Consistency between adiabatic and nonadiabatic geometric phases for nonselfadjoint hamiltonians

D. Viennot, A. Leclerc, G. Jolicard, J.P. Killingbeck

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David Viennot, Arnaud Leclerc & Georges Jolicard

Institut UTINAM (CNRS UMR 6213, Université de Franche-Comté), 41bis Avenue de l'Observatoire, BP1615, 25010 Besançon cedex, France.

John P. Killingbeck

Centre for Mathematics, University of Hull, Hull HU6 7RX, UK.

Abstract. We show that the adiabatic approximation for nonselfadjoint hamiltonians seems to induce two non-equal expressions for the geometric phase. The firts one is related to the spectral projector involved in the adiabatic theorem, the other one is the adiabatic limit of the nonadiabatic geometric phase. This apparent inconsistency is resolved by observing that the difference between the two expressions is compensate by a small deviation in the dynamical phases.

1. Introduction

The adiabatic approximation is currently a commonly used tool to study dynamical systems [1]. Recently Marzlin and Sanders have pointed out a possible inconsistency in the application of the adiabatic theorem [2]. Their work has produced some debate and controversy [3, 4, 5, 6] about the application of the adiabatic approximation to systems governed by selfadjoint Hamiltonians. Generalizations of the adiabatic theorem have also been proposed for nonselfadjoint Hamiltonians by Nenciu and Rasche [7], Abou Salem and Fröhlich [8, 9], Joye [10] and Avron *et al* [11, 12]. In the present work we show that another kind of apparent inconsistency arises for nonselfadjoint Hamiltonians. Like the Marzlin-Sanders inconsistency it is intimately associated with the geometric phase concept. The geometric phase usually used in the adiabatic approximation (so-called the Berry phase) does not coincide with the adiabatic limit of the geometric phase usually used in the nonadiabatic cyclic quantum dynamics (so-called the Aharonov-Anandan) [13, 14]. Section 2 is general considerations about the two possible expressions of the geometric phase. Section 3 is an analysis of the origin of the apparent inconsistency. We present how to treat correctly the adiabatic approximation of the nonadiabatic geometric phase to solve this inconsistency.

2. General considerations

Let $H(t)$ be a $\mathcal{C}^1$-time-dependent nonselfadjoint hamiltonian with $\frac{1}{2i}(H(t)-H(t)^\dagger) \leq 0$ ($H(t)$ generates a contraction). Let $\lambda_a(t) \in \mathbb{C}$ be an isolated non-degenerate eigenvalue

of H, $\phi_a(t)$ be the associated (right) $\mathcal{C}^1$-eigenvector and $\phi_a^*(t)$ be the associated biorthogonal (left) $\mathcal{C}^1$-eigenvector:

$$H\phi_a = \lambda_a\phi_a \tag{1}$$
$$H^\dagger\phi_a^* = \overline{\lambda_a}\phi_a^* \tag{2}$$

(here the ordinary complex conjugate is denoted by an overline rather than by a star) with

$$\langle\phi_a^*|\phi_b\rangle = \delta_{ab} \tag{3}$$

The adiabatic approximation states that the wave function $\psi(s)$, which is the solution of the Schrödinger equation $\frac{\imath\hbar}{T}\dot{\psi} = H\psi$ with $\psi(0) = \phi_a(0)$, remains approximately projected onto $\mathrm{Lin}(\phi_a)$ (Lin denotes the linear envelope). $s = \frac{t}{T}$ is the reduced time, T being the total duration. The dot denotes the derivative with respect to s. We should first point out that there are two "natural" projections onto $\mathrm{Lin}(\phi_a)$, the orthogonal projection:

$$P_o = \frac{|\phi_a\rangle\langle\phi_a|}{\langle\phi_a|\phi_a\rangle} \tag{4}$$

and the spectral (Riesz) projection:

$$P_s = \frac{1}{2\pi\imath}\oint_{\Gamma_{\lambda_a}}(H-z)^{-1}dz = |\phi_a\rangle\langle\phi_a^*| \tag{5}$$

where Γ_{λ_a} is a closed path in the complex plan surrounding only λ_a. We note that the two projectors satisfy $P_o^2 = P_o$, $P_s^2 = P_s$, $P_sP_o = P_o$ and $P_oP_s = P_s$ but $P_o^\dagger = P_o$ whereas $P_s^\dagger \neq P_s$. The adiabatic theorems of Nenciu-Rasche [7], Abou Salem-Fröhlich [8] and Joye [10] deal with the spectral projector:

$$U_T(s,0)P_s(0) = P_s(s)U_T(s,0) + \mathcal{O}(\frac{1}{T}) \tag{6}$$

where $U_T(s,0)$ is the evolution operator ($\frac{\imath\hbar}{T}\dot{U}_T(s,0) = H(s)U_T(s,0)$ with $U(0,0) = 1$). Equation 6 constitutes the fundamental assumption of this work. By construction we then have

$$\psi(s) = U_T(s,0)\phi_a(0) \tag{7}$$
$$= P_s(s)U_T(s,0)\phi_a(0) + \mathcal{O}(\frac{1}{T}) \tag{8}$$
$$= \underbrace{\langle\phi_a^*(s)|U_T(s,0)|\phi_a(0)\rangle}_{c(s)}\phi_a(s) + \mathcal{O}(\frac{1}{T}) \tag{9}$$

where $c(s) \in \mathbb{C}$ is a time-dependent complex coefficient (in contrast with the selfadjoint case, c is not just a phase, since the evolution is not unitary). By inserting the expression $\psi \simeq c\phi_a$ in the Schrödinger equation, we find that

$$\dot{c}\phi_a \simeq -(\imath\hbar^{-1}T\lambda_a\phi_a + \dot{\phi}_a)c \tag{10}$$

By projection of eq. 10 with $\langle\phi_a^*|$ we find that

$$\psi(s) \simeq e^{-\imath\hbar^{-1}T\int_0^s\lambda_a ds - \int_0^s\langle\phi_a^*|\dot{\phi}_a\rangle ds}\phi_a(s) \tag{11}$$
$$\equiv \psi_s(t)$$

This is the expression that we can find in the literature concerning the adiabatic geometric phases of nonselfadjoint hamiltonians [15, 16, 17, 18, 19]. Since the norms

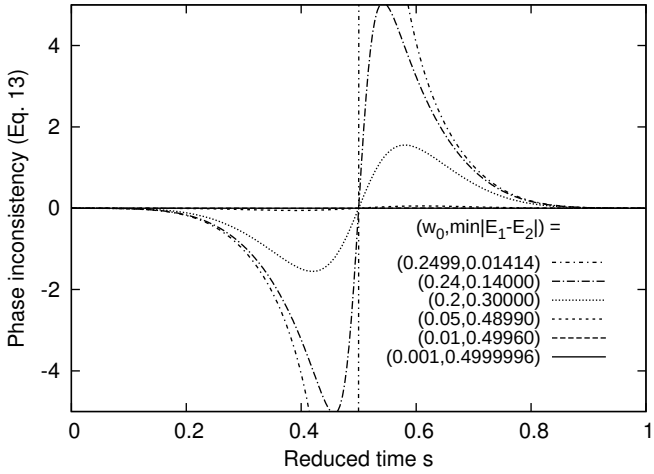

Figure 1. $\langle\phi_1^*|\dot{\phi}_1\rangle - \frac{\langle\phi_1|\dot{\phi}_1\rangle}{\langle\phi_1|\phi_1\rangle}$ for the hamiltonian $H(s) = \begin{pmatrix} 0 & \Omega(s) \\ \Omega(s) & -\imath\frac{\Gamma}{2} \end{pmatrix}$ with $\Omega(s) = \Omega_0 e^{-\frac{(s-s_0)^2}{2\sigma}}$, with different values of $w_0 = \frac{\Omega_0}{\Gamma}$. The gaussian parameters are $s = 0.5$ and $\sigma = 0.16$. $\min|E_1 - E_2|$ is the minimal distance during the evolution between the two eigenvalues (this distance is proportionnal to the inverse of nonadiabatic couplings). This model is associated with a quantum bound state coupled to a quantum resonance (with resonance width Γ) by a laser gaussian pulse.

are not fixed to 1, the gauge structure associated with the geometric phases in the nonselfadjoint case deals with changes of phase and norm ("geometric factor" in place of "geometric phase" could be a more appropriate expression). In other words, the pincipal bundle describing the geometric phases has not $U(1)$ as structure group but $\mathbb{C}^*$ (the group of non-zero complex number). The expression (11) seems to be quite natural, since the adiabatic theorems deal with the spectral projection. Nevertheless, nothing forbids the projection of eq. 10 with $\frac{\langle\phi_a|}{\langle\phi_a|\phi_a\rangle}$, and in this case we find that

$$\psi(s) \simeq e^{-\imath\hbar^{-1}T\int_0^s \lambda_a ds - \int_0^s \frac{\langle\phi_a|\dot{\phi}_a\rangle}{\langle\phi_a|\phi_a\rangle}ds}\phi_a(s) \tag{12}$$
$$\equiv \psi_o(s)$$

The apparent inconsistency arises from the adiabatic geometric phases:

$$\langle\phi_a^*|\dot{\phi}_a\rangle - \frac{\langle\phi_a|\dot{\phi}_a\rangle}{\langle\phi_a|\phi_a\rangle} = \langle\phi_a^*|\dot{P}_o|\phi_a\rangle \tag{13}$$
$$= -\frac{\langle\phi_a|\dot{P}_s|\phi_a\rangle}{\langle\phi_a|\phi_a\rangle} \tag{14}$$
$$\neq 0 \tag{15}$$

This problem does not arise for selfadjoint hamiltonians where $\phi_a = \phi_a^*$. This deviation is moreover proportionnal to the amplitude of the instantaneous nonadiabatic couplings, see fig. 1. It is then small where the nonadiabatic couplings are small, i.e. far from the eigenvalues crossings . The question is then: What is the correct adiabatic geometric phase to use for nonselfadjoint hamiltonians? An evident argument in favour of the "spectral adiabatic geometric phase" is that it is the

only one which is compatible with a late application of the adiabatic approximation. Indeed, let $\psi(s) = \sum_b c_b(s)\phi_b(s)$ (we suppose for the sake of simplicity that $H(s)$ is diagonalizable). By putting this expression in the Schrödinger equation and by projecting with $\langle\phi_a^*|$ we find

$$\dot{c}_a = -\imath\hbar^{-1}T\lambda_a c_a - \sum_b \langle\phi_a^*|\dot{\phi}_b\rangle c_b \tag{16}$$

and by then applying the following adiabatic approximation (for $b \neq a$):

$$\langle\phi_a^*|\dot{\phi}_b\rangle = \frac{\langle\phi_a^*|\dot{H}|\phi_b\rangle}{\lambda_b - \lambda_a} \tag{17}$$

$$\simeq 0 \tag{18}$$

we find again that $\psi(s) \simeq \psi_s(s)$. In contrast, by projecting with $\frac{\langle\phi_a|}{\langle\phi_a|\phi_a\rangle}$, since the eigenvectors are not orthogonal, we find

$$\sum_b \dot{c}_b \frac{\langle\phi_a|\phi_b\rangle}{\langle\phi_a|\phi_a\rangle} = -\imath\hbar^{-1}T\sum_b \lambda_b c_b \frac{\langle\phi_a|\phi_b\rangle}{\langle\phi_a|\phi_a\rangle} - \sum_b c_b \frac{\langle\phi_a|\dot{\phi}_b\rangle}{\langle\phi_a|\phi_a\rangle} \tag{19}$$

An adiabatic approximation seems to be not efficient to treat this expression and cannot be used to claim that $\psi(s) \simeq \psi_o(s)$. Is this argument sufficient to claim that the "orthogonal adiabatic geometric phase" is irrelevant? It seems that the answer is "no". First, the rigorously proved adiabatic theorems concern the approximation of eq. 6 and not that of eq. 18 (moreover the use of the approximation eq. 18 is not efficient even for some selfadjoint cases, see [6], and we can remark that the conditions of validity of eq. 18 are the same that to the deviation between geometric phases be small). But more importantly, the orthogonal adiabatic geometric phase is the adiabatic limit of the nonadiabatic geometric phase. Indeed consider a quantum dynamics $\frac{\imath\hbar}{T}\dot{\psi} = H(s)\psi(s)$ such that $\psi(1) = \mu\psi(0)$ with $\mu \in \mathbb{C}^*$ (the dynamics is said cyclic). Let $\underline{\psi}_T(s) \in \mathrm{Lin}(\psi(s))$ be such that $\underline{\psi}_T(1) = \underline{\psi}_T(0) = \psi(0)$ ($\underline{\psi}_T$ is an arbitrary choice in $\mathrm{Lin}(\psi(s))$ called a local section in the geometric language of the fibre bundle theory). By construction, it exists $f(s) \in \mathbb{C}^*$ such that $\underline{\psi}_T(s) = f(s)\psi(s)$. By inserting $\psi(s) = f(s)^{-1}\underline{\psi}_T(s)$ in the Schrödinger equation, we find

$$f^{-1}\dot{f}\underline{\psi}_T = \imath\hbar^{-1}TH\underline{\psi}_T + \dot{\underline{\psi}}_T \tag{20}$$

By projecting this equation on $\underline{\psi}_T$ we find

$$f^{-1}\dot{f} = \imath\hbar^{-1}T\frac{\langle\underline{\psi}_T|H|\underline{\psi}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle} + \frac{\langle\underline{\psi}_T|\dot{\underline{\psi}}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle} \tag{21}$$

Finally

$$\psi(s) = e^{-\imath\hbar^{-1}T\int_0^s \frac{\langle\underline{\psi}_T|H|\underline{\psi}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle}ds - \int_0^s \frac{\langle\underline{\psi}_T|\dot{\underline{\psi}}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle}ds}\underline{\psi}_T(s) \tag{22}$$

We can note that no approximation occurs in this last expression. $\frac{\langle\underline{\psi}_T|\dot{\underline{\psi}}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle}$ generates the nonadiabatic geometric phase. The nonadiabatic geometric phase has an important property concerning the non-unitary evolution. Since

$$\overline{\frac{\langle\underline{\psi}_T|\dot{\underline{\psi}}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle}} = -\frac{\langle\underline{\psi}_T|\dot{\underline{\psi}}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle} + \frac{d}{ds}\ln\langle\underline{\psi}_T|\underline{\psi}_T\rangle \tag{23}$$

we find that

$$\left|e^{-\int_0^s \frac{\langle \underline{\psi}_T|\dot{\underline{\psi}}_T\rangle}{\langle \underline{\psi}_T|\underline{\psi}_T\rangle}ds}\right|^2 = e^{-\int_0^s \frac{d}{ds}\ln\langle \underline{\psi}_T|\underline{\psi}_T\rangle} \tag{24}$$

$$= \frac{\langle\psi(0)|\psi(0)\rangle}{\langle \underline{\psi}_T(s)|\underline{\psi}_T(s)\rangle} \tag{25}$$

and then

$$\|\psi(s)\|^2 = \|\psi(0)\|^2 e^{2\hbar^{-1}T\int_0^s \mathrm{Im}\frac{\langle \underline{\psi}_T|H|\underline{\psi}_T\rangle}{\langle \underline{\psi}_T|\underline{\psi}_T\rangle}ds} \tag{26}$$

The evolution of the norm (and then the dissipation evolution) depends only on the dynamical phase. At the end of the evolution, the nonadiabatic geometric phase does not take part to the dissipation process, in this sense it is a good generalization of a "phase" for the nonselfadjoint dynamics. It is well a generalization since during the evolution it belongs to $\mathbb{C}^*$ and not $U(1)$ (it is not a pure phase), but it corresponds to a right "anholonomy" for the cyclicity of the dynamics independently on the dissipation. At the adiabatic limit $T \to +\infty$, it is clear that we can chose the local section such that $\lim_{T\to+\infty} \underline{\psi}_T(s) = \phi_a(s)$. We have then

$$\lim_{T\to+\infty} \frac{\langle \underline{\psi}_T|\dot{\underline{\psi}}_T\rangle}{\langle \underline{\psi}_T|\underline{\psi}_T\rangle} = \frac{\langle\phi_a|\dot{\phi}_a\rangle}{\langle\phi_a|\phi_a\rangle} \tag{27}$$

The orthogonal adiabatic geometric phase has the same property that the nonadiabatic geometric phase: it does not take part to the dissipation process and is then a good generalization of a "phase" for the nonselfadjoint adiabatic dynamics, in contrast to the spectral geometric phase for which we have

$$\|\psi(s)\|^2 \simeq \|\phi_a(0)\|^2 e^{2\hbar^{-1}T\int_0^s \mathrm{Im}\lambda_a ds}\left|e^{-\int_0^s \langle\phi_a^*|\dot{\phi}_a\rangle ds}\right|^2 \tag{28}$$

where

$$\left|e^{-\int_0^s \langle\phi_a^*|\dot{\phi}_a\rangle ds}\right|^2 = \left|e^{-\int_0^s \langle\phi_a^*|\dot{P}_o|\phi_a\rangle ds}\right|^2 \neq 1 \tag{29}$$

The spectral adiabatic geometric phase includes a geometric contribution to the dissipation, which is precisely its deviation from the orthogonal adiabatic geometric phase.

3. Consistency between the two adiabatic geometric phases

To solve the apparent inconsistency, we remark first that equation 20 can be projected onto $\forall\chi(s)$ such that $\langle\chi|\underline{\psi}_T\rangle \neq 0$:

$$f^{-1}\dot{f} = \imath\hbar^{-1}T\frac{\langle\chi|H|\underline{\psi}_T\rangle}{\langle\chi|\underline{\psi}_T\rangle} + \frac{\langle\chi|\dot{\underline{\psi}}_T\rangle}{\langle\chi|\underline{\psi}_T\rangle} \tag{30}$$

This induces no inconsistency since by construction $\forall\chi$ nonorthogonal to the dynamics we have

$$\imath\hbar^{-1}T\frac{\langle\chi|H|\underline{\psi}_T\rangle}{\langle\chi|\underline{\psi}_T\rangle} + \frac{\langle\chi|\dot{\underline{\psi}}_T\rangle}{\langle\chi|\underline{\psi}_T\rangle} = \imath\hbar^{-1}T\frac{\langle\underline{\psi}_T|H|\underline{\psi}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle} + \frac{\langle\underline{\psi}_T|\dot{\underline{\psi}}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle} \tag{31}$$

The modification of the geometric phase is compensed by a modification of the dynamical phase (it is the sum of the geometric and dynamical phases which is

invariant). We can note that $\frac{\langle\chi|\dot{\underline{\psi}}_T\rangle}{\langle\chi|\underline{\psi}_T\rangle}$ has not the good property of nonparticipating to the dissipation, and has no clear physical sense. Nevertheless we can choose $\chi(s)=\phi_a^*(s)$ (for T sufficiently large, by the adiabatic assumption, ϕ_a^* is not orthogonal to the dynamics), and have a geometric phase tending to the spectral adiabatic geometric phase. The inconsistency arises in the fact that all generators of dynamical phases tend to $\lambda_a(s)$.

To solve this problem it is important to note that the adiabatic theorem for nonselfadjoint hamiltonians, as the Joye theorem or the Nenciu-Rasche theorem, needs a "superadiabatic renormalization" [10, 7]. In other words, these theorems do not deal with ϕ_a but with $\phi_{aT}^{(1)}$ which is eigenvector of the superadiabatic renormalized Hamiltonian:

$$H_T^{(1)}(s)=H(s)-\frac{\imath\hbar}{T}\left(\dot{P}P+\sum_{b\neq a}\dot{Q}_bQ_b\right) \tag{32}$$

where $\{Q_b\}_b$ are the spectral projectors onto the other eigenspaces. We note that the demonstrations of the adiabatic theorems for nonselfadjoint hamiltonians need iterations of the superadiabatic renormalization ($H_T^{(n)}=H_T^{(n-1)}-\frac{\imath\hbar}{T}\left(\dot{P}_T^{(n-1)}P_T^{(n-1)}+\sum_{b\neq a}\dot{Q}_{bT}^{(n-1)}Q_{bT}^{(n-1)}\right)$), but for the present analysis the first step is sufficient. The adiabatic approximation is then

$$\underline{\psi}_T(s)\sim\phi_{aT}^{(1)} \tag{33}$$

where "$\sim$" denotes the equivalence for T in the neighbourhood of $+\infty$. By perturbative analysis we can write for T in the neighbourhood of $+\infty$,

$$\phi_{aT}^{(1)}=\phi_a-\frac{\imath\hbar}{T}\sum_{b\neq a}\frac{\langle\phi_b|\dot{P}|\phi_a\rangle}{\lambda_a-\lambda_b}\phi_b+\mathcal{O}(\frac{1}{T^2}) \tag{34}$$

$$\phi_{aT}^{*(1)}=\phi_a^*-\frac{\imath\hbar}{T}\sum_{b\neq a}\frac{\langle\phi_b^*|\dot{P}^\dagger|\phi_a^*\rangle}{\overline{\lambda_a}-\overline{\lambda_b}}\phi_b^*+\mathcal{O}(\frac{1}{T^2}) \tag{35}$$

We have then

$$\langle\phi_{aT}^{*(1)}|\dot{\phi}_{aT}^{(1)}\rangle=\langle\phi_a^*|\dot{\phi}_a\rangle+\mathcal{O}(\frac{1}{T}) \tag{36}$$

$$\frac{\langle\phi_a^{(1)}|\dot{\phi}_a^{(1)}\rangle}{\langle\phi_a^{(1)}|\phi_a^{(1)}\rangle}=\frac{\langle\phi_a|\dot{\phi}_a\rangle}{\langle\phi_a|\phi_a\rangle}+\mathcal{O}(\frac{1}{T}) \tag{37}$$

$$\langle\phi_{aT}^{*(1)}|H|\phi_{aT}^{(1)}\rangle=\lambda_a+\mathcal{O}(\frac{1}{T^2}) \tag{38}$$

$$\frac{\langle\phi_{aT}^{(1)}|H|\phi_{aT}^{(1)}\rangle}{\langle\phi_{aT}^{(1)}|\phi_{aT}^{(1)}\rangle}=\lambda_a+\frac{\imath\hbar}{T}\sum_{b\neq a}\frac{\langle\phi_a|\phi_b\rangle\langle\phi_b|\dot{P}|\phi_a\rangle}{\langle\phi_a|\phi_a\rangle}+\mathcal{O}(\frac{1}{T^2}) \tag{39}$$

$$=\lambda_a+\frac{\imath\hbar}{T}\frac{\langle\phi_a|(1-P)\dot{P}|\phi_a\rangle}{\langle\phi_a|\phi_a\rangle}+\mathcal{O}(\frac{1}{T^2}) \tag{40}$$

and since $P^2=P\Rightarrow\dot{P}P+P\dot{P}=\dot{P}\Rightarrow P\dot{P}P=0$, we have

$$\frac{\langle\phi_{aT}^{(1)}|H|\phi_{aT}^{(1)}\rangle}{\langle\phi_{aT}^{(1)}|\phi_{aT}^{(1)}\rangle}=\lambda_a+\frac{\imath\hbar}{T}\frac{\langle\phi_a|\dot{P}|\phi_a\rangle}{\langle\phi_a|\phi_a\rangle}+\mathcal{O}(\frac{1}{T^2}) \tag{41}$$

We see then that for the generator of the dynamical phase $\lim_{T\to+\infty}\frac{\langle\underline{\psi}_T|H|\underline{\psi}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle}=\lambda_a$, but

$$\imath\hbar^{-1}T\frac{\langle\underline{\psi}_T|H|\underline{\psi}_T\rangle}{\langle\underline{\psi}_T|\underline{\psi}_T\rangle}\sim\imath\hbar^{-1}T\lambda_a-\frac{\langle\phi_a|\dot{P}|\phi_a\rangle}{\langle\phi_a|\phi_a\rangle} \tag{42}$$

$$\sim\imath\hbar^{-1}T\lambda_{aT}^{eff} \tag{43}$$

with $\lambda_{aT}^{eff}=\lambda_a+\frac{\imath\hbar}{T}\frac{\langle\phi_a|\dot{P}|\phi_a\rangle}{\langle\phi_a|\phi_a\rangle}$. The deviation between the usual dynamical phase and the effective dynamical phase is precisely equal to the deviation between the adiabatic spectral and orthogonal geometric phases. We have then

$$\imath\hbar^{-1}T\lambda_a+\langle\phi_a^*|\dot{\phi}_a\rangle=\imath\hbar^{-1}T\lambda_{aT}^{eff}+\frac{\langle\phi_a|\dot{\phi}_a\rangle}{\langle\phi_a|\phi_a\rangle} \tag{44}$$

This solves the inconsistency, the adiabatic geomretric phases are not equal, but their deviation is compensed by a deviation between the dynamical phases if λ_{aT}^{eff} generates the dynamical phase associated with the orthogonal geometric phase. λ_{aT}^{eff} is well the correct equivalent of the dynamical phase associated with the nonadiabatic geometric phase. It is interesting to note that the geometric contribution to the dissipation $\left|e^{-\int_0^s\langle\phi_a^*|\dot{P}_o|\phi_a\rangle ds}\right|^2$ can be then interpreted as a contribution of the dynamical phase.

4. Conclusion

Even if the adiabatic spectral geometric phase seems to be more natural with respect to the adiabatic theorem, it is important to note that it is not the adiabatic limit of the nonadiabatic geometric phase and in consequences it contributes to the dissipation process. In contrast, the adiabatic orthogonal geometric phase does not contribute to the dissipation process and is then a good equivalent to a phase for the nonselfadjoint dynamics. This can be very important for experimental measurements of the geometric phase in dissipative quantum dynamics. It not evident that we can have access to a measurement of the adiabatic spectral geometric phase because of its implication in the quantum flow loss. The adiabatic orthogonal geometric phase could be more pertinent for an experimental measurement.

Finally, we can remark that we can also introduce "non-natural" geometric phases. Let $\chi(s)$ be a state such that $\langle\chi|\phi_a\rangle\neq 0$. $P_\chi=\frac{|\phi_a\rangle\langle\chi|}{\langle\chi|\phi_a\rangle}$ constitutes a projector onto $\mathrm{Lin}(\phi_a)$. A geometric phase generated by $\frac{\langle\chi|\dot{\phi}_a\rangle}{\langle\chi|\phi_a\rangle}$ is associated with this projection, and we have $\langle\phi_a^*|\dot{\phi}_a\rangle-\frac{\langle\chi|\dot{\phi}_a\rangle}{\langle\chi|\phi_a\rangle}=\langle\phi_a^*|\dot{P}_\chi|\phi_a\rangle=-\frac{\langle\chi|\dot{P}_s|\phi_a\rangle}{\langle\chi|\phi_a\rangle}$ which is small at the adiabatic limit. If the orthogonal geometric phase has a physical interpretation (it preserves the norm evolution), an interpretation of the non-natural geometric phases is not directly evident (note that the non-natural geometric phase are forbidden in the selfadjoint case, because of the requirement of the norm preservation). Nevertheless, we can say that the geometric phase can be transformed by an arbitrary new kind of gauge change of the form $\langle\phi_a^*|\dot{P}_\chi|\phi_a\rangle$ (the usual gauge change being of the form $g^{-1}\dot{g}$ where g is a non-zero complex number). This remark is particularly interesting since a previous work [19] has shown that for some geometric phases associated with a resonance, the geometric structure describing the geometric phase is not a principal bundle (where the only gauge changes are $g^{-1}\dot{g}$) but a gerbe (which includes also

another kind of gauge changes). In ref. [19], the other kind of gauge change is $\langle\phi_a^*|\Omega^{-1}\dot{\Omega}|\phi_a\rangle$ where Ω is a wave operator. We remark that in the present case we have $\Omega = P_s(P_\chi P_s P_\chi)^{-1} = P_\chi$ and $\Omega^{-1} = P_\chi P_s = P_s$ (where $(P_\chi P_s P_\chi)^{-1}$ is the inverse in the space spaned by P_χ and Ω^{-1} is the weak left inverse of Ω i.e. $\Omega^{-1}\Omega = P_\chi$). We have then $\langle\phi_a^*|\Omega^{-1}\dot{\Omega}|\phi_a\rangle = \langle\phi_a^*|\dot{P}_\chi|\phi_a\rangle$.

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