



**HAL**  
open science

# Powers of ideals and convergence of Green functions with colliding poles

Alexander Rashkovskii, Pascal J. Thomas

► **To cite this version:**

Alexander Rashkovskii, Pascal J. Thomas. Powers of ideals and convergence of Green functions with colliding poles. *International Mathematics Research Notices*, 2014, 2014, pp.1253-1272. 10.1093/imrn/rns251 . hal-00728273

**HAL Id: hal-00728273**

**<https://hal.science/hal-00728273>**

Submitted on 15 May 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# POWERS OF IDEALS AND CONVERGENCE OF GREEN FUNCTIONS WITH COLLIDING POLES

ALEXANDER RASHKOVSKII AND PASCAL J. THOMAS

ABSTRACT. Let  $\Omega$  be a bounded hyperconvex domain in  $\mathbb{C}^n$ , and  $\mathcal{I}_\varepsilon$  be a family of ideals of holomorphic functions on  $\Omega$  vanishing at  $N$  distinct points all tending to  $a \in \Omega$  as  $\varepsilon \rightarrow 0$ . As is known, convergence of the ideals  $\mathcal{I}_\varepsilon$  to an ideal  $\mathcal{I}$  does not guarantee the convergence of the pluricomplex Green functions  $G_{\mathcal{I}_\varepsilon}$  to  $G_{\mathcal{I}}$ ; moreover, the existence of the limit of the Green functions was unclear. Assuming that all the powers  $\mathcal{I}_\varepsilon^p$  converge to some ideals  $\mathcal{I}_{(p)}$ , we prove that the functions  $G_{\mathcal{I}_\varepsilon}$  converge, locally uniformly away from  $a$ , to a function which is essentially the upper envelope of the scaled Green functions  $p^{-1}G_{\mathcal{I}_{(p)}}$ ,  $p \in \mathbb{N}$ . As examples, we consider ideals generated by hyperplane sections of a holomorphic curve in  $\mathbb{C}^{n+1}$  near a singular point. In particular, our result explains the asymptotics for 3-point models from [10].

## 1. INTRODUCTION

Pluricomplex Green functions are fundamental solutions of the (complex) Monge-Ampère operator with zero boundary values [9]. Since the operator is non-linear, superposition does not work and it makes sense to consider pluricomplex Green functions with multiple poles [8]. We will always do this in the framework of a bounded hyperconvex domain  $\Omega \subset \mathbb{C}^n$ .

It is well known that a multipole Green function depends continuously on its poles, provided they do not collide. The convergence problem for the Green functions with simple logarithmic poles at finitely many points as the poles tend to the origin was considered in [10]. Let  $S_\varepsilon := \{a_1(\varepsilon), \dots, a_N(\varepsilon)\} \subset \Omega$  be our pole set. Assume that  $\lim_{\varepsilon \rightarrow 0} a_j(\varepsilon) = a \in \Omega$  for all  $j$ . The key to the analysis in [10] was to consider  $\mathcal{I}_\varepsilon := \{f \in \mathcal{O}(\Omega) : f(a_j(\varepsilon)) = 0, 1 \leq j \leq N\}$ , the radical ideal associated to  $S_\varepsilon$ , and its limit, taken in an appropriate sense (see Section 2 for a precise definition). There it was proved that, if the respective limits existed, with the limits of Green functions taken in  $L_{\text{loc}}^1(\Omega)$ , then

$$\lim_{\varepsilon \rightarrow 0} G_{\mathcal{I}_\varepsilon} \geq G_{\lim \mathcal{I}_\varepsilon}.$$

Furthermore, we have  $\lim_{\varepsilon \rightarrow 0} G_{\mathcal{I}_\varepsilon} = G_{\lim \mathcal{I}_\varepsilon}$  if and only if the ideal  $\lim \mathcal{I}_\varepsilon$  is a complete intersection, that is to say, admits exactly  $n$  generators (and in that case convergence

---

*Key words and phrases.* pluricomplex Green function, ideals of holomorphic functions, Hilbert-Samuel multiplicity, flat families.

2010 *Mathematics Subject Classification* 32U35, 32A27.

Part of this work was carried out during the stay of the second-named author in the University of Stavanger in October 2009 and during the stay of the first-named author in the Université Paul Sabatier in October 2011.

of the Green functions always does occur, and is uniform on compacta of  $\Omega \setminus \{a\}$ ). A detailed study of the example of families of three points showed that in cases where the limit ideal is not a complete intersection, the limits of the Green functions may exist, even though they are far from being the Green function of the limit ideal.

The complete intersection condition is equivalent to the fact that the codimension (or “length”) of the ideal equals its Hilbert-Samuel multiplicity, which is defined asymptotically from the lengths of the powers of the ideal. The main idea of the present paper is to take all powers of  $\mathcal{I}_\varepsilon$  before passing to the limit as  $\varepsilon \rightarrow 0$ , and use the infinite family of those limits to determine the limit of the Green functions.

Our main result is the following:

**Theorem 1.1.** *Let  $\{\mathcal{I}_\varepsilon\}_{\varepsilon \in A}$  be a family of ideals of holomorphic functions vanishing at distinct points  $a_1(\varepsilon), \dots, a_N(\varepsilon)$  of a bounded hyperconvex domain  $\Omega \subset \mathbb{C}^n$ , where  $A$  is a set in the complex plane,  $0 \in \overline{A} \setminus A$ . Assume that all  $a_j \rightarrow a \in \Omega$  and  $\mathcal{I}_\varepsilon^p \rightarrow \mathcal{I}_{(p)}$  for all  $p \in \mathbb{N}$  as  $\varepsilon \rightarrow 0$  along  $A$ . Then the limit of the Green functions  $G_{\mathcal{I}_\varepsilon}$  exists and equals essentially the upper envelope of the scaled Green functions of the limit ideals:*

$$\lim_{\varepsilon \rightarrow 0} G_{\mathcal{I}_\varepsilon}(z) = \limsup_{y \rightarrow z} \sup_{p \in \mathbb{N}} p^{-1} G_{\mathcal{I}_{(p)}}(y).$$

Observe that we have used the subset  $A$  in order to allow convergence along any partial set of parameters (subsequences for instance).

An ingredient in our proof which should be of interest in itself is Theorem 3.1, which proves that for families of pluricomplex Green functions with a fixed number of poles, all reasonable notions of convergence coincide (the weak convergence in local integrability implies the strong one, uniform on compacta).

We also provide some examples to show how the limits of Green functions for three points investigated in [10] and even [5] can be obtained much faster, and some of those results can be generalized to the higher-dimensional case. We also obtain results in the case of sections of holomorphic curves.

**Acknowledgements.** The second named author would like to thank Norman Levenberg and Nguyen Quang Dieu for stimulating discussions on the topic of this article.

## 2. SOME BASIC NOTIONS

Let  $\mathcal{O}(\Omega)$  be the space of all holomorphic functions on a bounded hyperconvex domain  $\Omega \subset \mathbb{C}^n$ . Given an ideal  $\mathcal{I} \subset \mathcal{O}(\Omega)$ ,  $V(\mathcal{I})$  denotes its zero variety:

$$V(\mathcal{I}) = \{z \in \Omega : f(z) = 0, \forall f \in \mathcal{I}\}.$$

In what follows, we always assume  $V(\mathcal{I})$  to be a finite set. Recall that the *length* of such an ideal is  $\ell(\mathcal{I}) = \dim \mathcal{O}/\mathcal{I} < \infty$ , and the *Hilbert-Samuel multiplicity* is

$$e(\mathcal{I}) = \lim_{k \rightarrow \infty} \frac{n!}{k^n} \ell(\mathcal{I}^k) < \infty.$$

It is known that  $e(\mathcal{I}) \geq \ell(\mathcal{I})$ , and the two values are equal if and only if  $\mathcal{I}$  is a complete intersection ideal, which means that it has precisely  $n$  generators [20, Ch. VIII, Theorem 23].

Let  $0 \in \bar{A} \setminus A \subset \mathbb{C}$  and let  $(\mathcal{I}_\varepsilon)_{\varepsilon \in A}$  be a family of finite length ideals in  $\mathcal{O}(\Omega)$ . Convergence of such ideals we will understand in the topology of the Douady space [4]. In particular, it implies

$$\ell(\lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \ell(\mathcal{I}_\varepsilon). \quad (1)$$

As was shown in [10], this convergence is equivalent to the one given in the definition below.

**Definition 2.1.** [10]

- (i)  $\liminf_{A \ni \varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$  is the ideal consisting of all  $f \in \mathcal{O}(\Omega)$  such that  $f_\varepsilon \rightarrow f$  locally uniformly on  $\Omega$ , as  $\varepsilon \rightarrow 0$ , where  $f_\varepsilon \in \mathcal{I}_\varepsilon$ .
- (ii)  $\limsup_{A \ni \varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$  is the ideal of  $\mathcal{O}(\Omega)$  generated by all functions  $f$  such that  $f_j \rightarrow f$  locally uniformly, as  $j \rightarrow \infty$ , for some sequence  $\varepsilon_j \rightarrow 0$  in  $A$  and  $f_j \in \mathcal{I}_{\varepsilon_j}$ .
- (iii) If the two limits are equal, we say that the family  $\mathcal{I}_\varepsilon$  converges and write  $\lim_{A \ni \varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$  for the common value of the upper and lower limits.

If it is clear from the context which set  $A$  we are referring to, then we just drop it from the subscript.

The main object of the note is the pluricomplex Green function for an ideal  $\mathcal{I}$  of  $\mathcal{O}(\Omega)$ , defined as follows.

**Definition 2.2.** [16] For each  $a \in \Omega$ , let  $(\psi_{a,i})_i$  be a (local) system of generators of  $\mathcal{I}$ . Then the Green function of  $\mathcal{I}$  is  $G_{\mathcal{I}}(z) = \sup\{u(z) : u \in \mathcal{F}_{\mathcal{I}}\}$ , where

$$\mathcal{F}_{\mathcal{I}} = \sup\{u(z) : u \in PSH_-(\Omega), u(z) \leq \max_i \log |\psi_{a,i}| + O(1), \forall a \in \Omega\}. \quad (2)$$

It was proved in [16] that the function  $G_{\mathcal{I}}$  belongs to the class  $\mathcal{F}_{\mathcal{I}}$  and, moreover,

$$G_{\mathcal{I}}(z) = \max_i \log |\psi_{a,i}| + O(1). \quad (3)$$

In addition, it satisfies  $(dd^c G_{\mathcal{I}})^n = 0$  on  $\Omega \setminus V(\mathcal{I})$  and, if  $V(\mathcal{I}) \Subset \Omega$ , it equals 0 on the boundary of  $\Omega$ . Furthermore, it is the only plurisubharmonic function with these properties. This implies, in particular, that for every power  $\mathcal{I}^p$  of  $\mathcal{I}$ ,

$$G_{\mathcal{I}^p} = p G_{\mathcal{I}}. \quad (4)$$

Note also that, in our setting of finite length ideals on bounded pseudoconvex domain, one can always choose global generators  $\psi_i \in \mathcal{O}(\Omega)$  and, when  $V(\mathcal{I}) = \{a\}$ , relation (3) implies that the residual Monge-Ampère mass of  $G_{\mathcal{I}}$  at  $a$  equals that of the function  $\frac{1}{2} \log \sum |\psi_i|^2$ , so by [3, Lemma 2.1],

$$(dd^c G_{\mathcal{I}})^n = e(\mathcal{I}) \delta_a. \quad (5)$$

A closely related (though technical) object is the *greenification* of a plurisubharmonic function near its singularity point.

**Definition 2.3.** [14] *Given a function  $\varphi \in PSH^-(\Omega)$ , its greenification at a point  $a \in \Omega$  is the upper regularization  $g_\varphi$  of the function  $\sup\{u \in PSH^-(\Omega) : u \leq \varphi + O(1) \text{ near } a\}$ .*

The function  $g_\varphi$  is maximal on  $\Omega \setminus \{\varphi = -\infty\}$ . If  $\varphi$  is locally bounded near the boundary of  $\Omega$ , then  $g_\varphi = 0$  on  $\partial\Omega$ . Obviously,  $\varphi \leq g_\varphi$ . Furthermore,  $\varphi = g_\varphi + O(1)$  near  $a$  if  $\varphi$  is locally bounded and maximal on a punctured neighborhood of  $a$ , and in this case it coincides with the *Green function for the singularity  $\varphi$*  introduced in [18], see also [19]. Note that the relation  $(dd^c g_\varphi)^n(a) = (dd^c \varphi)^n(a)$  remains true without the maximality assumption on  $\varphi$ .

### 3. MODES OF CONVERGENCE

**Theorem 3.1.** *Let  $S_\varepsilon = \{a_1^\varepsilon, \dots, a_N^\varepsilon\}$  with  $\lim_{\varepsilon \rightarrow 0} a_j^\varepsilon = a$ ,  $1 \leq j \leq N$ . Suppose that  $\lim_{\varepsilon \rightarrow 0} G_{S_\varepsilon} = g$  in  $L^1_{loc}(\Omega \setminus \{a\})$ . Then the convergence takes place uniformly on compacta of  $\Omega \setminus \{a\}$ , and  $(dd^c g)^n = N\delta_a$ ; in particular,  $g$  is maximal plurisubharmonic on  $\Omega \setminus \{a\}$ .*

Note that we may assume  $a = 0$  without loss of generality. We will use the well-known rough estimates of a multipole Green function:

$$\min_{a \in S} G_a \geq G_S \geq \sum_{a \in S} G_a.$$

The proof of the Theorem rests on the proof of the analogous fact in the special case of the ball, Proposition 3.4 below, and will be given at the end of this section.

In what follows,  $\|\cdot\|$  stands for the usual Euclidean norm,  $a \cdot b = \sum a_j b_j$ ,  $B(a, r) = \{z \in \mathbb{C}^n : \|z - a\| < r\}$ , and  $\mathbb{B}^n = B(0, 1)$ .

**Lemma 3.2.** *Let  $K \subset \mathbb{B}^n \setminus \{0\}$  be a compact set. Then for any  $\eta > 0$ , there exists  $\delta > 0$  depending only on  $\eta$ ,  $N$  and  $K$  such that if  $z_1, z_2 \in K$ ,  $\|z_1 - z_2\| \leq \delta$  and  $S \subset B(0, \delta)$ , then  $|G_S(z_1) - G_S(z_2)| < \eta$ .*

*Proof.* Since the roles of  $z_1$  and  $z_2$  are symmetric, it will be enough to show that

$$G_S(z_1) \geq G_S(z_2) - \eta$$

whenever  $z_2$  is close enough to  $z_1$ .

**Lemma 3.3.** *For any  $\eta_1 > 0$ , one can find  $\delta_1 > 0$  depending only on  $\eta_1$ ,  $\Omega$ ,  $N$  and  $K$  such that if  $z_1, z_2 \in K$ ,  $\|z_1 - z_2\| \leq \delta_1$  and  $S \subset B(0, \delta_1)$ , then there exists a holomorphic map  $\Phi$  defined on  $\mathbb{B}^n$  such that  $\Phi|_S = id|_S$ ,  $\Phi(z_1) = z_2$  and  $\Phi(\mathbb{B}^n) \subset B(0, 1 + \eta_1)$ .*

*Proof.* Let

$$P(z) := \prod_{a \in S} (z - a) \cdot \frac{\bar{z}_1}{\|z_1\|}, \Phi(z) := z + \frac{P(z)}{P(z_1)}(z_2 - z_1).$$

Take any  $\delta_1 \leq \frac{1}{2} \min_K \|z\|$ . Then  $|P(z_1)| \geq 2^{-N} \|z_1\|^N \geq 2^{-N} \min_K \|z\|^N$ . On the other hand,  $|P(z)| \leq 2^N$  for  $z \in \mathbb{B}^n$ . So the conclusion will hold whenever  $\|z_1 - z_2\| \leq 2^{-2N} \|z_1\|^N \eta_1 =: \delta_1$ .  $\square$

We need to construct a function plurisubharmonic and negative on  $B(0, 1 + \eta_1)$  that is a competitor for  $G_S$ . First recall that

$$G_S(z) \geq \sum_{a \in S} G_a(z) = \sum_{a \in S} \log \|\phi_a(z)\|,$$

where  $\phi_a$  is an automorphism of the unit ball exchanging  $a$  and 0. From an explicit formula for  $\phi_a$  [17], we know that

$$1 - \|\phi_a(z)\|^2 = \frac{(1 - \|a\|^2)(1 - \|z\|^2)}{|1 - z \cdot \bar{a}|^2}.$$

Since  $|1 - z \cdot \bar{a}|^2 \geq (1 - \|a\|)^2$ , we deduce, if  $\|a\| \leq \delta \leq \frac{1}{3}$ ,

$$\|\phi_a(z)\|^2 \geq 1 - \frac{1 + \|a\|}{1 - \|a\|}(1 - \|z\|^2) \geq 1 - 2(1 - \|z\|^2).$$

We may assume  $1 - \|z\|^2 \leq \frac{1}{4}$  so we have  $\|\phi_a(z)\|^2 \geq \frac{1}{2}$  and

$$\log \|\phi_a(z)\| \geq -(1 - \|\phi_a(z)\|^2) \geq -2(1 - \|z\|^2) \geq -4(1 - \|z\|) \geq 4 \log \|z\|.$$

Assume that  $\eta_1 < \frac{1}{4}$ . Let

$$\begin{aligned} v(z) &= G_S(z), \text{ for } \|z\| \leq e^{-2\eta_1}, \\ &= \max(G_S(z), \eta_1 + \log \|z\| + 4N \log \|z\|), \text{ for } e^{-2\eta_1} \leq \|z\| \leq 1, \\ &= \eta_1 + \log \|z\| + 4N \log \|z\|, \text{ for } 1 \leq \|z\| < 1 + \eta_1. \end{aligned}$$

Since  $G_S(z) > \eta_1 + \log \|z\| + 4N \log \|z\|$  for  $\|z\| = e^{-2\eta_1}$  and  $G_S(z) = 0 < \eta_1$  for  $\|z\| = 1$ , we have  $v \in PSH(B(0, 1 + \eta_1))$ . Let  $v_1 := v - (1 + 4N) \log(1 + \eta_1) - \eta_1 \in PSH_-(B(0, 1 + \eta_1))$ .

Clearly  $v_1 \circ \Phi \in PSH_-(\mathbb{B}^n)$ . Since  $v_1 = G_S + O(1)$  in a fixed ball containing  $S$ , and  $\Phi$  fixes  $S$ ,  $v_1 \circ \Phi \leq G_S$ . We apply this at the point  $z_1$ :

$$v_1(z_2) = v_1 \circ \Phi(z_1) \leq G_S(z_1).$$

Now we choose  $\eta_1$  small enough so that  $K \subset B(0, e^{-2\eta_1})$ , and so that

$$v_1(z_2) = G_S(z_2) - (1 + 2N) \log(1 + \eta_1) - \eta_1 > G_S(z_2) - \eta$$

for  $\|a\| \leq \delta(\eta, \eta_1, N) < \delta_1$ . □

**Proposition 3.4.** *Let  $S_\varepsilon = \{a_1^\varepsilon, \dots, a_N^\varepsilon\} \subset \mathbb{B}^n$  with  $\lim_{\varepsilon \rightarrow 0} a_j^\varepsilon = 0$ ,  $1 \leq j \leq N$ . Suppose that  $\lim_{\varepsilon \rightarrow 0} G_{S_\varepsilon} = g$  in  $L_{loc}^1(\mathbb{B}^n \setminus \{0\})$ . Then the convergence takes place uniformly on compacta of  $\mathbb{B}^n \setminus \{0\}$ ; in particular,  $g$  is maximal plurisubharmonic on  $\mathbb{B}^n \setminus \{0\}$ .*

*Proof.* Since the topology of uniform convergence on compacta is metrizable, it will be enough to show that any subsequence  $\{G_{S_{\varepsilon_j}}\}$  admits a convergent subsequence. First consider a fixed compact set  $K \subset \mathbb{B}^n \setminus \{0\}$ . Then  $\{G_{S_{\varepsilon_j}}\}$  converges in  $L^1(K)$ , so there exists a subsequence, which we denote by  $\{G_j\}$ , which converges almost everywhere on  $K$ . Note that the standard rough estimates on Green functions show that all  $G_{S_\varepsilon}$  are bounded by common bounds on  $K$ , and therefore so is  $g$  (where it is defined).

We want to show that the subsequence  $\{G_j\}$  satisfies the uniform Cauchy criterion. Let  $\delta > 0$ . Let  $\eta_0 := \min(\min_K \|z\|, 1 - \max_K \|z\|)$ . By Lemma 3.2 applied to  $\{z : \text{dist}(z, K) \leq \eta_0/2\}$ , there exists  $\eta = \eta(\delta) \leq \eta_0/2$  and  $J_1 = J_1(\delta)$  such that for any  $j \geq J_1$ , the oscillation of  $G_j$  on any ball of radius  $\eta$  is at most  $\delta/4$ .

We cover the compact set  $K$  by balls  $B(c_k, \eta)$ ,  $1 \leq k \leq m(\delta)$ . The almost everywhere convergence implies that for each  $k$ , there exists  $c'_k \in B(c_k, \eta)$  such that  $\lim_{j \rightarrow \infty} G_j(c'_k) = g(c'_k)$ . Since there is only a finite number of  $c'_k$ , there exists  $J_2 \geq J_1$  such that for any  $j \geq J_2$  and any  $k \leq m(\delta)$ ,  $|G_j(c'_k) - g(c'_k)| \leq \delta/4$ . The rest is routine: for any  $j, l \geq J_2$ , for any  $z \in K$ , we choose  $k$  such that  $z \in B(c_k, \eta)$  and we have

$$\begin{aligned} |G_l(z) - G_j(z)| &\leq |G_l(z) - G_l(c'_k)| + |G_l(c'_k) - g(c'_k)| \\ &\quad + |g(c'_k) - G_j(c'_k)| + |G_j(c'_k) - G_j(z)| \leq \delta. \end{aligned}$$

To get the uniform convergence on any compact set, we repeat this argument over an exhaustion sequence  $\{K_m\}$  of compacta, and perform a diagonal extraction.  $\square$

*Proof of Theorem 3.1.* To prove the uniform convergence, by [10, Lemma 4.5], it is enough to show that there exists  $g$  and  $\delta_0 > 0$  such that for any  $\delta < \delta_0$ ,  $|G_{S_\varepsilon}^\Omega - g| \leq C$  for  $\|z\| = \delta$  and  $|\varepsilon| < \varepsilon(\delta)$ . Of course we take  $\delta_0$  small enough so that  $B(0, \delta_0) \Subset \Omega$ . Then it is easy to see that on  $B(0, \delta_0)$ ,  $|G_{S_\varepsilon}^\Omega - G_{S_\varepsilon}^{B(0, \delta_0)}| \leq C(\delta_0, \Omega)$ . By Proposition 3.4,  $\{G_{S_\varepsilon}^{B(0, \delta_0)}\}$  converges uniformly on compacta of  $B(0, \delta_0) \setminus \{0\}$ , in particular on the sphere of radius  $\delta$ , and we have the desired property.

To prove the fact about Monge-Ampère measure, first notice that uniform convergence on compacta (or even pointwise convergence) clearly implies, using the maximum principle, that  $g$  is maximal plurisubharmonic on  $\Omega \setminus \{a\}$ , and so  $(dd^c)^n g = C\delta_a$ , with  $C \leq N$  by [10, Proposition 1.13]. To finish the proof, we only need to show that  $(dd^c)^n g(\Omega) = N$ .

For any non-positive function  $u$  on  $\Omega$  and  $m \in \mathbb{N}^*$ , let  $T_m(u) := \max(-m, u)$ . Then, the rough estimates imply that  $\{g \leq -m\} \Subset \Omega$  and  $\{G_{S_\varepsilon}^\Omega \leq -m\} \Subset \Omega$ , so

$$(dd^c)^n T_m(g)(\Omega) = (dd^c)^n g(\Omega), \quad (dd^c)^n T_m(G_{S_\varepsilon}^\Omega)(\Omega) = (dd^c)^n G_{S_\varepsilon}^\Omega(\Omega) = N.$$

Again by the rough estimates and the fact that all the  $a_j(\varepsilon)$  tend to  $a$ , for any fixed  $m$  there exists  $r_m > 0, \varepsilon_m > 0$  such that  $B(a, r_m) \subset \{g \leq -m-1\}$  and  $B(a, r_m) \subset \{G_{S_\varepsilon}^\Omega \leq -m-1\}$ , for  $|\varepsilon| \leq \varepsilon_m$ . For any compactum  $K \subset \Omega$ ,  $K \setminus B(a, r_m)$  is a compactum of  $\Omega \setminus \{a\}$ , so  $G_{S_\varepsilon}^\Omega$  converges uniformly to  $g$  on it, therefore  $T_m(G_{S_\varepsilon}^\Omega)$  converges uniformly to  $T_m(g)$  on  $K$ . This implies that

$$(dd^c)^n T_m(g)(\Omega) = \lim_{\varepsilon \rightarrow 0} (dd^c)^n T_m(G_{S_\varepsilon}^\Omega)(\Omega) = N.$$

Note that convergence of the Monge-Ampère measures can also be proved by noticing that uniform convergence on compacta of  $\Omega \setminus \{a\}$  implies convergence in capacity, and that type of convergence guarantees convergence of the corresponding Monge-Ampère measures, see [2] or [13].  $\square$

## 4. PROOF OF THE MAIN RESULT

Let  $0 \in \Omega$  and let  $\mathcal{I}_\varepsilon$  be a family of finite length ideals in  $\mathcal{O}(\Omega)$  such that  $V(\mathcal{I}_\varepsilon) \rightarrow \{0\}$  as  $\varepsilon \rightarrow 0$ . We assume that all the powers  $\mathcal{I}_\varepsilon^p$  converge (in the sense of Definition 2.1) to some limits  $\mathcal{I}_{(p)}$ ,  $p = 1, 2, \dots$ . Surely,  $V(\mathcal{I}_{(p)}) = \{0\}$ .

The crucial point is the following simple observation.

**Proposition 4.1.** *For any  $p, q \in \mathbb{N}$ ,*

$$\mathcal{I}_{(p)} \cdot \mathcal{I}_{(q)} \subset \mathcal{I}_{(p+q)}. \quad (6)$$

*Proof.* If  $f \in \mathcal{I}_{(p)}$  and  $g \in \mathcal{I}_{(q)}$ , then they are limits of certain functions  $f_\varepsilon \in \mathcal{I}_\varepsilon^p$  and  $g_\varepsilon \in \mathcal{I}_\varepsilon^q$ , respectively. Note that  $f_\varepsilon g_\varepsilon \in \mathcal{I}_\varepsilon^{p+q}$ . Therefore,

$$fg = \lim_{\varepsilon \rightarrow 0} f_\varepsilon g_\varepsilon \in \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{p+q} = \mathcal{I}_{(p+q)}.$$

□

Note that the inclusion in (6) can be strict.

**Example 4.2.** *3-point model in  $\mathbb{C}^2$ .*

Let  $a_1(\varepsilon) = (\varepsilon, 0)$ ,  $a_2(\varepsilon) = (0, \varepsilon)$ ,  $a_3(\varepsilon) = (0, 0)$ . The ideals

$$\mathcal{I}_\varepsilon = \langle z_1 z_2, z_1(z_1 - \varepsilon), z_2(z_2 - \varepsilon) \rangle$$

converge to the ideal  $\mathcal{I} = \langle z_1^2, z_1 z_2, z_2^2 \rangle$ , while the squares  $\mathcal{I}_\varepsilon^2$  converge to the ideal  $\mathcal{I}_{(2)}$  generated by  $\mathcal{I}^2$  and the function  $z_1 z_2(z_1 + z_2)$ . Indeed, the ideal

$$\mathcal{I}_\varepsilon^2 = \langle z_1^2 z_2^2, z_1^2(z_1 - \varepsilon)^2, z_2^2(z_2 - \varepsilon)^2, z_1^2 z_2(z_1 - \varepsilon), z_1 z_2^2(z_2 - \varepsilon), z_1 z_2(z_1 - \varepsilon)(z_2 - \varepsilon) \rangle$$

contains the function  $z_1 z_2(z_1 + z_2) - \varepsilon z_1 z_2 = \varepsilon^{-1}[z_1^2 z_2^2 - z_1 z_2(z_1 - \varepsilon)(z_2 - \varepsilon)]$ . □

Relation (6) means precisely that  $\{\mathcal{I}_{(p)}\}$  is a *graded family of ideals*. In particular, this implies that we have control over the Hilbert-Samuel multiplicities of these limit ideals:

**Proposition 4.3.** *There exists the limit*

$$e(\mathcal{I}_\bullet) := \lim_{p \rightarrow \infty} p^{-n} e(\mathcal{I}_{(p)}) = \inf_{p \in \mathbb{N}} p^{-n} e(\mathcal{I}_{(p)}) = \limsup_{p \rightarrow \infty} n! p^{-n} \ell(\mathcal{I}_{(p)}).$$

*Proof.* This follows directly from Theorem 1.7 of [11] valid for any graded family of zero dimensional ideals. □

The value  $e(\mathcal{I}_\bullet)$  is called the *volume* of the graded family  $\mathcal{I}_\bullet$ .

Proposition 4.1 implies

$$G_{\mathcal{I}_{(p)} \cdot \mathcal{I}_{(q)}} \leq G_{\mathcal{I}_{(p+q)}}, \quad (7)$$

and we are going to deduce from this a convergence result for  $G_{\mathcal{I}_{(p)}}$  – more precisely, for the *scaled Green functions*

$$\widehat{G}_{\mathcal{I}_{(p)}} = p^{-1} G_{\mathcal{I}_{(p)}}. \quad (8)$$



**Proposition 4.4.** *There exists the limit*

$$V(z) = \lim_{p \rightarrow \infty} \widehat{G}_{\mathcal{I}_{(p)}}(z) = \sup_{p \in \mathbb{N}} \widehat{G}_{\mathcal{I}_{(p)}}(z) \quad (9)$$

whose upper regularization  $G_{\mathcal{I}_\bullet}(z) = \limsup_{x \rightarrow z} V(x)$  is a plurisubharmonic function satisfying

$$(dd^c G_{\mathcal{I}_\bullet})^n = e(\mathcal{I}_\bullet) \delta_0. \quad (10)$$

Furthermore,  $\widehat{G}_{\mathcal{I}_{(p)}} \rightarrow G_{\mathcal{I}_\bullet}$  in  $L^p(\Omega)$  for all  $p \in [1, n]$ .

*Proof.* By (3), since the product of ideals is generated by pairwise products of their generators, we have

$$G_{\mathcal{I}_{(p)} \cdot \mathcal{I}_{(q)}} = G_{\mathcal{I}_{(p)}} + G_{\mathcal{I}_{(q)}} + O(1),$$

so the function  $G_{\mathcal{I}_{(p)}} + G_{\mathcal{I}_{(q)}}$  belongs to the class  $\mathcal{F}_{\mathcal{I}_{(p)} \cdot \mathcal{I}_{(q)}}$  defined in (2), and inequality (7) gives us

$$G_{\mathcal{I}_{(p)}} + G_{\mathcal{I}_{(q)}} \leq G_{\mathcal{I}_{(p+q)}}. \quad (11)$$

Relations (9) follow now from (11) by standard arguments; see, for example, [11, Lemma 1.4] applied to  $\alpha_p = -G_{\mathcal{I}_{(p)}}(z) \geq 0$ .

Now we turn to proving (10). By the Choquet lemma, one can find a sequence  $\widehat{G}_{\mathcal{I}_{(p_j)}}$  increasing almost everywhere to the function  $G_{\mathcal{I}_\bullet}$ ; actually, one can just choose  $\widehat{G}_{\mathcal{I}_{(p!)}}$ , cf. [15]. Indeed, relation (6) implies, in particular,  $\mathcal{I}_{(p)}^k \subset \mathcal{I}_{(kp)}$ , and using (4) we get

$$k G_{\mathcal{I}_{(p)}} = G_{\mathcal{I}_{(p)}^k} \leq G_{\mathcal{I}_{(kp)}}$$

for any  $k, p \in \mathbb{N}$ . Therefore,  $\widehat{G}_{\mathcal{I}_{(p)}} \leq \widehat{G}_{\mathcal{I}_{(kp)}}$ , so  $\widehat{G}_{\mathcal{I}_{(p!)}} \geq \widehat{G}_{\mathcal{I}_{(q)}}$  for all  $q \leq p$ .

By the monotone convergence theorem for the complex Monge-Ampère operator,

$$(dd^c \widehat{G}_{\mathcal{I}_{(p!)}})^n \rightarrow (dd^c G_{\mathcal{I}_\bullet})^n.$$

Since  $(dd^c \widehat{G}_{\mathcal{I}_{(p!)}})^n = (p!)^{-n} e(\mathcal{I}_{(p!)})$ , relation (10) follows from the first equality in Proposition 4.3.

Finally, since  $\widehat{G}_{\mathcal{I}_{(p)}} \leq G_{\mathcal{I}_\bullet}$  for any  $p$ , the last assertion follows from Lemma 4.5 below and Hölder's inequality.  $\square$

**Lemma 4.5.** *Let  $u, v \in \text{PSH}^-(\Omega)$  be maximal on  $\Omega \setminus \{x\}$ , equal to 0 on  $\partial\Omega$ , and  $u \leq v$  in  $\Omega$ . Then*

$$\int_{\Omega} (v - u)^n (dd^c w)^n \leq n! \int_{\Omega} w [(dd^c u)^n - (dd^c v)^n]$$

for any  $w \in \text{PSH}(\Omega)$ ,  $0 \leq w \leq 1$ .

*Proof.* This is a particular case of [12, Prop. 3.4].  $\square$

Next, we get a lower bound for the limit of Green functions. In order to state it without assumption on uniform convergence of the Green functions  $G_{\mathcal{I}_\varepsilon}$ , we will use here the notion of greenification of a plurisubharmonic function, see Definition 2.3.

**Proposition 4.6.** *Let  $\varphi$  be the largest plurisubharmonic minorant of the function  $\liminf_{\varepsilon \rightarrow 0} G_{\mathcal{I}_\varepsilon}$ . Then its greenification  $g_\varphi$  satisfies  $g_\varphi \geq G_{\mathcal{I}_\bullet}$ . Consequently, if  $G_{\mathcal{I}_\varepsilon}$  converge to  $\varphi$  locally uniformly outside 0, then  $\varphi \geq G_{\mathcal{I}_\bullet}$ .*

*Proof.* By [10, Proposition 1.5],

$$\varphi \geq G_{\liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon} + O(1)$$

for any family of zero-dimensional ideals  $\mathcal{J}_\varepsilon$  such that  $V(\liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon) = \{0\}$ . Applying this to  $\mathcal{J}_\varepsilon = \mathcal{I}_\varepsilon^p$  and taking into account (4) and (8), we get the inequalities

$$\varphi \geq \widehat{G}_{\mathcal{I}_{(p)}} + C_p$$

with some constants  $C_p$ . By passing to the greenifications, we deduce  $g_\varphi \geq \widehat{G}_{\mathcal{I}_{(p)}}$  and then, in view of Proposition 4.4,  $g_\varphi \geq G_{\mathcal{I}_\bullet}$ . If, in addition, the convergence of  $G_{\mathcal{I}_\varepsilon}$  to  $\varphi$  is locally uniform outside 0, then  $\varphi = g_\varphi$ , which completes the proof.  $\square$

From now on, we assume that the ideals  $\mathcal{I}_\varepsilon$  are intersections of maximal ideals. In this case we can compute the volume  $e(\mathcal{I}_\bullet)$  of the graded family  $\mathcal{I}_\bullet$ .

**Proposition 4.7.** *Let  $\mathcal{I}_\varepsilon$  be radical ideals with  $V(\mathcal{I}_\varepsilon)$  consisting of  $N$  different points  $a_1(\varepsilon), \dots, a_N(\varepsilon)$  for all  $\varepsilon \neq 0$  sufficiently small. Then  $e(\mathcal{I}_\bullet) = N$ .*

*Proof.* Since the length is stable under limit transitions,  $\ell(\mathcal{I}_{(p)}) = \ell(\mathcal{I}_\varepsilon^p)$ , so Proposition 4.3 gives us

$$e(\mathcal{I}_\bullet) = \limsup_{p \rightarrow \infty} n! p^{-n} \ell(\mathcal{I}_\varepsilon^p).$$

Each ideal  $\mathcal{I}_\varepsilon^p$  consists of all functions  $f \in \mathcal{O}(\Omega)$  satisfying

$$\frac{\partial^{|\beta|} f}{\partial z^\beta}(a_j(\varepsilon)) = 0, \quad |\beta| < p, \quad 1 \leq j \leq N,$$

so  $\ell(\mathcal{I}_\varepsilon^p) = \binom{p+n-1}{n} N$  and

$$e(\mathcal{I}_\bullet) = \lim_{p \rightarrow \infty} n! p^{-n} \binom{p+n-1}{n} N = N.$$

$\square$

Now we are ready to prove our main result. It will rest on the following domination principle.

**Lemma 4.8.** [14, Lemma 6.3] *Let  $u_1$  and  $u_2$  be two plurisubharmonic solutions of the Dirichlet problem  $(dd^c u)^n = \delta_0$ ,  $u|_{\partial\Omega} = 0$ . If  $u_1 \geq u_2$  on  $\Omega$ , then  $u_1 \equiv u_2$ .*

*Remark.* This can actually be deduced from the more advanced Lemma 4.5. A more general version of the domination principle can be found in [1] as well.

**Proposition 4.9.** *Let  $\{\mathcal{I}_\varepsilon\}_{\varepsilon \in A}$  be a family of ideals of holomorphic functions vanishing at distinct points  $a_1(\varepsilon), \dots, a_N(\varepsilon)$  of a bounded hyperconvex domain  $\Omega \subset \mathbb{C}^n$ , where  $A$  is a set in the complex plane,  $0 \in \overline{A} \setminus A$ . Assume that all  $a_j \rightarrow a \in \Omega$  and  $\mathcal{I}_\varepsilon^p \rightarrow \mathcal{I}_{(p)}$  for all  $p \in \mathbb{N}$  as  $\varepsilon \rightarrow 0$  along  $A$ . If the limit of the Green functions  $G_{\mathcal{I}_\varepsilon}$  for the ideals  $\mathcal{I}_\varepsilon$  exists, uniformly on compact subsets of  $\Omega \setminus \{a\}$ , then*

$$\lim_{\varepsilon \rightarrow 0} G_{\mathcal{I}_\varepsilon}(z) = \limsup_{y \rightarrow z} \sup_{p \in \mathbb{N}} p^{-1} G_{\mathcal{I}_{(p)}}(y).$$

*Proof.* Denote the limit of the Green functions  $G_{\mathcal{I}_\varepsilon}$  by  $\varphi$ . The uniform convergence implies  $(dd^c \varphi)^n = \lim (dd^c G_{\mathcal{I}_\varepsilon})^n = N\delta_0$ . By Proposition 4.6, we have  $g \geq G_{\mathcal{I}_\bullet}$ , and (10) implies  $g = G_{\mathcal{I}_\bullet}$  by Lemma 4.8.  $\square$

*Proof of Theorem 1.1.* First notice that the set  $\{G_{S_\varepsilon}, \varepsilon \in A\}$  is sequentially weakly compact in the dual of the space of bounded continuous functions on  $\Omega$ , since  $0 \geq G_{S_\varepsilon} \geq \sum_{j=1}^N G_{a_j(\varepsilon)}$ , and each of those functions has a uniformly bounded  $L^1$  norm when  $a_j(\varepsilon)$  is close to 0. From this standard arguments of measure theory can be used to show that a subsequence converging in  $L^1_{\text{loc}}$  can always be extracted. Or we can simply use [7, Theorem 3.2.12, p. 149], noticing that the case of a subsequence converging to  $-\infty$  is excluded by the estimate above.

Now suppose that  $\mathcal{I}_\varepsilon^p \rightarrow \mathcal{I}_{(p)}$  for all  $p \in \mathbb{N}$ , and suppose to get a contradiction that  $G_{S_\varepsilon}$  is not converging uniformly on compacta to  $G_{\mathcal{I}_\bullet}$ . Then we can get  $\{\varepsilon_j\} \subset A$ ,  $\varepsilon_j \rightarrow 0$ , such that  $\sup_K |G_{\varepsilon_j} - G_{\mathcal{I}_\bullet}| \geq \delta > 0$  for  $j$  large enough and for some compactum  $K \subset \Omega \setminus \{0\}$ . Then there is a subsequence of  $\{\varepsilon_j\}$ , which we denote again by  $\{\varepsilon_j\}$ , such that  $G_{\varepsilon_j}$  converges in  $L^1_{\text{loc}}$ , and by Theorem 3.1 uniformly on compacta. By Proposition 4.9, it must converge to  $G_{\mathcal{I}_\bullet}$ : a contradiction.  $\square$

Finally, we mention the following finiteness result.

**Theorem 4.10.** *If, in addition to the conditions of Theorem 1.1 (or Proposition 4.9), the Hilbert-Samuel multiplicity of some limit ideal  $\mathcal{I}_{(p)}$  equals  $p^n N$ , then the limit of the Green functions equals  $\widehat{G}_{\mathcal{I}_{(p)}}$ .*

*Proof.* The condition on multiplicity of  $\mathcal{I}_{(p)}$  means that  $(dd^c \widehat{G}_{\mathcal{I}_{(p)}})^n(0) = N$ . Since  $\widehat{G}_{\mathcal{I}_{(p)}} \leq G_{\mathcal{I}_\bullet}$  and the latter one has the same Monge-Ampère mass at 0, Lemma 4.8 gives us  $\widehat{G}_{\mathcal{I}_{(p)}} = G_{\mathcal{I}_\bullet}$ .  $\square$

In particular, this works when the limit ideal  $G_{\mathcal{I}_{(1)}}$  is a complete intersection, because in this case  $e(\mathcal{I}_{(1)}) = N$ . More advanced situations will be considered in the next section.

## 5. EXAMPLES AND QUESTIONS

There are several natural examples where our theorem works. In what follows,  $\mathfrak{m}_a \subset \mathcal{O}(\Omega)$  denotes the maximal ideal composed by the functions vanishing at the point  $a \in \Omega$ .

**Example 5.1.** *Two points in  $\mathbb{C}^n$ .*

Let  $a_1, a_2$  be continuous mappings of the unit disk  $\mathbb{D}$  into the unit polydisk  $\mathbb{D}^n$  such that  $a_1(\varepsilon) \neq a_2(\varepsilon)$  for all  $\varepsilon \in \mathbb{D} \setminus \{0\}$ , and  $a_i(0) = 0$ . Set  $\mathcal{I}_\varepsilon = \mathfrak{m}_{a_1(\varepsilon)} \cap \mathfrak{m}_{a_2(\varepsilon)}$ .

The family  $\{\mathcal{I}_\varepsilon\}_{\varepsilon \neq 0}$  need not have a limit as  $\varepsilon \rightarrow 0$ . By compactness, there is a sequence  $\varepsilon_k \rightarrow 0$  such that  $[a_1(\varepsilon_k) - a_2(\varepsilon_k)] \rightarrow \nu \in \mathbb{P}^{n-1}\mathbb{C}$ , where  $[z]$  denotes the class of  $z$  in  $\mathbb{P}^{n-1}\mathbb{C}$ , for  $z \in \mathbb{D}^n \setminus \{0\}$ . As is easy to see, the sequence  $\mathcal{I}_{\varepsilon_k}$  has a limit,  $\mathcal{I}_{(1)}$ , whose multiplicity equals 2 because it is a complete intersection.

According to Section 6.1 of [10], the corresponding Green functions  $G_{\mathcal{I}_{\varepsilon_k}}$  converge to a function whose Monge-Ampère mass at 0 equals 2. Therefore, by Theorems 1.1 and 4.10, the limit function coincides with  $G_{\mathcal{I}_{(1)}}$ .  $\square$

**Example 5.2.** *Complete intersection case of the 4-point ideals in  $\mathbb{C}^2$ .*

Consider the ideals  $\mathcal{I}_\varepsilon = \mathfrak{m}_0 \cap \mathfrak{m}_{(\varepsilon,0)} \cap \mathfrak{m}_{(0,\varepsilon)} \cap \mathfrak{m}_{(\varepsilon,\varepsilon)}$  in a bounded hyperconvex domain  $\Omega \subset \mathbb{C}^2$  containing the origin. They converge to  $\mathcal{I}_{(1)} = \langle z_1^2, z_2^2 \rangle$  whose Hilbert–Samuel multiplicity equals 4, so the limit of the Green functions is the Green function of  $\mathcal{I}_{(1)}$ . The existence of the limit is however quite a simple fact in this case, see [10]. (Note that a much stronger result was proved there. Namely, when the limit ideal  $\mathcal{I}_{(1)}$  is a complete intersection, then the limit of the Green functions exists and coincides with the Green function of  $\mathcal{I}_{(1)}$ .)  $\square$

**Example 5.3.** *3-point problem in  $\mathbb{C}^2$ .*

This more complicated problem is also treated in [10], where the following two cases were considered: the generic one modeled by the ideals  $\mathcal{I}_\varepsilon = \mathfrak{m}_0 \cap \mathfrak{m}_{(\varepsilon,0)} \cap \mathfrak{m}_{(0,\varepsilon)}$ , and the degenerate one modeled by  $\mathcal{I}_\varepsilon = \mathfrak{m}_0 \cap \mathfrak{m}_{(\rho(\varepsilon),0)} \cap \mathfrak{m}_{(0,\varepsilon)}$  with  $\rho(\varepsilon)/\varepsilon \rightarrow 0$ . Both families converge to  $\mathfrak{m}_0^2$ , however the limits of the corresponding Green functions were shown to be different.

In the first case, as was mentioned in Example 4.2, the squares  $\mathcal{I}_\varepsilon^2$  converge to the ideal  $\mathcal{I}_{(2)}$  generated by  $\mathfrak{m}_0^4$  and the function  $z_1 z_2 (z_1 + z_2)$ . Note that

$$\widehat{G}_{\mathcal{I}_{(2)}} = \frac{1}{2} G_{\mathcal{I}_{(2)}} = \max\{2 \log |z_1|, 2 \log |z_2|, \frac{1}{2} \log |z_1 z_2 (z_1 + z_2)|\} + O(1).$$

Comparing it with results from [10], we see that this is precisely the asymptotic of the limit function  $\lim_{\varepsilon \rightarrow 0} G_{\mathcal{I}_\varepsilon}$  and, therefore, in this case one has  $G_{\mathcal{I}_\bullet} = \widehat{G}_{\mathcal{I}_{(2)}}$ .

Another way to check this, according to Theorem 4.10, is to show that the Monge-Ampère mass of  $\widehat{G}_{\mathcal{I}_{(2)}}$  equals 3. This can be done easily, since the mass of  $G_{\mathcal{I}_{(2)}}$  can be computed as the Hilbert–Samuel multiplicity of the ideal  $\mathcal{I}_{(2)}$ , and the latter equals the multiplicity of generic mappings  $(f_1, f_2)$  for  $f_1, f_2 \in \mathcal{I}_{(2)}$ , which is 12.

In the second case, the limit ideal  $\mathcal{I}_{(2)}$  is monomial and contains, besides  $\mathfrak{m}_0^4$ , the function  $z_1^2 z_2$ . The Hilbert–Samuel multiplicity of  $\mathcal{I}_{(2)}$  easily computes to be 12, so the Monge-Ampère mass of  $\widehat{G}_{\mathcal{I}_{(2)}}$  equals 3 again. And the limit of the Green functions is indeed, up to a bounded term,  $\frac{1}{2} \log \max(|z_1^4|, |z_1^2 z_2|, |z_2^4|)$ .

Note that in [10] the convergence of the Green functions were established by using a sophisticated machinery of constructing special analytic disks, while now we get the results almost for free.

The case not treated in [10] was that when all the poles tend to 0 along the same asymptotic directions, or to be more precise, when one point is equal to  $(0, 0)$  (which is no loss of generality) and the other two verify  $\lim_{\varepsilon} a_2^{\varepsilon}/\|a_2^{\varepsilon}\| = a_3^{\varepsilon}/\|a_3^{\varepsilon}\| = v$ . This question is dealt with in [5], but there is no answer there for the limit of the Green functions when the limit ideal is not a complete intersection.

Recall that the situation may be reduced to  $a_1^{\varepsilon} = (0, 0)$ ,  $a_2^{\varepsilon} = (\varepsilon, 0)$ ,  $a_3^{\varepsilon} = (\rho, \delta\rho)$ , with  $\rho = \rho(\varepsilon)$ ,  $\delta = \delta(\varepsilon)$ ,  $|\varepsilon|^2 \geq |\varepsilon - \rho|^2 + |\rho|^2|\delta|^2 \geq |\rho|^2 + |\rho|^2|\delta|^2$ , so in particular  $|\varepsilon - \rho| \geq \frac{1}{2}|\varepsilon|$ .

Denote  $\alpha = \frac{\delta}{\rho - \varepsilon}$ . It is shown in [5, Theorem 1.5] that  $\lim_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon}$  is a complete intersection ideal when  $\lim_{\varepsilon \rightarrow 0} \alpha^{-1} \neq 0$ , and that  $\lim_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon} = \mathfrak{m}_0^2$  when  $\lim_{\varepsilon \rightarrow 0} \alpha^{-1} = 0$ . We want to settle the latter case.

From [5], there are polynomials in  $\mathcal{I}_{\varepsilon}$

$$\begin{aligned} Q_1^{\varepsilon}(z) &= z_1^2 - \varepsilon z_1 - \alpha^{-1} \delta z_2 \rightarrow z_1^2; \\ Q_2^{\varepsilon}(z) &= z_2(z_1 - \rho) \rightarrow z_1 z_2; \\ Q_3^{\varepsilon}(z) &= z_2(z_2 - \delta\rho) \rightarrow z_2^2, \end{aligned}$$

so  $\mathcal{I}_{\varepsilon}^2$  contains

$$Q_1^{\varepsilon}(z)Q_3^{\varepsilon}(z) - (Q_2^{\varepsilon}(z))^2 = -\alpha^{-1} (z_2^3 + \varepsilon\alpha z_1^2 z_2^2 + (1 - \varepsilon)\delta\rho\alpha z_1^2 z_2 - 2\rho\alpha z_1 z_2^2 + \varepsilon\rho\alpha z_2^2),$$

therefore  $z_2^3 \in \mathcal{I}_{(2)}$ . Clearly,  $\mathfrak{m}_0^4 \subset \mathcal{I}_{(2)}$ . So  $\widehat{G}_{\mathcal{I}_{(2)}} \geq \max(2 \log |z_1|, \frac{3}{2} \log |z_2|) + O(1)$ . Using the fact that the Monge-Ampère mass of this lower bound is 3, or that the multiplicity of the mapping  $f(z) = (z_1^4, z_2^3)$  is 12, we see that in fact, by Theorem 4.10,  $\lim_{\varepsilon} G_{S_{\varepsilon}} = \widehat{G}_{\mathcal{I}_{(2)}} = \max(2 \log |z_1|, \frac{3}{2} \log |z_2|) + O(1)$  (if  $\Omega = \mathbb{D}^2$ , we don't even need the  $O(1)$  term).  $\square$

**Example 5.4.** *Generic  $n + 1$  points in  $\mathbb{C}^n$ .*

Consider ideals  $\mathcal{I}_{\varepsilon}$  with  $S_{\varepsilon} = V(\mathcal{I}_{\varepsilon})$  consisting of the origin and the points  $\varepsilon e_k$  for basis vectors  $e_k$ ,  $1 \leq k \leq n$ . They are generated by the functions  $z_i z_j$  with  $i < j$  and  $z_i(z_i - \varepsilon)$  and they converge, as expected, to  $\mathfrak{m}_0^2$ .

**Proposition 5.5.** *The Green functions  $G_{S_{\varepsilon}}$  converge, uniformly on compacta of  $\Omega \setminus \{0\}$ , to  $w_n(z) + O(1)$ , where*

$$w_n(z) := \max \left\{ \frac{1}{\#A} \log \left| \left( \sum_{j \in A} z_j \right) \prod_{j \in A} z_j \right|, A \subset \{1, \dots, n\} \right\}.$$

Notice that for  $n = 2$ , this yields the previous result in the generic case of Example 5.3.

*Proof.* We will proceed by induction on the dimension. For  $n = 1$ ,  $w_1(z) = 2 \log |z|$  and the result is immediate by direct calculation (we can add up the Green functions for each pole in this case).

In general, notice that if  $L$  is an affine subspace of  $\mathbb{C}^n$ , and  $z = (z', z'')$  where  $L = \{(z', 0) : z' \in \mathbb{C}^k\}$  in suitable coordinates,  $G_S^\Omega(z) \geq G_{S \cap L}^{\Omega \cap L}(z')$ , simply because the latter function is a competitor in the supremum that defines the former.

We assume the result is true in  $\mathbb{C}^d$ , for any  $d \leq n - 1$ . Therefore, to prove that  $\liminf_\varepsilon G_{S_\varepsilon} \geq w_n$ , it will be enough to prove

$$\liminf_\varepsilon G_{S_\varepsilon} \geq \log \left| \left( \sum_{j=1}^n z_j \right) \prod_{j=1}^n z_j \right|, \quad (12)$$

and to use the induction hypothesis on all the subspaces  $\text{Span}\{e_j, j \in A\}$  for  $\#A \leq n - 1$ .

We will look at  $S_\varepsilon$  as a simplex in a space with one more dimension, formally let

$$\varphi_\varepsilon : (z_1, \dots, z_n) \mapsto (\varepsilon - \sum_j z_j, z_1, \dots, z_n) \in \mathbb{C}^{n+1},$$

where the coordinates in  $\mathbb{C}^{n+1}$  are  $(z_0, z_1, \dots, z_n)$ . Then  $S_\varepsilon = \varphi_\varepsilon^{-1}(\cup_{j=0}^n \mathbb{C}e_j)$ .

The ideal associated to  $\cup_{j=0}^n \mathbb{C}e_j$  is  $\mathcal{I} := \langle z_i z_j, 0 \leq i < j \leq n \rangle$ .

We are using multi-index notation,  $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$ .

**Lemma 5.6.**

$$\mathcal{I}^k = \langle z^\gamma : |\gamma| = 2k, \gamma_j \leq k, 0 \leq j \leq n \rangle.$$

*Proof.* The equality for  $k = 1$  is the very definition of  $\mathcal{I}$ . By taking products of generators, it is immediate that  $\mathcal{I}^k \subset \langle z^\gamma : |\gamma| = 2k, \gamma_j \leq k, 0 \leq j \leq n \rangle$ . Conversely, suppose the reverse inclusion is verified up to  $k - 1$ . For any  $\gamma$  involved in the right hand side, the definition implies that there are at least two distinct indices  $i, j$  such that  $\gamma_i, \gamma_j \geq 1$ , and no more than two indices such that  $\gamma_i, \gamma_j \geq k$ . Pick  $i, j$  so that  $\gamma_i, \gamma_j$  are maximal among the exponents. So  $z^\gamma = z_i z_j z^{\gamma'}$ , where  $|\gamma'| = 2k - 2$ , and  $\gamma'_l \leq k - 1$  for any  $l$ : the induction hypothesis implies that  $z^{\gamma'} \in \mathcal{I} \cdot \mathcal{I}^{k-1}$ .  $\square$

Now we want to prove that  $(\sum_{j=1}^n z_j) \prod_{j=1}^n z_j \in \mathcal{I}_{(n)}$ , which will prove (12) by Theorem 1.1 and the fact that  $\widehat{G}_{\mathcal{I}_{(n)}} \leq G_{\mathcal{I}}$ . By Lemma 5.6,  $z_0 \cdots z_n (z_0 + \cdots + z_n)^{n-1} \in \mathcal{I}^n$ , so using the pull-back,  $(\varepsilon - (z_1 + \cdots + z_n)) z_1 \cdots z_n \varepsilon^{n-1} \in \mathcal{I}_\varepsilon^n$ , so dividing by  $\varepsilon^{n-1}$  and taking the limit, we have the desired fact. So we have proved the “ $\geq$ ” part of our Proposition. To prove the reverse inequality, we must show that  $w_n$  has small enough Monge-Ampère mass.

**Lemma 5.7.**  $(dd^c)^n w_n(0) \leq n + 1$ .

*Proof.* Let  $t_k = t/k$ , where  $t$  is a common multiple of  $1, \dots, n$ . Consider the mapping  $f$  with components

$$f_k = \sum_{|A|=k} \left( \sum_{j \in A} z_j \right)^{t_k} \prod_{j \in A} z_j^{t_k}, \quad A \subset \{1, \dots, n\}, \quad k = 1, \dots, n.$$

First we check that it has an isolated zero at 0. If  $f_n = 0$ , then either  $z_k = 0$  for some  $k$ , or  $\sum_j z_j = 0$ . In both cases, the equation  $f_{n-1} = 0$  gives us then either  $z_l = 0$  for

some  $l \neq k$ , or  $\sum_{j \neq k} z_j = 0$ . Continuing this, we arrive at the last step to the unique solution  $z = 0$ .

Now, since the components of  $f$  are homogeneous polynomials of degrees  $(k+1)t_k$ , Bezout's theorem gives us the multiplicity of  $f$  equal to

$$\prod_k (k+1)t_k = \frac{(n+1)!t^n}{n!} = (n+1)t^n,$$

which shows that the Hilbert–Samuel multiplicity of the ideal generated by the functions  $(\sum_{j \in A} z_j)^{t_k} \prod_{j \in A} z_j^{t_k}$ ,  $A \subset \{1, \dots, n\}$ , is at most  $(n+1)t^n$ , so the Monge-Ampère mass of  $w_n$  at 0 is at most  $n+1$ .  $\square$

Now let  $u$  be any limit point in  $L_{\text{loc}}^1$  of  $\{G_{S_\varepsilon}\}$ . By Theorem 3.1, the convergence is in fact uniform on compacta of  $\Omega \setminus \{0\}$ , and  $(dd^c u)^n = (n+1)\delta_0$ . We also have  $u \geq w_n + O(1)$ , so passing to the greenifications, we have  $u = g_u \geq g_{w_n}$  and  $(dd^c)^n g_{w_n} = c\delta_0$ ,  $c \leq n+1$ , so that, by Lemma 4.8, the two functions must be in fact equal. Now by the usual reasoning (there is only one possible limit point), the limit of  $G_{S_\varepsilon}$  must be  $g_{w_n} = w_n + O(1)$ .  $\square$

**Example 5.8.** *Hyperplane sections of holomorphic curves.*

More generally, let us have a holomorphic curve (one-dimensional analytic variety)  $\Gamma$  such that  $0 \in \mathbb{C}^{n+1}$  is its singular point. By Thie's theorem, there exists a neighborhood  $U$  of 0 such that for a choice of coordinates  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ , the restriction of  $\Gamma$  to  $U$  lies in the cone  $\{|z| \leq C|w|\}$ ,  $C > 0$ . Let for  $\varepsilon \in \mathbb{C} \setminus \{0\}$ , sufficiently small,  $\mathcal{I}_\varepsilon$  be the ideal in  $O(\mathbb{D}^n)$  determined by the points  $a_k = a_k(\varepsilon)$  such that  $(a_k, \varepsilon) \in \Gamma$ . By Propositions III.4.7 and III.4.8 of [6], the collection  $\{\mathcal{I}_\varepsilon^p\}_\varepsilon$  with any  $p \in \mathbb{N}$  has a unique continuation to a flat family, so there exists a limit of  $\mathcal{I}_\varepsilon^p$  as  $\varepsilon \rightarrow 0$ . Therefore, the limit of the corresponding Green functions  $G_{\mathcal{I}_\varepsilon}$  exists and equals the function  $G_{\mathcal{I}_\bullet}$ .  $\square$

In the end, we would like to mention a few open questions.

1. In Example 5.4, we have found that  $G_{\mathcal{I}_\bullet} = \widehat{G}_{\mathcal{I}_{t(n)}}$  with  $t(n)$  equal to the least common multiple of  $1, \dots, n$  (in particular,  $t(n) \leq n!$ ). *What is (the asymptotic of) the best possible index  $p(n)$  such that  $G_{\mathcal{I}_\bullet} = \widehat{G}_{\mathcal{I}_{p(n)}}$ ?*
2. *Is it always true that  $G_{\mathcal{I}_\bullet} = \widehat{G}_{\mathcal{I}_{(p)}}$  for some  $p \in \mathbb{N}$ , at least in the setting of Example 5.8?*
3. *What can be said in the case of non-radical ideals  $\mathcal{I}_\varepsilon$  whose varieties tend to a single point?*

## REFERENCES

- [1] P. ÅHAG, U. CEGRELL, R. CZYZ, PHẠM HOÀNG HIỆP, *Monge-Ampère measures on pluripolar sets*, J. Math. Pures Appl. (9) **92** (2009), no. 6, 613–627.
- [2] U. CEGRELL, *Convergence in capacity*, Canad. Math. Bull. **55** (2012), no. 2, 242–248.

- [3] J.-P. DEMAILLY, *Estimates on Monge-Ampère operators derived from a local algebra inequality*, in: Complex Analysis and Digital Geometry. Proceedings from the Kiselmanfest, 2006, Uppsala University, 2009, 131–143.
- [4] A. DOUADY, *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier (Grenoble) **16** (1966), fasc. 1, 1–95.
- [5] DUONG QUANG HAI, P. J. THOMAS, *Limit Of Three-Point Green Functions : The Degenerate Case*, arXiv:1205.5899.
- [6] R. HARTSHORNE, Algebraic Geometry. Springer, 1997.
- [7] L. HÖRMANDER, Notions of Convexity. Birkhäuser, Boston, 1994, Progress in Mathematics no. 127.
- [8] P. LELONG, *Fonction de Green pluricomplexe et lemmes de Schwarz dans les espaces de Banach*, J. Math. Pures Appl. **68** (1989), 319–347.
- [9] L. LEMPERT, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981), no. 4, 427–474.
- [10] J. MAGNÚSSON, A. RASHKOVSKII, R. SIGURDSSON, AND P. THOMAS, *Limits of multipole pluricomplex Green functions*, Int. J. Math. **23** (2012), no. 6; available at <http://arxiv.org/abs/1103.2296>
- [11] M. MUSTATA, *On multiplicities of graded sequences of ideals*, J. Algebra **256** (2002), 229–249.
- [12] NGUYEN VAN KHUE AND PHAM HOÀNG HIỆP, *A comparison principle for the complex Monge-Ampère operator in Cegrell's classes and applications*, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5539–5554.
- [13] PHAM HOÀNG HIỆP, *Convergence in capacity and application*, Math. Scand. **107** (2010), no. 1, 90–102.
- [14] A. RASHKOVSKII, *Relative types and extremal problems for plurisubharmonic functions*, Int. Math. Res. Not. **2006** (2006), Article ID 76283, 26 p.
- [15] A. RASHKOVSKII, *Asymptotically analytic and other plurisubharmonic singularities*, <http://arxiv.org/abs/0910.2137>
- [16] A. RASHKOVSKII AND R. SIGURDSSON, *Green functions with singularities along complex spaces*, Int. J. Math. **16** (2005), no. 4, 333–355.
- [17] W. RUDIN, Function theory in the unit ball of  $\mathbb{C}^n$ . Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2008.
- [18] V.P. ZAHARIUTA, *Spaces of analytic functions and maximal plurisubharmonic functions*. D.Sci. Dissertation, Rostov-on-Don, 1984.
- [19] V.P. ZAHARIUTA, *Spaces of analytic functions and Complex Potential Theory*, Linear Topological Spaces and Complex Analysis **1** (1994), 74–146.
- [20] O. ZARISKI, P. SAMUEL, *Commutative algebra. Vol. II*, Reprint of the 1960 edition. Graduate Texts in Mathematics, Vol. 29. Springer-Verlag, New York-Heidelberg, 1975.

ALEXANDER RASHKOVSKII, FACULTY OF SCIENCE AND TECHNOLOGY, UNIVERSITY OF STAVANGER, N-4036 STAVANGER, NORWAY

*E-mail address:* alexander.rashkovskii@uis.no

PASCAL J. THOMAS, UNIVERSITÉ DE TOULOUSE, UPS, INSA, UT1, UTM, INSTITUT DE MATHÉMATIQUES DE TOULOUSE, F-31062 TOULOUSE, FRANCE

*E-mail address:* pascal.thomas@math.univ-toulouse.fr