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Weighted Sobolev spaces for the Laplace equation in periodic infinite strips

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Abstract

This paper establishes a range of isomorphisms for the Laplace operator in weighted Sobolev spaces. These spaces are similar to standard Sobolev spaces $W^{m,p}(\mathbb{R}^n)$, but they are endowed with weights prescribing the functions' growth or decay at infinity. Although well established in \mathbb{R}^n [14], these weighted results do not apply in the specific hypothesis of periodicity. We provide a complete framework in order to fill this gap. The weights, which arise naturally from Hardy's inequalities, permit to prove fundamental Poincaré inequalities relating functions' norms to their derivatives. The results presented here provide a unified framework for microscopic problems in the boundary layer theory [1, 2]. Computing a periodic Green function for the Laplace operator, we construct explicit weak solutions for a wide range of weights. We identify these convolutions with variational solutions as well.

Keywords:

weighted Sobolev spaces, Hardy inequality, isomorphisms of the Laplace operator, periodic infinite strip, Green function, convolution

1 Introduction

In this article, we solve the Laplace equation in a 1-periodic infinite strip in two space dimensions:

$$\Delta u = f, \quad \text{in } Z :=]0, 1[\times \mathbb{R}. \quad (1)$$

As the domain is infinite in the vertical direction, one introduces weighted Sobolev spaces describing solution's behaviour at infinity. This behaviour is related to weighted Sobolev properties of f .

The usual weights, when adapted to our problem, are polynomial functions at infinity and regular bounded functions in the neighborhood of the origin: they are powers of $\rho(y_2) := (1 + y_2^2)^{1/2}$ and, in some critical cases, higher order derivatives are completed by logarithmic functions ($\rho(y_2)^\alpha \log^\beta(\rho(y_2))$).

The literature on the weighted Sobolev spaces is wide [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and deals with various types of domains. To our knowledge, this type of weights has not been applied to problem (1). The choice of the physical domain comes from periodic singular problems : in [1, 13, 2], a zoom around a domain's periodic ϵ -perturbation leads to set an obstacle of size 1 in Z and to consider a microscopic problem called *a boundary layer*. This

paper is a first step towards a systematic analysis of such microscopic problems. We intend to give a standard framework to skip tedious and particular proofs related to the unboundedness of Z .

We provide a wide range of isomorphisms of the Laplace operator between weighted Sobolev spaces. It is the first step among results in the spirit of [4, 14, 15]. Since error estimates for boundary layer problems [1, 2, 13] are mostly performed in the H^s framework we focus here on weighted Sobolev spaces $W_{\alpha,\beta}^{m,p}(Z)$ with $p = 2$. There are three types of tools used here: arguments specifically related to weighted Sobolev spaces [14, 18], variational techniques from the homogenization literature [2, 16, 17] and some potential theory methods [24]. A general scheme might illustrate how these ideas relate one to each other :

- a Green function G specific to the periodic infinite strip is exhibited for the Laplace operator. The convolution of f with G provides an explicit solution to (1). A particular attention is provided to give the weakest possible meaning to the latter convolution under minimal requirements on f .
- variational *inf-sup* techniques provide *a priori* estimates, in an initial range of weights, for interior and exterior problems with Dirichlet boundary conditions. Then a lift is built in order to cancel the lineic Dirac mass that's introduced by artificial interfaces. This leads to first onto results.
- these arguments are then applied to weighted derivatives and give natural regularity shift results in $H_{\alpha,\#}^m(Z)$ for $m \geq 2$ (see below).
- by duality and appropriate use of generalized Poincaré estimates (leading to interactions - orthogonality or quotient spaces - with various polynomial families), one ends with generic isomorphism results either for derivation order (spaces $H_{\alpha,\#}^m(Z)$ where m varies and α is fixed) or for weights (α varies and m is fixed).
- for a wide range of values of α and m , we identify the explicit solutions obtained via convolution with the variational solutions.

We pay a specific attention to critical combinations of (m, α) leading to logarithmic anisotropic weighted Sobolev spaces. In this case also, isomorphisms are established as in [18].

The paper is organized as follows. In section 2, we define the basic functional framework and some preliminary results used all along the article. In section 3, we adapt weighted Poincaré estimates to our setting. Then (section 4), we introduce a mixed Fourier transform: it is a discrete Fourier transform in the horizontal direction and a continuous one in the vertical direction. With this operator we compute the Green function, and we derive weighted and standard estimates of the convolution with this fundamental solution. At this stage, we prove a series of isomorphisms in the non-critical case (section 5) by variational techniques and we associate variational and convolution solutions. In the last section, we focus on critical cases.

2 Notation, preliminary results and functional framework

2.1 Notation and preliminaries

We denote by Z the two-dimensional infinite strip defined by

$$Z :=]0, 1[\times \mathbb{R}.$$

We use bold characters for vector or matrix fields. A point in \mathbb{R}^2 is denoted by $\mathbf{y} = (y_1, y_2)$ and its distance to the origin by

$$r := |\mathbf{y}| = (y_1^2 + y_2^2)^{1/2}.$$

Let \mathbb{N} denote the set of non-negative integers, and \mathbb{Z} the set of all integers. We denote by $[k]$ the integer part of k . For any $j \in \mathbb{Z}$, \mathbb{P}'_j stands for the polynomial space of degree less than or equal to j that only depends on y_2 . If j is a negative integer, we set by convention $\mathbb{P}'_j = \{0\}$. The support of a function φ is denoted by $\text{supp}(\varphi)$. We recall that $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R}^2)$ are the spaces of C^∞ functions with compact support in \mathbb{R} and \mathbb{R}^2 respectively, $\mathcal{D}'(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R}^2)$ their dual spaces, namely the distributional spaces. We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of functions in $C^\infty(\mathbb{R})$ with rapid decrease at infinity, and by $\mathcal{S}'(\mathbb{R})$ its dual, i.e. the space of tempered distributions. We recall that H^m is the classical Sobolev space and we denote by $H^m_\#(Z)$ the functional space that belong to $H^m(Z)$ and that are 1-periodic in the y_1 direction. We introduce τ_λ the operator of translation of $\lambda \in \mathbb{Z}$ in the y_1 direction. If $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}$ is a function, then we have

$$(\tau_\lambda \Phi)(\mathbf{y}) := \Phi(y_1 - \lambda, y_2).$$

In this paper, we shall also apply the operator τ_λ to single-variable function. If Φ is a such function, it reduces to

$$(\tau_\lambda \Phi)(t) := \Phi(t - \lambda).$$

For any $\Phi \in \mathcal{D}(\mathbb{R}^2)$, we set

$$\bar{\omega}\Phi = \sum_{\lambda \in \mathbb{Z}} \tau_\lambda \Phi$$

which is the y_1 -periodical transform of Φ . The mapping $y_1 \mapsto \bar{\omega}\Phi(\mathbf{y})$ belongs to $C^\infty(\mathbb{R})$ and is 1-periodic. Observe that there exists a function θ satisfying

$$\theta \in \mathcal{D}(\mathbb{R}) \text{ and } \bar{\omega}\theta = 1.$$

The function θ is called a periodical $\mathcal{D}(\mathbb{R})$ -partition of unity.

If $T \in \mathcal{D}'(\mathbb{R}^2)$, then for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we set

$$\langle \tau_\lambda T, \varphi \rangle = \langle T, \tau_{-\lambda} \varphi \rangle, \quad \lambda \in \mathbb{Z}.$$

Similarly, if $T \in \mathcal{D}'(\mathbb{R}^2)$ has a compact support in the y_1 direction, then the y_1 -periodical transform of T , denoted by $\bar{\omega}T$, is defined by

$$\langle \bar{\omega}T, \varphi \rangle = \langle T, \bar{\omega}\varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^2).$$

These definitions are well-known, we refer for instance to [19].

Remark 2.1. Let Φ be in $\mathcal{D}(\mathbb{R}^2)$. Then we have

$$\tau_\lambda \partial_i \Phi = \partial_i (\tau_\lambda \Phi), \quad \forall \lambda \in \mathbb{Z}, \tag{2}$$

$$\bar{\omega}(\partial_i \Phi) = \partial_i (\bar{\omega}\Phi) \tag{3}$$

and

$$\|\tau_\lambda \Phi\|_{L^2(Z)} \leq \|\Phi\|_{L^2(\mathbb{R}^2)}. \tag{4}$$

As a consequence of equality (3), if $u \in \mathcal{D}'(\mathbb{R}^2)$, we also have

$$\bar{\omega}(\partial_i u) = \partial_i (\bar{\omega}u). \tag{5}$$

The next lemma is used to prove the density result of Proposition 2.2.

Lemma 2.1. *Let K be a compact of \mathbb{R}^2 . Let u be in $H^m(\mathbb{R}^2)$ and have a compact support included in K . Then we have*

$$\|\bar{\omega}u\|_{H^m(Z)} \leq N(K)\|u\|_{H^m(\mathbb{R}^2)},$$

where $N(K)$ is an integer only depending on K .

Proof. Let us first notice that, since K is compact, there is a finite number of $\lambda \in \mathbb{Z}$ such that $\text{supp}(\tau_\lambda u) \cap [0, 1] \times \mathbb{R}$ is not an empty set. This number is bounded by a finite integer $N(K)$ that only depends on K . It follows that $\bar{\omega}u$ is a finite sum and

$$\|\bar{\omega}u\|_{L^2(Z)} \leq N(K)\|u\|_{L^2(\mathbb{R}^2)}.$$

The end of the proof follows then from (5). □

We now define the following spaces

$$\begin{aligned} \mathcal{C}_\#^\infty(Z) &:= \{ \varphi : Z \mapsto \mathbb{R}, y_1 \mapsto \varphi(\mathbf{y}) \in \mathcal{C}^\infty([0, 1]) \text{ 1-periodic}, y_2 \mapsto \varphi(\mathbf{y}) \in \mathcal{C}^\infty(\mathbb{R}) \}, \\ \mathcal{D}_\#(Z) &:= \{ \varphi : Z \mapsto \mathbb{R}, y_1 \mapsto \varphi(\mathbf{y}) \in \mathcal{C}^\infty([0, 1]) \text{ 1-periodic}, y_2 \mapsto \varphi(\mathbf{y}) \in \mathcal{D}(\mathbb{R}) \}, \\ \mathcal{S}_\#(Z) &:= \{ \varphi : Z \mapsto \mathbb{R}, y_1 \mapsto \varphi(\mathbf{y}) \in \mathcal{C}^\infty([0, 1]) \text{ 1-periodic}, y_2 \mapsto \varphi(\mathbf{y}) \in \mathcal{S}(\mathbb{R}) \}. \end{aligned}$$

The dual spaces of $\mathcal{D}_\#(Z)$ and $\mathcal{S}_\#(Z)$ are denoted respectively by $\mathcal{D}'_\#(Z)$ and $\mathcal{S}'_\#(Z)$.

2.2 Weighted Sobolev spaces in an infinite strip

We introduce the weight

$$\rho \equiv \rho(y_2) := (1 + y_2^2)^{1/2}.$$

Observe that, for $\lambda \in \mathbb{N}$ and for $\gamma \in \mathbb{R}$, as $|y_2|$ tends to infinity, we have

$$\left| \frac{\partial^\lambda \rho^\gamma}{\partial y_2^\lambda} \right| \leq C\rho^{\gamma-\lambda}. \quad (6)$$

For $\alpha \in \mathbb{R}$, we define the weighted space

$$L_\alpha^2(Z) \equiv H_{\alpha, \#}^0(Z) = \{u \in \mathcal{D}'(Z), \rho^\alpha u \in L^2(Z)\},$$

which is a Banach space equipped with the norm

$$\|u\|_{L_\alpha^2(Z)} = \|\rho^\alpha u\|_{L^2(Z)}.$$

Proposition 2.1. *The space $\mathcal{D}(Z)$ is dense in $L_\alpha^2(Z)$.*

Proof. Let u be in $L_\alpha^2(Z)$. Then, by the definition of the space $L_\alpha^2(Z)$, it follows that $\rho^\alpha u \in L^2(Z)$. Therefore there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{D}(Z)$ such that u_n converges to $\rho^\alpha u$ in $L^2(Z)$ as $n \rightarrow \infty$. Thus, setting $v_n = \rho^{-\alpha} u_n$, we see that v_n converges to u in $L_\alpha^2(Z)$ as $n \rightarrow \infty$. □

Remark 2.2. Observe that we have the algebraic inclusion $\mathcal{D}(Z) \subset \mathcal{D}_\#(Z)$. It follows that the space $\mathcal{D}_\#(Z)$ is dense in $L_\alpha^2(Z)$.

For a non-negative integer m and $\alpha \in \mathbb{R}$, we set

$$k = k(m, \alpha) := \begin{cases} -1 & \text{if } \alpha \notin \{1/2, \dots, m - 1/2\} \\ m - 1/2 - \alpha & \text{if } \alpha \in \{1/2, \dots, m - 1/2\} \end{cases}$$

and we introduce the weighted Sobolev space

$$H_{\alpha, \#}^m(Z) := \{u \in \mathcal{D}'_{\#}(Z); \forall \lambda \in \mathbb{N}^2, 0 \leq |\lambda| \leq k, \rho^{-m+|\lambda|}(\ln(1 + \rho^2))^{-1} \partial^\lambda u \in L_\alpha^2(Z), \\ k + 1 \leq |\lambda| \leq m, \rho^{-m+|\lambda|} \partial^\lambda u \in L_\alpha^2(Z)\},$$

which is a Banach space when endowed with the norm

$$\|u\|_{H_{\alpha, \#}^m(Z)} := \left(\sum_{0 \leq |\lambda| \leq k} \|\rho^{-m+|\lambda|}(\ln(1 + \rho^2))^{-1} \partial^\lambda u\|_{L_\alpha^2(Z)}^2 + \sum_{k+1 \leq |\lambda| \leq m} \|\rho^{-m+|\lambda|} \partial^\lambda u\|_{L_\alpha^2(Z)}^2 \right)^{1/2}.$$

We define the semi-norm

$$|u|_{H_{\alpha, \#}^m(Z)} := \left(\sum_{|\lambda|=m} \|\partial^\lambda u\|_{L_\alpha^2(Z)}^2 \right)^{1/2}.$$

Observe that the logarithmic weight function only appears for the so-called critical cases $\alpha \in \{1/2, \dots, m - 1/2\}$. All the local properties of the spaces $H_{\alpha, \#}^m(Z)$ coincide with those of the classical Sobolev space $H^m(Z)$. When $\alpha \notin \{1/2, \dots, m - 1/2\}$ we have the following algebraic and topological inclusions :

$$H_{\alpha, \#}^m(Z) \subset H_{\alpha-1, \#}^{m-1}(Z) \subset \dots \subset L_{\alpha-m}^2(Z). \quad (7)$$

When $\alpha \in \{1/2, \dots, m - 1/2\}$, the logarithmic weight function appears, so that we only have the inclusions

$$H_{\alpha, \#}^m(Z) \subset \dots \subset H_{1/2, \#}^{m-\alpha+1/2}(Z). \quad (8)$$

Observe that the mapping

$$u \in H_{\alpha, \#}^m(Z) \mapsto \partial^\lambda u \in H_{\alpha, \#}^{m-|\lambda|}(Z) \quad (9)$$

is continuous for $\lambda \in \mathbb{N}^2$. Using (6), for $\alpha, \gamma \in \mathbb{R}$ such that $\alpha \notin \{1/2, \dots, m - 1/2\}$ and $\alpha - \gamma \notin \{1/2, \dots, m - 1/2\}$, the mapping

$$u \in H_{\alpha, \#}^m(Z) \mapsto \rho^\gamma u \in H_{\alpha-\gamma, \#}^m(Z) \quad (10)$$

is onto.

Let $P := y_2^q$ with $q \in \mathbb{Z}$, the greatest q such that $P \in H_{\alpha, \#}^m(Z)$ reads :

$$q \equiv q(m, \alpha) := \begin{cases} m - 3/2 - \alpha, & \text{if } \alpha + 1/2 \in \{i \in \mathbb{Z}; i \leq 0\}, \\ [m - 1/2 - \alpha], & \text{otherwise.} \end{cases} \quad (11)$$

Note that the greatest q such that $P \in L_\alpha^2(Z)$ reads :

$$q = q(0, \alpha) := \begin{cases} -3/2 - \alpha, & \text{if } \alpha = i/2, i \in \mathbb{Z}^*, \\ [-1/2 - \alpha], & \text{otherwise.} \end{cases} \quad (12)$$

Proposition 2.2. *The space $\mathcal{D}_{\#}(Z)$ is dense in $H_{\alpha,\#}^m(Z)$.*

Proof. The idea of the proof comes from [14] and [20]. Let u be in $H_{\alpha,\#}^m(Z)$.

- (i) We first approximate u by functions with compact support in the y_2 direction. Let $\Phi \in \mathcal{C}^\infty([0, \infty[)$ such that $\Phi(t) = 0$ for $0 \leq t \leq 1$, $0 \leq \Phi(t) \leq 1$, for $1 \leq t \leq 2$ and $\Phi(t) = 1$, for $t \geq 2$. For $\ell \in \mathbb{N}$, we introduce the function Φ_ℓ , defined by

$$\Phi_\ell(t) = \begin{cases} \Phi\left(\frac{\ell}{\ln t}\right) & \text{if } t > 1, \\ 1 & \text{otherwise.} \end{cases} \quad (13)$$

Note that we have $\Phi_\ell(t) = 1$ if $0 \leq t \leq e^{\ell/2}$, $0 \leq \Phi_\ell(t) \leq 1$ if $e^{\ell/2} \leq t \leq e^\ell$ and $\Phi_\ell(t) = 0$ if $t \geq e^\ell$. Moreover, for all $\ell \geq 2$, $t \in [e^{\ell/2}, e^\ell]$, $\lambda \in \mathbb{N}$, owing that $t \leq \sqrt{1+t^2} \leq \sqrt{2}t$ and $\ln t \leq \ln(2+t^2) \leq 3 \ln t$, we prove that (see [14], Lemma 7.1):

$$\left| \frac{d^\lambda}{dt^\lambda} \Phi\left(\frac{\ell}{\ln t}\right) \right| \leq \frac{C}{(1+t^2)^{\lambda/2} \ln(2+t^2)},$$

where C is a constant independent of ℓ . We set $u_\ell(\mathbf{y}) = u(\mathbf{y})\Phi_\ell(y_2)$. Then, proceeding as in [14] (Theorem 7.2), one checks easily that u_ℓ belongs to $H_{\alpha,\#}^m(Z)$, has a compact support in the y_2 direction, and that u_ℓ converges to u in $H_{\alpha,\#}^m(Z)$ as ℓ tends to ∞ .

Thus, the functions of $H_{\alpha,\#}^m(Z)$ with compact support in the y_2 direction are dense in $H_{\alpha,\#}^m(Z)$ and, we may assume that u has a compact support in the y_2 direction.

- (ii) Let θ be a periodical $\mathcal{D}(\mathbb{R})$ -partition of unity and let $\alpha_j \in \mathcal{D}(\mathbb{R}^2)$ be a mollifier. We set $w(\mathbf{y}) = \theta(y_1)u(\mathbf{y})$. Then w belongs to $H^m(\mathbb{R}^2)$ and has a compact support. Moreover, since $\bar{w}\theta = 1$, we have $\bar{w}w = \bar{w}(\theta u) = (\bar{w}\theta)u = u$. We define $\varphi_j = w * \alpha_j$. Then φ_j belongs to $\mathcal{D}(\mathbb{R}^2)$ and converges to w in $H^m(\mathbb{R}^2)$ as j tends to ∞ . Let $\psi_j = \bar{w}\varphi_j$, then ψ_j belongs to $\mathcal{D}_{\#}(Z)$ and thanks to Lemma 2.1, ψ_j converges to $\bar{w}w = u$ in $H_{\alpha,\#}^m(Z)$ as j tends to ∞ . □

The above proposition implies that the dual space of $H_{\alpha,\#}^m(Z)$ denoted by $H_{-\alpha,\#}^{-m}(Z)$ is a subspace of $\mathcal{D}'_{\#}(Z)$.

3 Weighted Poincaré estimates

Let R be a positive real number. For $\beta \neq -1$, one uses the standard Hardy estimates:

$$\int_R^\infty |f(r)|^2 r^\beta dr \leq \left(\frac{2}{\beta+1}\right)^2 \int_R^\infty |f'(r)|^2 r^{\beta+2} dr, \quad (14)$$

while for the specific case when $\beta = -1$, one switches to

$$\int_R^\infty \frac{|f(r)|^2}{(\ln r)^2 r} dr \leq \left(\frac{4}{3}\right)^2 \int_R^\infty |f'(r)|^2 r dr.$$

We now introduce the truncated domain

$$Z_R :=]0, 1[\times (]-\infty, -R[\cup]R, +\infty[).$$

Using the above Hardy inequalities in the y_2 direction, we can easily prove the following lemma.

Lemma 3.1. *Let $\alpha, R > 1$ be real numbers and let $m \geq 1$ be an integer. Then there exists a constant C_α^m such that*

$$\forall \varphi \in \mathcal{D}_\#(Z_R), \quad \|\varphi\|_{H_{\alpha,\#}^m(Z_R)} \leq C_\alpha^m \|\partial_2 \varphi\|_{H_{\alpha,\#}^{m-1}(Z_R)}. \quad (15)$$

As a consequence, we also have

$$\forall \varphi \in \mathcal{D}_\#(Z_R), \quad \|\varphi\|_{H_{\alpha,\#}^m(Z_R)} \leq C_\alpha^m |\varphi|_{H_{\alpha,\#}^m(Z_R)}. \quad (16)$$

For the particular case when $\alpha \in \{1/2, \dots, m - 1/2\}$, (16) cannot hold without introducing logarithmic weights in the definition of the space $H_{\alpha,\#}^m(Z)$.

Proceeding as in [14] (Theorem 8.3), we have the Poincaré-type inequalities :

Theorem 3.1. *Let α be a real number and $m \geq 1$ an integer. Assume that $j := \min(q, m - 1)$ where q is defined by (11). Then there exists a constant $C > 0$, such that for any $u \in H_{\alpha,\#}^m(Z)$, we have*

$$\inf_{\lambda \in \mathbb{P}'_j} \|u + \lambda\|_{H_{\alpha,\#}^m(Z)} \leq C |u|_{H_{\alpha,\#}^m(Z)}. \quad (17)$$

In other words, the semi-norm $|\cdot|_{H_{\alpha,\#}^m(Z)}$ defines on $H_{\alpha,\#}^m(Z)/\mathbb{P}'_j$ a norm which is equivalent to the quotient norm.

In order to present the last results of the section, we introduce the space:

$$V_\alpha(Z) = \{\mathbf{v} \in L_\alpha^2(Z), \operatorname{div} \mathbf{v} = 0\}.$$

Then, as a consequence of Theorem 3.1, we have the following isomorphism results on the gradient and divergence operators, whose proof is similar to the proof of Proposition 4.1 in [14].

Proposition 3.1. *Assume that $\alpha \in \mathbb{R}$ and let $j = \min(q, 0)$ where q is defined by (11).*

1. *The gradient operator is an isomorphism:*

$$\nabla : H_{\alpha,\#}^1(Z)/\mathbb{P}'_j \mapsto L_\alpha^2(Z) \perp V_{-\alpha}(Z).$$

2. *The divergence operator is an isomorphism:*

$$\operatorname{div} : L_{-\alpha}^2(Z)/V_{-\alpha}(Z) \mapsto H_{-\alpha,\#}^{-1}(Z) \perp \mathbb{P}'_j.$$

4 The mixed Fourier transform (MFT) and the Green function

The purpose of this section is twofold:

- (i) we look for the fundamental solution for the Laplace equation in Z ,
- (ii) we estimate the convolution with this solution in weighted Sobolev spaces.

To achieve this goal, we define the MFT and an adequate functional setting.

4.1 Rapidly decreasing functions

Let set $\Gamma := \mathbb{Z} \times \mathbb{R}$ and let us write the locally convex linear topological spaces :

$$\begin{aligned} \tilde{\mathcal{D}}(\Gamma) &:= \{ \tilde{\varphi} \in c_0(\mathbb{Z}; C^\infty(\mathbb{R})) \text{ s.t. } \tilde{\varphi}(k, \cdot) \in \mathcal{D}(\mathbb{R}) \quad \forall k \in \mathbb{Z} \\ &\quad \text{and } \exists (k_0, k_1) \in \mathbb{Z}^2 \text{ s.t. } \tilde{\varphi}(k, \cdot) \equiv 0 \quad \forall k \notin \{k_0, \dots, k_1\} \}, \\ \tilde{\mathcal{S}}(\Gamma) &:= \left\{ \tilde{\varphi} \in c_0(\mathbb{Z}; C^\infty(\mathbb{R})) \text{ s.t. } \sup_{k \in \mathbb{Z}, l \in \mathbb{R}} \left| k^\alpha l^\beta \partial_{l'}^\gamma \tilde{\varphi}(k, l) \right| < \infty, \forall (\alpha, \beta, \gamma) \in \mathbb{N}^3 \right\}, \end{aligned}$$

where

$$c_0(\mathbb{Z}) := \{ (z_n)_{n \in \mathbb{Z}} \text{ s.t. } \sup_{n \in \mathbb{Z}} |z_n| < \infty \}.$$

The space $\tilde{\mathcal{S}}(\Gamma)$ is endowed with the semi-norms

$$|\tilde{\varphi}|_{\alpha, \beta, \gamma} := \sup_{k \in \mathbb{Z}, l \in \mathbb{R}} \left| k^\alpha l^\beta \partial_{l'}^\gamma \tilde{\varphi} \right|, \quad \forall (\alpha', \beta', \gamma') \leq (\alpha, \beta, \gamma).$$

We define also

$$l^2(\mathbb{Z}; L^2(\mathbb{R})) := \left\{ u \in \tilde{\mathcal{D}}'(\Gamma) \text{ s.t. } \sum_{k \in \mathbb{Z}} \|u(k, \cdot)\|_{L^2(\mathbb{R})}^2 < \infty \right\}.$$

Proposition 4.1. *Basic density results can be proven, i.e. :*

- i) $\tilde{\mathcal{D}}(\Gamma)$ is dense in $\tilde{\mathcal{S}}(\Gamma)$.
- ii) $\tilde{\mathcal{D}}(\Gamma)$ is dense in $l^2(\mathbb{Z}; L^2(\mathbb{R}))$.
- iii) $\tilde{\mathcal{S}}(\Gamma)$ is dense in $l^2(\mathbb{Z}; L^2(\mathbb{R}))$.

Proof. As in Proposition 3 p. 146 [21], the first claim is obvious : setting $\psi_\epsilon(k, l) = 1$ when $k \in \{-[1/\epsilon], 0, [1/\epsilon]\}$ and $|l| < 1/\epsilon$, one defines $\tilde{\varphi}_\epsilon := \tilde{\varphi} \psi_\epsilon$ which belongs to $\tilde{\mathcal{D}}(\Gamma)$. The rest follows the same arguments. Let then $u \in l^2(\mathbb{Z}; L^2(\mathbb{R}))$. Firstly, one truncates u in the discrete variable in order to get a “finite support” in \mathbb{Z} using the same truncation function as above $u_\epsilon := \psi_\epsilon u$, and then one regularizes u_ϵ . This ends the proof of the second claim. Combining i) and ii) gives the third claim by inclusion. \square

Definition 4.1. *We define the MFT operator $\mathcal{F} : \mathcal{S}_\#(Z) \rightarrow \tilde{\mathcal{S}}(\Gamma)$ as*

$$\mathcal{F}(\varphi)(k, l) := \int_Z \varphi(\mathbf{y}) e^{-i(\tilde{k}y_1 + \tilde{l}y_2)} d\mathbf{y}, \quad \forall (k, l) \in \Gamma,$$

where $\tilde{k} := 2\pi k$, and the same holds for \tilde{l} .

We list below some basic tools concerning the MFT needed in the rest of the paper. The claims can be easily proved following classical arguments ([21], [22], [23]).

Proposition 4.2. (i) *The operator $\mathcal{F} : \mathcal{S}_\#(Z) \rightarrow \tilde{\mathcal{S}}(\Gamma)$ is onto and the inverse operator is explicit:*

$$\mathcal{F}^{-1}(\tilde{\varphi})(\mathbf{y}) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \tilde{\varphi}(k, l) e^{i(\tilde{k}y_1 + \tilde{l}y_2)} dl, \quad \forall \tilde{\varphi} \in \tilde{\mathcal{S}}(\Gamma), \quad \forall \mathbf{y} \in Z.$$

(ii) *One has in a classical way, for any $(f, g) \in \mathcal{S}_\#(Z) \times \mathcal{S}_\#(Z)$, that*

$$\begin{aligned} \int_Z f \cdot \bar{g} d\mathbf{y} &= \sum_k \int_{\mathbb{R}} \mathcal{F}(f)(k, l) \cdot \overline{\mathcal{F}(g)(k, l)} dl, \\ \mathcal{F}(f * g) &= \mathcal{F}(f)\mathcal{F}(g), \quad \forall (k, l) \in \Gamma. \end{aligned}$$

4.2 Tempered distributions and MFT

In a quite natural manner we extend the concepts introduced above to tempered distributions.

Definition 4.2. (i) A linear form \tilde{T} acting on $\tilde{\mathcal{S}}(\Gamma)$ is s.t. $\exists(\alpha, \beta, \gamma) \in \mathbb{N}^3$

$$\left| \tilde{T}(\tilde{\varphi}) \right| \leq C |\tilde{\varphi}|_{\alpha, \beta, \gamma}, \quad \forall \tilde{\varphi} \in \tilde{\mathcal{S}}(\Gamma).$$

(ii) The MFT associated with the tempered distribution $T \in \mathcal{S}'_{\#}(Z)$ is defined as

$$\langle \mathcal{F}(T), \tilde{\varphi} \rangle_{\tilde{\mathcal{D}}', \tilde{\mathcal{D}}} = \mathcal{F}(T)(\tilde{\varphi}) := T(\mathcal{F}^T(\tilde{\varphi})) = \langle T, \mathcal{F}^T(\tilde{\varphi}) \rangle_{\mathcal{S}'_{\#} \times \mathcal{S}_{\#}}, \quad \forall \tilde{\varphi} \in \tilde{\mathcal{D}}(\Gamma),$$

where

$$\mathcal{F}^T(\tilde{\varphi}) := \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \tilde{\varphi}(k, l) e^{-i(\tilde{k}y_1 + \tilde{l}y_2)} dl, \quad \forall \tilde{\varphi} \in \tilde{\mathcal{S}}(\Gamma), \quad \forall \mathbf{y} \in Z.$$

Similarly we define the reciprocal operator denoted \mathcal{F}^{-1} which associates to each distribution $\tilde{T} \in \tilde{\mathcal{S}}'(\Gamma)$ a distribution $\mathcal{F}^{-1}(\tilde{T})$ s.t.

$$\langle \mathcal{F}^{-1}(\tilde{T}), \varphi \rangle_{\mathcal{D}'(Z) \times \mathcal{D}(Z)} := \langle \tilde{T}, \check{\mathcal{F}}(\varphi) \rangle_{\tilde{\mathcal{S}}'(\Gamma) \times \tilde{\mathcal{S}}(\Gamma)}$$

where $\check{\varphi}(k, l) = \tilde{\varphi}(-k, -l)$.

Again we sum up properties extending classical results of Fourier analysis to the MFT case.

Proposition 4.3. (i) If $T \in \mathcal{S}'_{\#}(Z)$ then $\mathcal{F}(T) \in \tilde{\mathcal{S}}'(\Gamma)$. Similarly, if $\tilde{T} \in \tilde{\mathcal{S}}'(\Gamma)$ then $\mathcal{F}^{-1}(\tilde{T}) \in \mathcal{S}'_{\#}(Z)$.

(ii) The Dirac measure belongs to $\mathcal{S}'_{\#}(Z)$ and :

$$\mathcal{F}(\delta_0) = 1, \quad \forall k \in \mathbb{Z}, \quad a.e. \ell \in \mathbb{R}.$$

(iii) the MFT acts on the derivatives in a polynomial fashion:

$$\forall T \in \mathcal{S}'_{\#}(Z), \quad \mathcal{F}(\partial^{|\alpha|} T) = i^{|\alpha|} (\tilde{k}^{\alpha} + \tilde{l}^{\alpha}) \mathcal{F}(T),$$

for any multi-index $\alpha \in \mathbb{N}^2$.

(iv) Plancherel's Theorem : if $f \in L^2(Z)$ then the Fourier transform $\mathcal{F}(T_f)$ is defined by a function $\mathcal{F}(f) \in l^2(\mathbb{Z}, L^2(\mathbb{R}))$ i.e.

$$\mathcal{F}(T_f) = T_{\mathcal{F}(f)} \quad \forall f \in L^2(Z) \text{ and } \|\mathcal{F}(f)\|_{l^2(\mathbb{Z}; L^2(\mathbb{R}))} = \|f\|_{L^2(Z)}.$$

4.3 The Green function for periodic strips

We want to solve the fundamental equation:

$$-\Delta G = \delta_0 \quad \text{in } Z, \tag{18}$$

stated here in the sense of tempered distributions. Then G should satisfy, in $\tilde{\mathcal{S}}(\Gamma)'$:

$$(\tilde{k}^2 + \tilde{l}^2) \mathcal{F}(G) = 1.$$

Proposition 4.4. *The Green function G solving (18) is a tempered distribution and it reads*

$$G(\mathbf{y}) = \frac{1}{2} \left\{ \sum_{k \in \mathbb{Z}^*} \frac{e^{-|\tilde{k}||y_2| + i\tilde{k}y_1}}{|\tilde{k}|} - |y_2| \right\} =: G_1(\mathbf{y}) + G_2(\mathbf{y}), \quad \forall \mathbf{y} \in Z.$$

Moreover, one has $G_1 \in L^1(Z) \cap L^2(Z)$ and thus $G \in L^2_{\text{loc}}(Z)$. The Green function can be written in a more compact expression :

$$G(\mathbf{y}) = -\frac{1}{4\pi} \ln(2(\cosh(2\pi y_2) - \cos(2\pi y_1))).$$

Proof. The formal inverse integral is singular in k ; we define:

$$\begin{cases} G = \lim_{N \rightarrow \infty} G_N = \lim_{N \rightarrow \infty} \sum_{|k| < N} J_k e^{i\tilde{k}y_1}, \\ J_k := \int_{\mathbb{R}_l} \frac{e^{i\tilde{l}y_2}}{\tilde{k}^2 + \tilde{l}^2} dl = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{i\tilde{l}|y_2|}}{\tilde{k}^2 + \tilde{l}^2} dl. \end{cases}$$

The absolute value in the last right hand side is added as follows:

$$\int_{\mathbb{R}} \frac{e^{i\tilde{l}y_2}}{\tilde{k}^2 + \tilde{l}^2} dl = \int_0^\infty \frac{e^{i\tilde{l}y_2} + e^{-i\tilde{l}y_2}}{\tilde{k}^2 + \tilde{l}^2} dl = 2 \int_0^\infty \frac{\cos(\tilde{l}|y_2|)}{\tilde{k}^2 + \tilde{l}^2} dl = 2 \int_0^\infty \frac{\cos(\tilde{l}|y_2|)}{\tilde{k}^2 + \tilde{l}^2} dl.$$

Proceeding as in example 1 p. 58 [23], one has, extending the integral to the complex plane, that

$$\begin{aligned} J_k &\equiv \lim_{R \rightarrow \infty} \oint_{C_R} \frac{e^{i\tilde{z}|y_2|}}{\tilde{k}^2 + \tilde{z}^2} dz = \lim_{R \rightarrow \infty} \oint_{C_R} \frac{e^{i\tilde{z}|y_2|}}{(\tilde{z} + i|\tilde{k}|)(\tilde{z} - i|\tilde{k}|)} dz \\ &= \lim_{R \rightarrow \infty} \frac{1}{2\pi} \oint_{C_R} \frac{\mathfrak{f}(z)}{z - i|\tilde{k}|} dz = i\mathfrak{f}(i|\tilde{k}|) = \frac{e^{-|\tilde{k}||y_2|}}{2|\tilde{k}|}, \end{aligned}$$

where $C_R := \{z \in \mathbb{C}; |z| = R \text{ and } \Im(z) > 0\} \cup ([-R; R] \times \{0\})$ as depicted in fig. 1 and $\mathfrak{f}(z) := e^{i\tilde{z}|y_2|}/(\tilde{z} + i|\tilde{k}|)$, being a holomorphic function inside C_R . The 1D-Fourier transform

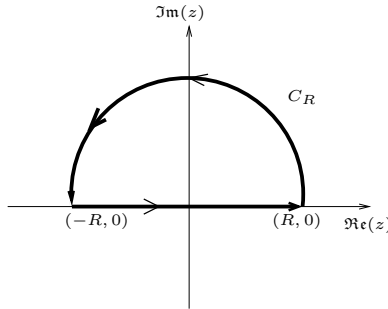


Figure 1: Path of integration in the complex plane

of the tempered distribution $|y|$ is $-2/\tilde{l}^2$, thus one concludes formally that

$$J_0 = \int_{\mathbb{R}_l} \frac{e^{i\tilde{l}y_2}}{\tilde{l}^2} dl = -\frac{1}{2}|y_2|.$$

The L^2 bound is achieved thanks to the Parseval formula

$$\|G_1\|_{L^2(Z)}^2 = \|\mathcal{F}(G_1)\|_{\ell^2(\mathbb{Z}; L^2(\mathbb{R}))}^2 = \sum_{k \in \mathbb{Z}^*} \left\| \frac{1}{\tilde{k}^2 + \tilde{\ell}^2} \right\|_{L^2_{\tilde{\ell}}(\mathbb{R})}^2 \equiv C \sum_{k \in \mathbb{Z}^*} \frac{1}{|\tilde{k}|^3} \leq C'.$$

One recalls the expansion in series of the logarithm :

$$\ln(1 - z) = - \sum_{k=1}^{\infty} \frac{z^k}{k}, \quad \forall z \in \mathbb{C} : |z| < 1.$$

Thus for $\mathbf{y} \neq 0$

$$\begin{aligned} G_1 &= -\frac{1}{2\pi} \Re \ln \left\{ 1 - e^{2\pi(-|y_2| + iy_1)} \right\} = -\frac{1}{2\pi} \ln \left| 1 - e^{2\pi(-|y_2| + iy_1)} \right| \\ &= -\frac{1}{2\pi} \ln \left\{ e^{-\pi|y_2|} \sqrt{2} (\cosh(2\pi|y_2|) - \cos(2\pi y_1)) \right\} \\ &= \frac{|y_2|}{2} - \frac{1}{4\pi} \ln(2 (\cosh(2\pi y_2) - \cos(2\pi y_1))), \end{aligned}$$

which gives the desired result for G . □

One can easily obtain the asymptotic behavior of the Green function. As $|y_2|$ tends to infinity, we have

$$|G(\mathbf{y})| \leq C|y_2|, \quad |\nabla G(\mathbf{y})| \leq C \quad \text{and} \quad |\partial_{y_2}^2 G(\mathbf{y})| \leq C e^{-2\pi|y_2|} \leq C \rho(y_2)^{-\sigma}, \quad \forall \sigma \in \mathbb{R}. \quad (19)$$

Moreover, as $|y_2|$ tends to infinity, we also have

$$|G_1(\mathbf{y})| \leq C \rho(y_2)^{-\sigma}, \quad \forall \sigma > 0. \quad (20)$$

4.4 Convolution with the fundamental solution in weighted spaces

Before proving weighted estimates on the Green function, we shall define the convolution if $f \in \mathcal{D}_{\#}(Z)$, i.e.

$$G * f = \int_Z G(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y} = \int_Z G(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) d\mathbf{y}.$$

So defined $G * f$ belongs to $\mathcal{C}_{\#}^{\infty}(Z)$. Moreover, thanks to (18), we have $\Delta(G * f) = f$ in Z .

4.5 Homogeneous estimates

Definition 4.3. For any function f in $L^2(Z)$ the horizontal average \bar{f} reads :

$$\bar{f}(y_2) := \int_0^1 f(y_1, y_2) dy_1.$$

Lemma 4.1. For any $h \in \mathcal{S}_{\#}(Z)$ such that $\bar{h} \equiv 0$ one has :

$$\partial^{\alpha} G_2 * h = 0, \quad \text{a.e. in } Z, \quad \alpha \in \{0, 1, 2\}.$$

Proof. G_2 is a tempered distribution, the convolution with $h \in \mathcal{S}_\#(Z)$ makes sense :

$$\begin{aligned} & \langle G_2 * h, \varphi \rangle_{\mathcal{S}'_\#(Z) \times \mathcal{S}_\#(Z)} = \langle G_2, (h \overset{\vee}{*} \varphi) \rangle_{\mathcal{S}'_\#(Z) \times \mathcal{S}_\#(Z)} \\ & = -\frac{1}{2} \int_Z |x_2| \int_Z h(\mathbf{x} + \mathbf{y}) \varphi(\mathbf{y}) d\mathbf{y} d\mathbf{x} = -\frac{1}{2} \int_{\mathbb{R}} |x_2| \int_Z \bar{h}(x_2 + y_2) dz \varphi(\mathbf{y}) d\mathbf{y} dx_2 = 0. \end{aligned}$$

The same proof holds for derivatives as well. As this is true for every $\varphi \in \mathcal{S}_\#(Z)$, the result is proved. \square

Proposition 4.5. *Let $m \in \mathbb{R}$ and $f \in H_\#^{m-2}(Z)$. Then there exists a constant independent on f s.t.*

$$\|G * (f - \bar{f})\|_{H_\#^m(Z)} \leq C \|f\|_{H_\#^{m-2}(Z)},$$

where the Sobolev spaces $H_\#^m(Z)$, defined thanks to the MFT, coincide with the classical Sobolev spaces for integer values of m .

Proof. We show this lemma for the particular case when $m \equiv 2$, the proof is similar for $m \in \mathbb{R}$. First we show the lemma for $f \in \mathcal{S}_\#(Z)$, then by density the result is extended to $L^2(Z)$ functions. We set $h := f - \bar{f}$. Thanks to lemma above $\partial^\alpha G * h = \partial^\alpha G_1 * h$. As G_1 belongs to $L^2(Z)$, one is allowed to apply the MFT in the strong sense. It provides :

$$\begin{aligned} \|\partial^\alpha G * h\|_{L^2}^2 &= \|\partial^\alpha G_1 * h\|_{L^2}^2 = \left\| (i\tilde{k})^{\alpha_1} (i\tilde{\ell})^{\alpha_2} \hat{G}_1 \hat{h} \right\|_{l^2 L^2}^2 = \sum_{k \in \mathbb{Z}^*} \int_{\mathbb{R}} \frac{|\tilde{k}|^{2\alpha_1} |\tilde{\ell}|^{2\alpha_2}}{(\tilde{k}^2 + \tilde{\ell}^2)^2} \left| \hat{h}(k, \ell) \right|^2 d\ell \\ &\leq \sum_{k \in \mathbb{Z}^*} \int_{\mathbb{R}} |\tilde{k}|^{2|\alpha|-4} \left| \hat{h}(k, \ell) \right|^2 d\ell \leq \sum_{k \in \mathbb{Z}^*} \int_{\mathbb{R}} \left| \hat{h}(k, \ell) \right|^2 d\ell \equiv \left\| \hat{h} \right\|_{l^2(\mathbb{Z}; L^2(\mathbb{R}))}^2 = \|h\|_{L^2(Z)}^2 \end{aligned}$$

where α is a multi index s.t. $|\alpha| \in \{0, 1, 2\}$. The summation in k is performed in \mathbb{Z}^* , this is crucial when estimating $|\tilde{k}|^{2\alpha_1} |\tilde{\ell}|^{2\alpha_2} / (\tilde{k}^2 + \tilde{\ell}^2)^2$ in the expression above. \square

4.6 Non-homogeneous estimates

We extend here the above results to the weighted context. We start by stating three lemmas.

Lemma 4.2. *We define the horizontal Fourier transform, given $f \in L^2(Z)$*

$$\mathcal{F}_k(f)(k, y_2) := \int_0^1 f(y_1, y_2) e^{-i\tilde{k}y_1} dy_1, \quad \forall k \in \mathbb{Z}, \quad a.e. y_2 \in \mathbb{R}.$$

If we define the convolution operators : for $g \in L^1(Z) \cap L^2(Z)$ and $f \in L^2(Z)$

$$g *_y f := \int_Z g(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \quad (g *_y f)(x_1, x_2, z_1) := \int_{\mathbb{R}} g(x_1, x_2 - y_2) f(z_1, y_2) dy_2,$$

then one has

$$\mathcal{F}_k(g *_y f)(k, x_2) = (\mathcal{F}_k g *_y \mathcal{F}_k f)(k, x_2), \quad \forall k \in \mathbb{Z}, \quad a.e. x_2 \in \mathbb{R}.$$

Proof. If $f \in L^2(Z)$ and $g \in (L^1 \cap L^2)(Z)$ then the convolution with respect to both variables is well defined. Setting

$$(f *_{y_1} g)(x_1, x_2, z_2) := \int_{\mathbb{R}} f(x_1 - y_1, x_2) g(y_1, z_2) dy_1,$$

almost everywhere in y_2 and for every $k \in \mathbb{Z}$ one has

$$\mathcal{F}_k(f *_{y_1} g(x_1, x_2 - y_2, y_2)) = (\mathcal{F}_k f)(k, x_2 - y_2) (\mathcal{F}_k g)(k, y_2).$$

Integrals in the vertical direction commute with the horizontal Fourier transform due to Fubini's theorem. So one shall write :

$$\int_{\mathbb{R}} \mathcal{F}_k(f *_{y_1} g)(k, x_2 - y_2, y_2) dy_2 = \mathcal{F}_k \left(\int_{\mathbb{R}} (f *_{y_1} g)(k, x_2 - y_2, y_2) dy_2 \right) = \mathcal{F}_k(f *_{y_1} g)(k, x_2),$$

which ends the proof. \square

Lemma 4.3. For $f \in L^2_{\beta}(Z)$, one has

$$\|f\|_{L^2_{\beta}(Z)}^2 = \sum_{k \in Z} \|\mathcal{F}_k(f)\|_{L^2_{\beta}(\mathbb{R})}^2, \quad \forall \beta \in \mathbb{R}.$$

Lemma 4.4. Let $a \in \mathbb{R}$ s.t. $|a| \geq 1$ then for any real b , one has

$$I(x) := \int_{\mathbb{R}} e^{-|a||x-y|} \rho(y)^{-b} dy \leq \frac{C \rho(x)^{-b}}{|a|}, \quad x \in \mathbb{R}.$$

Proof. We set $|x| = R$ and let R_0 be a real such that $R_0 > 1$. We define three regions of the real line :

$$D_1 := B(0, R/2), \quad D_2 := B(0, 2R) \setminus B(0, R/2), \quad D_3 := \mathbb{R} \setminus B(0, 2R)$$

Decomposing the convolution integral in these three parts, one gets :

$$I(x) = \sum_{i=1}^3 I_i(x), \quad \text{where } I_i(x) = \int_{D_i} \rho(x-y)^{-a} \rho(y)^{-b} dy.$$

We have two cases to investigate : $R > R_0$ and $R \leq R_0$.

(a) If $R > R_0$

(i) On $|y| \leq R/2$, $|x-y| \sim R$ so that

$$I_1(x) \sim e^{-|a|R} \int_{|y| \leq \frac{R}{2}} (1+|y|)^{-b} dy.$$

According to the value of b , one has

$$I_1(x) \leq C e^{-|a|R} \begin{cases} 1 & \text{if } b > 1, \\ R & \text{if } 0 \leq b \leq 1, \\ R^{1-b} & \text{if } b < 0. \end{cases}$$

As $R > 1$, there holds

$$e^{-|a|R} \leq \frac{([\sigma] + 1)!}{(|a|R)^\sigma}.$$

Setting

$$\sigma(b) := b \mathbb{1}_{1,+\infty}(b) + (1+b) \mathbb{1}_{[0,1]}(b) + \mathbb{1}_{]-\infty,0]}(b),$$

one recovers the claim in that case.

(ii) On $R/2 \leq |y| \leq 2R$, $|y| \sim R$

$$I_2(x) \sim R^{-b} \int_{\frac{R}{2} \leq |y| \leq 2R} e^{-|a||x-y|} dy,$$

because $|x-y| \leq 3R$, one has $I_2(x) \sim R^{-b}/|a|$.

(iii) On the rest of the line $|y| > 2R$, $|x-y| \sim |y|$, so that

$$I_3(x) \sim \int_{|y| > 2R} e^{-|a||y|} |y|^{-b} dy.$$

Two situations occur :

- either $b \leq 0$ then we set $\tilde{b} := [-b] + 1$ and one has

$$\begin{aligned} \int_{|y| > 2R} e^{-|a||y|} |y|^{-b} dy &\leq C \frac{e^{-|a|R}}{|a|} \left\{ R^{\tilde{b}} \sum_{p=0}^{\tilde{b}} \frac{\tilde{b}!}{(\tilde{b}-p)!} (|a|R)^{-p} \right\} \\ &\leq C \frac{e^{-|a|R}}{|a|} R^{\tilde{b}} \tilde{b}! \tilde{b} \leq \frac{C(b)R^{-b}}{|a|}, \end{aligned}$$

where in the latter inequality we used again the decreasing properties of the exponential function in order to fix the correct behavior for large R .

- either $b > 0$ and one has directly that

$$\int_{|y| > 2R} e^{-|a||y|} |y|^{-b} dy \leq R^{-b} \int_{|y| > R} e^{-|a||y|} dy.$$

(b) Otherwise, if $R < R_0$ then $\rho(x-y) \sim \rho(y)$, and thus

$$I(x) \sim \int_{\mathbb{R}} e^{-|a|\rho(y)} \rho(y)^{-b} dy < \frac{C}{|a|}.$$

□

Proposition 4.6. *Let us set β a positive real number. Then for every $f \in L_{\text{loc}}^2(Z)$ s.t. $f - \bar{f} \in L_{\beta}^2(Z)$*

$$\|\partial^\alpha G * (f - \bar{f})\|_{L_{\beta'}^2(Z)} \leq C \|f - \bar{f}\|_{L_{\beta}^2(Z)}, \quad \forall \beta' < \beta - \frac{1}{2}, \quad |\alpha| \in \{0, 1\},$$

where the constant C is independent of the data.

Proof. Set again $h = f - \bar{f}$ and suppose that $h \in \mathcal{S}_\#(Z)$. By Lemma 4.3 :

$$\begin{aligned} \|\partial^\alpha G * h\|_{L^2_{\beta'}(Z)}^2 &= \|\partial^\alpha G_1 * h\|_{L^2_{\beta'}(Z)}^2 = \sum_{k \in \mathbb{Z}^*} \|\mathcal{F}_k(\partial_{\mathbf{y}}^\alpha G_1 *_{\mathbf{y}} h)\|_{L^2_{\beta'}(\mathbb{R})}^2 \\ &= \sum_{k \in \mathbb{Z}^*} \left\| (i\tilde{k})^{\alpha_1} \mathcal{F}_k(\partial_{y_2}^{\alpha_2} G_1) *_{y_2} \mathcal{F}_k(h) \right\|_{L^2_{\beta'}(\mathbb{R})}^2. \end{aligned}$$

For all $k \in \mathbb{Z}^*$, thanks to Lemma 4.4, a.e. \mathbf{x} in Z ,

$$\begin{aligned} \left| (i\tilde{k})^{\alpha_1} \mathcal{F}_k(\partial_{y_2}^{\alpha_2} G_1) *_{y_2} \mathcal{F}_k(h) \right| &\leq \int_{\mathbb{R}} \frac{e^{-|\tilde{k}||y_2-x_2|}}{|\tilde{k}|^{1-|\alpha|}} \mathcal{F}_k(h)(k, y_2) dy_2 \\ &\leq C \|\mathcal{F}_k(h)(k, \cdot)\|_{L^2_{\beta}(\mathbb{R})} \frac{\rho(x_2)^{-\beta}}{|\tilde{k}|^{\frac{3}{2}-|\alpha|}} \leq C \|\mathcal{F}_k(h)(k, \cdot)\|_{L^2_{\beta}(\mathbb{R})} \rho(x_2)^{-\beta}, \end{aligned}$$

for α multi-index s.t. $|\alpha| \in \{0, 1\}$. Then

$$\|\partial^\alpha G * h\|_{L^2_{\beta'}(Z)}^2 \leq \sum_{k \in \mathbb{Z}^*} \|\mathcal{F}_k(h)(k, \cdot)\|_{L^2_{\beta}(\mathbb{R})}^2 \int_{\mathbb{R}} \rho(x_2)^{2\beta' - 2\beta} dx_2 \leq C \|f\|_{L^2_{\beta}(Z)},$$

if $\beta' < \beta - 1/2$. As in Proposition 4.5, the result above shall be extended to $L^2(Z)$ functions. \square

Theorem 4.1. *Let $f \in L^2_{\alpha}(Z) \perp \mathbb{P}'_1$, for any $\alpha > \frac{3}{2}$, one has*

$$\|G_2 * f\|_{L^2_{\alpha-2-\epsilon}(Z)} \leq C \|f\|_{L^2_{\alpha}(Z)}.$$

Proof. The proof is essentially 1D, so we consider functions in \mathbb{R} , the extension to Z is straightforward. We choose $f \in L^2_{\alpha}(\mathbb{R}) \perp \mathbb{P}'_1$ and $(\varphi_{\delta})_{\delta} \in \mathcal{S}_\#(\mathbb{R})$ s.t. $\varphi_{\delta} \rightarrow \varphi$ in $L^2_{2-\alpha+\epsilon}(\mathbb{R})$, then $|x| * \varphi_{\delta}$ is infinitely differentiable and one can apply the Taylor expansion with the integral rest :

$$\begin{aligned} &\int_{\mathbb{R}} (|x| * \varphi_{\delta})(x) f(x) dx \\ &= \int_{\mathbb{R}} \left\{ (|x| * \varphi_{\delta})(0) + (|x| * \varphi_{\delta})'(0) \cdot x + \int_0^x (|x| * \varphi_{\delta})''(s)(x-s) ds \right\} f(x) dx \\ &= \int_{\mathbb{R}} f(x) \int_0^x (|x| * \varphi_{\delta})''(s)(x-s) ds dx = 2 \int_{\mathbb{R}} f(x) \int_0^x \varphi_{\delta}(s)(x-s) ds dx \\ &\leq C \|\varphi_{\delta}\|_{L^2_{\beta}(\mathbb{R})} \int_{\mathbb{R}} \rho^{\frac{3}{2}-\beta}(x) |f(x)| dx \leq C \|\varphi_{\delta}\|_{L^2_{\beta}(\mathbb{R})} \|f\|_{L^2_{2-\beta+\epsilon}(\mathbb{R})}. \end{aligned}$$

In the second line we used the orthogonality condition on f in order to cancel the first two terms in the Taylor expansion. An easy computation shows that if $\varphi_{\delta} \rightarrow \varphi$

$$\ell := \lim_{\delta \rightarrow 0} \int_{\mathbb{R}} (|x| * \varphi_{\delta})(x) f(x) dx = 2 \int_{\mathbb{R}} f(x) \int_0^x \varphi(s)(x-s) ds dx, \quad \forall f \in L^2_{\alpha}(\mathbb{R}) \perp \mathbb{P}'_1.$$

By Fubini theorem, one is then allowed to write :

$$\begin{aligned}
& 2 \int_{\mathbb{R}} f(x) \int_0^x \varphi(s)(x-s) ds dx \\
&= 2 \left\{ \int_{\mathbb{R}_s^+} \varphi(s) \int_s^\infty f(x)(x-s) dx ds - \int_{\mathbb{R}_s^-} \varphi(s) \int_{-\infty}^s f(x)(x-s) dx ds \right\} \\
&= -2 \int_{\mathbb{R}_s} \varphi(s) \int_{-\infty}^s f(x)(x-s) dx ds \\
&= \int_{\mathbb{R}_s} \varphi(s) \left\{ \int_s^\infty f(x)(x-s) dx - \int_{-\infty}^s f(x)(x-s) dx \right\} ds \\
&= \int_{\mathbb{R}} \varphi(s) \int_{\mathbb{R}_x} f(x)|x-s| dx ds \equiv \int_{\mathbb{R}} (|x| * f)(s) \varphi(s) ds,
\end{aligned}$$

which ends the proof. \square

Lemma 4.5. *If $\alpha > \frac{1}{2}$, and if $f \in L_\alpha^2(Z) \perp \mathbb{R}$, then*

$$\|\operatorname{sgn} * \bar{f}\|_{L_{\alpha-1-\epsilon}^2(Z)} \leq C \|f\|_{L_\alpha^2(Z)}.$$

Proof. Under sufficient integrability conditions, one writes :

$$(\operatorname{sgn}(x_2) * \bar{f})(\mathbf{x}) = - \int_{y_2 < x_2} \bar{f} dy_2 + \int_{y_2 > x_2} \bar{f} dy_2.$$

Remark that for x_2 tending to infinity, $\bar{f} * \operatorname{sgn}(x_2)$ behaves as $\operatorname{sgn}(x_2) \int_{\mathbb{R}} \bar{f}$. Indeed

$$\begin{aligned}
& \left| (\operatorname{sgn}(x_2) * \bar{f})(\mathbf{x}) - \operatorname{sgn}(x_2) \int_{\mathbb{R}} \bar{f} \right| = 2 \left| \left(\int_{-\infty}^{x_2} \bar{f}(s) ds \right) \mathbb{1}_{\mathbb{R}_-}(x_2) - \left(\int_{x_2}^{\infty} \bar{f}(s) ds \right) \mathbb{1}_{\mathbb{R}_+}(x_2) \right| \\
& \leq 2 \|f\|_{L_\alpha^2(Z)} \left(\left\{ \int_{-\infty}^{x_2} \rho^{-2\alpha}(y_2) dy_2 \right\}^{\frac{1}{2}} \mathbb{1}_{\mathbb{R}_-}(x_2) + \left\{ \int_{x_2}^{\infty} \rho^{-2\alpha}(y_2) dy_2 \right\}^{\frac{1}{2}} \mathbb{1}_{\mathbb{R}_+}(x_2) \right) \\
& \leq C \|f\|_{L_\alpha^2(Z)} \rho^{\frac{1}{2}-\alpha}(x_2).
\end{aligned}$$

Then

$$\left\| \partial_{y_2} G * \bar{f} - \operatorname{sgn}(x) \int_{\mathbb{R}} \bar{f} \right\|_{L_\beta^2(Z)}^2 \leq C \|f\|_{L_\alpha^2(Z)}^2 \int_{\mathbb{R}} \rho^{1-2\alpha+2\beta}(x_2) dx_2,$$

which is bounded, provided that $\beta < \alpha - 1$. \square

Similarly to Theorem 4.1, thanks to the previous lemma, one gets that

Theorem 4.2. *Let $f \in L_\alpha^2(Z) \perp \mathbb{R}$ then for any $\alpha \in]\frac{1}{2}, \frac{3}{2}]$ one has*

$$\inf_{\lambda \in \mathbb{R}} \|G_2 * f - \lambda\|_{L_{\alpha-2-\epsilon}^2(Z)} \leq C \|f\|_{L_\alpha^2(Z)}.$$

Proof. As previously, the proof is essentially 1D. Let us choose $f \in L_\alpha^2(\mathbb{R}) \perp \mathbb{R}$ and $(\varphi_\delta)_\delta \in \mathcal{S}_\#(\mathbb{R}) \perp \mathbb{R}$ s.t. $\varphi_\delta \rightarrow \varphi$ in $L_{2-\alpha+\epsilon}^2(\mathbb{R}) \perp \mathbb{R}$ then $|x| * \varphi_\delta$ is infinitely differentiable and one can apply the Taylor expansion with the integral rest :

$$\begin{aligned}
& \int_{\mathbb{R}} (|x| * \varphi_\delta)(x) f(x) dx \\
&= \int_{\mathbb{R}} \left\{ (|x| * \varphi_\delta)(0) + \int_0^x (\varphi_\delta * \operatorname{sgn})(s) ds \right\} f(x) dx \equiv \int_{\mathbb{R}} f(x) \int_0^x (\varphi_\delta * \operatorname{sgn})(s) ds dx
\end{aligned}$$

But if $\beta := 2 - \alpha + \epsilon > \frac{1}{2}$, then by Lemma 4.5, one has

$$\left| \int_0^x \varphi_\delta * \operatorname{sgn}(s) ds \right| \leq \|\varphi_\delta\|_{L_\beta^2(Z)} \rho^{\frac{3}{2}-\beta}(x),$$

leading to

$$\left\| \int_0^x (\varphi_\delta * \operatorname{sgn})(s) ds \right\|_{L_{-\alpha}^2(\mathbb{R})} \leq C \|\varphi_\delta\|_{L_{2-\alpha+\epsilon}^2(\mathbb{R})},$$

with $\alpha < \frac{3}{2} + \epsilon$. Thus one has

$$\left| \int_{\mathbb{R}} (|x| * \varphi_\delta)(x) f(x) dx dy \right| \leq C \|f\|_{L_\alpha^2(\mathbb{R})} \|\varphi_\delta\|_{L_{2-\alpha+\epsilon}^2(\mathbb{R})}.$$

Moreover, one needs that $\mathbb{R} \in L_{-\alpha}^2(\mathbb{R})$, which is true if $\alpha > \frac{1}{2}$. It is not difficult to prove, using Lebesgue theorem, that

$$\lim_{\delta \rightarrow 0} \left\{ \int_{\mathbb{R}} f(x) \int_0^x (\varphi_\delta * \operatorname{sgn})(s) ds dx \right\} = \int_{\mathbb{R}} f(x) \int_0^x (\varphi * \operatorname{sgn})(s) ds dx.$$

As in the proof of Lemma 4.5

$$\int_0^x (\operatorname{sgn} * \varphi)(s) ds = 2 \int_0^x \left\{ \int_{-\infty}^s \varphi(t) dt \mathbb{1}_{s < 0}(s) - \int_s^\infty \varphi(t) dt \mathbb{1}_{s > 0}(s) \right\} ds =: 2 \int_0^x \mathbf{g}(s) ds.$$

As \mathbf{g} is a C^1 function on any compact set in \mathbb{R} , one can integrate by parts on $(0, x)$:

$$\int_0^x \mathbf{g}(s) ds = [\mathbf{g}(s)s]_0^x - \int_0^x \mathbf{g}'(s) s ds.$$

Using this in the right hand side of the previous limit, one writes

$$\begin{aligned} J &:= \int_{\mathbb{R}} f(x) \int_0^x (\varphi * \operatorname{sgn})(s) ds dx = 2 \int_{\mathbb{R}} f(x) \left\{ \mathbf{g}(x)x - \int_0^x \mathbf{g}'(s) s ds \right\} dx \\ &= 2 \left\{ \int_{\mathbb{R}_-} f(x) x \int_{-\infty}^x \varphi(t) dt dx - \int_{\mathbb{R}_+} f(x) x \int_x^\infty \varphi(t) dt dx - \int_{\mathbb{R}} f(x) \int_0^x \varphi(s) s ds \right\} \\ &= 2 \int_{\mathbb{R}} f(x) x \int_{-\infty}^x \varphi(t) dt dx - 2 \int_{\mathbb{R}} f(x) \int_0^x \varphi(s) s ds dx =: A - B. \end{aligned}$$

By Hölder estimates the integrals above are well defined. Fubini's theorem allows then to switch integration order. Using the orthogonality condition on φ and on f , one may easily show that

$$\left\{ \begin{aligned} A &= 2 \int_{\mathbb{R}_x} f(x) x \left(\int_{-\infty}^x \varphi(t) dt \right) dx = \int_{\mathbb{R}} f(x) x \left\{ \int_{-\infty}^x \varphi(s) ds - \int_x^\infty \varphi(s) ds \right\} dx \\ &= \int_{\mathbb{R}_s} \varphi(s) \left\{ \int_s^\infty x f(x) dx - \int_{-\infty}^s x f(x) dx \right\} ds, \\ B &= 2 \left\{ \int_{\mathbb{R}_+} f(x) \int_0^x s \varphi(s) ds dx - \int_{\mathbb{R}_-} f(x) \int_x^0 s \varphi(s) ds dx \right\} \\ &= 2 \left\{ \int_{\mathbb{R}_+} s \varphi(s) \int_s^\infty f(x) dx ds - \int_{\mathbb{R}_-} s \varphi(s) \int_{-\infty}^s f(x) dx ds \right\} \\ &= -2 \int_{\mathbb{R}} s \varphi(s) \int_{-\infty}^s f(x) dx ds = \int_{\mathbb{R}} s \varphi(s) \left\{ \int_s^\infty f(x) dx ds - \int_{-\infty}^s f(x) dx \right\} ds. \end{aligned} \right.$$

These computations give :

$$\begin{aligned} A - B &= \int_{\mathbb{R}_s} \varphi(s) \left\{ \int_s^\infty (x-s)f(x)dx - \int_{-\infty}^s (x-s)f(x)dx \right\} ds \\ &= \int_{\mathbb{R}} \varphi(s) \int_{\mathbb{R}} |x-s|f(x)dx ds = \int_{\mathbb{R}} \varphi(s)(|x| * f)(s) ds. \end{aligned}$$

And because

$$\inf_{\lambda \in \mathbb{R}} \||x| * f + \lambda\|_{L^2_{\alpha-(2+\epsilon)}(\mathbb{R})} = \sup_{\varphi \in L^2_{(2+\epsilon)-\alpha}(\mathbb{R}) \perp \mathbb{R}} \frac{(|x| * f, \varphi)}{\|\varphi\|_{L^2_{(2+\epsilon)-\alpha}(\mathbb{R})}},$$

the final claim follows. \square

Theorem 4.3. *Assume $\alpha > 1/2$ and recall that $q(0, \alpha)$ is defined by (12). Then the operator defined by the convolution with the fundamental solution G is a mapping from $L^2_\alpha(Z) \perp \mathbb{P}'_{q(0, \alpha)} \Delta$ on $(H^1_{\alpha-\epsilon, \#}(Z) \cap H^2_{\alpha-2-\epsilon, \#}(Z)) / \mathbb{P}'_{[3/2-\alpha]} \Delta$ for any $\epsilon > 0$.*

Proof. One follows the same lines as in the proof of Theorem 1 p. 786 in [24]. Indeed, by Theorems 4.1 and 4.2, one has that $K \cdot := G * \cdot$ is a bounded mapping from $L^2_\alpha(Z) \perp \mathbb{P}'_{[\alpha-1/2]} \Delta$ on $L^2_{\alpha-2-\epsilon, \#}(Z) / \mathbb{P}'_{[3/2-\alpha]} \Delta$ for every $\alpha > 1/2$. Thus for $f \in L^2_\alpha(Z) \perp \mathbb{P}'_{[\alpha-1/2]} \Delta$, let $u = Kf \in L^2_{\alpha-2-\epsilon, \#}(Z) / \mathbb{P}'_{[3/2-\alpha]} \Delta$. Then for any $\phi \in \mathcal{D}_\#(Z)$, $\langle \Delta Kf, \phi \rangle = \langle f, K\Delta\phi \rangle = \langle f, \phi \rangle$ and so $\Delta u = f$ in the distributional sense. But then, as $u \in L^2_{\alpha-2-\epsilon, \#}(Z) / \mathbb{P}'_{[3/2-\alpha]} \Delta$ and $\Delta u \in L^2_\alpha(Z) \perp \mathbb{P}'_{[\alpha-1/2]} \Delta$, using a dyadic partition of unity [25] and standard interior regularity results (see for instance a similar proof in Theorem 3.1 in [26] or chap I in [27]), one has that :

$$\sum_{|\alpha|=2} \|\rho^{-\alpha_1} D^\alpha u\|_{L^2_{\alpha-\epsilon}(Z)}^2 \leq C \left(\|\Delta u\|_{L^2_{\alpha-\epsilon}(Z)}^2 + \|u\|_{H^1_{\alpha-1-\epsilon, \#}(Z)}^2 \right),$$

where the constant does not depend on u . These estimates are not isotropic with respect to the weight. Taking the lowest weight in front of the second order derivative ends the proof. \square

4.7 Convolution and duality

We generalize the latter convolutions to weak data, namely, when $f \in H^{-1}_{\alpha, \#}(Z)$.

Proposition 4.7. *For $\beta < -\frac{3}{2}$ and $f \in H^{-1}_{\beta, \#}(Z)$ one has :*

$$\|G * f\|_{L^2_{\beta-1-\epsilon}(Z) / \mathbb{P}'_1} \leq C \|f\|_{H^{-1}_{\beta, \#}(Z)},$$

where C is independent on f .

Proof. Let $\varphi \in L^2_\alpha(Z) \perp \mathbb{P}'_1$ s.t. $\alpha > \frac{3}{2}$:

$$\begin{aligned} \langle f, G^* \varphi \rangle_{H^{-1}_{(1+\epsilon)-\alpha, \#}(Z) \times H^1_{\alpha-(1+\epsilon), \#}(Z)} &\leq \|f\|_{H^{-1}_{(1+\epsilon)-\alpha, \#}(Z)} \|G^* \varphi\|_{H^1_{\alpha-(1+\epsilon), \#}(Z)} \\ &\leq C \|f\|_{H^{-1}_{(1+\epsilon)-\alpha, \#}(Z)} \|\varphi\|_{L^2_\alpha(Z) \perp \mathbb{P}'_1}. \end{aligned}$$

After taking the supremum over all $\varphi \in L^2_\alpha(Z) \perp \mathbb{P}'_1$, one has

$$\|G * f\|_{L^2_{-\alpha}(Z)/\mathbb{P}'_1} \leq C \|f\|_{H^{-1}_{(1+\epsilon)-\alpha, \#}(Z)}.$$

Choosing $\beta := (1 + \epsilon) - \alpha$ one gets the estimate on the convolution. \mathbb{P}'_1 belongs to $L^2_\beta(Z)$ when $\beta < -\frac{3}{2}$, which ends the proof. \square

Similarly, one has

Proposition 4.8. *For $\beta \in [-\frac{3}{2}, -\frac{1}{2}[$ and $f \in H^{-1}_{\beta, \#}(Z)$ one has :*

$$\|G * f\|_{L^2_{\beta-1-\epsilon}(Z)/\mathbb{P}'_0} \leq C \|f\|_{H^{-1}_{\beta, \#}(Z)},$$

where C is independent on f .

5 The Laplace equation in a periodic infinite strip

In this section we study the problem

$$\begin{cases} -\Delta u = f \text{ in } Z, \\ u \text{ is 1-periodic in the } y_1 \text{ direction.} \end{cases} \quad (21)$$

in the variational context. Firstly, we characterize of the kernel of the Laplace operator.

Proposition 5.1. *Let $m \geq 1$ be an integer, α be a real number and $j = \min\{q, 1\}$ where q is defined by (11). A function $u \in H^m_{\alpha, \#}(Z)$ satisfies $\Delta u = 0$ if and only if $u \in \mathbb{P}'_j$.*

Proof. Since $j \leq 1$, it is clear that if $u \in \mathbb{P}'_j$, then $\Delta u = 0$. Conversely, let $u \in \mathcal{S}'_{\#}(Z)$ satisfies $\Delta u = 0$. We apply the MFT : in the sense defined in Definition 4.2, one has

$$(\tilde{k}^2 + \tilde{l}^2)\mathcal{F}(u) = 0, \quad \forall k \in \mathbb{Z}, \text{ a.e. } l \in \mathbb{R},$$

which implies that

$$\mathcal{F}(u) = \begin{cases} \sum_{j=0}^p \delta_{y_2=0}^j(l) & \text{if } k \equiv 0 \\ 0 & \text{otherwise.} \end{cases}$$

A simple computation shows that $\mathcal{F}^{-1}(\mathcal{F}(u)) = \sum_{j=0}^p (iy_2)^j$, indeed

$$\begin{aligned} \langle \mathcal{F}^{-1}(\mathcal{F}(u)), \varphi \rangle_{\mathcal{S}'_{\#}, \mathcal{S}_{\#}} &= \langle \mathcal{F}(u), \check{\mathcal{F}}(\varphi) \rangle_{\tilde{\mathcal{S}}'(\Gamma), \tilde{\mathcal{S}}(\Gamma)} = \left\langle \sum_{j=0}^p \delta_0^j, \check{\mathcal{F}}(\varphi)(0, \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})} \\ &= \sum_{j=0}^p (-1)^j \langle \delta_0, \partial_j \hat{\varphi} \rangle_{\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})} = \sum_{j=0}^p (-1)^j \langle \delta_0, ((iy_2)^j \varphi) \rangle_{\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})} \\ &= \sum_{j=0}^p (-1)^j \int_{\mathbb{R}} (iy_2)^j \varphi(\mathbf{y}) d\mathbf{y} = \left\langle \sum_{j=0}^p (-iy_2)^j, \varphi \right\rangle_{\mathcal{S}'_{\#} \times \mathcal{S}_{\#}}, \end{aligned}$$

where $\hat{\varphi}$ denotes the usual 1-dimensional Fourier transform on \mathbb{R} . \square

We establish a Poincaré-Wirtinger's type inequality.

Lemma 5.1. *For every $u \in H_{\alpha, \#}^1(Z)$ with $\alpha \in \mathbb{R}$, one has*

$$\|u - \bar{u}\|_{L_\alpha^2(Z)} \leq |u|_{H_{\alpha, \#}^1(Z)},$$

where

$$\bar{u}(y_2) := \int_0^1 u(y_1, y_2) dy_1, \quad \text{a.e. } y_2 \in \mathbb{R}.$$

Proof. If $u \in H_{\alpha, \#}^1(Z)$ then $\partial^\lambda u \in L_{\text{loc}}^2(Z)$ for all $0 \leq |\lambda| \leq 1$. In particular, almost everywhere in y_2 one has $\partial^\lambda u(\cdot, y_2)$ in $L^2(0, 1)$. Applying Parseval in the y_1 direction one gets :

$$\|u - \bar{u}\|_{L^2(0,1)}^2 = \sum_{k \in \mathbb{Z}^*} |\hat{u}(k)|^2 \quad \text{and} \quad \|\partial_1 u\|_{L^2(0,1)}^2 = \sum_{k \in \mathbb{Z}^*} |k|^2 |\hat{u}(k)|^2$$

for almost every $y_2 \in \mathbb{R}$. This obviously gives

$$\|u - \bar{u}\|_{L^2(0,1)}^2 \leq \|\partial_1 u\|_{L^2(0,1)}^2 \quad \text{a.e. } y_2 \in \mathbb{R}.$$

Integrating then with respect to the vertical weight, one gets the desired estimates. \square

In order to state existence and uniqueness results for solutions of (21), we deal with the Laplace operator in the truncated domain $Z_R :=]0, 1[\times (]-\infty, -R[\cup]R, +\infty[)$:

$$\begin{cases} -\Delta u = f & \text{in } Z_R, \\ u = 0 & \text{on } y_2 = \pm R. \end{cases} \quad (22)$$

Lemma 5.2. *Let α be any real number. If $f \in H_{\alpha, \#}^{-1}(Z)$ there exists $R(\alpha)$ large enough such that problem (22) has a unique solution $u \in H_{\alpha, \#}^1(Z_{R(\alpha)})$.*

Proof. The proof relies on an *inf-sup* argument similar to the one used in [16] (the main difference being the nature of weights : in [16] the author derives similar estimates for exponential weights on a half strip). Indeed, we set

$$v := \omega^2(u - \bar{u}) + \psi(u), \quad \omega := \rho^\alpha, \quad \text{and} \quad \psi(u) := \begin{cases} \int_R^{y_2} \omega^2 \partial_{y_2} \bar{u}(s) ds & \text{if } y_2 > R \\ 0 & \text{if } |y_2| \leq R \\ - \int_{y_2}^{-R} \omega^2 \partial_{y_2} \bar{u}(s) ds & \text{otherwise.} \end{cases}$$

First of all, we check that $v \in H_{-\alpha, \#}^1(Z_R)$.

$$\nabla v = \nabla(\omega^2)(u - \bar{u}) + \omega^2 \nabla \bar{u}.$$

An application of Lemma 5.1 proves that

$$\|\nabla v\|_{L_{-\alpha}^2(Z_R)} \leq k|u|_{H_{\alpha, \#}^1(Z_R)}.$$

which guarantees, through Hardy estimates, that indeed v belongs to $H_{-\alpha, \#}^1(Z_R)$. Secondly, we write

$$a(u, v) = \int_{Z_R} \nabla u \cdot \nabla v \, d\mathbf{y} = \int_{Z_R} |\nabla u|^2 \omega^2 \, d\mathbf{y} + \int_{Z_R} \nabla u \cdot \nabla(\omega^2)(u - \bar{u}) \, d\mathbf{y}.$$

Using Cauchy-Schwarz, we estimate the formula above as :

$$\int_{Z_R} \nabla u \cdot \nabla(\omega^2)(u - \bar{u}) \, d\mathbf{y} \leq 2 \left(\int_{Z_R} |\nabla u|^2 \omega^2 \, d\mathbf{y} \right)^{\frac{1}{2}} \left(\int_{Z_R} |u - \bar{u}|^2 |\nabla \omega|^2 \, d\mathbf{y} \right)^{\frac{1}{2}}.$$

Since $|\nabla \omega| < 1$, one also has that

$$\frac{|\nabla \omega|^2}{\omega^2} \leq \frac{\alpha^2}{\rho^2(R)}, \quad \forall \mathbf{y} \in Z_R,$$

which gives then

$$\int_{Z_R} \nabla u \cdot \nabla \omega(u - \bar{u}) \, d\mathbf{y} \leq \frac{|\alpha|}{\rho(R)} \int_{Z_R} |\nabla u|^2 \omega \, d\mathbf{y}.$$

This in turn implies that

$$a(u, v) \geq \left(1 - 2 \frac{|\alpha|}{\rho(R)} \right) |u|_{H_{\alpha, \#}^1(Z_R)}^2 \geq k |u|_{H_{\alpha, \#}^1(Z_R)} |v|_{H_{-\alpha, \#}^1(Z_R)},$$

which proves surjectivity. Injectivity of the transposed operator is obtained in the same way (but with $\omega = \rho^{-\alpha}$ above): showing that for all $u \in H_{-\alpha, \#}^1(Z_R)$ there exists a $v \in H_{\alpha, \#}^1(Z_R)$ s.t.

$$a(u, v) \geq \left(1 - 2 \frac{|\alpha|}{\rho(R)} \right) |u|_{H_{-\alpha, \#}^1(Z_R)}^2 \geq k |u|_{H_{-\alpha, \#}^1(Z_R)} |v|_{H_{\alpha, \#}^1(Z_R)}.$$

and one concludes using the classical Banach-Babuška-Nečas result (see Theorem 2.6 p. 85 in [28]). \square

We next solve the Laplace equation in the domain $Z_R \cup Z^R$.

Lemma 5.3. *Provided that $f \in H_{\alpha, \#}^{-1}(Z)$, there exists a unique solution $u_0 \in H_{\alpha, \#}^1(Z)$ solving:*

$$\begin{cases} -\Delta u_0 = f & \text{in } Z_R \cup Z^R, \\ u_0 = 0 & \text{on } \{y_2 = \pm R\}. \end{cases} \quad (23)$$

Moreover, if $f \perp \mathbb{R}$, and setting

$$\bar{h}_{\pm} := \langle [\partial_{y_2} u_0], 1 \rangle_{H^{-\frac{1}{2}}(\{y_2 = \pm R\}), H^{\frac{1}{2}}(\{y_2 = \pm R\})},$$

where the brackets $[\cdot]$ denote the jump across interfaces $\{y_2 = \pm R\}$, i.e.

$$[\partial_{y_2} u_0] = \lim_{y_2 \rightarrow R^+} \partial_{y_2} u_0(y_1, y_2) - \lim_{y_2 \rightarrow R^-} \partial_{y_2} u_0(y_1, y_2),$$

then a compatibility condition holds on h :

$$\bar{h}_+ + \bar{h}_- \equiv 0.$$

Proof. By Lemma 5.2 there exists a unique u_0 in $H_{\alpha,\#}^1(Z)$ for every $\alpha \in \mathbb{R}$. By truncation and approximation (following the same steps as in Lemma 6.2 and Proposition 6.1), we set

$$f_\delta = (f\Phi_\delta) * \alpha_\delta, \quad \tilde{f}_\delta := f_\delta - \bar{f}_\delta,$$

where Φ_δ and α_δ are chosen as in Proposition 2.2. An easy check shows that

$$\tilde{f}_\delta \in \mathcal{D}_\#(Z), \quad \tilde{f}_\delta \perp \mathbb{R} = 0.$$

We then compute $u_{0,\delta}$ solving (23) with the data \tilde{f}_δ . One has

$$\begin{aligned} \|u_{0,\delta}\|_{H_{\alpha,\#}^1(Z)} &= \|u_{0,\delta}\|_{H_{\alpha,\#}^1(Z_R \cup Z^R)} \leq C \|\tilde{f}_\delta\|_{H_{\alpha,\#}^{-1}(Z_R \cup Z^R)} \\ &\leq C \|\tilde{f}_\delta\|_{H_{\alpha,\#}^{-1}(Z)} \leq C \|f\|_{H_{\alpha,\#}^{-1}(Z)}, \\ \|u_{0,\delta} - u_0\|_{H_{\alpha,\#}^1(Z)} &\leq C \|\tilde{f}_\delta - f\|_{H_{\alpha,\#}^{-1}(Z)} \\ \|[\partial_{y_2} u_{0,\delta}] - [\partial_{y_2} u_0]\|_{H^{-\frac{1}{2}}(\{y_2 = \pm R\})} &\leq C \|u_{0,\delta} - u_0\|_{H_{\alpha,\#}^1(Z)} \leq C \|\tilde{f}_\delta - f\|_{H_{\alpha,\#}^{-1}(Z)}. \end{aligned} \tag{24}$$

Moreover, for every $\varphi \in \mathcal{D}_\#(Z)$, one might write

$$\begin{aligned} \langle -\Delta u_{0,\delta}, \varphi \rangle &= -\langle u_{0,\delta}, \Delta \varphi \rangle = - \int_{Z_R \cup Z^R} u_{0,\delta} \Delta \varphi d\mathbf{y} \\ &= - \int_{(0,1)} [\partial_{y_2} u_{0,\delta}]_{\{y_2 = \pm R\}} \varphi(y_1, \pm R) dy_1 + \langle \tilde{f}_\delta, \varphi \rangle, \end{aligned}$$

the latter equality being true because $u_{0,\delta}$ is in the domain of the operator : the Green formula holds. By density, one extends the above equality to test functions in $H_{-\alpha,\#}^1(Z)$, giving :

$$-\langle u_{0,\delta}, \Delta \varphi \rangle = -\langle [\partial_{y_2} u_{0,\delta}]_{\{y_2 = \pm R\}}, \varphi \rangle + \langle \tilde{f}_\delta, \varphi \rangle, \quad \forall \varphi \in H_{-\alpha,\#}^1(Z).$$

As α is chosen s.t. $\mathbb{R} \subset H_{-\alpha,\#}^1(Z)$, one takes $\varphi \equiv 1$ in the above formula. This leads to :

$$\bar{h}_{+,\delta} + \bar{h}_{-,\delta} := \langle [\partial_{y_2} u_{0,\delta}], 1 \rangle_{\{y_2 = R\}} + \langle [\partial_{y_2} u_{0,\delta}], 1 \rangle_{\{y_2 = -R\}} \equiv 0$$

where the brackets denote the $H^{-\frac{1}{2}}, H^{\frac{1}{2}}(\{y_2 = \pm R\})$ duality product. Thanks to (24) and by continuity, the result holds when passing to the limit with respect to δ . \square

When extending u_0 on Z , it solves

$$-\Delta u_0 = f + \delta_{\{y_2 = \pm R\}} h_\pm, \quad \text{in } Z. \tag{25}$$

In the next paragraph, we compute a lift that cancels the lineic Dirac mass in the right hand side of (25).

Lemma 5.4. *Let u_0 be the unique solution of problem (23). For $\alpha \leq 0$ there exists a solution w in $H_{\alpha,\#}^1(Z)$ of*

$$\begin{cases} -\Delta w = 0 & \text{in } Z_R \cup Z^R \\ [\partial_{y_2} w] = -[\partial_{y_2} u_0] =: -h_\pm & \text{on } \{y_2 = \pm R\}. \end{cases} \tag{26}$$

Proof. Setting

$$\begin{aligned} w &:= G * (h_{\pm} \delta_{\{y_2 = \pm R\}}) = G * ((h_{\pm} - \bar{h}_{\pm}) \delta_{\{y_2 = \pm R\}}) + G * (\bar{h}_{\pm} \delta_{\{y_2 = \pm R\}}) \\ &= G_1 * (h_{\pm} \delta_{\{y_2 = \pm R\}}) + G_2 * (\bar{h}_{\pm} \delta_{\{y_2 = \pm R\}}), \end{aligned}$$

which is a tempered distribution ($G \in \mathcal{S}'_{\#}(Z)$ and $h_{\pm} \delta_{\{y_2 = \pm R\}} \in \mathcal{E}'_{\#}(Z)$), w is explicit after some computations and reads:

$$w = \sum_{\pm, k \in \mathbb{Z}^*} \frac{\mathcal{F}_k(h_{\pm}) e^{ik y_1 - |k| |y_2 \pm R|}}{|k|} - \left\{ \frac{\bar{h}_{\pm}}{2} |y_2 - R| + \frac{\bar{h}_{\mp}}{2} |y_2 + R| \right\} =: I_1 + I_2.$$

One computes the $L^2_{\alpha-1}(Z)$ norm of I_1 :

$$\begin{aligned} \int_Z I_1^2 \rho(y_2)^{2\alpha-2} d\mathbf{y} &\leq \int_Z I_1^2 d\mathbf{y} \leq C \sum_{\pm, k \in \mathbb{Z}^*} \frac{|\mathcal{F}_k(h_{\pm})|^2}{(1+|k|^2)^{\frac{1}{2}}} \sum_{k \in \mathbb{Z}^*} \int_{\mathbb{R}} \frac{e^{-2|k| |y_2 \pm R|}}{|k|} d\mathbf{y} \\ &\leq C' \sum_{\pm} \|h_{\pm}\|_{H^{-\frac{1}{2}}(\{y_2 = \pm R\})}^2 \sum_{k \in \mathbb{Z}^*} \frac{1}{|k|^2} \leq C'' \|f\|_{H^{-1}_{\alpha, \#}(Z)}. \end{aligned}$$

Then using the Fourier transform one writes :

$$\mathcal{F}(\nabla G_1 * (h_{\pm} \delta_{\{y_2 = \pm R\}})) = \begin{cases} i \frac{(k, l)^*}{(k^2 + l^2)} \mathcal{F}_k(h_{\pm}), & \text{if } k \in \mathbb{Z}^*, \quad \text{a.e. } l \in \mathbb{R}, \\ 0 & \text{for } k = 0. \end{cases}$$

Taking the $l^2(\mathbb{Z}; L^2(Z))$ norm of the above expression gives then, by Parseval :

$$\begin{aligned} \|\partial_{y_1} G_1 * (h_{\pm} \delta_{\{y_2 = \pm R\}})\|_{L^2(Z)}^2 &= \sum_{k \in \mathbb{Z}^*} \int_{\mathbb{R}} \frac{k^2}{(k^2 + l^2)^2} dl |\mathcal{F}_k(h_{\pm})|^2 \\ &= \sum_{k \in \mathbb{Z}^*} \frac{C}{|k|} |\mathcal{F}_k(h_{\pm})|^2 = \|h_{\pm}\|_{H^{-\frac{1}{2}}(\{y_2 = \pm R\})}^2. \end{aligned}$$

A similar computation gives the derivative with respect to y_2 . Suppose from now on, that $\alpha \leq 0$. Due to the inclusion $L^2(Z) \subset L^2_{\alpha}(Z)$, this gives

$$\|I_1\|_{H^1_{\alpha, \#}(Z)} \leq C \|f\|_{H^{-1}_{\alpha, \#}(Z)}.$$

Thanks to Lemma 5.3, one might rewrite I_2 s.t.

$$I_2 = \mp h_{\pm} R \text{ when } |y_2| > R.$$

This means that the orthogonality condition of the data implies convergence to constants at infinity (instead of a linear growth). \square

We are now ready to state the first isomorphism result of the Laplace operator in Z .

Theorem 5.1. *Let α be a real number such that $-1/2 \leq \alpha \leq 1/2$. Then the Laplace operator defined by*

$$\Delta : H^1_{\alpha, \#}(Z)/\mathbb{R} \mapsto H^{-1}_{\alpha, \#}(Z) \perp \mathbb{R}, \quad (27)$$

is an isomorphism.

Proof. Let $f \in H_{\alpha, \#}^{-1}(Z)$. Observe that if $u \in H_{\alpha, \#}^1(Z)$ satisfies $\Delta u = f$, then due to the density of $\mathcal{D}_{\#}(Z)$ in $H_{-\alpha, \#}^1(Z)$, for any $\varphi \in H_{-\alpha, \#}^1(Z)$, we have

$$\langle u, \Delta \varphi \rangle = \langle f, \varphi \rangle.$$

Thus, we easily see that, under the assumptions on α , for any $p \in \mathbb{R} \subset H_{-\alpha, \#}^1(Z)$, the datum f satisfies the necessarily compatibility condition

$$\langle f, p \rangle = 0.$$

Now it is also clear that the Laplace operator defined by (27) is linear and continuous. It is also injective since $\Delta u = 0$ and $u \in H_{\alpha, \#}^1(Z)$ imply that u is a constant (see Proposition 5.1). Remains to prove that the operator is onto. Let f be in $H_{\alpha, \#}^{-1}(Z) \perp \mathbb{R}$ and $R > 0$ be a real number. Thanks to Lemmae 5.3 and 5.4, one sets

$$u := u_0 + w,$$

where u_0 solves (23) and w lifts the jumps of $\partial_{y_2} u_0$ on $\{y_2 = \pm R\}$ and thus solves (26). Results on u_0 and w end the proof. At this stage, the claim is proved for $\alpha \leq 0$. By duality the results is true also for $\alpha > 0$ which ends the proof. \square

Remains to extend the isomorphism result (27) to any $\alpha \in \mathbb{R}$. For this sake, we state some regularity results for the Laplace operator in Z .

Theorem 5.2. *Let α be a real number such that $-1/2 < \alpha < 1/2$ and let ℓ be an integer. Then the Laplace operator defined by*

$$\Delta : H_{\alpha+\ell, \#}^{1+\ell}(Z)/\mathbb{R} \mapsto H_{\alpha+\ell, \#}^{-1+\ell}(Z) \perp \mathbb{R} \quad (28)$$

is onto.

Proof. Owing to Theorem 5.1, the statement of the theorem is true for $\ell = 0$. Assume that it is true for $\ell = k$ and let us prove that it is still true for $\ell = k + 1$. The Laplace operator defined by (28) is clearly linear and continuous. It is also injective : if $u \in H_{\alpha+k+1, \#}^{k+2}(Z)$ and $\Delta u = 0$ then u is constant. To prove that it is onto, let f be given in $H_{\alpha+k+1, \#}^k(Z) \perp \mathbb{R}$. According to (7), f belongs to $H_{\alpha+k, \#}^{-1+k}(Z) \perp \mathbb{R}$. Then the induction assumption implies that there exists $u \in H_{\alpha+k, \#}^{1+k}(Z)$ such that $\Delta u = f$. Next, we have

$$\Delta(\rho \partial_i u) = \rho \partial_i f + \partial_i u \Delta \rho + 2 \nabla \rho \nabla(\partial_i u). \quad (29)$$

Using (6), (7), (9) and (10), all the terms of the right-hand side belong to $H_{\alpha+k, \#}^{-1+k}(Z)$. This implies that $\Delta(\rho \partial_i u)$ belongs to $H_{\alpha+k, \#}^{-1+k}(Z)$. Therefore, $\Delta(\rho \partial_i u)$ also belongs to $H_{\alpha+k-1, \#}^{-2+k}(Z)$. Moreover, u belonging to $H_{\alpha+k, \#}^{1+k}(Z)$ implies that $\rho \partial_i u$ belongs $H_{\alpha+k-1, \#}^k(Z)$ and for any $\varphi \in H_{-\alpha-k+1, \#}^{2-k}(Z)$, we have

$$\langle \Delta(\rho \partial_i u), \varphi \rangle_{H_{\alpha+k-1, \#}^{-2+k}(Z) \times H_{-\alpha-k+1, \#}^{2-k}(Z)} = \langle \rho \partial_i u, \Delta \varphi \rangle_{H_{\alpha+k-1, \#}^k(Z) \times H_{-\alpha-k+1, \#}^{-k}(Z)}.$$

Since $\mathbb{R} \subset H_{-\alpha-k+1, \#}^{2-k}(Z)$, we can take $\varphi \in \mathbb{R}$ which implies that $\Delta \varphi = 0$. It follows that $\Delta(\rho \partial_i u)$ belongs to $H_{\alpha+k, \#}^{-1+k}(Z) \perp \mathbb{R}$. Thanks to the induction assumption, there exists v in $H_{\alpha+k, \#}^{1+k}(Z)$ such that

$$\Delta v = \Delta(\rho \partial_i u).$$

Hence, the difference $v - \rho \partial_i u$ is a constant. Since the constants are in $H_{\alpha+k, \#}^{1+k}(Z)$, we deduce that $\rho \partial_i u$ belongs to $H_{\alpha+k, \#}^{1+k}(Z)$ which implies that u belongs to $H_{\alpha+k+1, \#}^{2+k}(Z)$. \square

Remark 5.1. Let us point out that the above theorem excludes the values $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$: due to logarithmic weights, the space $H_{1/2, \#}^1(Z)$ is not included in $L_{-1/2}^2(Z)$. By duality, this also implies that the space $L_{1/2}^2(Z)$ is not included in $H_{-1/2, \#}^{-1}(Z)$. Therefore, in (29) for $\alpha = 1/2$, the term $\partial_i u \Delta \rho$ does not belong to $H_{1/2, \#}^{-1}(Z)$. Thus, in this section, the extension of (27) to any $\alpha \in \mathbb{R}$ will exclude some critical values of α . We deal with these critical values in Section 6.

Remark 5.2. As an application of the above theorem for the particular case when $\ell = 1$, we derive a weighted version of Calderón-Zygmund inequalities [29]. More precisely, let α be a real number such that $1/2 < \alpha < 3/2$, then there exists a constant $C > 0$, such that for any $u \in \mathcal{D}_{\#}(Z)$:

$$\|\partial_i \partial_j u\|_{L_{\alpha}^2(Z)} \leq C \|\Delta u\|_{L_{\alpha}^2(Z)}. \quad (30)$$

Using (30) and thanks to the Closed Range Theorem of Banach, we prove that :

Theorem 5.3. *Let α be a real number satisfying $1/2 < \alpha < 3/2$. For $m \in \mathbb{N}$, $m \geq 3$, the following operator*

$$\Delta : H_{\alpha, \#}^m(Z)/\mathbb{P}'_{m-2} \mapsto H_{\alpha, \#}^{m-2}(Z)/\mathbb{P}'_{m-4} \quad (31)$$

is an isomorphism.

The next result is then a straightforward consequence of the latter result.

Theorem 5.4. *Let α be a real number satisfying $1/2 < \alpha < 3/2$. For $m \in \mathbb{N}$, $m \geq 3$, the following operator*

$$\Delta : H_{\alpha, \#}^m(Z)/\mathbb{P}'_{m-2} \mapsto H_{\alpha, \#}^{m-2}(Z) \quad (32)$$

is an isomorphism.

The next theorem extends Theorem 5.1.

Theorem 5.5. *Let α be a real number satisfying $1/2 < \alpha < 3/2$ and let $\ell \geq 1$ be an integer. Then the Laplace operators defined by*

$$\Delta : H_{-\alpha+\ell, \#}^1(Z)/\mathbb{P}'_{1-\ell} \mapsto H_{-\alpha+\ell, \#}^{-1}(Z) \perp \mathbb{P}'_{-1+\ell} \quad (33)$$

and

$$\Delta : H_{\alpha-\ell, \#}^1(Z)/\mathbb{P}'_{-1+\ell} \mapsto H_{\alpha-\ell, \#}^{-1}(Z) \perp \mathbb{P}'_{1-\ell} \quad (34)$$

are isomorphism.

Proof. Observe first that when $\ell = 1$, the result is proved in Theorem 5.1. Observe next that if $m \geq 2$ is an integer, the mapping

$$\Delta : H_{\alpha, \#}^m(Z)/\mathbb{P}'_{m-2} \mapsto H_{\alpha, \#}^{m-2}(Z) \perp \mathbb{P}'_{-m+2}$$

is onto. Indeed, if $m = 2$, this isomorphism is exactly defined by (28) with $\ell = 1$. If $m \geq 3$, it is defined by (32). Now through duality and transposition, the mapping

$$\Delta : H_{-\alpha, \#}^{-m+2}(Z)/\mathbb{P}'_{-m+2} \mapsto H_{-\alpha, \#}^{-m}(Z) \perp \mathbb{P}'_{m-2}$$

is onto. Next, using the same arguments as in the proof of Theorem 5.2, we are able to show that for any integer $\ell \geq 1$, the operator

$$\Delta : H_{-\alpha+\ell, \#}^{-m+2+\ell}(Z)/\mathbb{P}'_{-m+2} \mapsto H_{-\alpha+\ell, \#}^{-m+\ell}(Z) \perp \mathbb{P}'_{m-2} \Delta$$

is an isomorphism. Choosing $\ell = m - 1$, the operator defined by (33) is an isomorphism. By duality and transposition, the mapping defined by (34) is onto as well. \square

Remark 5.3. Summarizing theorems 5.1 and 5.5, we deduce that the mapping

$$\Delta : H_{\alpha, \#}^1(Z)/\mathbb{P}'_{[1/2-\alpha]} \Delta \mapsto H_{\alpha, \#}^{-1}(Z) \perp \mathbb{P}'_{[1/2+\alpha]} \Delta \quad (35)$$

is onto for any $\alpha \in \mathbb{R}$ such that $\alpha \neq \pm(\frac{1}{2} + k)$, $k \in \mathbb{N}^*$. As a consequence, the mapping

$$\Delta : H_{\alpha+1, \#}^2(Z)/\mathbb{P}'_{[1/2-\alpha]} \Delta \mapsto L_{\alpha+1}^2(Z) \perp \mathbb{P}'_{[1/2+\alpha]} \Delta \quad (36)$$

is onto for any $\alpha \in \mathbb{R}$ such that $\alpha \neq \pm(\frac{1}{2} + k)$, $k \in \mathbb{N}^*$. This result is obtained by proceeding as in Theorem 5.2.

Hereafter, we show connexions between variational solution above and explicit solutions of the previous section (see for instance Theorem 4.3 and Propositions 4.7 and 4.8).

Theorem 5.6. *Let α be a real number satisfying $\alpha \neq \pm(\frac{1}{2} + k)$, $k \in \mathbb{N}^*$.*

1. *If $\alpha > -1/2$ and $f \in L_{\alpha+1}^2(Z) \perp \mathbb{P}'_{[1/2+\alpha]} \Delta$, then $G * f \in H_{\alpha+1, \#}^2(Z)$ is a solution of the Laplace equation (21) unique up to a constant. Moreover, we have the estimate*

$$\inf_{\lambda \in \mathbb{R}} \|G * f + \lambda\|_{H_{\alpha+1, \#}^2(Z)} \leq \|f\|_{L_{\alpha+1}^2(Z)}.$$

2. *If $\alpha > 1/2$ and $f \in H_{\alpha, \#}^{-1}(Z) \perp \mathbb{P}'_{[1/2+\alpha]} \Delta$, then $G * f \in H_{\alpha, \#}^1(Z)$ is the unique solution of the Laplace equation (21). Moreover, we have the estimate*

$$\|G * f\|_{H_{\alpha, \#}^1(Z)} \leq \|f\|_{H_{\alpha, \#}^{-1}(Z)}.$$

Proof. The first claim of the theorem is obtained thanks to uniqueness results from Propositions 3.1 and 5.1. We detail the second part of the claim. Using the first onto result just proved and by duality arguments, as in Propositions 4.7 and 4.8, for any real $\beta < -1/2$, one gets

$$\|G * f\|_{L_{\beta-1}^2(Z)/\mathbb{P}'_{[1/2-\beta]}} \leq C \|f\|_{H_{\beta, \#}^{-1}(Z)}.$$

We detail the way to obtain the norm of the gradient. Let take $\varphi \in \mathcal{S}_{\#}(Z) \perp \mathbb{P}'_{[1/2-\beta]}$ then

$$\begin{aligned} \langle \nabla(G * f), \varphi \rangle_{\mathcal{S}'_{\#}(Z), \mathcal{S}_{\#}(Z)} &= \langle G * f, \operatorname{div} \varphi \rangle_{\mathcal{S}'_{\#}(Z), \mathcal{S}_{\#}(Z)} = \langle f, G^{\check{*}} \operatorname{div} \varphi \rangle_{H_{\beta, \#}^{-1}(Z), H_{-\beta, \#}^1(Z)} \\ &\leq \|f\|_{H_{\beta, \#}^{-1}(Z)} \|\operatorname{div} G^{\check{*}} \varphi\|_{H_{-\beta, \#}^1(Z)} \leq \|f\|_{H_{\beta, \#}^{-1}(Z)} \left\{ \inf_{\lambda \in \mathbb{R}} \|G^{\check{*}} \varphi + \lambda\|_{H_{-\beta, \#}^2(Z)} \right\} \\ &\leq C \|f\|_{H_{\beta, \#}^{-1}(Z)} \|\varphi\|_{L_{-\beta}^2(Z) \perp \mathbb{P}'_{[1/2-\beta]}}. \end{aligned}$$

We conclude by density arguments. At this stage we have, if $\beta < -\frac{1}{2}$:

$$\|G * \varphi\|_{H_{\beta, \#}^1(Z)/\mathbb{P}'_{[1/2-\beta]}} \leq C \|\varphi\|_{H_{\beta, \#}^{-1}(Z)}, \quad \forall \varphi \in H_{\beta, \#}^{-1}(Z).$$

Moreover, we know that $G * \varphi$ is solution of the Laplace problem. By uniqueness, we identify $G * \varphi$ with the variational solution. The convolution with G is then isomorphic from $H_{\beta, \#}^{-1}(Z)$ onto $H_{\beta, \#}^1(Z)/\mathbb{P}'_{[1/2-\beta]}$. For $f \in H_{-\beta, \#}^{-1}(Z) \perp \mathbb{P}'_{[1/2-\beta]}$, we set

$$\begin{aligned} \langle G * f, \varphi \rangle_{H_{-\beta, \#}^1(Z), H_{\beta, \#}^{-1}(Z)} &:= \langle f, G^{\check{*}} \varphi \rangle_{H_{-\beta, \#}^{-1}(Z) \perp \mathbb{P}'_{[1/2-\beta]}, H_{\beta, \#}^1(Z)/\mathbb{P}'_{[1/2-\beta]}} \\ &\leq C \|f\|_{H_{-\beta, \#}^{-1}(Z)} \|G^{\check{*}} \varphi\|_{H_{\beta, \#}^1(Z)/\mathbb{P}'_{[1/2-\beta]}} \leq C \|f\|_{H_{-\beta, \#}^{-1}(Z)} \|\varphi\|_{H_{\beta, \#}^{-1}(Z)}. \end{aligned}$$

We reassume results above writing that, for every f in $H_{-\beta, \#}^{-1}(Z) \perp \mathbb{P}'_{[1/2-\beta]}$,

$$\|G * f\|_{H_{-\beta, \#}^1(Z)} \leq C \|f\|_{H_{-\beta, \#}^{-1}(Z)}.$$

Setting $\alpha := -\beta$ gives the claim. \square

6 Isomorphism results for the critical cases

Note that the onto results stated in Theorem 5.1 and Theorem 5.5 are not valid for $\alpha = \pm(\frac{1}{2} + k)$, $k \in \mathbb{N}^*$. In order to extend them to these cases, we redefine the weighted spaces. Therefore, proceeding as in [18], for $m \in \mathbb{Z}$, $p \in \mathbb{N}$, and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$, we introduce the space

$$\begin{aligned} X_{\alpha+p, \#}^{m+p}(Z) &:= \{u \in H_{\alpha, \#}^m(Z), \quad \forall \lambda \in \mathbb{N}, \quad 1 \leq \lambda \leq p, \\ &\quad y_2^\lambda u \in H_{\alpha, \#}^{m+\lambda}(Z), \quad u \in H_{loc, \#}^{m+p}(Z)\}, \end{aligned}$$

where $H_{loc, \#}^{m+p}(Z)$ stands for the space of functions that belong to $H_{\#}^{m+p}(Z)$ with compact support in the y_2 direction. The space $X_{\alpha+p, \#}^{m+p}(Z)$ is a Banach space when endowed with the norm

$$\|u\|_{X_{\alpha+p, \#}^{m+p}(Z)} = \left(\sum_{0 \leq \lambda \leq p} \|y_2^\lambda u\|_{H_{\alpha, \#}^{m+\lambda}(Z)}^p + \|u\|_{H_{\#}^{m+p}(Z^1)}^p \right)^{1/p},$$

where $Z^1 :=]0, 1[\times]-1, 1[$.

Remark 6.1. We restricted the definition of the space $X_{\alpha+p, \#}^{m+p}(Z)$ to $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$, but this definition holds for all α in \mathbb{R} . For $\alpha \in \mathbb{R} \setminus \frac{1}{2}\mathbb{Z}$, since no logarithmic weights appear in the definition of $H_{\alpha+p, \#}^{m+p}(Z)$, we are able to prove that the spaces $X_{\alpha+p, \#}^{m+p}(Z)$ and $H_{\alpha+p, \#}^{m+p}(Z)$ coincide.

The next proposition is dedicated to density of $\mathcal{D}_{\#}(Z)$ in $X_{\alpha+p, \#}^{m+p}(Z)$. In order to prove it, we need to establish two lemmas giving additional properties on the space $H_{\alpha, \#}^m(Z)$. The first lemma characterizes the dual space $H_{-\alpha, \#}^{-m}(Z)$.

Lemma 6.1. *Let $\alpha \in \mathbb{R}$ and $f \in H_{-\alpha, \#}^{-m}(Z)$. Then for any $v \in H_{\alpha, \#}^m(Z)$, we have*

$$\langle f, v \rangle_{H_{-\alpha, \#}^{-m}(Z) \times H_{\alpha, \#}^m(Z)} = \sum_{0 \leq |\mu| \leq m} \int_Z g_\mu \partial^\mu v \, dy,$$

where $g_m \in L^2_{-\alpha}(Z)$ and for any $0 \leq |\mu| < m$,

- $g_\mu \in L^2_{-\alpha+m-|\mu|}(Z)$ if $\alpha \notin \{\frac{1}{2}, \dots, m - \frac{1}{2}\}$,
- $(\ln(1 + \rho^2))^{1/2} g_\mu \in L^2_{-\alpha+m-|\mu|}(Z)$ if $\alpha \in \{\frac{1}{2}, \dots, m - \frac{1}{2}\}$.

Proof. We shall only prove the statement for the case $\alpha \notin \{\frac{1}{2}, \dots, m - \frac{1}{2}\}$. The proof for the other case which corresponds to the critical values of α is similar. We set $\mathbf{E} = \prod_{0 \leq |\mu| \leq m} L^2_{-\alpha+m-|\mu|}(Z)$ such that for any $\psi = (\psi_\mu)_{0 \leq |\mu| \leq m} \in \mathbf{E}$,

$$\|\psi\|_{\mathbf{E}} = \sum_{0 \leq |\mu| \leq m} \|\psi_\mu\|_{L^2_{-\alpha+m-|\mu|}(Z)}.$$

The operator T defined by $T : v \in H^m_{\alpha, \#}(Z) \mapsto (\partial^\mu v)_{0 \leq |\mu| \leq m} \in \mathbf{E}$ is isometric. We now set $\mathbf{G} := T(H^m_{\alpha, \#}(Z))$ and $S := T^{-1} : \mathbf{G} \mapsto H^m_{\alpha, \#}(Z)$. The mapping $L : \mathbf{h} \in \mathbf{G} \mapsto \langle f, S\mathbf{h} \rangle_{H^{-m}_{-\alpha, \#}(Z) \times H^m_{\alpha, \#}(Z)}$ is linear and continuous. Therefore thanks to the Hahn-Banach theorem, there exists \tilde{L} , a linear and continuous extension of L on \mathbf{E} such that $\|\tilde{L}\|_{\mathbf{E}'} = \|f\|_{H^{-m}_{-\alpha, \#}(Z)}$. Thanks to the Riesz representation theorem, there exists $g_\mu \in L^2_{-\alpha+m-|\mu|}(Z)$ for any $0 \leq |\mu| \leq m$, such that

$$\forall v \in H^m_{\alpha, \#}(Z), \quad \langle \tilde{L}, v \rangle = \sum_{0 \leq |\mu| \leq m} \int_Z g_\mu \partial^\mu v \, d\mathbf{y}.$$

□

The second lemma concerns the density of functions with compact support.

Lemma 6.2. *Let $m \in \mathbb{Z}$, $\lambda \in \mathbb{N}$, $\alpha \in \mathbb{R}$, Φ_ℓ be defined by (13) and $u \in H^{m+\lambda}_{\alpha, \#}(Z)$. We set $u_\ell(\mathbf{y}) = \Phi_\ell(y_2)u(\mathbf{y})$ a.e. $\mathbf{y} \in Z$. Then we have*

$$\lim_{\ell \rightarrow \infty} \|u - u_\ell\|_{H^{m+\lambda}_{\alpha, \#}(Z)} = 0. \quad (37)$$

Proof. It is clear that for the case $m + \lambda \geq 0$, (37) is satisfied. We shall now focus on the case $m + \lambda < 0$. We have

$$\|u - u_\ell\|_{H^{m+\lambda}_{\alpha, \#}(Z)} = \sup_{\substack{v \in H^{-m-\lambda}_{-\alpha, \#}(Z) \\ v \neq 0}} \frac{|\langle u - u_\ell, v \rangle|}{\|v\|},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^{m+\lambda}_{\alpha, \#}(Z)$ and $H^{-m-\lambda}_{-\alpha, \#}(Z)$. Besides, thanks to Lemma 6.1 there exist g_μ such that $g_{-m-\lambda} \in L^2_\alpha(Z)$ and for any $0 \leq |\mu| < -m - \lambda$,

- $g_\mu \in L^2_{\alpha-m-\lambda-|\mu|}(Z)$ if $-\alpha \notin \{1/2, \dots, -m - \lambda - 1/2\}$
- $(\ln(1 + \rho^2))^{1/2} g_\mu \in L^2_{\alpha-m-\lambda-|\mu|}(Z)$ if $-\alpha \in \{1/2, \dots, -m - \lambda - 1/2\}$

satisfying for any $v \in H^{-m-\lambda}_{-\alpha, \#}(Z)$,

$$\langle u, v \rangle = \sum_{0 \leq |\mu| \leq -m-\lambda} \int_Z g_\mu \partial^\mu v \, d\mathbf{y}.$$

Therefore, we can write

$$\langle u - u_\ell, v \rangle = \sum_{0 \leq |\mu| \leq -m-\lambda} \int_Z g_\mu (\partial^\mu v - \partial^\mu (\Phi_\ell v)) d\mathbf{y}.$$

It follows that

$$\begin{aligned} |\langle u - u_\ell, v \rangle| &\leq \sum_{0 \leq |\mu| \leq -m-\lambda} \int_Z |g_\mu (1 - \Phi_\ell) \partial^\mu v| d\mathbf{y} \\ &\quad + C \sum_{0 \leq |\mu| \leq -m-\lambda} \sum_{0 < |k| \leq |\mu|} \int_Z g_\mu \partial^k \Phi_\ell \partial^{\mu-k} v d\mathbf{y}. \end{aligned}$$

We deduce that

$$\begin{aligned} |\langle u - u_\ell, v \rangle| &\leq \left(\sum_{0 \leq |\mu| \leq -m-\lambda} \|g_\mu (1 - \Phi_\ell)\|_{L^2_{\alpha-m-\lambda-|\mu|}(Z)} \right. \\ &\quad \left. + C \sum_{0 \leq |\mu| \leq -m-\lambda} \sum_{0 < |k| \leq |\mu|} \|g_\mu \partial^k \Phi_\ell\|_{L^2_{\alpha-m-\lambda-|\mu|}(Z)} \right) \|v\|_{H^{-m-\lambda}_{-\alpha, \#}(Z)}. \end{aligned}$$

The first two terms in the right hand side tend to zero as ℓ tends to infinity. This proves (37) for $m + \lambda < 0$. \square

We proceed with three results showing some properties of $X_{\alpha, \#}^{m,p}(Z)$ spaces and the functions belonging to them.

Proposition 6.1. *Let $m \in \mathbb{Z}$, $p \in \mathbb{N}$ and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$, then the space $\mathcal{D}_\#(Z)$ is dense in $X_{\alpha+p, \#}^{m+p}(Z)$.*

Proof. Let u be in $X_{\alpha+p, \#}^{m+p}(Z)$. Then according to Lemma 6.2, u can be approximated by $u_\ell \in X_{\alpha+p, \#}^{m+p}(Z)$ with compact support in the y_2 direction. Therefore u_ℓ belongs to $H_{\alpha, \#}^{m+p}(Z)$, and thanks to Proposition 2.2, u_ℓ can be approximated by $\psi_\ell \in \mathcal{D}_\#(Z)$. This ends the proof. \square

For $m \in \mathbb{Z}$, $p \in \mathbb{N}$ and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$, we also define the space $X_{-\alpha-p, \#}^{m-p}(Z)$ that is the dual space of $X_{\alpha+p, \#}^{-m+p}(Z)$. It is therefore a subspace of $\mathcal{D}'_\#(Z)$.

Proposition 6.2. *Let $m, p \in \mathbb{Z}$ and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$. $X_{\alpha+p+1, \#}^{m+p+1}(Z)$ is densely embedded in $X_{\alpha+p, \#}^{m+p}(Z)$.*

Proof. If $p \in \mathbb{N}$, due to the definition of the space $X_{\alpha+p+1, \#}^{m+p+1}(Z)$, the imbedding is straightforward. Moreover the density of $\mathcal{D}_\#(Z)$ in $X_{\alpha+p, \#}^{m+p}(Z)$ implies the density of $X_{\alpha+p+1, \#}^{m+p+1}(Z)$ in $X_{\alpha+p, \#}^{m+p}(Z)$. Hence, if $p \leq 0$, the dense imbedding holds :

$$X_{-\alpha-p, \#}^{-m-p}(Z) \subset X_{-\alpha-p-1, \#}^{-m-p-1}(Z).$$

By duality, the desired result follows. \square

Proposition 6.3. *Let $m, p \in \mathbb{Z}$ and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$. If u belongs to $X_{\alpha+p, \#}^{m+p}(Z)$, then $\partial_j u$ belongs to $X_{\alpha+p, \#}^{m-1+p}(Z)$, for any $j = 1, 2$.*

Proof. (i) Assume that $p \geq 0$. If $u \in X_{\alpha+p, \#}^{m+p}(Z)$, then $u \in H_{\alpha, \#}^m(Z)$ and $\partial_j u \in H_{\alpha, \#}^{m-1}(Z)$. Besides, for any $\lambda \in \mathbb{N}$, such that $1 \leq \lambda \leq p$, we have $y_2^\lambda(\partial_j u) = \partial_j(y_2^\lambda u) - \partial_j(y_2^\lambda)u$. Since $y_2^\lambda u \in H_{\alpha, \#}^{m+\lambda}(Z)$, it follows that $\partial_j(y_2^\lambda u) \in H_{\alpha, \#}^{m-1+\lambda}(Z)$. Thus it is clear that $y_2^\lambda(\partial_j u)$ belongs to $H_{\alpha, \#}^{m-1+\lambda}(Z)$. Furthermore, we have $\partial_2(y_2^\lambda)u = \lambda y_2^{\lambda-1}u$ and since $0 \leq \lambda - 1 \leq p - 1$, $y_2^{\lambda-1}u$ belongs to $H_{\alpha, \#}^{m+\lambda-1}(Z)$. We deduce that for any $\lambda \in \mathbb{N}$, such that $1 \leq \lambda \leq p$, $y_2^\lambda(\partial_j u)$ belongs to $H_{\alpha, \#}^{m-1+\lambda}(Z)$. Finally it is straightforward that if u is in $H_{\#}^{m+p}(Z^1)$, then $\partial_j u$ is in $H_{\#}^{m+p-1}(Z^1)$.

(ii) Let now $p < 0$. If $u \in X_{\alpha+p, \#}^{m+p}(Z)$, then for any $\varphi \in \mathcal{D}_{\#}(Z)$, we can write

$$\langle \partial_j u, \varphi \rangle_{\mathcal{D}'_{\#}(Z) \times \mathcal{D}_{\#}(Z)} = -\langle u, \partial_j \varphi \rangle_{\mathcal{D}'_{\#}(Z) \times \mathcal{D}_{\#}(Z)}.$$

Thanks to (i), for any $\varphi \in X_{-\alpha-p, \#}^{-m+1-p}(Z)$, $\partial_j \varphi$ belongs to $X_{-\alpha-p, \#}^{-m-p}(Z)$ which ends the proof. \square

Proposition 6.4. *Let $m, p \in \mathbb{Z}$ and $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$. If u belongs to $X_{\alpha+p, \#}^{m+p}(Z)$, then $y_2 u$ belongs to $X_{\alpha+p-1, \#}^{m+p}(Z) \equiv X_{\alpha+p-1, \#}^{m+1+p-1}(Z)$.*

Proof. (i) Assume that $p \geq 1$. If u belongs to $X_{\alpha+p, \#}^{m+p}(Z)$, then u belongs to $H_{\alpha, \#}^m(Z)$ and $y_2 u$ belongs to $H_{\alpha, \#}^{m+1}(Z)$. Next, for any $\lambda \in \mathbb{N}$, such that $1 \leq \lambda \leq p - 1$, we have $y_2^\lambda(y_2 u) = y_2^{\lambda+1}u \in H_{\alpha, \#}^{m+\lambda+1}(Z)$. Finally if u belongs to $H_{\#}^{m+p}(Z^1)$, then $y_2 u$ also belongs to $H_{\#}^{m+p}(Z^1)$. We thus deduce that $y_2 u \in X_{\alpha+p-1, \#}^{m+p}(Z)$.

(ii) Assume now $p \leq 0$. For any $\varphi \in \mathcal{D}_{\#}(Z)$, we can write

$$\langle y_2 u, \varphi \rangle_{\mathcal{D}'_{\#}(Z) \times \mathcal{D}_{\#}(Z)} = \langle u, y_2 \varphi \rangle_{\mathcal{D}'_{\#}(Z) \times \mathcal{D}_{\#}(Z)}.$$

Thanks to (i), for any $\varphi \in X_{-\alpha-p+1, \#}^{-m-p}(Z) \equiv X_{-\alpha-p+1, \#}^{-m-1-p+1}(Z)$, $y_2 \varphi$ belongs to $X_{-\alpha-p, \#}^{-m-p}(Z)$ which ends the proof. \square

The above properties allow to extend Theorem 5.2.

Theorem 6.1. *Let $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$ and $\ell \in \mathbb{N}$. Then the operator*

$$\Delta : X_{\alpha+\ell, \#}^{1+\ell}(Z)/\mathbb{R} \mapsto X_{\alpha+\ell, \#}^{-1+\ell}(Z) \perp \mathbb{R} \quad (38)$$

is onto.

Proof. Observe first that since $X_{\alpha, \#}^1(Z) = H_{\alpha, \#}^1(Z)$ and $X_{\alpha, \#}^{-1}(Z) = H_{\alpha, \#}^{-1}(Z)$, the case $\ell = 0$ is proved in Theorem 5.1. Now we proceed by induction, as in the proof of Theorem 5.2. Assume that the statement is true for $\ell = k$ and let us prove that it is then true for $\ell = k + 1$. The Laplace operator defined by (38) is clearly linear and continuous. It is also injective : if u belongs to $X_{\alpha+k+1, \#}^{2+k}(Z) \subset H_{\alpha, \#}^1(Z)$ and is harmonic then it is a constant. Let us prove that it is onto. Let $f \in X_{\alpha+k+1, \#}^k(Z) \perp \mathbb{R}$. Then $f \in X_{\alpha+k, \#}^{-1+k}(Z) \perp \mathbb{R}$ and the induction assumption implies that there exists $u \in X_{\alpha+k+1, \#}^{1+k}(Z)$ such that $\Delta u = f$. We now consider the equality

$$\Delta(y_2 \partial_i u) = y_2 \partial_i f + 2\partial_2 \partial_i u. \quad (39)$$

Using Proposition 6.3 and Proposition 6.4, we deduce that $\Delta(y_2\partial_i u)$ belongs to $X_{\alpha+k,\#}^{-1+k}(Z) \subset X_{\alpha+k-1,\#}^{-2+k}(Z)$. Moreover, thanks to Propositions 6.3 and 6.4, if $u \in X_{\alpha+k,\#}^{1+k}(Z)$ then $y_2\partial_i u \in X_{\alpha+k-1,\#}^k(Z)$ and for any $\varphi \in X_{-\alpha-k+1,\#}^{2-k}(Z)$, we have

$$\langle \Delta(y_2\partial_i u), \varphi \rangle_{X_{\alpha+k-1,\#}^{-2+k}(Z) \times X_{-\alpha-k+1,\#}^{2-k}(Z)} = \langle y_2\partial_i u, \Delta\varphi \rangle_{X_{\alpha+k-1,\#}^k(Z) \times X_{-\alpha-k+1,\#}^{-k}(Z)}.$$

Since $\mathbb{R} \subset X_{-\alpha-k+1,\#}^{2-k}(Z)$, we shall take $\varphi \in \mathbb{R}$. This shows that $\Delta(y_2\partial_i u) \in X_{\alpha+k,\#}^{-1+k}(Z) \perp \mathbb{R}$. Thanks to the induction assumption, there exists $v \in X_{\alpha+k,\#}^{1+k}(Z)$ satisfying $\Delta v = \Delta(y_2\partial_i u)$. Hence $v - y_2\partial_i u \in \mathbb{R}$ and since $\mathbb{R} \subset X_{\alpha+k,\#}^{1+k}(Z)$, it follows that $y_2\partial_i u \in X_{\alpha+k,\#}^{1+k}(Z)$ which in turn yields $u \in X_{\alpha+k+1,\#}^{2+k}(Z)$. \square

Remark 6.2. The above theorem extends the Calderón-Zygmund inequality (30) to $\alpha \in \{\frac{1}{2}, \frac{3}{2}\}$.

Proceeding as in the non-critical cases, we can extend theorems 5.3, 5.4 and 5.5. More precisely, we have :

Theorem 6.2. *Let α be a real number satisfying $1/2 \leq \alpha \leq 3/2$. For $m \in \mathbb{N}$, $m \geq 3$, the mapping*

$$\Delta : H_{\alpha,\#}^m(Z)/\mathbb{P}'_{m-2} \mapsto H_{\alpha,\#}^{m-2}(Z)/\mathbb{P}'_{m-4} \quad (40)$$

is onto.

Theorem 6.3. *Let α be a real number satisfying $1/2 \leq \alpha \leq 3/2$. For $m \in \mathbb{N}$, $m \geq 3$, the mapping*

$$\Delta : H_{\alpha,\#}^m(Z)/\mathbb{P}'_{m-2} \mapsto H_{\alpha,\#}^{m-2}(Z) \quad (41)$$

is onto.

Theorem 6.4. *Let $\alpha \in \{\frac{1}{2}, \frac{3}{2}\}$ and let $\ell \geq 1$ be an integer. Then the Laplace operators defined by*

$$\Delta : X_{-\alpha+\ell,\#}^1(Z)/\mathbb{P}'_{1-\ell} \mapsto X_{-\alpha+\ell,\#}^{-1}(Z) \perp \mathbb{P}'_{-1+\ell} \quad (42)$$

and

$$\Delta : X_{\alpha-\ell,\#}^1(Z)/\mathbb{P}'_{-1+\ell} \mapsto X_{\alpha-\ell,\#}^{-1}(Z) \perp \mathbb{P}'_{1-\ell} \quad (43)$$

are onto.

As for the non-critical cases, the last results show that there exists a connexion between Theorems 4.3 and 6.1. Let us first note that we might compare the spaces $X_{\alpha+p,\#}^{m+p}(Z)$ and $H_{\alpha+p,\#}^{m+p}(Z)$ for $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$.

Proposition 6.5. *Let m, p be two integers and let $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$. Then*

$$X_{\alpha+p,\#}^{m+p}(Z) \subset H_{\alpha+p,\#}^{m+p}(Z). \quad (44)$$

If, moreover $\alpha = \frac{1}{2}$, then

$$X_{\frac{1}{2}+p,\#}^{m+p}(Z) = H_{\frac{1}{2}+p,\#}^{m+p}(Z). \quad (45)$$

The proof of the above proposition is based on a partition of unity in the y_2 direction and direct computations. Note that (45) is not valid for $\alpha = -\frac{1}{2}$ and $p \geq 1$ since the space $H_{-1/2+p,\#}^{m+p}(Z)$ is not included in $H_{-1/2,\#}^m(Z)$ (see (8)).

Theorem 6.5. *Let $f \in L_{3/2}^2(Z) \perp \mathbb{R}$, then $G * f \in H_{3/2,\#}^2(Z)$ is a solution of (21) unique up to a constant. Moreover, we have the estimate*

$$\inf_{\lambda \in \mathbb{R}} \|G * f + \lambda\|_{H_{3/2,\#}^2(Z)} \leq \|f\|_{L_{3/2}^2(Z)}.$$

7 Conclusion

This paper is a first attempt towards a systematic analysis of boundary layer problems in periodic strips. This framework provide generic spaces avoiding tedious *a priori* definitions of solutions' behaviour at infinity, among other advantages.

Nevertheless, in the homogenization theory, solutions of boundary layer problems often converge exponentially to some polynomial at infinity (when $y_2 \rightarrow \infty$) [16]. In a forecoming work, our goal is to provide an adequate framework of these results in the weighted context, and to consider exterior periodic domains [4].

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