(z) > 0\} \cup ([-R;R] \times \{0\})$ as depicted in fig. 1 and $\mathfrak{f}(z) := e^{i\tilde{z}|y_2|}/(\tilde{z}+i|\tilde{k}|)$, being a holomorphic function inside C_R. The 1D-Fourier transform

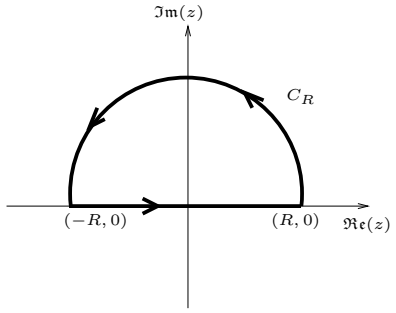

Figure 1: Path of integration in the complex plane

of the tempered distribution $|y|$ is $-2/\tilde{l}^2$, thus one concludes formally that

$$J_0 = \int_{\mathbb{R}} \frac{e^{i\tilde{l}y_2}}{\tilde{l}^2} dl = -\frac{1}{2}|y_2|.$$

The L^2 bound is achieved thanks to the Parseval formula

$$\|G_1\|^2_{L^2(Z)} = \|\mathcal{F}(G_1)\|^2_{\ell^2(\mathbb{Z};L^2(\mathbb{R}))} = \sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}} \frac{1}{(\tilde{k}^2+\tilde{\ell}^2)^2} d\ell \le C \sum_{k\in\mathbb{Z}^*} \frac{1}{|\tilde{k}|^3} \le C'.$$

One recalls the expansion in series of the logarithm :

$$\ln(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}, \quad \forall z \in \mathbb{C} : |z| < 1.$$

Thus for $\boldsymbol{y} \neq 0$

$$G_1(\boldsymbol{y}) = -\frac{1}{2\pi} \mathfrak{Re} \ln\left\{1 - e^{2\pi(-|y_2|+iy_1)}\right\} = -\frac{1}{2\pi} \ln\left|1 - e^{2\pi(-|y_2|+iy_1)}\right|$$
$$= -\frac{1}{2\pi} \ln\left\{e^{-\pi|y_2|}\sqrt{2}\,(\cosh(2\pi|y_2|) - \cos(2\pi y_1))^{\frac{1}{2}}\right\}$$
$$= \frac{|y_2|}{2} - \frac{1}{4\pi} \ln(2\,(\cosh(2\pi y_2) - \cos(2\pi y_1))),$$

which gives the desired result for G. □

One can easily obtain the asymptotic behavior of the Green function. As $|y_2|$ tends to infinity, we have

$$|G(\boldsymbol{y})| \le C|y_2|, \ \ |\nabla G(\boldsymbol{y})| \le C \ \text{ and } \ |\partial^2_{\boldsymbol{y}^2} G(\boldsymbol{y})| \le Ce^{-2\pi|y_2|} \le C\rho(y_2)^{-\sigma}, \ \ \forall \sigma \in \mathbb{R}. \qquad (19)$$

Moreover, as $|y_2|$ tends to infinity, we also have

$$|G_1(\boldsymbol{y})| \leq C\rho(y_2)^{-\sigma}, \ \ \forall \sigma > 0, \quad \forall y_1 \in (0,1). \tag{20}$$

4.4. Convolution with the fundamental solution in weighted spaces

Before proving weighted estimates on the Green function, we shall define the convolution if $f \in \mathcal{D}_{\#}(Z)$, i.e.

$$G * f = \int_Z G(\boldsymbol{x} - \boldsymbol{y})\, f(\boldsymbol{y})\, d\boldsymbol{y} = \int_Z G(\boldsymbol{y})\, f(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{y}.$$

Thus the convolution $G*f$ belongs to $\mathcal{C}^{\infty}_{\#}(Z)$. Moreover, thanks to (18), we have $\Delta(G*f) = f$ in Z.

4.5. Homogeneous estimates

Definition 4.3. *For any function f in $L^2(Z)$ the horizontal average $\overline{f}$ reads :*

$$\overline{f}(y_2) := \int_0^1 f(y_1, y_2) dy_1.$$

Lemma 4.1. *For any $h \in \mathcal{S}_{\#}(Z)$ such that $\overline{h} \equiv 0$ one has :*

$$\partial^{\alpha} G_2 * h = 0, \ \ a.e. \ in \ Z, \quad \alpha \in \{0,1,2\}.$$

Proof. G_2 is a tempered distribution, the convolution with $h \in \mathcal{S}_{\#}(Z)$ makes sense :

$$< G_2 * h, \varphi >_{\mathcal{S}'_{\#}(Z) \times \mathcal{S}_{\#}(Z)} = \frac{1}{2} \int_{\mathbb{R}} |x_2| \int_Z \overline{h}(x_2 + y_2) dz \ \varphi(\boldsymbol{y}) d\boldsymbol{y} dx_2 = 0.$$

The same proof holds for derivatives as well. As this is true for every $\varphi \in \mathcal{S}_{\#}(Z)$, the result is proved. □

Proposition 4.5. *Let $f \in L^2(Z)$. Then there exists a constant independent on f s.t.*

$$\left\| G * (f - \overline{f}) \right\|_{H^2_{\#}(Z)} \leq C \|f\|_{H^{m-2}_{\#}(Z)},$$

where C is a constant independent on f.

Proof. First we show the lemma for $f \in \mathcal{S}_{\#}(Z)$, then by density the result is extended to $L^2(Z)$ functions. We set $h := f - \overline{f}$. Thanks to lemma above $\partial^{\alpha} G * h = \partial^{\alpha} G_1 * h$. As G_1 belongs to $L^2(Z)$, one is allowed to apply the MFT in the strong sense. It provides :

$$\|\partial^{\alpha} G * h\|^2_{L^2} = \|\partial^{\alpha} G_1 * h\|^2_{L^2} = \left\| (i\tilde{k})^{\alpha_1} (i\ell)^{\alpha_2} \hat{G}_1 \hat{h} \right\|^2_{l^2 L^2} = \sum_{k \in \mathbb{Z}^*} \int_{\mathbb{R}} \frac{|\tilde{k}|^{2\alpha_1} |\tilde{\ell}|^{2\alpha_2}}{(\tilde{k}^2 + \tilde{\ell}^2)^2} \left| \hat{h}(k,\ell) \right|^2 d\ell$$

$$\leq C \sum_{k \in \mathbb{Z}^*} \int_{\mathbb{R}} \left| \hat{h}(k,\ell) \right|^2 d\ell \equiv C \left\| \hat{h} \right\|^2_{l^2(\mathbb{Z}; L^2(\mathbb{R}))} = C \|h\|^2_{L^2(Z)}$$

where α is a multi index s.t. $|\alpha| \in \{0,1,2\}$. The summation in k is performed in $\mathbb{Z}^*$, this is crucial when estimating $|\tilde{k}|^{2\alpha_1} |\tilde{\ell}|^{2\alpha_2} / (\tilde{k}^2 + \tilde{\ell}^2)^2$ by a constant. □

4.6. Non-homogeneous estimates

We extend here the above results to the weighted context. We start by stating three lemmas.

Lemma 4.2. *We define the horizontal Fourier transform, given $f \in L^2(Z)$*

$$\mathcal{F}_k(f)(k, y_2) := \int_0^1 f(y_1, y_2) e^{-i\tilde{k}y_1} dy_1, \quad \forall k \in \mathbb{Z}, \quad a.e\ y_2 \in \mathbb{R}.$$

If we define the convolution operators : for $g \in L^1(Z) \cap L^2(Z)$ and $f \in L^2(Z)$

$$g *_{\boldsymbol{y}} f := \int_Z g(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{y}, \quad (g *_{y_2} f)(x_1, x_2, z_1) := \int_{\mathbb{R}} g(x_1, x_2 - y_2) f(z_1, y_2) dy_2,$$

then one has

$$\mathcal{F}_k(g *_{\boldsymbol{y}} f)(k, x_2) = (\mathcal{F}_k g *_{y_2} \mathcal{F}_k f)(k, x_2), \quad \forall k \in \mathbb{Z}, \quad a.e.\ x_2 \in \mathbb{R}.$$

Proof. If $f \in L^2(Z)$ and $g \in (L^1 \cap L^2)(Z)$ then the convolution with respect to both variables is well defined. Setting

$$(f *_{y_1} g)(x_1, x_2, z_2) := \int_{\mathbb{R}} f(x_1 - y_1, x_2) g(y_1, z_2) dy_1,$$

almost everywhere in y_2 and for every $k \in \mathbb{Z}$ one has

$$\mathcal{F}_k \left(f *_{y_1} g(x_1, x_2 - y_2, y_2) \right) = (\mathcal{F}_k f)(k, x_2 - y_2)(\mathcal{F}_k g)(k, y_2).$$

Integrals in the vertical direction commute with the horizontal Fourier transform due to Fubini's theorem. So one shall write :

$$\int_{\mathbb{R}} \mathcal{F}_k \left(f *_{y_1} g \right)(k, x_2 - y_2, y_2) dy_2 = \mathcal{F}_k \left(\int_{\mathbb{R}} (f *_{y_1} g)(k, x_2 - y_2, y_2) dy_2 \right) = \mathcal{F}_k(f *_{\boldsymbol{y}} g)(k, x_2),$$

which ends the proof. □

Lemma 4.3. *For $f \in L^2_\beta(Z)$, one has*

$$\|f\|^2_{L^2_\beta(Z)} = \sum_{k \in Z} \|\mathcal{F}_k(f)\|^2_{L^2_\beta(\mathbb{R})}, \quad \forall \beta \in \mathbb{R}.$$

Lemma 4.4. *Let $a \in \mathbb{R}$ s.t. $|a| \geq 1$ then for any real b, one has*

$$I(x) := \int_{\mathbb{R}} e^{-|a||x-y|} \rho(y)^{-b} dy \leq \frac{C\rho(x)^{-b}}{|a|}, \quad x \in \mathbb{R}.$$

Proof. We set $|x| = R$ and let R_0 be a real such that $R_0 > 1$. We define three regions of the real line :

$$D_1 := B(0, R/2), \quad D_2 := B(0, 2R) \setminus B(0, R/2), \quad D_3 := \mathbb{R} \setminus B(0, 2R)$$

Decomposing the convolution integral in these three parts, one gets :

$$I(x) = \sum_{i=1}^{3} I_i(x), \quad \text{where } I_i(x) = \int_{D_i} e^{-|a||x-y|} \rho(y)^{-b}\, dy.$$

We have two cases to investigate : $R > R_0$ and $R \leq R_0$.

(a) If $R > R_0$

(i) On $|y| \leq R/2$, $|x-y| \sim R$ so that

$$I_1(x) \sim e^{-|a|R} \int_{|y| \leq \frac{R}{2}} (1+|y|)^{-b} dy.$$

According to the value of b, one has

$$I_1(x) \leq C e^{-|a|R} \begin{cases} 1 & \text{if } b > 1, \\ R & \text{if } 0 \leq b \leq 1, \\ R^{1-b} & \text{if } b < 0. \end{cases}$$

As $R > 1$, there holds

$$e^{-|a|R} \leq \frac{([\sigma]+1)!}{(|a|R)^{\sigma}}.$$

Setting

$$\sigma(b) := b\, \mathbb{1}_{]1,+\infty[}(b) + (1+b)\, \mathbb{1}_{[0,1]}(b) + \mathbb{1}_{]-\infty,0[}(b),$$

one recovers the claim in that case.

(ii) On $R/2 \leq |y| \leq 2R$, $|y| \sim R$

$$I_2(x) \sim R^{-b} \int_{\frac{R}{2} \leq |y| \leq 2R} e^{-|a||x-y|}\, dy,$$

because $|x-y| \leq 3R$, one has $I_2(x) \sim R^{-b}/|a|$.

(iii) On the rest of the line $|y| > 2R$, $|x-y| \sim |y|$, so that

$$I_3(x) \sim \int_{|y|>2R} e^{-|a||y|} |y|^{-b} dy.$$

Two situations occur :

- either $b \leq 0$ then we set $\tilde{b} := [-b] + 1$ and one has

$$\begin{aligned} \int_{|y|>2R} e^{-|a||y|} |y|^{-b} dy &\leq C \frac{e^{-|a|R}}{|a|} \left\{ R^{\tilde{b}} \sum_{p=0}^{\tilde{b}} \frac{\tilde{b}!}{(\tilde{b}-p)!} (|a|R)^{-p} \right\} \\ &\leq C \frac{e^{-|a|R}}{|a|} R^{\tilde{b}}\, \tilde{b}!\, \tilde{b} \leq \frac{C(b) R^{-b}}{|a|}. \end{aligned}$$

  In the latter inequality we used decreasing properties of the exponential function in order to fix the correct behavior for large R.

- either $b > 0$ and one has directly that

$$\int_{|y|>2R} e^{-|a||y|} |y|^{-b} dy \leq R^{-b} \int_{|y|>R} e^{-|a||y|} dy.$$

(b) Otherwise, if $R < R_0$ then $\rho(x-y) \sim \rho(y)$, and thus

$$I(x) \sim \int_{\mathbb{R}} e^{-|a||y|} \rho(y)^{-b} dy < \frac{C}{|a|}.$$

□

Proposition 4.6. *Let us set β a positive real number. Then for every $f \in L^2_{\rm loc}(Z)$ s.t. $f - \overline{f} \in L^2_\beta(Z)$*

$$\left\|\partial^\alpha G * (f - \overline{f})\right\|_{L^2_{\beta'}(Z)} \leq C\left\|f - \overline{f}\right\|_{L^2_\beta(Z)}, \quad \forall \beta' < \beta - \frac{1}{2}, \quad |\alpha| \in \{0, 1\},$$

where the constant C is independent of the data.

Proof. Set again $h = f - \overline{f}$ and suppose that $h \in \mathcal{S}_\#(Z)$. By Lemma 4.3 :

$$\begin{aligned}\|\partial^\alpha G * h\|^2_{L^2_{\beta'}(Z)} &= \|\partial^\alpha G_1 * h\|^2_{L^2_{\beta'}(Z)} = \sum_{k\in\mathbb{Z}^*} \left\|\mathcal{F}_k(\partial^\alpha_{\boldsymbol{y}} G_1 *_{\boldsymbol{y}} h)\right\|^2_{L^2_{\beta'}(\mathbb{R})} \\ &= \sum_{k\in\mathbb{Z}^*} \left\|(i\tilde{k})^{\alpha_1}\mathcal{F}_k(\partial^{\alpha_2}_{y_2} G_1) *_{y_2} \mathcal{F}_k(h)\right\|^2_{L^2_{\beta'}(\mathbb{R})}.\end{aligned}$$

For all $k \in \mathbb{Z}^*$, thanks to Lemma 4.4, a.e. $\boldsymbol{x}$ in Z,

$$\begin{aligned}\left|(i\tilde{k})^{\alpha_1}\mathcal{F}_k(\partial^{\alpha_2}_{y_2} G_1) *_{y_2} \mathcal{F}_k(h)\right| &\leq \int_{\mathbb{R}} \frac{e^{-|\tilde{k}||y_2 - x_2|}}{|\tilde{k}|^{1-|\alpha|}} \mathcal{F}_k(h)(k, y_2) dy_2 \\ &\leq C\|\mathcal{F}_k(h)(k,\cdot)\|_{L^2_\beta(\mathbb{R})} \frac{\rho(x_2)^{-\beta}}{|\tilde{k}|^{\frac{3}{2}-|\alpha|}} \leq C\|\mathcal{F}_k(h)(k,\cdot)\|_{L^2_\beta(\mathbb{R})}\rho(x_2)^{-\beta},\end{aligned}$$

for α multi-index s.t. $|\alpha| \in \{0, 1\}$. Then

$$\|\partial^\alpha G * h\|^2_{L^2_{\beta'}(Z)} \leq \sum_{k\in\mathbb{Z}^*} \|\mathcal{F}_k(h)(k,\cdot)\|^2_{L^2_\beta(\mathbb{R})} \int_{\mathbb{R}} \rho(x_2)^{2\beta'-2\beta} dx_2 \leq C\|f\|_{L^2_\beta(Z)},$$

if $\beta' < \beta - 1/2$. As in Proposition 4.5, the result above shall be extended to $L^2(Z)$ functions by density arguments. □

Theorem 4.1. *Let $f \in L^2_\alpha(Z) \perp \mathbb{P}'_1$, for any $\alpha > \frac{3}{2}$, one has*

$$\|G_2 * f\|_{L^2_{\alpha-2-\epsilon}(Z)} \leq C\|f\|_{L^2_\alpha(Z)}.$$

Proof. Since the proof is essentially 1D, we consider functions defined in $\mathbb{R}$. The extension to Z is straightforward. We proceed by duality, namely we expect that

$$\sup_{\varphi\in L^2_{2-\alpha+\epsilon}(Z)} \frac{(G * f, \varphi)}{\|\varphi\|_{L^2_{2-\alpha+\epsilon}(Z)}} < \infty.$$

We choose $f \in L^2_\alpha(\mathbb{R}) \perp \mathbb{P}'_1$ and $(\varphi_\delta)_\delta \in \mathcal{S}_\#(\mathbb{R})$ s.t. $\varphi_\delta \to \varphi$ in $L^2_{2-\alpha+\epsilon}(\mathbb{R})$, then $|x| * \varphi_\delta$ is infinitely differentiable and one can apply the Taylor expansion with an integral remainder :

$$\begin{aligned}&\int_{\mathbb{R}} (|x| * \varphi_\delta)(x) f(x) dx \\ &= \int_{\mathbb{R}} \left\{(|x| * \varphi_\delta)(0) + (|x| * \varphi_\delta)'(0).x + \int_0^x (|x| * \varphi_\delta)''(s)(x - s) ds\right\} f(x) dx \\ &= \int_{\mathbb{R}} f(x) \int_0^x (|x| * \varphi_\delta)''(s)(x - s) ds dx = 2\int_{\mathbb{R}} f(x) \int_0^x \varphi_\delta(s)(x - s) ds dx \\ &\leq C\|\varphi_\delta\|_{L^2_{2-\alpha+\epsilon}(\mathbb{R})} \int_{\mathbb{R}} \rho^{\alpha-\frac{1}{2}-\epsilon}(x)|f(x)| dx \leq C\|\varphi_\delta\|_{L^2_{2-\alpha+\epsilon}(\mathbb{R})}\|f\|_{L^2_\alpha(\mathbb{R})}.\end{aligned}$$

In the second line we used the orthogonality condition on f in order to cancel the first two terms in the Taylor expansion. An easy computation shows that if $\varphi_\delta \to \varphi$

$$\ell := \lim_{\delta\to 0}\int_{\mathbb{R}}(|x|*\varphi_\delta)(x)f(x)dx = 2\int_{\mathbb{R}} f(x)\int_0^x \varphi(s)(x-s)dsdx, \quad \forall f \in L^2_\alpha(\mathbb{R})\perp\mathbb{P}_1'.$$

By Fubini theorem, one is then allowed to write :

$$\begin{aligned}
&2\int_{\mathbb{R}} f(x)\int_0^x \varphi(s)(x-s)dsdx\\
&= 2\left\{\int_{\mathbb{R}_s^+}\varphi(s)\int_s^\infty f(x)(x-s)dxds - \int_{\mathbb{R}_s^-}\varphi(s)\int_{-\infty}^s f(x)(x-s)dxds\right\}\\
&= -2\int_{\mathbb{R}_s}\varphi(s)\int_{-\infty}^s f(x)(x-s)dxds\\
&= \int_{\mathbb{R}_s}\varphi(s)\left\{\int_s^\infty f(x)(x-s)dx - \int_{-\infty}^s f(x)(x-s)dx\right\}ds\\
&= \int_{\mathbb{R}}\varphi(s)\int_{\mathbb{R}_x} f(x)|x-s|dx\,ds \equiv \int_{\mathbb{R}}(|x|*f)(s)\varphi(s)ds,
\end{aligned}$$

which ends the proof. □

Lemma 4.5. *If $\alpha > \frac{1}{2}$, and if $f \in L^2_\alpha(Z)\perp\mathbb{R}$, then*

$$\left\|\mathrm{sgn}*\overline{f}\right\|_{L^2_{\alpha-1-\epsilon}(Z)} \le C\|f\|_{L^2_\alpha(Z)}.$$

Proof. In a first step f does not satisfy the polar condition. Under sufficient integrability conditions, one writes :

$$(\mathrm{sgn}(x_2)*\overline{f})(\boldsymbol{x}) = -\int_{y_2<x_2}\overline{f}dy_2 + \int_{y_2>x_2}\overline{f}dy_2.$$

For x_2 tending to infinity, $\overline{f}*\mathrm{sgn}(x_2)$ behaves as $\mathrm{sgn}(x_2)\int_{\mathbb{R}}\overline{f}$. Indeed

$$\begin{aligned}
\left|(\mathrm{sgn}(x_2)*\overline{f})(\boldsymbol{x}) - \mathrm{sgn}(x_2)\int\overline{f}\right| &= 2\left|\left(\int_{-\infty}^{x_2}\overline{f}(s)ds\right)\mathbb{1}_{\mathbb{R}_-}(x_2) - \left(\int_{x_2}^\infty\overline{f}(s)ds\right)\mathbb{1}_{\mathbb{R}_+}(x_2)\right|\\
&\le 2\|f\|_{L^2_\alpha(Z)}\left(\left\{\int_{-\infty}^{x_2}\rho^{-2\alpha}(y_2)dy_2\right\}^{\frac{1}{2}}\mathbb{1}_{\mathbb{R}_-}(x_2) + \left\{\int_{x_2}^\infty\rho^{-2\alpha}(y_2)dy_2\right\}^{\frac{1}{2}}\mathbb{1}_{\mathbb{R}_+}(x_2)\right)\\
&\le C\|f\|_{L^2_\alpha(Z)}\rho^{\frac{1}{2}-\alpha}(x_2).
\end{aligned}$$

Then taking the square, multiplying by $\rho^{2\beta}$ and integrating wrt x

$$\left\|\mathrm{sgn}(x_2)*\overline{f} - \mathrm{sgn}(x)\int\overline{f}\right\|^2_{L^2_\beta(Z)} \le C\|f\|^2_{L^2_\alpha(Z)}\int_{\mathbb{R}}\rho^{1-2\alpha+2\beta}(x_2)dx_2,$$

which is bounded, provided that $\beta < \alpha - 1$. Taking into account the polar condition $f\perp\mathbb{R}$ gives then the claim. □

Similarly to Theorem 4.1, thanks to the previous lemma, one gets that

Theorem 4.2. *Let $f \in L^2_\alpha(Z)\perp\mathbb{R}$ then for any $\alpha \in]\frac{1}{2}, \frac{3}{2}]$ one has*

$$\|G_2 * f\|_{L^2_{\alpha-2-\epsilon}(Z)/\mathbb{R}} \leq C\|f\|_{L^2_\alpha(Z)}.$$

Proof. As previously, the proof is essentially 1D. Let us choose $f \in L^2_\alpha(\mathbb{R})\perp\mathbb{R}$ and $(\varphi_\delta)_\delta \in \mathcal{S}_\#(\mathbb{R})\perp\mathbb{R}$ s.t. $\varphi_\delta \to \varphi$ in $L^2_{2-\alpha+\epsilon}(\mathbb{R})\perp\mathbb{R}$ then $|x| * \varphi_\delta$ is infinitely differentiable and one can apply the Taylor expansion with the integral rest :

$$\begin{aligned}
&\int_{\mathbb{R}} (|x| * \varphi_\delta)(x) f(x) dx \\
&= \int_{\mathbb{R}} \left\{ (|x| * \varphi_\delta)(0) + \int_0^x (\varphi_\delta * \mathrm{sgn})(s) ds \right\} f(x) dx \equiv \int_{\mathbb{R}} f(x) \int_0^x (\varphi_\delta * \mathrm{sgn})(s) ds dx.
\end{aligned}$$

Proceeding as in the proof of Lemma 4.5, one has

$$\left| \int_0^x \varphi_\delta * \mathrm{sgn}(s)\, ds \right| \leq \|\varphi_\delta\|_{L^2_{2-\alpha+\epsilon}(Z)} \rho^{\alpha-\frac{1}{2}-\epsilon}(x),$$

leading to

$$\left\| \int_0^x (\varphi_\delta * \mathrm{sgn})(s) ds \right\|_{L^2_{-\alpha}(\mathbb{R})} \leq C\|\varphi_\delta\|_{L^2_{2-\alpha+\epsilon}(\mathbb{R})},$$

with $\alpha < \frac{3}{2} + \epsilon$. Thus one has

$$\left| \int_{\mathbb{R}} (|x| * \varphi_\delta)(x) f(x) dx dy \right| \leq C\|f\|_{L^2_\alpha(\mathbb{R})} \|\varphi_\delta\|_{L^2_{2-\alpha+\epsilon}(\mathbb{R})}.$$

Moreover, one needs that $\mathbb{R} \in L^2_{-\alpha}(\mathbb{R})$, which is true if $\alpha > \frac{1}{2}$. It is not difficult to prove by similar arguments that

$$\lim_{\delta\to 0} \int_0^x (\varphi_\delta * \mathrm{sgn})(s) ds = \int_0^x (\varphi * \mathrm{sgn})(s) ds$$

strongly in the $L^2_{-\alpha}(\mathbb{R})$ topology. As in the proof of Lemma 4.5, by Fubini,

$$\int_0^x (\mathrm{sgn} * \varphi)(s) ds = 2 \int_0^x \left\{ \int_{-\infty}^s \varphi(t) dt \mathbb{1}_{s<0}(s) - \int_s^\infty \varphi(t) dt \mathbb{1}_{s>0}(s) \right\} ds =: 2 \int_0^x \mathfrak{g}(s) ds.$$

As $\mathfrak{g}$ is a C^1 function on any compact set in $\mathbb{R}$, one can integrate by parts on $(0, x)$:

$$\int_0^x \mathfrak{g}(s) ds = [\mathfrak{g}(s)s]_0^x - \int_0^x \mathfrak{g}'(s)\, s\, ds.$$

Using this in the right hand side of the previous limit, one writes

$$\begin{aligned}
J &:= \int_{\mathbb{R}} f(x) \int_0^x (\varphi * \mathrm{sgn})(s) ds dx = 2 \int_{\mathbb{R}} f(x) \left\{ \mathfrak{g}(x)x - \int_0^x \mathfrak{g}'(s)\, s\, ds \right\} dx \\
&= 2 \left\{ \int_{\mathbb{R}_-} f(x)\, x \int_{-\infty}^x \varphi(t) dt dx - \int_{\mathbb{R}_+} f(x) x \int_x^\infty \varphi(t) dt dx - \int_{\mathbb{R}} f(x) \int_0^x \varphi(s)\, s\, ds \right\} \\
&= 2 \int_{\mathbb{R}} f(x)\, x \int_{-\infty}^x \varphi(t) dt dx - 2 \int_{\mathbb{R}} f(x) \int_0^x \varphi(s)\, s\, ds\, dx =: A - B.
\end{aligned}$$

By Hölder estimates the integrals above are well defined. Fubini's theorem allows then to switch integration order. Using the orthogonality condition on φ and on f, one may easily show that

$$\left\{\begin{array}{l} A = 2\displaystyle\int_{\mathbb{R}_x} f(x)\, x \left(\int_{-\infty}^{x} \varphi(t)dt\right) dx = \int_{\mathbb{R}} f(x)x \left\{\int_{-\infty}^{x} \varphi(s)ds - \int_{x}^{\infty} \varphi(s)ds\right\} dx \\ \quad = \displaystyle\int_{\mathbb{R}_s} \varphi(s) \left\{\int_{s}^{\infty} xf(x)dx - \int_{-\infty}^{s} xf(x)dx\right\} ds, \\ B = 2\left\{\displaystyle\int_{\mathbb{R}_+} f(x) \int_0^x s\,\varphi(s)\, ds\, dx - \int_{\mathbb{R}_-} f(x) \int_x^0 s\,\varphi(s)\, ds\, dx\right\} \\ \quad = 2\left\{\displaystyle\int_{\mathbb{R}_+} s\,\varphi(s) \int_s^{\infty} f(x)dxds - \int_{\mathbb{R}_-} s\,\varphi(s) \int_{-\infty}^{s} f(x)dxds\right\} \\ \quad = -2\displaystyle\int_{\mathbb{R}} s\,\varphi(s) \int_{-\infty}^{s} f(x)dxds = \int_{\mathbb{R}} s\,\varphi(s) \left\{\int_{s}^{\infty} f(x)dxds - \int_{-\infty}^{s} f(x)dx\right\} ds. \end{array}\right.$$

These computations give :

$$\begin{aligned} A - B &= \int_{\mathbb{R}_s} \varphi(s) \left\{\int_{s}^{\infty} (x-s)f(x)dx - \int_{-\infty}^{s} (x-s)f(x)dx\right\} ds \\ &= \int_{\mathbb{R}} \varphi(s) \int_{\mathbb{R}} |x-s|f(x)dxds = \int_{\mathbb{R}} \varphi(s)(|x| * f)(s)ds. \end{aligned}$$

And because

$$\inf_{\lambda\in\mathbb{R}} \| |x| * f + \lambda \|_{L^2_{\alpha-(2+\epsilon)}(\mathbb{R})} = \sup_{\varphi\in L^2_{(2+\epsilon)-\alpha}(\mathbb{R})\perp\mathbb{R}} \frac{(|x| * f, \varphi)}{\|\varphi\|_{L^2_{(2+\epsilon)-\alpha}(\mathbb{R})}},$$

the final claim follows. □

Theorem 4.3. *Assume $\alpha > 1/2$ and recall that $q(0,-\alpha)$ is defined by* (12)*. Then the operator defined by the convolution with the fundamental solution G is a mapping from $L^2_\alpha(Z)\perp\mathbb{P}'^{\Delta}_{q(0,-\alpha)}$ on $H^1_{\alpha-1-\epsilon,\#}(Z) \cap H^2_{\alpha-2-\epsilon,\#}(Z)/\mathbb{P}'^{\Delta}_{[3/2-\alpha]}$ for any $\epsilon > 0$.*

Proof. One follows the same lines as in the proof of Theorem 1 p. 786 in [20]. More precisely, by Theorems 4.1 and 4.2, Prop. 4.6 and Lemma 4.5, the convolution with G maps also $L^2_\alpha(Z)\perp\mathbb{P}'^{\Delta}_{q(0,-\alpha)}$ on $H^1_{\alpha-1-\epsilon,\#}(Z)/\mathbb{P}'^{\Delta}_{[3/2-\alpha]}$. Thus for $f \in L^2_\alpha(Z)\perp\mathbb{P}'^{\Delta}_{q(0,-\alpha)}$, let $u = G * f \in L^2_{\alpha-2-\epsilon,\#}(Z)/\mathbb{P}'^{\Delta}_{[3/2-\alpha]}$. Then for any $\phi \in \mathcal{D}_\#(Z)$, $\langle \Delta(G*f), \phi\rangle = \langle f, G * \Delta\phi\rangle = \langle f, \phi\rangle$ which implies that $\Delta u = f$ in the sense of distribution. But since $u \in L^2_{\alpha-2-\epsilon,\#}(Z)/\mathbb{P}'^{\Delta}_{[3/2-\alpha]}$ and $\Delta u \in L^2_\alpha(Z)\perp\mathbb{P}'^{\Delta}_{q(0,-\alpha)}$, using a dyadic partition of unity [27] and standard inner regularity results (see for instance a similar proof in Theorem 3.1 in [28] or chap I in [29]), one has that:

$$\sum_{|\gamma|=2} \left\|\rho^{-\gamma} D^\gamma u\right\|^2_{L^2_{\alpha-\epsilon}(Z)} \le C\left(\|\Delta u\|^2_{L^2_{\alpha-\epsilon}(Z)} + \|u\|^2_{H^1_{\alpha-1-\epsilon,\#}(Z)}\right),$$

where the constant does not depend on u. These estimates are not isotropic with respect to the weight. Taking the lowest weight in front of the second order derivative ends the proof. □

4.7. Convolution and duality

We generalize the latter convolutions to weak data, namely, when $f \in H^{-1}_{\alpha,\#}(Z)$.

Proposition 4.7. *For $\beta < \frac{1}{2}$ and $f \in H^{-1}_{\beta,\#}(Z) \perp \mathbb{P}'^{\Delta}_{[\frac{1}{2}+\beta]}$ one has :*

$$\|G * f\|_{L^2_{\beta-1-\epsilon}(Z)/\mathbb{P}'^{\Delta}_{[\frac{1}{2}-\beta]}} \leq C\|f\|_{H^{-1}_{\beta,\#}(Z)},$$

where C is independent on f.

Proof. Using Theorem 4.3 with $\alpha = (1+\epsilon) - \beta$, which is possible since $\beta < 1/2$, one has

$$\begin{aligned}\langle f, G * \varphi\rangle_{H^{-1}_{\beta,\#}(Z)\perp\mathbb{P}'^{\Delta}_{[\frac{1}{2}+\beta]} \times H^1_{-\beta,\#}(Z)/\mathbb{P}'^{\Delta}_{[\frac{1}{2}+\beta]}} &\leq \|f\|_{H^{-1}_{\beta,\#}(Z)}\|G * \varphi\|_{H^1_{-\beta,\#}(Z)/\mathbb{P}'^{\Delta}_{[\frac{1}{2}+\beta]}} \\ &\leq C\|f\|_{H^{-1}_{\beta,\#}(Z)}\|\varphi\|_{L^2_{(1+\epsilon)-\beta}(Z)\perp\mathbb{P}'^{\Delta}_{q(0,\beta-(1+\epsilon))}}.\end{aligned}$$

as ϵ is positive and arbitrarily small $q(0, \beta-(1+\epsilon)) = [1/2-\beta]$. Hence taking the supremum over all $\varphi \in L^2_\alpha(Z)\perp\mathbb{P}'^{\Delta}_{[1/2-\beta]}$, ends the proof. □

5. The Laplace equation in a periodic infinite strip

In this section we study the problem

$$\begin{cases} -\Delta u = f \text{ in } Z, \\ \quad u \text{ is 1-periodic in the } y_1 \text{ direction} \end{cases} \tag{21}$$

in the variational context. Firstly, we characterize of the kernel of the Laplace operator.

Proposition 5.1. *Let $m \geq 1$ be an integer, α be a real number and $j = \min\{q(m,\alpha), 1\}$ where $q(m,\alpha)$ is defined by (11). A function $u \in H^m_{\alpha,\#}(Z)$ satisfies $\Delta u = 0$ if and only if $u \in \mathbb{P}'^{\Delta}_j$.*

Proof. Since $j \leq 1$, it is clear that if $u \in \mathbb{P}'^{\Delta}_j$, then $\Delta u = 0$. Conversely, let $u \in \mathcal{S}'_\#(Z)$ satisfies $\Delta u = 0$. We apply the MFT : in the sense defined in Definition 4.2, one has

$$(\tilde{k}^2 + \tilde{l}^2)\mathcal{F}(u) = 0, \quad \forall k \in \mathbb{Z}, \text{ a.e. } l \in \mathbb{R},$$

which implies that

$$\mathcal{F}(u) = \begin{cases} \sum_{j=0}^p \delta^j_{y_2=0}(l) & \text{if } k \equiv 0 \\ 0 & \text{otherwise} \end{cases}$$

where p is a non negative integer. A simple computation shows that $\mathcal{F}^{-1}(\mathcal{F}(u)) = \sum_{j=0}^p (iy_2)^j$, indeed

$$\begin{aligned}\left\langle \mathcal{F}^{-1}\left(\mathcal{F}(u)\right), \varphi\right\rangle_{\mathcal{S}'_\#,\mathcal{S}_\#} &= \left\langle \mathcal{F}(u), \breve{\mathcal{F}}(\varphi)\right\rangle_{\tilde{\mathcal{S}}'(\Gamma),\tilde{\mathcal{S}}(\Gamma)} = \left\langle \sum_{j=0}^p \delta^j_0, \breve{\mathcal{F}}(\varphi)(0,\cdot)\right\rangle_{\mathcal{S}'(\mathbb{R})\times\mathcal{S}(\mathbb{R})} \\ &= \sum_{j=0}^p (-1)^j \langle \delta_0, \partial_j\hat{\varphi}\rangle_{\mathcal{S}'(\mathbb{R})\times\mathcal{S}(\mathbb{R})} = \sum_{j=0}^p (-1)^j \left\langle \delta_0, \widehat{((iy_2)^j\varphi)}\right\rangle_{\mathcal{S}'(\mathbb{R})\times\mathcal{S}(\mathbb{R})} \\ &= \sum_{j=0}^p (-1)^j \int_{\mathbb{R}} (iy_2)^j \varphi(\boldsymbol{y})d\boldsymbol{y} \quad = \left\langle \sum_{j=0}^p (-iy_2)^j, \varphi\right\rangle_{\mathcal{S}'_\#\times\mathcal{S}_\#},\end{aligned}$$

where $\hat{\varphi}$ denotes the usual 1-dimensional Fourier transform on $\mathbb{R}$. □

We establish a Poincaré-Wirtinger's type inequality.

Lemma 5.1. *For every $u \in H^1_{\alpha,\#}(Z)$ with $\alpha \in \mathbb{R}$, one has*

$$\|u - \overline{u}\|_{L^2_\alpha(Z)} \leq |u|_{H^1_{\alpha,\#}(Z)},$$

Proof. If $u \in H^1_{\alpha,\#}(Z)$ then $\partial^\lambda u \in L^2_{\rm loc}(Z)$ for all $0 \leq |\lambda| \leq 1$. In particular, almost everywhere in y_2 one has $\partial^\lambda u(\cdot, y_2)$ in $L^2(0,1)$. Applying Parseval in the y_1 direction one gets :

$$\|u - \overline{u}\|^2_{L^2(0,1)} = \sum_{k \in \mathbb{Z}^*} |\hat{u}(k)|^2 \text{ and } \|\partial_1 u\|^2_{L^2(0,1)} = \sum_{k \in \mathbb{Z}^*} |k|^2 |\hat{u}(k)|^2$$

for almost every $y_2 \in \mathbb{R}$. This obviously gives

$$\|u - \overline{u}\|^2_{L^2(0,1)} \leq \|\partial_1 u\|^2_{L^2(0,1)} \quad \text{a.e. } y_2 \in \mathbb{R}.$$

Integrating then with respect to the vertical weight, one gets the desired estimates. □

In order to state existence and uniqueness results for solutions of (21), we first deal with the Laplace operator in the truncated domain $Z_R :=]0,1[\times(]-\infty, -R[\cup]R, +\infty[)$:

$$\begin{cases} -\Delta u = f & \text{in } Z_R, \\ u = 0 & \text{on } y_2 = \pm R. \end{cases} \tag{22}$$

Lemma 5.2. *Let α be any real number. If $f \in H^{-1}_{\alpha,\#}(Z)$ there exists $R(\alpha)$ large enough such that problem (22) has a unique solution $u \in H^1_{\alpha,\#}(Z_{R(\alpha)})$.*

Proof. First we notice that $H^{-1}_{\alpha,\#}(Z) \subset H^{-1}_{\alpha,\#}(Z_R)$. Then the proof relies on an *inf-sup* argument similar to the one used in [18] (the main difference being the nature of weights : in [18] the author derives similar estimates for exponential weights on a half strip). Indeed, we set

$$v := \omega^2(u - \overline{u}) + \psi(u), \quad \omega := \rho^\alpha, \text{ and } \psi(u) := \begin{cases} \displaystyle\int_R^{y_2} \omega^2 \partial_{y_2} \overline{u}(s)\, ds & \text{if } y_2 > R \\ 0 & \text{if } |y_2| \leq R \\ \displaystyle -\int_{y_2}^{-R} \omega^2 \partial_{y_2} \overline{u}(s)\, ds & \text{otherwise.} \end{cases}$$

First of all, we check that $v \in H^1_{-\alpha,\#}(Z_R)$.

$$\nabla v = \nabla(\omega^2)(u - \overline{u}) + \omega^2 \nabla \overline{u}.$$

An application of Lemma 5.1 proves that

$$\|\nabla v\|_{L^2_{-\alpha}(Z_R)} \leq C |u|_{H^1_{\alpha,\#}(Z_R)}.$$

which guarantees, through Hardy estimates, that indeed v belongs to $H^1_{-\alpha,\#}(Z_R)$. Secondly, we write

$$a(u,v) = \int_{Z_R} \nabla u \cdot \nabla v \, d\boldsymbol{y} = \int_{Z_R} |\nabla u|^2 \omega^2 \, d\boldsymbol{y} + \int_{Z_R} \nabla u \cdot \nabla(\omega^2)\, (u - \overline{u})\, d\boldsymbol{y}\,.$$

Using Cauchy-Schwarz, we estimate the formula above as :

$$\int_{Z_R} \nabla u \cdot \nabla(\omega^2)\ (u-\overline{u})\ d\boldsymbol{y} \leq 2\left(\int_{Z_R} |\nabla u|^2\omega^2\ d\boldsymbol{y}\right)^{\frac{1}{2}}\left(\int_{Z_R} |u-\overline{u}|^2|\nabla\omega|^2\ d\boldsymbol{y}\right)^{\frac{1}{2}}.$$

Since $|\nabla\rho| < 1$, one also has that

$$\frac{|\nabla\omega|^2}{\omega^2} \leq \frac{\alpha^2}{\rho^2(R)}, \quad \forall \boldsymbol{y} \in Z_R,$$

which gives then

$$\int_{Z_R} \nabla u \cdot \nabla(\omega^2)\ (u-\overline{u})\ d\boldsymbol{y} \leq \frac{|\alpha|}{\rho(R)}\int_{Z_R} |\nabla u|^2\omega^2\ d\boldsymbol{y}.$$

This in turn implies that

$$a(u,v) \geq \left(1-2\frac{|\alpha|}{\rho(R)}\right)|u|^2_{H^1_{\alpha,\#}(Z_R)} \geq C(R,\alpha)|u|_{H^1_{\alpha,\#}(Z_R)}|v|_{H^1_{-\alpha,\#}(Z_R)}.$$

For every fixed α there exists R large enough s.t. $C(R,\alpha) > 0$. This proves that the operator $a(\cdot,v)$ is onto from $H^1_{\alpha,\#}(Z)$ to $H^{-1}_{\alpha,\#}(Z)$. In the same way, the adjoint operator is injective. Indeed (taking $\omega = \rho^{-\alpha}$ above), for all $u \in H^1_{-\alpha,\#}(Z_R)$ there exists a $v \in H^1_{\alpha,\#}(Z_R)$ s.t.

$$a(u,v) \geq \left(1-2\frac{|\alpha|}{\rho(R)}\right)|u|^2_{H^1_{-\alpha,\#}(Z_R)} \geq k|u|_{H^1_{-\alpha,\#}(Z_R)}|v|_{H^1_{\alpha,\#}(Z_R)}.$$

and one concludes using the classical Banach-Babuška-Nečas result (see Theorem 2.6 p. 85 in [30]). □

We next solve the Laplace equation in the domain $Z_R \cup Z^R$.

Lemma 5.3. *Provided that $f \in H^{-1}_{\alpha,\#}(Z)$, there exists a unique solution $u_0 \in H^1_{\alpha,\#}(Z)$ solving:*

$$\begin{cases} -\Delta u_0 = f & \text{in } Z_R \cup Z^R, \\ u_0 = 0 & \text{on } \{y_2 = \pm R\}. \end{cases} \tag{23}$$

Moreover, if $f \perp \mathbb{R}$, and setting

$$\overline{h}_\pm := \langle [\partial_{y_2} u_0]\,, 1\rangle_{H^{-\frac{1}{2}}_{\#}(\{y_2=\pm R\}),H^{\frac{1}{2}}_{\#}(\{y_2=\pm R\})},$$

where the brackets $[\cdot]$ denote the jump across interfaces $\{y_2 = \pm R\}$, i.e.

$$[\partial_{y_2} u_0] = \lim_{y_2 \to R^+} \partial_{y_2} u_0(y_1,y_2) - \lim_{y2 \to R^-} \partial_{y_2} u_0(y_1,y_2),$$

then h satisfies :

$$\overline{h}_+ + \overline{h}_- \equiv 0. \tag{24}$$

Proof. By Lemma 5.2 there exists a unique u_0 in $H^1_{\alpha,\#}(Z)$ for every $\alpha \in \mathbb{R}$. By truncation and approximation (following the same steps as in Lemma 6.2 and Proposition 6.1), we set

$$f_\delta = (f\Phi_\delta) * \alpha_\delta, \quad \tilde{f}_\delta := f_\delta - \overline{f}_\delta,$$

where Φ_δ and α_δ are chosen as in Proposition 2.2. An easy check shows that

$$\tilde{f}_\delta \in \mathcal{D}_\#(Z), \quad \tilde{f}_\delta \perp \mathbb{R} = 0.$$

We then compute $u_{0,\delta}$ solving (23) with the data $\tilde{f}_\delta$. One has

$$\begin{aligned} \|u_{0,\delta}\|_{H^1_{\alpha,\#}(Z)} = \|u_{0,\delta}\|_{H^1_{\alpha,\#}(Z_R\cup Z^R)} &\le C\left\|\tilde{f}_\delta\right\|_{H^{-1}_{\alpha,\#}(Z_R\cup Z^R)} \\ &\le C\left\|\tilde{f}_\delta\right\|_{H^{-1}_{\alpha,\#}(Z)} \le C\|f\|_{H^{-1}_{\alpha,\#}(Z)}, \\ \|u_{0,\delta} - u_0\|_{H^1_{\alpha,\#}(Z)} &\le C\left\|\tilde{f}_\delta - f\right\|_{H^{-1}_{\alpha,\#}(Z)} \\ \|[\partial_{y_2} u_{0,\delta}] - [\partial_{y_2} u_0]\|_{H^{-\frac{1}{2}}(\{y_2=\pm R\})} &\le C\|u_{0,\delta} - u_0\|_{H^1_{\alpha,\#}(Z)} \le C\left\|\tilde{f}_\delta - f\right\|_{H^{-1}_{\alpha,\#}(Z)}. \end{aligned} \tag{25}$$

Moreover, for every $\varphi \in \mathcal{D}_\#(Z)$, one might write

$$\begin{aligned} \langle -\Delta u_{0,\delta}, \varphi\rangle = -\langle u_{0,\delta}, \Delta\varphi\rangle &= -\int_{Z_R\cup Z^R} u_{0,\delta}\Delta\varphi d\boldsymbol{y} \\ &= -\int_{(0,1)} [\partial_{y_2} u_{0,\delta}]_{\{y_2=\pm R\}}\, \varphi(y_1, \pm R)\, dy_1 + \langle \tilde{f}_\delta, \varphi\rangle, \end{aligned}$$

the latter equality being true because $u_{0,\delta}$ is in the domain of the operator : the Green formula holds. By density, one extends the above equality to test functions in $H^1_{-\alpha,\#}(Z)$, giving :

$$-\langle u_{0,\delta}, \Delta\varphi\rangle = -\langle [\partial_{y_2} u_{0,\delta}], \varphi\rangle_{\{y_2=\pm R\}} + \langle \tilde{f}_\delta, \varphi\rangle, \quad \forall \varphi \in H^1_{-\alpha,\#}(Z).$$

As α is chosen s.t. $\mathbb{R} \subset H^1_{-\alpha,\#}(Z)$, one takes $\varphi \equiv 1$ in the above formula. This leads to :

$$\overline{h}_{+,\delta} + \overline{h}_{-,\delta} := \langle [\partial_{y_2} u_{0,\delta}], 1\rangle_{\{y_2=R\}} + \langle [\partial_{y_2} u_{0,\delta}], 1\rangle_{\{y_2=-R\}} \equiv 0,$$

where the brackets denote the $H^{-\frac{1}{2}}_\#, H^{\frac{1}{2}}_\#$ duality product. Thanks to (25) and by continuity, the result holds when passing to the limit with respect to δ. □

When extending u_0 on Z, it solves

$$-\Delta u_0 = f + \delta_{\{y_2=\pm R\}} h_\pm, \quad \text{in } Z. \tag{26}$$

In the next paragraph, we compute a lift that cancels the lineic Dirac mass in the right hand side of (26).

Lemma 5.4. *Let u_0 be the unique solution of problem (23). For $\alpha \le 0$ there exists a solution w in $H^1_{\alpha,\#}(Z)$ of*

$$\begin{cases} -\Delta w = 0 & \text{in } Z_R \cup Z^R, \\ [\partial_{y_2} w] = -[\partial_{y_2} u_0] =: -h_\pm & \text{on } \{y_2 = \pm R\}. \end{cases} \tag{27}$$

Moreover, we have

$$\|w\|_{H^1_{\alpha,\#}(Z)} \le C \sum_\pm \left(\|h_\pm\|^2_{H^{-\frac{1}{2}}_\#(\{y_2=\pm R\})} \right)^{1/2}.$$

Proof. Setting

$$w := G * (h_\pm \delta_{\{y_2=\pm R\}}) = G * ((h_\pm - \overline{h}_\pm)\delta_{\{y_2=\pm R\}}) + G * (\overline{h}_\pm \delta_{\{y_2=\pm R\}}) \\ = G_1 * (h_\pm \delta_{\{y_2=\pm R\}}) + G_2 * (\overline{h}_\pm \delta_{\{y_2=\pm R\}}),$$

which is a tempered distribution ($G \in \mathcal{S}'_\#(Z)$ and $h_\pm \delta_{\{y_2=\pm R\}} \in \mathcal{E}'_\#(Z)$), w is explicit after some computations and reads:

$$w = \sum_{\pm, k\in\mathbb{Z}^*} \frac{\mathcal{F}_k(h_\pm) e^{iky_1 - |k||y_2 \pm R|}}{|k|} - \left\{ \frac{\overline{h}_+}{2}|y_2 - R| + \frac{\overline{h}_-}{2}|y_2 + R| \right\} = I_1 + I_2,$$

where

$$I_1 := \sum_{\pm, k\in\mathbb{Z}^*} \frac{\mathcal{F}_k(h_\pm) e^{iky_1 - |k||y_2 \pm R|}}{|k|} \text{ and } I_2 := -\frac{\overline{h}_+}{2}|y_2 - R| - \frac{\overline{h}_-}{2}|y_2 + R|.$$

One computes the $L^2_{\alpha-1}(Z)$ norm of I_1:

$$\int_Z I_1^2 \rho(y_2)^{2\alpha-2} d\boldsymbol{y} \le \int_Z I_1^2 d\boldsymbol{y} \le C \sum_{\pm, k\in Z^*} \frac{|\mathcal{F}_k(h_\pm)|^2}{(1+|k|^2)^{\frac{1}{2}}} \sum_{k\in Z^*} \int_{\mathbb{R}} \frac{e^{-2|k||y_2\pm R|}}{|k|} d\boldsymbol{y}, \\ \le C' \sum_\pm \|h_\pm\|^2_{H^{-\frac{1}{2}}_\#(\{y_2=\pm R\})} \sum_{k\in\mathbb{Z}^*} \frac{1}{|k|^2} \le C'' \sum_\pm \|h_\pm\|^2_{H^{-\frac{1}{2}}_\#(\{y_2=\pm R\})}.$$

In the above relations, the first inequality holds since $\alpha \le 0$. Then using the Fourier transform one writes :

$$\mathcal{F}\left(\nabla G_1 * (h_\pm \delta_{\{y_2=\pm R\}})\right) = \begin{cases} i\dfrac{(k,l)^*}{(k^2+l^2)} \mathcal{F}_k(h_\pm), & \text{if } k \in \mathbb{Z}^*, \quad \text{a.e. } l \in \mathbb{R}, \\ 0 & \text{for } k = 0. \end{cases}$$

Next, taking the $l^2(\mathbb{Z}; L^2(Z))$ norm of the above expression, using again the fact that $\alpha \le 0$ and by Parseval, one gets:

$$\int_Z \rho(y_2)^{2\alpha} |\partial_{y_1} I_1|^2 d\boldsymbol{y} \le \left\|\partial_{y_1} G_1 * \left(h_\pm \delta_{\{y_2=\pm R\}}\right)\right\|^2_{L^2(Z)} = \sum_{k\in\mathbb{Z}^*} \int_R \frac{k^2}{(k^2+l^2)^2}\, dl |\mathcal{F}_k(h_\pm)|^2, \\ = \sum_{k\in\mathbb{Z}^*} \frac{C}{|k|} |\mathcal{F}_k(h_\pm)|^2 = \|h_\pm\|^2_{H^{-\frac{1}{2}}_\#(\{y_2=\pm R\})}.$$

A similar computation gives the derivative with respect to y_2. It follows that

$$\|I_1\|_{H^1_{\alpha,\#}(Z)} \le C \sum_\pm \left(\|h_\pm\|^2_{H^{-\frac{1}{2}}_\#(\{y_2=\pm R\})} \right)^{1/2}.$$

Thanks to Lemma 5.3, one can reduce I_2 to

$$I_2 = \mp \overline{h}_\pm R \text{ when } |y_2| > R.$$

Thus the claims follows since I_1 and I_2 both belong to $H^1_{\alpha,\#}(Z)$. □

We are now ready to state the first isomorphism result of the Laplace operator in Z.

Theorem 5.1. *Let α be a real number such that $-1/2 \leq \alpha \leq 1/2$. Then the Laplace operator defined by*

$$\Delta \, : \, H^1_{\alpha,\#}(Z)/\mathbb{R} \mapsto H^{-1}_{\alpha,\#}(Z) \perp \mathbb{R}, \tag{28}$$

is an isomorphism.

Proof. Let $f \in H^{-1}_{\alpha,\#}(Z)$. Observe that if $u \in H^1_{\alpha,\#}(Z)$ satisfies $\Delta u = f$, then due to the density of $\mathcal{D}_{\#}(Z)$ in $H^1_{-\alpha,\#}(Z)$, for any $\varphi \in H^1_{-\alpha,\#}(Z)$, we have

$$\langle u, \Delta\varphi \rangle = \langle f, \varphi \rangle.$$

Thus, we easily see that, under the assumptions on α, for any $p \in \mathbb{R} \subset H^1_{-\alpha,\#}(Z)$, the datum f satisfies the necessarily compatibility condition

$$\langle f, p \rangle = 0.$$

Now it is also clear that the Laplace operator defined by (28) is linear and continuous. It is also injective since $\Delta u = 0$ and $u \in H^1_{\alpha,\#}(Z)$ imply that u is a constant (see Proposition 5.1). It remains to prove that the operator is onto. Let f be in $H^{-1}_{\alpha,\#}(Z) \perp \mathbb{R}$ and $R > 0$ be a real number. Thanks to Lemmas 5.3 and 5.4, one sets

$$u := u_0 + w,$$

where u_0 solves (23) and w lifts the jumps of $\partial_{y_2} u_0$ on $\{y_2 = \pm R\}$ and thus solves (27). Results on u_0 and w end the proof. At this stage, the claim is proved for $\alpha \leq 0$. By duality the results is true also for $\alpha > 0$ which ends the proof. □

It remains to extend the isomorphism result (28) to any $\alpha \in \mathbb{R}$ and to any $m \in \mathbb{Z}$. For this sake, we state some regularity results for the Laplace operator in Z.

Theorem 5.2. *Let α be a real number such that $-1/2 < \alpha < 1/2$ and let ℓ be an integer. Then the Laplace operator defined by*

$$\Delta \, : \, H^{1+\ell}_{\alpha+\ell,\#}(Z)/\mathbb{R} \mapsto H^{-1+\ell}_{\alpha+\ell,\#}(Z) \perp \mathbb{R} \tag{29}$$

is onto.

Proof. Owing to Theorem 5.1, the statement of the theorem is true for $\ell = 0$. Assume that it is true for $\ell = k$ and let us prove that it is still true for $\ell = k + 1$. The Laplace operator defined by (29) is clearly linear and continuous. It is also injective : if $u \in H^{k+2}_{\alpha+k+1,\#}(Z)$ and $\Delta u = 0$ then u is constant. To prove that it is onto, let f be given in $H^{k}_{\alpha+k+1,\#}(Z) \perp \mathbb{R}$. According to (7), f belongs to $H^{-1+k}_{\alpha+k,\#}(Z) \perp \mathbb{R}$. Then the induction assumption implies that there exists $u \in H^{1+k}_{\alpha+k,\#}(Z)$ such that $\Delta u = f$. Next, we have

$$\Delta(\rho\partial_i u) = \rho\partial_i f + \partial_i u \Delta\rho + 2\nabla\rho\nabla(\partial_i u). \tag{30}$$

Using (6), (7), (9) and (10), all the terms of the right-hand side belong to $H^{-1+k}_{\alpha+k,\#}(Z)$. This implies that $\Delta(\rho\partial_i u)$ belongs to $H^{-1+k}_{\alpha+k,\#}(Z)$. Therefore, $\Delta(\rho\partial_i u)$ also belongs to $H^{-2+k}_{\alpha+k-1,\#}(Z)$.

Moreover, u belonging to $H^{1+k}_{\alpha+k,\#}(Z)$ implies that $\rho\partial_i u$ belongs $H^{k}_{\alpha+k-1,\#}(Z)$ and for any $\varphi \in H^{2-k}_{-\alpha-k+1,\#}(Z)$, we have

$$\langle \Delta(\rho\partial_i u), \varphi\rangle_{H^{-2+k}_{\alpha+k-1,\#}(Z)\times H^{2-k}_{-\alpha-k+1,\#}(Z)} = \langle \rho\partial_i u, \Delta\varphi\rangle_{H^{k}_{\alpha+k-1,\#}(Z)\times H^{-k}_{-\alpha-k+1,\#}(Z)}.$$

Since $\mathbb{R} \subset H^{2-k}_{-\alpha-k+1,\#}(Z)$, we can take $\varphi \in \mathbb{R}$ which implies that $\Delta\varphi = 0$. It follows that $\Delta(\rho\partial_i u)$ belongs to $H^{-1+k}_{\alpha+k,\#}(Z)\perp\mathbb{R}$. Thanks to the induction assumption, there exists v in $H^{1+k}_{\alpha+k,\#}(Z)$ such that

$$\Delta v = \Delta(\rho\partial_i u).$$

Hence, the difference $v - \rho\partial_i u$ is a constant. Since the constants are in $H^{1+k}_{\alpha+k,\#}(Z)$, we deduce that $\rho\partial_i u$ belongs to $H^{1+k}_{\alpha+k,\#}(Z)$ which implies that u belongs to $H^{2+k}_{\alpha+k+1,\#}(Z)$.

□

Remark 5.1. Let us point out that the above theorem excludes the values $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$: due to logarithmic weights, the space $H^1_{1/2,\#}(Z)$ is not included in $L^2_{-1/2}(Z)$. By duality, this also implies that the space $L^2_{1/2}(Z)$ is not included in $H^{-1}_{-1/2,\#}(Z)$. Therefore, in (30) for $\alpha = 1/2$, the term $\partial_i u\Delta\rho$ does not belong to $H^{-1}_{1/2,\#}(Z)$. Thus, in this section, the extension of (28) to any $\alpha \in \mathbb{R}$ will exclude some critical values of α. We deal with these critical values in Section 6.

Remark 5.2. As an application of the above theorem for the particular case when $\ell = 1$, we derive a weighted version of Calderón-Zygmund inequalities [31]. More precisely, let α be a real number such that $1/2 < \alpha < 3/2$, then there exists a constant $C > 0$, such that for any $u \in \mathcal{D}_{\#}(Z)$:

$$\|\partial_i\partial_j u\|_{L^2_\alpha(Z)} \leq C\|\Delta u\|_{L^2_\alpha(Z)}. \tag{31}$$

Using (31) and thanks to the Closed Range Theorem of Banach, we prove that :

Theorem 5.3. *Let α be a real number satisfying $1/2 < \alpha < 3/2$. For $m \in \mathbb{N}$, $m \geq 3$, the following operator*

$$\Delta \,:\, H^m_{\alpha,\#}(Z)/\mathbb{P}'_{m-2} \mapsto H^{m-2}_{\alpha,\#}(Z)/\mathbb{P}'_{m-4} \tag{32}$$

is an isomorphim.

The next result is then a straightforward consequence of the latter result.

Theorem 5.4. *Let α be a real number satisfying $1/2 < \alpha < 3/2$. For $m \in \mathbb{N}$, $m \geq 3$, the following operator*

$$\Delta \,:\, H^m_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{m-2} \mapsto H^{m-2}_{\alpha,\#}(Z) \tag{33}$$

is an isomorphim.

The next theorem extends Theorem 5.1.

Theorem 5.5. *Let α be a real number satisfying $1/2 < \alpha < 3/2$ and let $\ell \geq 1$ be a given integer. Then the Laplace operators defined by*

$$\Delta \,:\, H^1_{-\alpha+\ell,\#}(Z)/\mathbb{P}'^{\Delta}_{1-\ell} \mapsto H^{-1}_{-\alpha+\ell,\#}(Z)\perp\mathbb{P}'^{\Delta}_{-1+\ell} \tag{34}$$

and

$$\Delta \,:\, H^1_{\alpha-\ell,\#}(Z)/\mathbb{P}'^{\Delta}_{-1+\ell} \mapsto H^{-1}_{\alpha-\ell,\#}(Z)\perp\mathbb{P}'^{\Delta}_{1-\ell} \tag{35}$$

are isomorphism.

Proof. Observe first that when $\ell = 1$, the result is proved in Theorem 5.1. Observe next that if $m \geq 2$ is an integer, the mapping

$$\Delta \,:\, H^m_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{m-2} \mapsto H^{m-2}_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{-m+2}$$

is onto. Indeed, if $m = 2$, this isomorphism is exactly defined by (29) with $\ell = 1$. If $m \geq 3$, it is defined by (33). Now through duality and transposition, the mapping

$$\Delta \,:\, H^{-m+2}_{-\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{-m+2} \mapsto H^{-m}_{-\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{m-2}$$

is onto. Next, using the same arguments as in the proof of Theorem 5.2, we are able to show that for any integer $\ell \geq 1$, the operator

$$\Delta \,:\, H^{-m+2+\ell}_{-\alpha+\ell,\#}(Z)/\mathbb{P}'^{\Delta}_{-m+2} \mapsto H^{-m+\ell}_{-\alpha+\ell,\#}(Z) \perp \mathbb{P}'^{\Delta}_{m-2}$$

is an isomorphism. Choosing $m = \ell + 1$, the operator defined by (34) is an isomorphism. By duality and transposition, the mapping defined by (35) is onto as well.

□

Remark 5.3. Summarizing theorems 5.1 and 5.5, we deduce that, for any $\alpha \in \mathbb{R}$ such that $\alpha \neq \pm(\frac{1}{2} + k)$, $k \in \mathbb{N}^*$, the mapping

$$\Delta \,:\, H^1_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[1/2-\alpha]} \mapsto H^{-1}_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{[1/2+\alpha]} \tag{36}$$

is an isomorphism . As a consequence, for any $m \in \mathbb{Z}$, for any $\alpha \in \mathbb{R}$ such that $\alpha \neq \pm(\frac{1}{2} + k)$, $k \in \mathbb{N}^*$, the mapping

$$\Delta \,:\, H^{m+2}_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[m+3/2-\alpha]} \mapsto H^m_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{[-m-1/2+\alpha]} \tag{37}$$

is an isomorphism.

6. Isomorphism results for the critical cases

Note that the isomorphism result (37) is not valid for $\alpha = \pm(\frac{1}{2} + k)$, $k \in \mathbb{N}^*$. In order to extend it to these cases, we redefine the weighted spaces. Therefore, proceeding as in [17], for $m \in \mathbb{Z}$, $p \in \mathbb{N}$, and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$, we introduce the space

$$\begin{aligned} X^{m+p}_{\alpha+p,\#}(Z) := \{ & u \in H^m_{\alpha,\#}(Z),\ \ \forall \lambda \in \mathbb{N},\ \ 1 \leq \lambda \leq p, \\ & y_2^\lambda u \in H^{m+\lambda}_{\alpha,\#}(Z),\ \ u \in H^{m+p}_{\mathrm{loc},\#}(Z)\}, \end{aligned}$$

where $H^{m+p}_{\mathrm{loc},\#}(Z)$ stands for the space of functions that belong to $H^{m+p}_{\#}(Z)$ with compact support in the y_2 direction. The space $X^{m+p}_{\alpha+p,\#}(Z)$ is a Banach space when endowed with the norm

$$\|u\|_{X^{m+p}_{\alpha+p,\#}(Z)} = \left(\sum_{0 \leq \lambda \leq p} \|y_2^\lambda u\|^p_{H^{m+\lambda}_{\alpha,\#}(Z)} + \|u\|^p_{H^{m+p}_{\#}(Z^1)} \right)^{1/p},$$

where $Z^1 :=]0,1[\times]-1,1[$.

Remark 6.1. We restricted the definition of the space $X^{m+p}_{\alpha+p,\#}(Z)$ to $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$, but this definition holds for all α in $\mathbb{R}$. For $\alpha \in \mathbb{R} \setminus \frac{1}{2}\mathbb{Z}$, since no logarithmic weights appear in the definition of $H^{m+p}_{\alpha+p,\#}(Z)$, the spaces $X^{m+p}_{\alpha+p,\#}(Z)$ and $H^{m+p}_{\alpha+p,\#}(Z)$ coincide algebraically and topologically [17].

The next proposition is dedicated to density of $\mathcal{D}_{\#}(Z)$ in $X^{m+p}_{\alpha+p,\#}(Z)$. In order to prove it, we establish two lemmas giving additional properties on the space $H^{m}_{\alpha,\#}(Z)$. The first lemma characterizes the dual space $H^{-m}_{-\alpha,\#}(Z)$.

Lemma 6.1. *Let $\alpha \in \mathbb{R}$ and $f \in H^{-m}_{-\alpha,\#}(Z)$. Then for any $v \in H^{m}_{\alpha,\#}(Z)$, we have*

$$\langle f, v\rangle_{H^{-m}_{-\alpha,\#}(Z)\times H^{m}_{\alpha,\#}(Z)} = \sum_{0\le|\mu|\le m} \int_Z g_\mu \, \partial^\mu v \, d\boldsymbol{y},$$

where $g_m \in L^2_{-\alpha}(Z)$ and for any $0 \le |\mu| < m$,

- *$g_\mu \in L^2_{-\alpha+m-|\mu|}(Z)$ if $\alpha \notin \{\frac{1}{2}, ..., m-\frac{1}{2}\}$,*
- *$(\ln(1+\rho^2))^{1/2} g_\mu \in L^2_{-\alpha+m-|\mu|}(Z)$ if $\alpha \in \{\frac{1}{2}, ..., m-\frac{1}{2}\}$.*

Proof. We shall only prove the statement for $\alpha \notin \{\frac{1}{2}, ..., m-\frac{1}{2}\}$. The proof corresponding to critical values of α is similar. We set $\boldsymbol{E} = \prod_{0\le|\mu|\le m} L^2_{-\alpha+m-|\mu|}(Z)$ such that for any $\boldsymbol{\psi} = (\psi_\mu)_{0\le|\mu|\le m} \in \boldsymbol{E}$,

$$\|\boldsymbol{\psi}\|_{\boldsymbol{E}} = \sum_{0\le|\mu|\le m} \|\psi_\mu\|_{L^2_{-\alpha+m-|\mu|}(Z)}.$$

The operator T defined by $T : v \in H^{m}_{\alpha,\#}(Z) \mapsto (\partial^\mu v)_{0\le|\mu|\le m} \in \boldsymbol{E}$ is isometric. We now set $\boldsymbol{G} := T(H^{m}_{\alpha,\#}(Z))$ and $S := T^{-1} : \boldsymbol{G} \mapsto H^{m}_{\alpha,\#}(Z)$. The mapping $L : \boldsymbol{h} \in \boldsymbol{G} \mapsto \langle f, S\boldsymbol{h}\rangle_{H^{-m}_{-\alpha,\#}(Z)\times H^{m}_{\alpha,\#}(Z)}$ is linear and continuous. Therefore thanks to the Hahn-Banach theorem, there exists $\tilde{L}$, a linear and continuous extension of L on $\boldsymbol{E}$ such that $\|\tilde{L}\|_{\boldsymbol{E}'} = \|f\|_{H^{-m}_{-\alpha,\#}(Z)}$. Thanks to the Riesz representation theorem, there exists $g_\mu \in L^2_{-\alpha+m-|\mu|}(Z)$ for any $0 \le |\mu| \le m$, such that

$$\forall v \in H^{m}_{\alpha,\#}(Z), \;\; \langle \tilde{L}, v\rangle = \sum_{0\le|\mu|\le m} \int_Z g_\mu \, \partial^\mu v \, d\boldsymbol{y}\,.$$

□

The second lemma concerns the density of functions with compact support.

Lemma 6.2. *Let $m \in \mathbb{Z}$, $\lambda \in \mathbb{N}$, $\alpha \in \mathbb{R}$, Φ_ℓ be defined by (13) and $u \in H^{m+\lambda}_{\alpha,\#}(Z)$. We set $u_\ell(\boldsymbol{y}) = \Phi_\ell(y_2)u(\boldsymbol{y})$ a.e. $\boldsymbol{y} \in Z$. Then we have*

$$\lim_{\ell\to\infty} \|u - u_\ell\|_{H^{m+\lambda}_{\alpha,\#}(Z)} = 0. \tag{38}$$

Proof. It is clear that for the case $m+\lambda \geq 0$, (38) is satisfied. We shall now focus on the case $m+\lambda < 0$. We have

$$\|u-u_\ell\|_{H^{m+\lambda}_{\alpha,\#}(Z)} = \sup_{\substack{v\in H^{-m-\lambda}_{-\alpha,\#}(Z)\\ v\neq 0}} \frac{|\langle u-u_\ell, v\rangle|}{\|v\|},$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^{m+\lambda}_{\alpha,\#}(Z)$ and $H^{-m-\lambda}_{-\alpha,\#}(Z)$. Besides, thanks to Lemma 6.1 there exist g_μ such that $g_{-m-\lambda} \in L^2_\alpha(Z)$ and for any $0 \leq |\mu| < -m-\lambda$,

- $g_\mu \in L^2_{\alpha-m-\lambda-|\mu|}(Z)$ if $-\alpha \notin \{1/2, ..., -m-\lambda-1/2\}$
- $(\ln(1+\rho^2))^{1/2} g_\mu \in L^2_{\alpha-m-\lambda-|\mu|}(Z)$ if $-\alpha \in \{1/2, ..., -m-\lambda-1/2\}$

satisfying for any $v \in H^{-m-\lambda}_{-\alpha,\#}(Z)$,

$$\langle u, v\rangle = \sum_{0\leq|\mu|\leq -m-\lambda} \int_Z g_\mu\, \partial^\mu v\, d\boldsymbol{y}\,.$$

Therefore, we can write

$$\langle u-u_\ell, v\rangle = \sum_{0\leq|\mu|\leq -m-\lambda} \int_Z g_\mu(\partial^\mu v - \partial^\mu(\Phi_\ell\, v))d\boldsymbol{y}\,.$$

It follows that

$$|\langle u-u_\ell, v\rangle| \leq \sum_{0\leq|\mu|\leq -m-\lambda} \int_Z |g_\mu(1-\Phi_\ell)\partial^\mu v| d\boldsymbol{y} + C \sum_{0\leq|\mu|\leq -m-\lambda} \sum_{0<|k|\leq|\mu|} \int_Z g_\mu \partial^k \Phi_\ell\, \partial^{\mu-k} v d\boldsymbol{y}\,.$$

We deduce that

$$|\langle u-u_\ell, v\rangle| \leq \left(\sum_{0\leq|\mu|\leq -m-\lambda} \|g_\mu(1-\Phi_\ell)\|_{L^2_{\alpha-m-\lambda-|\mu|}(Z)} \right.$$
$$\left. + C \sum_{0\leq|\mu|\leq -m-\lambda} \sum_{0<|k|\leq|\mu|} \|g_\mu\, \partial^k \Phi_\ell\|_{L^2_{\alpha-m-\lambda-|\mu|}(Z)} \right) \|v\|_{H^{-m-\lambda}_{-\alpha,\#}(Z)}.$$

The first two terms in the right hand side tend to zero as ℓ tends to infinity. This proves (38) for $m+\lambda < 0$. □

We proceed with three results showing some properties of the spaces $X^{m,p}_{\alpha,\#}(Z)$.

Proposition 6.1. *Let $m \in \mathbb{Z}$, $p \in \mathbb{N}$ and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$, then the space $\mathcal{D}_\#(Z)$ is dense in $X^{m+p}_{\alpha+p,\#}(Z)$.*

Proof. Let u be in $X^{m+p}_{\alpha+p,\#}(Z)$. Then according to Lemma 6.2, u can be approximated by $u_\ell \in X^{m+p}_{\alpha+p,\#}(Z)$ with compact support in the y_2 direction. Therefore u_ℓ belongs to $H^{m+p}_{\alpha,\#}(Z)$, and thanks to Proposition 2.2, u_ℓ can be approximated by $\psi_\ell \in \mathcal{D}_\#(Z)$. This ends the proof. □

For $m \in \mathbb{Z}$, $p \in \mathbb{N}$ and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$, we also define the space $X^{m-p}_{-\alpha-p,\#}(Z)$ that is the dual space of $X^{-m+p}_{\alpha+p,\#}(Z)$. It is therefore a subspace of $\mathcal{D}'_{\#}(Z)$.

Proposition 6.2. *Let $m, p \in \mathbb{Z}$ and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$. $X^{m+p+1}_{\alpha+p+1,\#}(Z)$ is densely embedded in $X^{m+p}_{\alpha+p,\#}(Z)$.*

Proof. If $p \in \mathbb{N}$, due to the definition of the space $X^{m+p+1}_{\alpha+p+1,\#}(Z)$, the imbedding is straightforward. Moreover the density of $\mathcal{D}_{\#}(Z)$ in $X^{m+p}_{\alpha+p,\#}(Z)$ implies the density of $X^{m+p+1}_{\alpha+p+1,\#}(Z)$ in $X^{m+p}_{\alpha+p,\#}(Z)$. Hence, if $p \leq 0$, the dense imbedding holds :

$$X^{-m-p}_{-\alpha-p,\#}(Z) \subset X^{-m-p-1}_{-\alpha-p-1,\#}(Z).$$

By duality, the desired result follows. □

Proposition 6.3. *Let $m, p \in \mathbb{Z}$ and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$. If u belongs to $X^{m+p}_{\alpha+p,\#}(Z)$, then $\partial_j u$ belongs to $X^{m-1+p}_{\alpha+p,\#}(Z)$, for any $j = 1, 2$.*

Proof. (i) Assume that $p \geq 0$. If $u \in X^{m+p}_{\alpha+p,\#}(Z)$, then $u \in H^m_{\alpha,\#}(Z)$ and $\partial_j u \in H^{m-1}_{\alpha,\#}(Z)$. Besides, for any $\lambda \in \mathbb{N}$, such that $1 \leq \lambda \leq p$, we have $y_2^\lambda(\partial_j u) = \partial_j(y_2^\lambda u) - \partial_j(y_2^\lambda)u$. Since $y_2^\lambda u \in H^{m+\lambda}_{\alpha,\#}(Z)$, it follows that $\partial_j(y_2^\lambda u) \in H^{m-1+\lambda}_{\alpha,\#}(Z)$. Thus it is clear that $y_2^\lambda(\partial_1 u)$ belongs to $H^{m-1+\lambda}_{\alpha,\#}(Z)$. Furthermore, we have $\partial_2(y_2^\lambda)u = \lambda y_2^{\lambda-1}u$ and since $0 \leq \lambda - 1 \leq p - 1$, $y_2^{\lambda-1}u$ belongs to $H^{m+\lambda-1}_{\alpha,\#}(Z)$. We deduce that for any $\lambda \in \mathbb{N}$, such that $1 \leq \lambda \leq p$, $y_2^\lambda(\partial_j u)$ belongs $H^{m-1+\lambda}_{\alpha,\#}(Z)$. Finally it is straightforward that if u is in $H^{m+p}_{\#}(Z^1)$, then $\partial_j u$ is in $H^{m+p-1}_{\#}(Z^1)$.
(ii) Let now $p < 0$. If $u \in X^{m+p}_{\alpha+p,\#}(Z)$, then for any $\varphi \in \mathcal{D}_{\#}(Z)$, we can write

$$\langle \partial_j u, \varphi \rangle_{\mathcal{D}'_{\#}(Z)\times\mathcal{D}_{\#}(Z)} = -\langle u, \partial_j \varphi \rangle_{\mathcal{D}'_{\#}(Z)\times\mathcal{D}_{\#}(Z)}.$$

Thanks to (i), for any $\varphi \in X^{-m+1-p}_{-\alpha-p,\#}(Z)$, $\partial_j \varphi$ belongs to $X^{-m-p}_{-\alpha-p,\#}(Z)$ which ends the proof. □

Proposition 6.4. *Let $m, p \in \mathbb{Z}$ and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$. If u belongs to $X^{m+p}_{\alpha+p,\#}(Z)$, then $y_2 u$ belongs to $X^{m+p}_{\alpha+p-1,\#}(Z) \equiv X^{m+1+p-1}_{\alpha+p-1,\#}(Z)$.*

Proof. (i) Assume that $p \geq 1$. If u belongs to $X^{m+p}_{\alpha+p,\#}(Z)$, then u belongs to $H^m_{\alpha,\#}(Z)$ and $y_2 u$ belongs to $H^{m+1}_{\alpha,\#}(Z)$. Next, for any $\lambda \in \mathbb{N}$, such that $1 \leq \lambda \leq p - 1$, we have $y_2^\lambda(y_2 u) = y_2^{\lambda+1}u \in H^{m+\lambda+1}_{\alpha,\#}(Z)$. Finally if u belongs to $H^{m+p}_{\#}(Z^1)$, then $y_2 u$ also belongs to $H^{m+p}_{\#}(Z^1)$. We thus deduce that $y_2 u \in X^{m+p}_{\alpha+p-1,\#}(Z)$.
(ii) Assume now $p \leq 0$. For any $\varphi \in \mathcal{D}_{\#}(Z)$, we can write

$$\langle y_2 u, \varphi \rangle_{\mathcal{D}'_{\#}(Z)\times\mathcal{D}_{\#}(Z)} = \langle u, y_2 \varphi \rangle_{\mathcal{D}'_{\#}(Z)\times\mathcal{D}_{\#}(Z)}.$$

Thanks to (i), for any $\varphi \in X^{-m-p}_{-\alpha-p+1,\#}(Z) \equiv X^{-m-1-p+1}_{-\alpha-p+1,\#}(Z)$, $y_2\varphi$ belongs to $X^{-m-p}_{-\alpha-p,\#}(Z)$ which ends the proof. □

The above properties allow to extend Theorem 5.2.

Theorem 6.1. *Let $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$ and $\ell \in \mathbb{N}$. Then the operator defined by*

$$\Delta \,:\, X^{1+\ell}_{\alpha+\ell,\#}(Z)/\mathbb{R} \mapsto X^{-1+\ell}_{\alpha+\ell,\#}(Z) \perp \mathbb{R} \tag{39}$$

is an isomorphism.

Proof. Observe first that since $X^1_{\alpha,\#}(Z) = H^1_{\alpha,\#}(Z)$ and $X^{-1}_{\alpha,\#}(Z) = H^{-1}_{\alpha,\#}(Z)$, the case $\ell = 0$ is proved in Theorem 5.1. Now we proceed by induction, as in the proof of Theorem 5.2. Assume that the statement is true for $\ell = k$ and let us prove that it is then true for $\ell = k+1$. The Laplace operator defined by (39) is clearly linear and continuous. It is also injective : if u belongs to $X^{2+k}_{\alpha+k+1,\#}(Z) \subset H^1_{\alpha,\#}(Z)$ and is harmonic then it is a constant. Let us prove that it is onto. Let $f \in X^k_{\alpha+k+1,\#}(Z) \perp \mathbb{R}$. Then $f \in X^{-1+k}_{\alpha+k,\#}(Z) \perp \mathbb{R}$ and the induction assumption implies that there exists $u \in X^{1+k}_{\alpha+k,\#}(Z)$ such that $\Delta u = f$. We now consider the equality

$$\Delta(y_2 \partial_i u) = y_2 \partial_i f + 2\partial_2 \partial_i u. \tag{40}$$

Using Proposition 6.3 and Proposition 6.4, we deduce that $\Delta(y_2\partial_i u)$ belongs to $X^{-1+k}_{\alpha+k,\#}(Z) \subset X^{-2+k}_{\alpha+k-1,\#}(Z)$. Moreover, thanks to Propositions 6.3 and 6.4, if $u \in X^{1+k}_{\alpha+k,\#}(Z)$ then $y_2\partial_i u \in X^k_{\alpha+k-1,\#}(Z)$ and for any $\varphi \in X^{2-k}_{-\alpha-k+1,\#}(Z)$, we have

$$\langle \Delta(y_2\partial_i u), \varphi \rangle_{X^{-2+k}_{\alpha+k-1,\#}(Z) \times X^{2-k}_{-\alpha-k+1,\#}(Z)} = \langle y_2 \partial_i u, \Delta\varphi \rangle_{X^k_{\alpha+k-1,\#}(Z) \times X^{-k}_{-\alpha-k+1,\#}(Z)}.$$

Since $\mathbb{R} \subset X^{2-k}_{-\alpha-k+1,\#}(Z)$, we shall take $\varphi \in \mathbb{R}$. This shows that $\Delta(y_2\partial_i u) \in X^{-1+k}_{\alpha+k,\#}(Z) \perp \mathbb{R}$. Thanks to the induction assumption, there exists $v \in X^{1+k}_{\alpha+k,\#}(Z)$ satisfying $\Delta v = \Delta(y_2\partial_i u)$. Hence $v - y_2\partial_i u \in \mathbb{R}$ and since $\mathbb{R} \subset X^{1+k}_{\alpha+k,\#}(Z)$, it follows that $y_2\partial_i u \in X^{1+k}_{\alpha+k,\#}(Z)$. Therefore summarizing, we obtained that $u \in H^1_{\alpha,\#}(Z)$, for any $1 \le \lambda \le k+1$ $y_2 u \in H^{1+\lambda}_{\alpha,\#}(Z)$. It remains to prove that $u \in H^{k+2}_{\mathrm{loc},\#}(Z)$. Let $\chi \in \mathcal{D}(\mathbb{R})$ and consider the function $\boldsymbol{y} \mapsto \chi(y_2)u(\boldsymbol{y})$ for almost every $\boldsymbol{y} \in Z$. Then $\Delta(\chi u) = \chi f + 2\nabla\chi\nabla u + u\Delta\chi$ belongs to $H^k_{\mathrm{loc},\#}(Z)$. It follows from standard inner elliptic regularity results that χu belongs to $H^{2+k}_{\mathrm{loc},\#}(Z)$ which in turn yields $u \in H^{2+k}_{\mathrm{loc},\#}(Z)$. □

Remark 6.2. The above theorem extends the Calderón-Zygmund inequality (31) to $\alpha \in \{\frac{1}{2}, \frac{3}{2}\}$.

Proceeding as in the non-critical cases, we can extend Theorems 5.3, 5.4 and 5.5. More precisely, we have :

Theorem 6.2. *Let α be a real number satisfying $1/2 \le \alpha \le 3/2$. For $m \in \mathbb{N}$, $m \ge 3$, the mapping*

$$\Delta \,:\, H^m_{\alpha,\#}(Z)/\mathbb{P}'_{m-2} \mapsto H^{m-2}_{\alpha,\#}(Z)/\mathbb{P}'_{m-4} \tag{41}$$

is an isomorphism.

Theorem 6.3. *Let α be a real number satisfying $1/2 \le \alpha \le 3/2$. For $m \in \mathbb{N}$, $m \ge 3$, the mapping*

$$\Delta \,:\, H^m_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{m-2} \mapsto H^{m-2}_{\alpha,\#}(Z) \tag{42}$$

is an isomorphism.

Theorem 6.4. *Let $\alpha \in \{\frac{1}{2}, \frac{3}{2}\}$ and let $\ell \geq 1$ be an integer. Then the Laplace operators defined by*

$$\Delta \, : \, X^1_{-\alpha+\ell,\#}(Z)/\mathbb{P}'^{\Delta}_{1-\ell} \mapsto X^{-1}_{-\alpha+\ell,\#}(Z) \perp \mathbb{P}'^{\Delta}_{-1+\ell} \tag{43}$$

and

$$\Delta \, : \, X^1_{\alpha-\ell,\#}(Z)/\mathbb{P}'^{\Delta}_{-1+\ell} \mapsto X^{-1}_{\alpha-\ell,\#}(Z) \perp \mathbb{P}'^{\Delta}_{1-\ell} \tag{44}$$

are isomorphisms.

Remark 6.3. At this step we are able to state Thereom 1.1 that we recall here. Let $m \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$, then the Laplace operator defined by

$$\Delta \, : X^{m+2}_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{q(m+2,\alpha)} \mapsto X^{m}_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{q(-m,-\alpha)} \tag{45}$$

is an isomorphism.

In order to prove our main result stated in Theorem 1.2, let us first note that we might compare the spaces $X^{m+p}_{\alpha+p,\#}(Z)$ and $H^{m+p}_{\alpha+p,\#}(Z)$ for $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$.

Proposition 6.5. *Let m, p be two integers and let $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$. Then*

$$X^{m+p}_{\alpha+p,\#}(Z) \subset H^{m+p}_{\alpha+p,\#}(Z). \tag{46}$$

If, moreover $\alpha = \frac{1}{2}$, then

$$X^{m+p}_{\frac{1}{2}+p,\#}(Z) = H^{m+p}_{\frac{1}{2}+p,\#}(Z). \tag{47}$$

The proof is based on a partition of unity in the y_2 direction and direct computations. Note that (47) is not valid for $\alpha = -\frac{1}{2}$ and $p \geq 1$ since the space $H^{m+p}_{-1/2+p,\#}(Z)$ is not included in $H^m_{-1/2,\#}(Z)$ (see (8)).

Sketch of the proof of Theorem 1.2. We shall decompose it into three successive steps:

1. If $\alpha > -1/2$ and $f \in L^2_{\alpha+1}(Z) \perp \mathbb{P}'^{\Delta}_{[1/2+\alpha]}$, then $G * f \in X^2_{\alpha+1,\#}(Z)$ is a solution of the Laplace equation (21) unique up to a polynomial of $\mathbb{P}'^{\Delta}_{q(2,\alpha+1)}$. Moreover, we have the estimate
$$\|G * f\|_{X^2_{\alpha+1,\#}(Z)/\mathbb{P}'^{\Delta}_{q(2,\alpha+1)}} \leq C \|f\|_{L^2_{\alpha+1}(Z)}.$$
This result is a straightforward consequence of the isomorphism result (45) for $m = 0$, Theorem 4.3 and Proposition 5.1.
2. If $\alpha \in \mathbb{R}$ and $f \in H^{-1}_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{q(1,-\alpha)}$, then $G * f \in H^1_{\alpha,\#}(Z)$ is a solution of the Laplace equation (21) unique up to a polynomial of $\mathbb{P}'^{\Delta}_{q(1,\alpha)}$. Moreover, we have the estimate
$$\|G * f\|_{H^1_{\alpha,\#}(Z)} \leq C \|f\|_{H^{-1}_{\alpha,\#}(Z)}.$$
This second claim is obtained using the first step and duality arguments. Then the isomorphism result (45) for $m = -1$ and Proposition 5.1 allows to identify the variational solution with the solution by convolution.

3. In the third step we first assume $m \geq 0$, $\alpha \in \mathbb{R}$ and let $f \in X^m_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{q(-m,-\alpha)}$. Then it is clear that $f \in H^{-1}_{\alpha-m-1,\#}(Z) \perp \mathbb{P}'^{\Delta}_{q(1,-\alpha+m+1)}$. Note that by definition $\mathbb{P}'^{\Delta}_{q(-m,-\alpha)}$ and $\mathbb{P}'^{\Delta}_{q(1,-\alpha+m+1)}$ coincide. On the one hand, from the second step $G * f$ belongs to $H^1_{\alpha-m-1,\#}(Z)/\mathbb{P}'^{\Delta}_{q(1,\alpha-m-1)}$. On the other hand, from (45), there exists a unique u in $X^{m+2}_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{q(m+2,\alpha)}$, a subset of $H^1_{\alpha-m-1,\#}(Z)/\mathbb{P}'^{\Delta}_{q(m+2,\alpha)}$. Then using Proposition 5.1 we have $u - G * f \in \mathbb{P}'^{\Delta}_{q(m+2,\alpha)}$.
Finally using again a duality argument we prove the statement for $m \leq 0$ and $\alpha \in \mathbb{R}$.

□

7. Conclusion

This paper is a first attempt towards a systematic analysis of boundary layer problems in periodic strips. This framework provides generic spaces avoiding tedious *a priori* definitions of solutions' behavior at infinity, among other advantages.

Nevertheless, in the homogenization theory, solutions of boundary layer problems often converge exponentially to some polynomial at infinity (when $y_2 \to \infty$) [18]. In a forecoming work, our goal is to provide an adequate framework of these results in the weighted context, and to consider exterior periodic domains [5].

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