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Robustness of stochastic discrete-time switched linear systems with application to control with shared resources

L. Greco, A. Chaillet and E. Panteley

Abstract—Motivated by control applications relying on shared resources (such as computation time or bandwidth), we analyze the stability and robustness of discrete-time switched linear systems driven by a stochastic process. The class of systems we study encompasses Markov Chain and independent and identically distributed switching processes. For these systems, we recall existing definitions of stability and robustness, by relying on the input-to-state (ISS) property. We show that, for the class of systems under concern, δ -moment stability is equivalent to its ISS counterpart and that they both imply almost sure ISS. Several sufficient conditions are provided to guarantee these properties. All the concepts are illustrated by the anytime control of a mechanical system.

I. INTRODUCTION

The increasing demand of control functionalities in modern embedded systems (e.g. in the automotive domain [1]) calls for an increasingly heavy utilization of hardware resources. On the other hand, the need for keeping under control both hardware costs and system complexity clearly asks for an intense resource sharing. The same trend can be recognized in networked control systems (NCS), where economical concerns and ease of maintenance are forcing the adoption of a unique Ethernet-like network for the whole industrial plant, progressively eliminating specific subnets or fieldbuses [2]. The price to pay for resource sharing is a reduced predictability of the timing behavior: an application receives a different availability of resources and suffer time-varying delays depending on the “interference” suffered from other applications. This aspect becomes particularly apparent when the shared resource is the computation time provided by an embedded platform to many concurrent tasks. Traditional hard real-time approaches, based on worst case execution time (WCET) estimates, fall short of catching the large variability in computation time required by each task. Moreover, they suffer of an intrinsic conservativeness leading to poor performance, hardware underexploitation, and cost inefficiencies. A current trend in embedded system design is to relax hard schedulability constraints and introduce “softer” models of computation. Probabilistic modeling of real-time (RT) systems is by now a widely accepted approach to avoid overconservatism of deterministic (WCET-based) models, to which an ample and growing literature [3], [4], [5] is devoted. Within stochastic models of RT systems, the use of Markov chains (MC) (cf. e.g. [6], [7], [8]) is one

of the most promising avenues to accurately compute the response time distribution and deadline miss probabilities of different tasks. This direction is also followed by a series of papers [9], [7], [10] of Lemmon and co-workers, where they consider performance of NCSs and embedded systems in a Firm RT systems framework and introduce a stochastic model to describe the task dropout process based on MCs.

All the previous models belong to the class of Stochastic Jump Systems (SJS), namely switching systems driven by a stochastic process. Commonly adopted stochastic processes are MCs and independent and identically distributed (i.i.d.) processes. Recently (cf. e.g. [11], [12], [13], [14]) the need for a more general model has also emerged. As customary in control domain, finding conditions for ensuring the stability is the first concern for SJSs too. There exists a wide gamut of stability definitions for stochastic systems (see for instance [15], [16], [17], [18]), whose comparison is usually a problematic task. Since our interest is mainly devoted to stochastic jump linear systems (SJLS) we will refer to the widely adopted stability definitions in [19], [20], [21], [22], [23], [24]. However, these definitions do not explicitly take into account a prominent problem in control: ensuring robustness of the stability property with respect to perturbations or model inaccuracies. Traditional extensions of the deterministic input-to-state stability (ISS) approach [25] to stochastic systems mainly focus on continuous-time systems with a stochastic input (see for instance [26], [27], [28]). Recently an important first step towards stochastic ISS for continuous-time SJS has appeared in [29], even if the general framework there considered prevents from obtaining tight results.

The goal of the present paper is twofold. First, we provide a formal description of a model of SJLS encompassing as special cases switching systems driven by MCs and i.i.d. processes. We briefly present some concrete cases, where such model turns out essential to catch the complex and varying behavior of shared resource availability. Second, we make a connection between deterministic ISS results and the stochastic stability definitions in [19], [20], [21], [22], [23], [24]. In particular we provide stochastic ISS definitions and easily testable conditions for the aforementioned more general model of discrete-time SJLSs. These definitions and conditions enjoy the nice property to coincide with the classical definitions and conditions in [19], [20], [21], [22], [23], [24], when they are specialized to SJLSs driven by MCs or i.i.d. processes and the disturbance is considered to be zero. We finally show the effectiveness of our robust stability conditions on an interesting SJLS taken from anytime control

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literature.

II. BASIC DEFINITIONS AND MOTIVATIONS

A. System studied in this paper

In this note we focus on the following stochastic jump linear system (SJLS)

$$x(k+1) = A_{\varphi(k)}x(k) + B(x, k)d(k), \quad (1)$$

where $k \in \mathbb{Z}_{\geq 0}$, $x(0) = x_0 \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n \times n}$, $i \in J_\varphi \triangleq \{1, \dots, N\}$, $N \in \mathbb{Z}_{>0}$ and $d: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^p$ is a deterministic, locally bounded function. We assume that $B(x, k) \in \mathbb{R}^{p \times n}$ is a time-varying vector field for which¹ there exists $\bar{b} > 0$ such that

$$\|B(x, k)\| \leq \bar{b} \quad \forall k \in \mathbb{Z}_{\geq 0}, \quad \forall x \in \mathbb{R}^n. \quad (2)$$

The stochastic process $\{\varphi(\cdot)\}$, governing the index switching, takes values in the set J_φ and is defined on the probability space $(\Omega_\varphi, \mathcal{F}_\varphi, P_\varphi)$ where Ω_φ is the space of elementary events, \mathcal{F}_φ is the associated σ -algebra and P_φ is the probability measure. In order to fully describe the process $\{\varphi(\cdot)\}$, let us introduce a discrete-time finite-state homogeneous Markov Chain (FSH-MC) $\{\sigma(\cdot)\}$ taking values in the set $J_\sigma \triangleq \{1, \dots, M\}$, $M \geq N$ and defined on the probability space $(\Omega_\sigma, \mathcal{F}_\sigma, P_\sigma)$. Due to the finite cardinality of the sets J_σ and J_φ , the probability distributions associated to the processes $\{\varphi(\cdot)\}$ and $\{\sigma(\cdot)\}$ can be compactly represented at each step $k \in \mathbb{Z}_{\geq 0}$ by the row vectors $\pi_\sigma(k) \in S^{M-1}$ and $\pi_\varphi(k) \in S^{N-1}$, where $S^{N-1} \triangleq \left\{s = (s_1, \dots, s_N) \in [0, 1]^N \mid \sum_{i=1}^N s_i = 1\right\}$ is the $(N-1)$ -dimensional canonical stochastic simplex (and similarly for M). The joint process $\{\sigma(\cdot), \varphi(\cdot)\}$, taking values in the set $J_\sigma \times J_\varphi$, is defined on the probability space (Ω, \mathcal{F}, P) , where $\Omega \triangleq \Omega_\sigma \times \Omega_\varphi$, \mathcal{F} is the σ -algebra generated by the measurable rectangles $A \times B$ with $A \in \mathcal{F}_\sigma$ and $B \in \mathcal{F}_\varphi$ and P is a probability measure guaranteeing $P\{A \times B\} = P_\sigma\{A\}P_\varphi\{B\}$ (see [30] page 231). Let us also denote with $E\{\cdot\}$ the associated expectation operator.

A complete description of the FSH-MC $\{\sigma(\cdot)\}$ is given in terms of its transition probability matrix $P = (p_{li})_{l,i=1,\dots,M}$, with $p_{li} \triangleq P\{\sigma(k+1) = i \mid \sigma(k) = l\}$, and its initial probability distribution $\pi_\sigma(0) = \pi_{\sigma 0}$. We consider a process $\{\varphi(\cdot)\}$ whose probability distribution $\pi_\varphi \in S^{N-1}$ is linearly related to the probability distribution $\pi_\sigma \in S^{M-1}$ of $\{\sigma(\cdot)\}$ by means of a row stochastic matrix $L = (\ell_{lj})_{\substack{l=1,\dots,M \\ j=1,\dots,N}}$, with $\ell_{lj} \triangleq P\{\varphi(k) = j \mid \sigma(k) = l\}$. Therefore, the probability distribution of the joint process $\{\sigma(\cdot), \varphi(\cdot)\}$ evolves according to the following dynamics

$$\pi_\sigma(k+1) = \pi_\sigma(k)P \quad (3)$$

$$\pi_\varphi(k) = \pi_\sigma(k)L. \quad (4)$$

The equations (3)-(4) can be regarded as the description of a discrete-time LTI system without control, where π_σ is the state vector and π_φ is the output. It is worth noting that the present stochastic model encompasses as special

cases both FSH-MCs (if $J_\varphi = J_\sigma$ and L is chosen as the identity matrix) and independent and identically distributed (i.i.d.) processes. Moreover, if the FSH-MC $\{\sigma(\cdot)\}$ is also irreducible and aperiodic (FSHIA-MC), then there exists a unique invariant probability distribution (i.p.d.) $\bar{\pi}_\sigma$ such that $\lim_{k \rightarrow \infty} \pi_\sigma(k) = \bar{\pi}_\sigma$ for any $\pi_\sigma(0)$. Namely, $\{\sigma(\cdot)\}$ is an ergodic process with a unique ergodic class (see [30, Section 24] and [31, Section 8.4]). In this case the process $\{\varphi(\cdot)\}$ inherits the ergodic property of $\{\sigma(\cdot)\}$ and a unique i.p.d. $\bar{\pi}_\varphi \triangleq \bar{\pi}_\sigma L$ exists for that process.

B. Motivating examples

The stochastic model (3)-(4) describes a wide range of different systems. It is especially relevant in modeling a limited resource shared among competing users. The MC $\{\sigma(\cdot)\}$ can represent the quantized amount of resource, in the finite set J_σ , available at each instant $k \in \mathbb{Z}_{\geq 0}$, while the process $\{\varphi(\cdot)\}$ represent the index of the user using the resource in that instant. With such an interpretation $\pi_\varphi(k)$ is the probability of each user, in the finite set J_φ , to exclusively use the resource at time k and the (i, j) -th entry of L represents the probability of the j -th user to need an amount of resource equal to the i -th value in the set J_σ .

Example 1: In [11] a system fitting the previous description is presented. There the shared resource is the computation time offered by a multitasking embedded unit for the execution of a periodic control task. In order to cope with the fluctuations in the available computation time, the control task is designed according to the anytime paradigm. Namely, it is made up of a finite set of subroutines Γ_j , $j \in J_\varphi$, guaranteeing increasing performance but requiring increasing computation time. Due to the lack of deterministic information on the computation time availability in each period, the subroutines are forced to execute sequentially, i.e. the computation of Γ_i cannot start until the computation of Γ_{i-1} has terminated. The ensuing closed-loop system is described by the SJLS (1), (3)-(4). In this setting the FSHIA-MC $\{\sigma(\cdot)\}$ describes the amount of available computation time and the process $\{\varphi(\cdot)\}$ the highest index of a schedulable controller, i.e. the subroutine whose execution time is the largest among those shorter than the available time. The execution time T^j required to compute sequentially the subroutines from Γ_1 to Γ_j is modeled as a discrete time i.i.d. process taking values in the set J_σ and the cumulative distributions of all the T^j , $j \in J_\varphi$, are used to derive the matrix L in (4). Thus the (i, j) -th entry of L represents the probability of executing all subroutines until the j -th one if an amount i of computation time is available. As a consequence if $\varphi(k) = j$ then in the k -th period all controllers Γ_i , $i \leq j$, but no controller Γ_k , $k > i$, can be executed.

The model (3)-(4) is open to a dual interpretation. Still in the case of a shared resource among different users, the MC $\{\sigma(\cdot)\}$ can represent the index of the user exploiting the resource at any $k \in \mathbb{Z}_{\geq 0}$, while the process $\{\varphi(\cdot)\}$ represents the amount of resource used at time k .

¹An interesting case is obtained for $B(x, k) = B_{\varphi(k)}$.

Example 2: In Section IV of [12] this model is used to describe the overall computation time used by a set of concurrent periodic tasks. For sake of simplicity, let us focus on a single task only. Such a task is assumed to have a finite number of working modes. In each period k the task can be in one of M possible modes defined by the set J_σ . The change in time of the modes is assumed to be ruled by a FSHIA-MC $\{\sigma(\cdot)\}$. According to each mode, the task can require a certain amount of computation time assuming values in the finite set J_φ and described by an i.i.d. process. In this case, the (i, j) -th entry of L represents the probability of the task to require a computation time j during the mode i . In the cited paper this model is used to provide a stochastic description of the computation time left available by higher priority tasks to the execution of a lower priority control task.

The model is flexible enough to also encompass embedded control systems affected by variable computation delays (see for instance [13], [14]) and networked systems. In the latter case the MC $\{\sigma(\cdot)\}$ can describe changes in network load conditions and $\{\varphi(\cdot)\}$ the transmission delay experienced by a measurement or a control packet.

C. Stability and Robustness Definitions

Classical stochastic stability definitions such as those in [19], [20], [21] can be adapted to fit the SJLS system (1) driven by the stochastic process $\{\sigma(\cdot), \varphi(\cdot)\}$. We denote with $x(k, d)$ the stochastic process solution $x(k, x_0, d(k), \omega)$, $\omega \in \Omega$ of the difference equation (1) with initial condition $x(0) = x_0 \in \mathbb{R}^n$ and input signal $d : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^p$.

Definition 3: Given a constant $\delta \in \mathbb{R}_{>0}$ and the set $\Phi \triangleq \{(\pi_{\sigma_0}, \pi_{\varphi_0}) \in S^{N+M-2} \mid \pi_{\varphi_0} = \pi_{\sigma_0} L, \pi_{\sigma_0} \in S^{M-1}\}$, the system (1) with $d = 0$ and driven by the stochastic process $\{\sigma(\cdot), \varphi(\cdot)\}$ with dynamics (3)-(4) is said to be globally:

- 1) δ -moment globally asymptotically stable (δ -GAS) if for any $x_0 \in \mathbb{R}^n$ and any initial distribution $(\pi_{\sigma_0}, \pi_{\varphi_0}) \in \Phi$ of $\{\sigma(\cdot), \varphi(\cdot)\}$

$$\lim_{k \rightarrow \infty} \mathbb{E} \left\{ \|x(k, 0)\|^\delta \right\} = 0 ;$$

- 2) δ -moment globally exponentially stable (δ -GES) if there exist constants $a, b \in \mathbb{R}_{>0}$ such that for any $x_0 \in \mathbb{R}^n$ and any initial distribution $(\pi_{\sigma_0}, \pi_{\varphi_0}) \in \Phi$ of $\{\sigma(\cdot), \varphi(\cdot)\}$

$$\mathbb{E} \left\{ \|x(k, 0)\|^\delta \right\} \leq a \|x_0\|^\delta e^{-bk}$$

for all $k \in \mathbb{Z}_{\geq 0}$;

- 3) almost surely (with probability one) globally stable (as-GS) if for any $x_0 \in \mathbb{R}^n$ and any initial distribution $(\pi_{\sigma_0}, \pi_{\varphi_0}) \in \Phi$ of $\{\sigma(\cdot), \varphi(\cdot)\}$

$$\mathbb{P} \left\{ \lim_{k \rightarrow \infty} \|x(k, 0)\| = 0 \right\} = 1.$$

The definitions of δ -GAS and as-GS actually concern convergence properties of the system (1), but for stochastic linear systems convergence and stability properties are equivalent (see for instance the equivalence between δ -GAS and δ -GES in [22], [23], [24]). We provide the following robust stability

definitions, which hold in general for the following nonlinear system $(x(k, d))$ is defined accordingly:

$$x(k+1) = f_{\varphi(k)}(k, x(k), d(k)), \quad (5)$$

where $f_i : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ are continuous functions satisfying $f_i(k, 0, 0) = 0$, for each $i \in J_\varphi$.

Definition 4: Given a constant $\delta \in \mathbb{R}_{>0}$ and the set $\Phi \triangleq \{(\pi_{\sigma_0}, \pi_{\varphi_0}) \in S^{N+M-2} \mid \pi_{\varphi_0} = \pi_{\sigma_0} L, \pi_{\sigma_0} \in S^{M-1}\}$, the system (5) driven by the stochastic process $\{\sigma(\cdot), \varphi(\cdot)\}$ with dynamics (3)-(4) is said to be:

- 1) δ -moment input-to-state stable (δ -ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for any $x_0 \in \mathbb{R}^n$, any initial distribution $(\pi_{\sigma_0}, \pi_{\varphi_0}) \in \Phi$ of $\{\sigma(\cdot), \varphi(\cdot)\}$ and any locally bounded $d : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^p$

$$\mathbb{E} \left\{ \|x(k, d)\|^\delta \right\} \leq \beta(\|x_0\|, k) + \gamma(\|d\|_\infty)$$

for all $k \in \mathbb{Z}_{\geq 0}$;

- 2) δ -moment exponentially input-to-state stable (δ -EISS) if there exist constants $a, b \in \mathbb{R}_{>0}$ and a function $\gamma \in \mathcal{K}_\infty$ such that for any $x_0 \in \mathbb{R}^n$, any initial distribution $(\pi_{\sigma_0}, \pi_{\varphi_0}) \in \Phi$ of $\{\sigma(\cdot), \varphi(\cdot)\}$ and any locally bounded $d : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^p$

$$\mathbb{E} \left\{ \|x(k, d)\|^\delta \right\} \leq a \|x_0\|^\delta e^{-bk} + \gamma(\|d\|_\infty)$$

for all $k \in \mathbb{Z}_{\geq 0}$;

- 3) almost surely (with probability one) input-to-state stable (as-ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for any $x_0 \in \mathbb{R}^n$, any initial distribution $(\pi_{\sigma_0}, \pi_{\varphi_0}) \in \Phi$ of $\{\sigma(\cdot), \varphi(\cdot)\}$ and any locally bounded $d : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^p$

$$\mathbb{P} \left\{ \|x(k, d)\| \leq \beta(\|x_0\|, k) + \gamma(\|d\|_\infty) \right\} = 1$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Remark 5: Due to the special structure of Φ , which imposes a linear relation between π_{φ_0} and π_{σ_0} , in the following we will slightly abuse the notation and refer to the probability measure $\mathbb{P}_{(\pi_{\sigma_0}, \pi_{\varphi_0})}\{\cdot\}$, induced by any initial distribution $(\pi_{\sigma_0}, \pi_{\varphi_0}) \in \Phi$, as $\mathbb{P}_{\pi_{\sigma_0}}\{\cdot\}$.

As pointed out in [23], the definitions 3 are stronger than the weak stability definitions in [15], as they require to be independent from the initial distribution. Definitions 4 inherit the same property. It is also worth noting that our δ -ISS definition is close to the ISS in L_1 estimate at switching instants introduced for continuous time switching systems in [29].

III. MAIN RESULTS

In this section we focus on the SJLS (1) in order to provide easily computable criteria for robust stability.

A. δ -ISS and 2-ISS

Asymptotic second moment stability (2-GAS) and exponential second moment stability (2-GES), for systems without input and driven by a Markov chain, have been shown to be equivalent and to imply the almost sure stability (as-GS) ([22], [23], [24]). Similar properties hold for δ -GAS

and δ -GES (see for instance [32]). In this section we prove similar equivalences for our system.

Theorem 6: For the system (1) driven by the stochastic process $\{\sigma(\cdot), \varphi(\cdot)\}$, whose distributions are described by the evolutions (3) and (4), δ -GAS, δ -GES, δ -ISS and δ -EISS are equivalent.

Proof: We will prove the following chain of implications: δ -GAS \Rightarrow δ -GES \Rightarrow δ -EISS \Rightarrow δ -ISS \Rightarrow δ -GAS. The latter two implications are easily verified, thus we need to prove only the first two: δ -GAS \Rightarrow δ -GES \Rightarrow δ -EISS. For any $k \in \mathbb{Z}_{\geq 0}$ and any $x_0 \in \mathbb{R}^n$ we can write

$$\begin{aligned} x(k, d) &= A_{\varphi(k-1)} \cdots A_{\varphi(0)} x_0 + B(k-1)d(k-1) \\ &\quad + A_{\varphi(k-2)} B(k-2)d(k-2) + \cdots \\ &\quad + A_{\varphi(k-2)} \cdots A_{\varphi(0)} B(0)d(0). \end{aligned} \quad (6)$$

Notice that the constant \bar{b} in (2) can be assumed greater than 1 without loss of generality. Therefore, we get

$$\begin{aligned} \|x(k, d)\|^\delta &\leq \|A_{\varphi(k-1)} \cdots A_{\varphi(0)}\|^\delta \|x_0\|^\delta \\ &\quad + \bar{b}^\delta \|d\|_\infty^\delta \left(1 + \|A_{\varphi(k-2)}\|^\delta + \cdots\right. \\ &\quad \left.+ \|A_{\varphi(k-2)} \cdots A_{\varphi(0)}\|^\delta\right). \end{aligned}$$

Hence, recalling that d is deterministic, we have

$$\begin{aligned} \mathbb{E} \left\{ \|x(k, d)\|^\delta \right\} &\leq \mathbb{E} \left\{ \|A_{\varphi(k-1)} \cdots A_{\varphi(0)}\|^\delta \right\} \|x_0\|^\delta \\ &\quad + \bar{b}^\delta \|d\|_\infty^\delta \left(1 + \mathbb{E} \left\{ \|A_{\varphi(k-2)}\|^\delta \right\} + \cdots\right. \\ &\quad \left.+ \mathbb{E} \left\{ \|A_{\varphi(k-2)} \cdots A_{\varphi(0)}\|^\delta \right\} \right). \end{aligned} \quad (7)$$

Let us write any initial distribution $\pi_{\sigma 0} \in S^{M-1}$ as $\pi_{\sigma 0} \triangleq (\pi_{\sigma 0,1}, \dots, \pi_{\sigma 0,M})$ and recall the corresponding distribution $\pi_{\varphi 0} = \pi_{\sigma 0} L$. We can then write

$$\begin{aligned} \mathbb{E}_{\pi_{\sigma 0}} \left\{ \|A_{\varphi(k-1)} \cdots A_{\varphi(0)}\|^\delta \right\} \\ &= \sum_{i_0, \dots, i_{k-1}} \sum_{j_0, \dots, j_{k-1}} \pi_{\sigma 0, i_0} p_{i_0 i_1} \cdots p_{i_{k-2} i_{k-1}} \times \\ &\quad \times \ell_{i_0 j_0} \cdots \ell_{i_{k-1} j_{k-1}} \|A_{j_{k-1}} \cdots A_{j_0}\|^\delta, \end{aligned} \quad (8)$$

with $\mathbb{E}_{\pi_{\sigma 0}} \{\cdot\}$ denoting the expectation value with respect to the probability measure $\mathbb{P}_{\pi_{\sigma 0}} \{\cdot\}$ (see Remark 5). According to Definition 3-1 and recalling (6) with $d = 0$, we have that

$$\lim_{k \rightarrow \infty} \mathbb{E}_{\pi_{\sigma 0}} \left\{ \|A_{\varphi(k-1)} \cdots A_{\varphi(0)}\|^\delta \right\} = 0.$$

We can argue by the previous limit and the compactness of S^{M-1} that, for any $\varepsilon \in (0, 1)$, there exists $Q = Q(\varepsilon) \in \mathbb{Z}_{>0}$ such that, for every initial distribution $\pi_{\sigma 0} \in S^{M-1}$,

$$\mathbb{E}_{\pi_{\sigma 0}} \left\{ \|A_{\varphi(q-1)} \cdots A_{\varphi(0)}\|^\delta \right\} \leq \varepsilon, \quad \forall q \geq Q.$$

The previous inequality is satisfied by any sequence $\varphi(0), \dots, \varphi(q-1)$ generated by any initial distribution $\pi_{\sigma 0} \in S^{M-1}$. Due to the homogeneity property of the composite process $\{\sigma(\cdot), \varphi(\cdot)\}$, there exist suitable initial distributions

such that the same sequence can be generated starting from any instant $s \in \mathbb{Z}_{>0}$. Hence, we have

$$\mathbb{E}_{\pi_{\sigma 0}} \left\{ \|A_{\varphi(q+s-1)} \cdots A_{\varphi(s)}\|^\delta \right\} \leq \varepsilon, \quad \forall s \geq 0, \quad \forall q \geq Q. \quad (9)$$

We can also define a constant $c \in \mathbb{R}_{>0}$ such that we have

$$\mathbb{E}_{\pi_{\sigma 0}} \left\{ \|A_{\varphi(h+s-1)} \cdots A_{\varphi(s)}\|^\delta \right\} \leq c, \quad \forall h, s \in \mathbb{Z}_{>0}, \quad h \in (0, Q). \quad (10)$$

Inequalities (9) and (10) are given in terms of arbitrary initial distributions, hence they hold in particular for $\pi_{\sigma 0} = p_i$, $i \in J_\sigma$ with p_i i -th row of the matrix P in (3). As in the proof of Theorem 2.3.1 in [19] and using again the homogeneity property of $\{\sigma(\cdot), \varphi(\cdot)\}$, the expectation $\mathbb{E}_{\pi_{\sigma 0}} \left\{ \|A_{\varphi(k-1)} \cdots A_{\varphi(0)}\|^\delta \right\}$ for any $k = rQ + h$, $h, r \in \mathbb{Z}_{>0}$, $h < Q$, can be sliced in r pieces of length Q and one piece of length h . Therefore, we have from (9) and (10)

$$\mathbb{E}_{\pi_{\sigma 0}} \left\{ \|A_{\varphi(k-1)} \cdots A_{\varphi(0)}\|^\delta \right\} \leq c\varepsilon^r = ae^{-bk}, \quad (11)$$

with $a \triangleq c\varepsilon^{-\frac{1}{Q}}$ and $b \triangleq -\frac{1}{Q} \log \varepsilon$. This proves the first implication: δ -GAS \Rightarrow δ -GES.

Applying similar arguments we also have

$$\begin{aligned} 1 + \mathbb{E} \left\{ \|A_{\varphi(k-2)}\|^\delta \right\} + \cdots + \mathbb{E} \left\{ \|A_{\varphi(k-2)} \cdots A_{\varphi(0)}\|^\delta \right\} \\ &\leq 1 + cQ + Q\varepsilon + Q\varepsilon^2 + \cdots + (Q+h-1)\varepsilon^r \\ &\leq 1 + c(Q+h-1) \sum_{i=0}^r \varepsilon^i \\ &\leq 1 + c(Q+h-1) \sum_{i=0}^{\infty} \varepsilon^i \\ &= 1 + \frac{c(Q+h-1)}{1-\varepsilon}. \end{aligned} \quad (12)$$

By inequalities (7), (11) and (12) we have that $\mathbb{E} \left\{ \|x(k, d)\|^\delta \right\} \leq a \|x_0\|^\delta e^{-bk} + \gamma (\|d\|_\infty)^\delta$ with the class \mathcal{K}_∞ function $\gamma(z) \triangleq \bar{b}^\delta \left(1 + \frac{c(Q+h-1)}{1-z}\right) z^\delta$ for all $z \geq 0$, which proves the second implication: δ -GES \Rightarrow δ -EISS. ■

We are now ready to provide sufficient conditions for the δ -EISS (hence also for δ -GAS, δ -GES and δ -ISS) of the system (1). But, before we introduce a Lyapunov characterization for the δ -EISS of the nonlinear system (5).

Lemma 7: The system (5) is δ -EISS if there exist functions $V_i : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, constants $\underline{\alpha}_i, \bar{\alpha}_i, \alpha \in \mathbb{R}_{>0}$, $i \in J_\varphi$, and a function $\chi \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^p$

- i) $\underline{\alpha}_i \|x\|^\delta \leq V_i(x) \leq \bar{\alpha}_i \|x\|^\delta$ for all $i \in J_\varphi$;
- ii) $\mathbb{E} \left\{ V_{\varphi(k+1)}(f_{\varphi(k)}(k, x, d)) - V_{\varphi(k)}(x) \right\} \leq -\alpha \|x\|^\delta + \chi(\|d\|)$.

Proof: Let us pick $\alpha_1 \triangleq \min_{i \in J_\varphi} \underline{\alpha}_i$ and $\alpha_2 \triangleq \max_{i \in J_\varphi} \bar{\alpha}_i$. Applying the expectation to both members of

inequality ii) we have

$$\begin{aligned} \mathbf{E} \left\{ \mathbf{E} \left\{ V_{\varphi(k+1)}(f_{\varphi(k)}(k, x, d)) - V_{\varphi(k)}(x) \right\} \right\} &\leq \\ &- \alpha \mathbf{E} \left\{ \|x\|^\delta \right\} + \mathbf{E} \left\{ \chi(\|d\|) \right\} \\ \mathbf{E} \left\{ V_{\varphi(k+1)}(x(k+1, d(k+1))) \right\} &\leq \\ \left(1 - \frac{\alpha}{\alpha_2} \right) \mathbf{E} \left\{ V_{\varphi(k)}(x(k, d(k))) \right\} &+ \mathbf{E} \left\{ \chi(\|d(k)\|) \right\}. \end{aligned}$$

By recursively solving the previous relation, we have

$$\begin{aligned} \mathbf{E} \left\{ V_{\varphi(k)}(x(k, d(k))) \right\} &\leq \rho^k \mathbf{E} \left\{ V_{\varphi(0)}(x_0) \right\} \\ &+ \sum_{j=0}^k \rho^{k-j} \mathbf{E} \left\{ \chi(\|d(j)\|) \right\} \end{aligned}$$

with $\rho \triangleq \left(1 - \frac{\alpha}{\alpha_2} \right) < 1$. Whereby, recalling that d is a deterministic input,

$$\alpha_1 \mathbf{E} \left\{ \|x(k, d)\|^\delta \right\} \leq \alpha_2 \rho^k \|x_0\|^\delta + \frac{1}{1-\rho} \chi(\|d\|_\infty).$$

Definition 4-2 is satisfied with

$$\begin{aligned} \alpha &= \frac{\alpha_2}{\alpha_1}, \quad \beta = -\log \left(1 - \frac{\alpha}{\alpha_2} \right) \\ \gamma(\cdot) &= \frac{\alpha_2}{\alpha_1 \alpha} \chi(\cdot). \end{aligned}$$

■

Lemma 8: The system (1) driven by the stochastic process $\{\sigma(\cdot), \varphi(\cdot)\}$, whose distributions are described by the evolutions (3) and (4), is δ -EISS if there exist M matrices $R_l = R_l^T > 0$ and $\tilde{\alpha}_3 \in \mathbb{R}_{>0}$ such that for all $x \in \mathbb{R}^n$ and for all $l \in J_\sigma$

$$\left[\sum_{h=1}^N \sum_{j=1}^M p_{lj} \ell_{lh} (x^T A_h^T R_j A_h x)^{\frac{\delta}{2}} \right] - (x^T R_l x)^{\frac{\delta}{2}} \leq -\tilde{\alpha}_3 \|x\|^\delta, \quad (13)$$

where we recall that $p_{li} \triangleq \mathbf{P}\{\sigma(k+1) = i \mid \sigma(k) = l\}$ and $\ell_{lj} \triangleq \mathbf{P}\{\varphi(k) = j \mid \sigma(k) = l\}$.

Proof: In the light of Theorem 6 we can simply prove that the system is δ -GES. Let us define the following candidate Lyapunov functions $V_l(x) \triangleq (x^T R_l x)^{\frac{\delta}{2}}$, $l \in J_\sigma$. They satisfy condition i) of Lemma 7 with $\alpha_1 = \min_{j \in J_\sigma} \lambda_{\min}(R_j)$ and $\alpha_2 = \max_{j \in J_\sigma} \lambda_{\max}(R_j)$. In order to prove the thesis, we need to verify condition ii) of the same Lemma considering $d \equiv 0$. We will show that the following conditions hold: for any $l \in J_\sigma$ and any $x \in \mathbb{R}^n$

$$\begin{aligned} \mathbf{E} \left\{ V_{\sigma(k+1)}(x(k+1)) - V_{\sigma(k)}(x(k)) \mid \sigma(k) = l, x(k) = x \right\} \\ \leq -\tilde{\alpha}_3 \|x\|^\delta \end{aligned} \quad (14)$$

with $x(k+1)$ given by (1). This can be shown as follows

$$\begin{aligned} \mathbf{E} \left\{ V_{\sigma(k+1)}(x(k+1)) - V_{\sigma(k)}(x(k)) \mid \sigma(k) = l, x(k) = x \right\} \\ = \mathbf{E} \left\{ (x(k+1)^T R_{\sigma(k+1)} x(k+1))^{\frac{\delta}{2}} \right. \\ \left. - (x(k)^T R_{\sigma(k)} x(k))^{\frac{\delta}{2}} \mid \sigma(k) = l, x(k) = x \right\} \\ = \mathbf{E} \left\{ \left(x^T A_{\varphi(k)}^T R_{\sigma(k+1)} A_{\varphi(k)} x \right)^{\frac{\delta}{2}} \mid \sigma(k) = l \right\} - (x^T R_l x)^{\frac{\delta}{2}} \\ = \left[\sum_{h=1}^N \sum_{j=1}^M \mathbf{P}\{\sigma(k+1) = j, \varphi(k) = h \mid \sigma(k) = l\} \times \right. \\ \left. \times (x^T A_h^T R_j A_h x)^{\frac{\delta}{2}} \right] - (x^T R_l x)^{\frac{\delta}{2}} \\ = \left[\sum_{h=1}^N \sum_{j=1}^M p_{lj} \ell_{lh} (x^T A_h^T R_j A_h x)^{\frac{\delta}{2}} \right] - (x^T R_l x)^{\frac{\delta}{2}} \\ \leq -\tilde{\alpha}_3 \|x\|^\delta, \end{aligned}$$

where we used inequality (13) and, recalling (3),

$$\begin{aligned} \mathbf{P}\{\sigma(k+1) = j, \varphi(k) = h \mid \sigma(k) = l\} \\ = \mathbf{P}\{\sigma(k+1) = j \mid \varphi(k) = h, \sigma(k) = l\} \mathbf{P}\{\varphi(k) = h \mid \sigma(k) = l\} \\ = \mathbf{P}\{\sigma(k+1) = j \mid \sigma(k) = l\} \mathbf{P}\{\varphi(k) = h \mid \sigma(k) = l\} \\ = p_{lj} \ell_{lh}. \end{aligned}$$

■

The previous lemma provides sufficient conditions for the δ -EISS of system (1), but they are not very easy to verify. Nonetheless, such a lemma leads to simple LMI conditions if it is specialized to 2-EISS (just let $\delta = 2$). It is worth noting that if L is the identity matrix, these conditions become those for the second moment stability [22], [33], [34].

Theorem 9: The system (1) driven by the stochastic process $\{\sigma(\cdot), \varphi(\cdot)\}$, whose distributions are described by the evolutions (3) and (4), is 2-EISS if there exist M matrices $R_l = R_l^T > 0$ such that

$$\sum_{h=1}^N \ell_{lh} A_h^T \tilde{R}_l A_h - R_l < 0, \quad \forall l \in J_\sigma \quad (15)$$

with

$$\tilde{R}_l \triangleq \sum_{j=1}^M p_{lj} R_j, \quad \forall l \in J_\sigma. \quad (16)$$

B. Almost Sure ISS

In this section we will exploit Lemma 8 to find sufficient conditions for as-ISS of system (1). As a first step we prove the following theorem.

Theorem 10: For the system (1) driven by the stochastic process $\{\sigma(\cdot), \varphi(\cdot)\}$, whose distributions are described by the evolutions (3) and (4), any property among δ -GAS, δ -GES, δ -ISS and δ -EISS implies as-ISS.

Proof: It is well known (see for instance [32], [20]) that δ -GAS \Rightarrow as-GS, hence we can assume that

$$\lim_{k \rightarrow \infty} \|x(k, d)\| = 0 \quad \text{almost surely (a.s.).}$$

This property, together with (6), implies that, for any $\varepsilon \in (0, 1)$, there exists $Q = Q(\varepsilon) \in \mathbb{Z}_{>0}$ such that for every $q \geq Q$ and every initial distribution $\pi_{\sigma 0} \in S^{M-1}$,

$$\|A_{\varphi(q-1)} \cdots A_{\varphi(0)}\| \leq \varepsilon \quad \text{a.s.}$$

The previous inequality is satisfied by almost any sequence $\varphi(0), \dots, \varphi(q-1)$ generated by any initial distribution $\pi_{\sigma 0} \in S^{M-1}$. Due to the homogeneity property of the composite process $\{\sigma(\cdot), \varphi(\cdot)\}$, there exist suitable initial distributions such that the same sequence can be generated starting from any instant $s \in \mathbb{Z}_{>0}$. Or, equivalently, almost all sequences of length $q \geq Q$ of product of dynamic matrices satisfy such inequality. Hence, we have

$$\left\| \prod_{j=0}^{q-1} A_{i_j} \right\| \leq \varepsilon \quad \text{a.s.} \quad \forall i_j \in J_{\varphi}.$$

Moreover, there exists $c \in \mathbb{R}_{>0}$ such that, for any $h \in \mathbb{Z}_{>0}$ with $h < Q$, we have

$$\left\| \prod_{j=0}^{h-1} A_{i_j} \right\| \leq c \quad \text{a.s.} \quad \forall i_j \in J_{\varphi}.$$

Using again the homogeneity property of $\{\sigma(\cdot), \varphi(\cdot)\}$, we can write for any $k = rQ + h$, $h, r \in \mathbb{Z}_{>0}$, $h < Q$

$$\|A_{\varphi(k-1)} \cdots A_{\varphi(0)}\| \leq c\varepsilon^r = ae^{-\eta k}, \quad (17)$$

with $a \triangleq c\varepsilon^{-\frac{h}{Q}}$ and $\eta \triangleq -\frac{1}{Q} \log \varepsilon$. Proceeding in a similar way as for the proof of Theorem 6, we obtain the inequality $\|x(k, d)\| \leq \beta(k, \|x_0\|) + \gamma(\|d\|_{\infty})$ almost surely, with the class \mathcal{KL} function $\beta(k, s) \triangleq ae^{-\eta k} s$ and the class \mathcal{K}_{∞} function $\gamma(s) \triangleq \bar{b} \left(1 + \frac{c(Q+h-1)}{1-\varepsilon}\right) s$. ■

Exploiting the previous theorem and Lemma 8 and proceeding as in Theorem 2.1 in [35], we can provide sufficient conditions for the as-ISS of our system. In case L is the identity matrix, such conditions turn to those provided in [35] for as-GS.

Corollary 11: The system (1) driven by the stochastic process $\{\sigma(\cdot), \varphi(\cdot)\}$, whose distributions are described by the evolutions (3) and (4), is as-ISS if there exist M matrices $R_l = R_l^T > 0$ such that one of the following conditions is verified

- $$\max_{\|x\|=1} \prod_{h=1}^N \prod_{j=1}^M \left(\frac{x^T A_h^T R_j A_h x}{x^T R_l x} \right)^{p_{lj} \ell_{lh}} < 1, \quad \forall l \in J_{\sigma}$$
- $$\prod_{h=1}^N \prod_{j=1}^M \lambda_{\max} (A_h^T R_j A_h R_l^{-1})^{p_{lj} \ell_{lh}} < 1, \quad \forall l \in J_{\sigma},$$

where $\lambda_{\max}(H)$ denotes the largest eigenvalue of H .

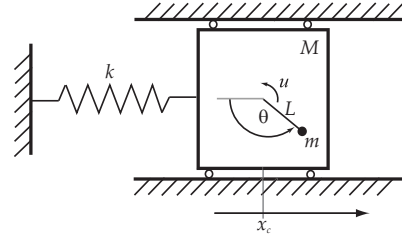


Fig. 1. Model of a Translational Oscillator/Rotational Actuator (TORA) system ([36]).

As remarked in Section III, if $\{\sigma(\cdot)\}$ is a FSHIA-MC, thus an ergodic process with a unique ergodic class and a unique i.p.d. $\bar{\pi}_{\sigma}$, the process $\{\varphi(\cdot)\}$ inherits the same ergodic property and it has a unique i.p.d. $\bar{\pi}_{\varphi} = \bar{\pi}_{\sigma} L$. Therefore, for ergodic processes we have the following condition (see also [20], [35]).

Corollary 12: The system (1) driven by the stochastic ergodic process $\{\sigma(\cdot), \varphi(\cdot)\}$, whose distributions are described by the evolutions (3) and (4) and whose i.p.d. is given by $(\bar{\pi}_{\sigma}, \bar{\pi}_{\varphi})$ with $\bar{\pi}_{\varphi} \triangleq [\bar{\pi}_{\varphi_1}, \dots, \bar{\pi}_{\varphi_M}] = \bar{\pi}_{\sigma} L$, is as-ISS if there exists an induced matrix norm $\|\cdot\|$ such that

$$\prod_{j=1}^M \|A_j\|^{\bar{\pi}_{\varphi_j}} < 1.$$

IV. CASE STUDY

In this section we illustrate the robustness properties guaranteed by the 2-EISS conditions in Theorem 9. To this aim, we consider a tracking problem for the benchmark mechanical system in Figure 1.

A. Plant and control description

The Translational Oscillator/Rotational Actuator (TORA) system in Figure 1 (see [36] for further details) can be described as follows

$$\begin{bmatrix} \dot{\theta} \\ \ddot{x}_c \end{bmatrix} = \frac{1}{\Delta(\theta)} \begin{bmatrix} m + M & -mL \cos \theta \\ -mL \cos \theta & I + mL^2 \end{bmatrix} \begin{bmatrix} u + d \\ mL\dot{\theta}^2 \sin \theta - kx_c \end{bmatrix}$$

$$y = \theta + n$$

where $\Delta(\theta) = (I + mL^2)(m + M) - mL^2 \cos^2 \theta > 0$, M is the mass of the translational oscillator, m and I are the mass and the inertia of the rotational actuator located at a distance L from the center of rotation, k is the stiffness of the spring, θ is the angle of the actuator, x_c is the horizontal displacement of the oscillator, u is the control torque, d is the torque disturbance and n is the measurement noise. The previous parameters are assumed to take the following values throughout this section: $m = 1$ kg, $M = 5$ kg, $L = 0.1$ m, $I = 0.01$ kg m² and $k = 20$ N/m. After linearization about the origin and sampled-time discretization with sampling time $T = 0.1$ s, we obtain the following transfer function from u to θ

$$G(z) = \frac{0.27266(z+1)(z^2 - 1.967z + 1)}{(z-1)^2(z^2 - 1.964z + 1)}.$$

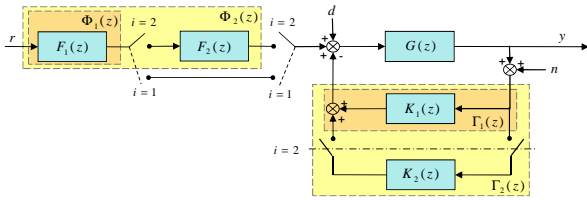


Fig. 2. Schematic representation of the system and the anytime control. When the switches are in the $i = 1$ ($i = 2$) position the controller Γ_1 (Γ_2) and the prefilter Φ_1 (Φ_2) are active.

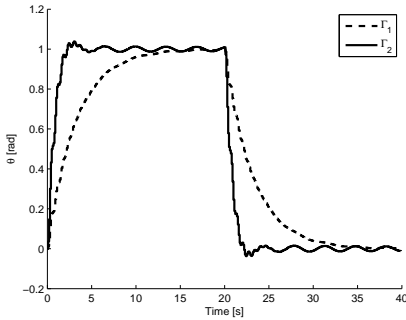


Fig. 3. Outputs of the TORA system in closed loop with (Γ_1, Φ_1) and (Γ_2, Φ_2) for a square wave reference of period 40 s, duty cycle 50% and amplitude 1 rad.

We assume measurements to be acquired at the beginning of each period and control inputs to be released at the end, thus the controller is not affected by jitter but experiences a constant unit delay. In order to account for the unit delay, controllers are designed for the transfer function $G(z)^{\frac{1}{z}}$ instead of $G(z)$.

We assume here that the control task is designed according to the anytime paradigm briefly illustrated in Example 1 and to have the special *modular structure* represented in Figure 2 (see [11, Sec. VII] for further information on anytime control implementation and tracking problems). In particular, the controller $\Gamma_1(z) = K_1(z) = \frac{3.04(z-0.97)}{z+0.9}$ is designed to ensure only stability requirement, while the second controller $\Gamma_2(z) = K_1(z) + K_2(z)$ with $K_2(z) = \frac{-0.021(z-2)}{z-0.76}$ to enhance performance in terms of rise time and settling time (see Figure 3 for a graphical comparison). The prefilters $\Phi_1(z) = F_1(z) = 0.0480$ and $\Phi_2(z) = F_1(z)F_2(z) = 0.1351$ are used to adapt the steady-state gain and ensure static requirements. According to the anytime paradigm, controllers are implemented in Jordan state-space realization. The two closed loop matrices for the feedback connection of the linearized system and the two controllers have the structure described in [11, Sec. VII].

B. Computation platform stochastic description

In this case study we have two controllers, hence $J_\varphi \triangleq \{1, 2\}$, and four possible values of available computation time labeled from 1 to 4, hence $J_\sigma \triangleq \{1, \dots, 4\}$. We assume the FSH-MC $\{\sigma(\cdot)\}$ to be described by the following

transition probability matrix

$$P = \begin{bmatrix} 0.5 & 0.1 & 0.4 & 0 \\ 0.2 & 0.1 & 0.5 & 0.2 \\ 0.2 & 0.2 & 0.4 & 0.2 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{bmatrix}$$

and the following initial probability distribution $\pi_{\sigma 0} = \frac{1}{4}[1, 1, 1, 1]$.

In the light of the modular structure chosen for the implementation of the anytime control, the discrete-time i.i.d. process T^j , $j \in J_\varphi$ describes here the execution time required to compute sequentially all component controllers from K_1 to K_j and not all subroutines from Γ_1 to Γ_j as in Example 1. The stationary probability distribution of T^j is represented by the row vector $\pi_{T^j} = [\pi_{T^j_1}, \dots, \pi_{T^j_M}]$ and the associated cumulative probability distribution by $\kappa_{T^j} = [\pi_{T^j_1}, \pi_{T^j_1} + \pi_{T^j_2}, \dots, \sum_{k=1}^M \pi_{T^j_k}]$. The matrix L in (4) has the following structure (see [11])

$$L = \begin{bmatrix} \kappa_{T^1} - \kappa_{T^2} \\ \vdots \\ \kappa_{T^{M-1}} - \kappa_{T^M} \\ \kappa_{T^M} \end{bmatrix}^T.$$

We consider the following numerical values $\pi_{T^1} = [1, 0, 0, 0]$ and $\pi_{T^2} = [0, 0.33, 0.5, 0.17]$ and we get

$$L = \begin{bmatrix} 1 & 0 \\ 0.67 & 0.33 \\ 0.17 & 0.83 \\ 0 & 1 \end{bmatrix}.$$

C. Simulation results

The closed loop SJLS with the computation platform described in the previous sections satisfies the LMI conditions of Theorem 9, thus it turns out to be 2-EISS. For simulation purposes we have considered a tracking problem with a square wave reference of period 40 s, duty cycle 50% and amplitude 1 rad. We assume also that the input disturbance d and the measurement noise n affecting the SJLS (see the scheme in Figure 2) are independent white noises such that $|d| \leq 0.05$ N m and $|n| \leq 0.05$ rad. Figure 4 shows a simulation run for the TORA system in closed loop with (Γ_1, Φ_1) , with (Γ_2, Φ_2) and with the anytime control (SJLS), thus verifying the robustness property.

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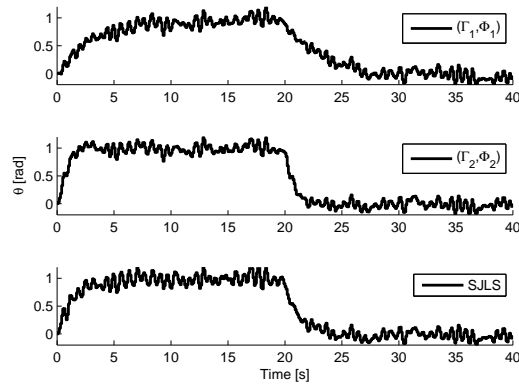


Fig. 4. Outputs of the TORA system in closed loop with (Γ_1, Φ_1) , (Γ_2, Φ_2) and with the anytime control for a square wave reference of period 40 s, duty cycle 50% and amplitude 1 rad. The system is affected by input disturbance d and measurement noise n modeled as independent white noises with $|d| \leq 0.05$ N m and $|n| \leq 0.05$ rad.

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