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# Existence of Ground State of an Electron in the BDF Approximation.

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## Abstract

The Bogoliubov-Dirac-Fock (BDF) model allows to describe relativistic electrons interacting with the Dirac sea. It can be seen as a mean-field approximation of Quantum Electro-dynamics (QED) where photons are neglected. This paper treats the case of an electron together with the Dirac sea in absence of any external field. Such a system is described by its one-body density matrix, an infinite rank, self-adjoint operator which is a compact perturbation of the negative spectral projector of the free Dirac operator. We prove the existence of minimizers of the BDF-energy under the charge constraint of one electron assuming that the coupling constant  $\alpha$  and the quantity  $L = \alpha \log(\Lambda)$  are small where  $\Lambda > 0$  is the ultraviolet cut-off and chosen very large.

We then study the non-relativistic limit of such a system in which the speed of light  $c$  tends to infinity (or equivalently  $\alpha$  tends to zero) with  $L$  fixed: after rescaling the electronic solution tends to the Choquard-Pekar ground state.

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## 1 Introduction

We study an approximation of no-photon Quantum Electrodynamics (QED) allowing to describe the behavior of relativistic electrons in an external field interacting with the virtual electrons of the Dirac sea via the electrostatic potential in a mean-field type theory. Here there will be one "real" electron and no external field.

We use relativistic units  $\hbar = c = 1$  and set the bare particle mass equal to 1 and  $\alpha = e^2/(4\pi)$ . We denote by  $D^0 = -i\boldsymbol{\alpha} \cdot \nabla + \beta$  the free Dirac operator acting on the Hilbert space  $\mathfrak{H} = L^2(\mathbf{R}^3, \mathbf{C}^4)$  and by  $P^0 = \chi_{(-\infty, 0)}(D^0)$  the projector on its negative spectral subspace. Later on we will use a modified Dirac operator  $\mathcal{D}^0$  together with the free vacuum  $\mathcal{P}_-^0$  introduced in [8, 11] instead of  $D^0$  and  $P^0$ .

In the BDF model a system is described by a Hartree-Fock state  $\Omega$  in Fock space completely characterized by its one-body density matrix  $P$  (an orthogonal projector for pure states) containing both "real" and "virtual" electrons. It is infinite-rank. To manipulate such a system and in particular to define properly its density we consider the difference between  $P$  and the free vacuum  $\mathcal{P}_-^0$ , that is  $Q = P - \mathcal{P}_-^0$ . Moreover an ultraviolet cut-off  $\Lambda$  is needed, restricting our study to the Hilbert space

$$\mathfrak{H}_\Lambda = \{f \in \mathfrak{H} : \text{supp } \hat{f} \subset B(0, \Lambda)\}.$$

Note that  $\mathfrak{H}_\Lambda \subset H^1(\mathbf{R}^3, \mathbf{C}^4)$  is  $D^0$  and  $\mathcal{D}^0$  invariant. Indeed  $\mathcal{P}_-^0$  is a translation-invariant projector on  $\mathfrak{H}_\Lambda$  satisfying the Euler-Lagrange equation

$$\begin{cases} \mathcal{P}_-^0 = \chi_{-\infty, 0}(D^0), \\ \mathcal{D}^0 = D^0 - \alpha \frac{(\mathcal{P}_-^0 - 1/2)(x, y)}{|x-y|}. \end{cases} \quad (1)$$

In Fourier space  $\mathcal{D}^0$  takes the following form

$$\widehat{\mathcal{D}^0}(p) = \boldsymbol{\alpha} \cdot \omega_p g_1(|p|) + g_0(|p|)\beta, \quad \omega_p = \frac{p}{|p|}, \quad (2)$$

where  $g_0$  and  $g_1$  are real and smooth functions satisfying

$$x \leq g_1(x) \leq x g_0(x).$$

In the regime  $L := \alpha \log(\Lambda) = O(1)$  following [11] we will be able to get further information on them *via* their self-consistent equation (that we have written in (55)).

We consider then  $\tilde{\mathcal{Q}}_\Lambda := \{Q \in \mathfrak{S}_2(\mathfrak{H}_\Lambda) : Q^* = Q, 0 \leq Q + \mathcal{P}_-^0 \leq 1\}$  where  $\mathfrak{S}_p(\mathfrak{H}_\Lambda)$  denotes the usual Schatten class of compact operators  $A$  on  $\mathfrak{H}_\Lambda$  such that  $\text{Tr}(|A|^p) < \infty$ . The charge of  $Q \in \mathcal{Q}_\Lambda$  is defined by its  $\mathcal{P}_-^0$ -trace that is by

$$\text{Tr}_{\mathcal{P}^0}(Q) = \text{Tr}(\mathcal{P}_-^0 Q \mathcal{P}_-^0) + \text{Tr}(\mathcal{P}_+^0 Q \mathcal{P}_+^0), \quad \mathcal{P}_+^0 := 1 - \mathcal{P}_-^0;$$

it is known (cf [5]) that  $\mathcal{P}_-^0 Q \mathcal{P}_-^0$  and  $\mathcal{P}_+^0 Q \mathcal{P}_+^0$  are trace-class when  $Q = P - \mathcal{P}_-^0$  and we introduce the set of  $\mathcal{P}_-^0$ -trace class operators

$$\mathfrak{S}_1^{\mathcal{P}_-^0}(\mathfrak{H}_\Lambda) = \mathfrak{S}_2(\mathfrak{H}_\Lambda) \cap \{Q : Q^{++} := \mathcal{P}_+^0 Q \mathcal{P}_+^0, Q^{--} := \mathcal{P}_-^0 Q \mathcal{P}_-^0 \in \mathfrak{S}_1(\mathfrak{H}_\Lambda)\},$$

so we will work in

$$\mathcal{Q}_\Lambda := \tilde{\mathcal{Q}}_\Lambda \cap \mathfrak{S}_1^{\mathcal{P}_-^0}(\mathfrak{H}_\Lambda). \quad (3)$$

The density of  $\Omega_P$  is represented by  $\rho_{(P-\mathcal{P}_-^0)}(x) = \text{Tr}_{\mathbb{C}^4}((P - \mathcal{P}_-^0)(x, x))$  (which makes sense as  $Q$  is locally trace-class). Its Fourier transform is:

$$\widehat{\rho_Q}(k) := \frac{1}{(2\pi)^{3/2}} \int_{|u+\frac{k}{2}|, |u-\frac{k}{2}| \leq \Lambda} \text{Tr}_{\mathbb{C}^4}(\hat{Q}(u + \frac{k}{2}, u - \frac{k}{2})) du, \quad (4)$$

The energy functional is defined on  $\mathcal{Q}_\Lambda$  by

$$\mathcal{E}(Q) := \text{Tr}_{\mathcal{P}^0}(\mathcal{D}^0 Q) + \frac{\alpha}{2} \left( D(\rho_Q, \rho_Q) - \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{|Q(x, y)|^2}{|x - y|} dx dy \right), \quad (5)$$

where

$$D(f, g) := 4\pi \int_p \frac{\widehat{f}(p) \overline{\widehat{g}(p)}}{|p|^2} dp$$

coincides with  $\iint \frac{\widehat{f}(x) \overline{\widehat{g}(y)}}{|x - y|} dx dy$  for sufficiently smooth functions. Here  $Q(x, y)$  denotes the kernel of  $\hat{Q}$ . The trace part is the kinetic energy while the two others are respectively the *direct term* and the *exchange term*. Moreover there holds [5],[8, 1]

$$\text{Tr}_{\mathcal{P}^0}(\mathcal{D}^0 Q) = \text{Tr}(|\mathcal{D}^0|(Q^{++} - Q^{--})) \geq \text{Tr}(|\mathcal{D}^0|Q^2), \quad (6a)$$

$$\iint \frac{|Q(x, y)|^2}{|x - y|} dx dy \leq \frac{\pi}{2} \text{Tr}(|\mathcal{D}^0|Q^2), \quad (6b)$$

we will assume that  $\alpha < \frac{4}{\pi}$ .

We introduce

$$\mathcal{C} := \{\rho \in \mathcal{S}'(\mathbf{R}^3) : D(\rho, \rho) < \infty\},$$

along with its norm  $\|\rho\|_{\mathcal{C}} := \sqrt{D(\rho, \rho)}$ . Moreover we introduce the following notations concerning the Dirac operator:

*Notation 1.1.* We note  $\tilde{E}(p) := \sqrt{g_0(p)^2 + g_1(p)^2} = |\mathcal{D}^0(p)|$  and

$$E(p) := \sqrt{1 + |p|^2} = |\mathcal{D}^0(p)|.$$

We will designate by  $g_0$  (respectively  $g_1$ ) both functions  $g_* : x \in [0, \Lambda] \rightarrow g_*(x) \in \mathbf{R}^+$  and  $g_* : p \in B(0, \Lambda) \rightarrow g_*(|p|) \in \mathbf{R}^+$ . The  $(g_0)$ 's are  $\mathcal{C}^\infty$  while  $g_1 \in \mathcal{C}^1(B(0, \Lambda))$  (cf Appendix A).

At last we note

$$\begin{cases} \mathbf{g}_1 : p \in B(0, \Lambda) \rightarrow g_1(|p|)\omega_p \in \mathbf{R}^3 \\ \mathbf{g} : p \in B(0, \Lambda) \rightarrow \begin{pmatrix} g_0(p) \\ \mathbf{g}_1(p) \end{pmatrix} \in \mathbf{R}^4. \end{cases}$$

*Notation 1.2.*  $C_1 > 0$  denotes a constant verifying  $g_1(r) \leq C_1|r|$  and  $|g_0|_\infty \leq C_1$ .

*Notation 1.3.* A recurrent function of this problem is

$$B_\Lambda(k) := \frac{1}{\pi^2 |k|^2} \int_{|p=l+\frac{k}{2}|, |q=l-\frac{k}{2}| \leq \Lambda} \frac{\tilde{E}(p)\tilde{E}(q) - \mathbf{g}(p) \cdot \mathbf{g}(q)}{\tilde{E}(p)\tilde{E}(q)(\tilde{E}(p) + \tilde{E}(q))} dl. \quad (7)$$

We define  $\alpha_r(k)$  by

$$\alpha_r(k) := \frac{\alpha B_\Lambda(k)}{1 + \alpha B_\Lambda(k)}. \quad (8)$$

In Appendix A it is shown that  $B_\Lambda(k) = O(\log(\Lambda))$  and that for  $L \ll 1$  there holds  $B_\Lambda(0) = \frac{2}{3\pi} \log(\Lambda) + O(L \log(\Lambda) + 1)$ .

*Notation 1.4.* Throughout this paper we work in the regime

$$\alpha \rightarrow 0, \quad \Lambda \rightarrow +\infty, \quad \alpha \log(\Lambda) = L \leq \varepsilon_0, \quad \alpha(\log(\Lambda))^3 \geq \varepsilon_1 > 0 \quad (9)$$

so whenever we write  $o(\cdot)$  and  $O(\cdot)$  without specifying the limit it is understood that it holds in the regime (9).

Moreover,  $K$  denotes a constant which is independent of  $\alpha$  and  $\Lambda$ . It is understood that  $\lesssim$  refers to such a constant.

## 2 Main results

Here we restrict our study to states  $Q \in \mathcal{Q}_\Lambda$  such that  $\text{Tr}_{\mathcal{P}^0}(Q) = 1$ : is there a minimizer on the surface of charge constraint 1? Following [7] it suffices to show that the energy function

$$E(q) := \inf_{Q \in \mathcal{Q}_\Lambda, \text{Tr}_{\mathcal{P}^0}(Q)=q} (\mathcal{E}(Q))$$

satisfies binding inequalities at level 1 that is

$$E(1) < E(1 - q) + E(q), \quad \forall q \in \mathbf{R} \setminus \{0, 1\}. \quad (10)$$

We will show that it is the case in the regime (9).

The difficult case of (2) is  $0 < q < 1$ , it will be a corollary of the fact that  $E(1) < g_0(0) := m(\alpha) = \min(\sigma(|\mathcal{D}^0|))$ . The inequality  $E(q) \leq |q|m(\alpha)$  is proven in [7]. For  $0 < q < 1$  it is straightforward: it suffices to take trial tests of the form  $Q = q|\psi\rangle\langle\psi|$  with  $\psi \in \text{Ran}(\mathcal{P}_+^0)$ .

Indeed the first step will be to show

**Theorem 1.** *There exist three constants  $\alpha_0, L_0, \Lambda_0 > 0$  such that for  $\alpha \leq \alpha_0, L \leq L_0, \Lambda \geq \Lambda_0$  there holds*

$$E(1) \leq m(\alpha) + \frac{(\alpha\alpha_r(0))^2 m(\alpha)}{2g_1'(0)^2} E_{\text{CP}} + o((\alpha\alpha_r(0))^2), \quad (11)$$

where  $E_{\text{CP}}$  is the Choquard-Pekar energy

$$E_{\text{CP}} := \inf_{\phi \in H^1(\mathbf{R}^3): \|\phi\|_{L^2} = 1} \left\{ \int |\nabla \phi|^2 dx - D(|\phi|^2, |\phi|^2) \right\} < 0.$$

*Remark 2.1.* For sufficiently small  $L$  there holds  $g_1'(0) > \varepsilon > 0$ . More generally all the results we need about  $g_0$  and  $g_1$  are proven in Appendix A.

*Remark 2.2.* The condition  $\alpha \log(\Lambda)^3 \gtrsim 1$  of (9) is not needed for this theorem.

We consider along with the authors of [7] that such a minimizer  $Q$  should satisfy a self-consistent equation of the form (with  $Q = \gamma + |\psi\rangle\langle\psi|$ )

$$\gamma + \mathcal{P}_-^0 = \chi_{(-\infty, 0)}(\mathcal{D}_Q), \quad \mathcal{D}_Q := \mathcal{D}^0 + \alpha \left( \rho_Q \star |\cdot|^{-1} - \frac{Q(x, y)}{|x - y|} \right), \quad (12)$$

and  $|\psi\rangle\langle\psi| = \chi_{[0, \mu]}(\mathcal{D}_Q)$  where  $\mu < m(\alpha)$  can be chosen such that  $\mathcal{D}_Q \psi = \mu \psi$ .

We thus take a trial test of the following form: let us first take  $\phi'_1$  the *unique* minimizer of the Choquard-Pekar energy (*cf* Theorem 1.), then consider

$\phi_1 := \frac{P_{\mathfrak{H}_\Lambda} \phi'_1}{\|P_{\mathfrak{H}_\Lambda} \phi'_1\|_{L^2}}$  where  $P_{\mathfrak{H}_\Lambda}$  is the projector onto  $\mathfrak{H}_\Lambda$  and form the spinor

$\psi_1 := \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}$ . For  $\lambda^{-1} := \frac{\alpha\alpha_r(0)m(\alpha)}{g'_1(0)^2}$  we take  $\psi_\lambda := \lambda^{-3/2}\psi_1(\lambda^{-1}(\cdot))$  to form

$N = N_\lambda := |\psi_\lambda\rangle\langle\psi_\lambda|$  and  $n_\lambda := |\psi_\lambda|^2 = \rho_N$ . We define  $\Gamma$  by:

$\Gamma := N' + \gamma$  with

$$\gamma = \chi_{(-\infty,0)}\left(\mathcal{D}^0 + \alpha((\rho_\gamma + n) \star |\cdot|^{-1} - \frac{\gamma(x,y)+N(x,y)}{|x-y|})\right) - \mathcal{P}_-, \quad (13a)$$

$$\pi = \gamma + \mathcal{P}_-, \quad N' = \frac{|(1-\pi)\psi_\lambda\rangle\langle(1-\pi)\psi_\lambda|}{1 - \|\pi\psi_\lambda\|_{L^2}^2}. \quad (13b)$$

It is not obvious that such a trial test exists: in fact the fixed point method of [5] can be adapted to prove it. This last paper treats the case of  $D^0$ , in Appendix A it is shown that taking  $\mathcal{D}^0$  does not change anything.

Then we calculate the energy of  $\Gamma$ .

Decomposing each term of the energy and considering that an electron does not see its own field (that is here  $D(|\psi|^2, |\psi|^2) - \iint \frac{|\psi(x)|^2|\psi(y)|^2}{|x-y|} dx dy = 0$ ) we can write

$$\mathcal{E}(\Gamma) = T + \frac{\alpha}{2}(I - J) \quad (14)$$

with  $T = \text{Tr}_{\mathcal{P}^0}(\mathcal{D}^0\Gamma)$  the kinetic energy and

$I = D(\rho_\Gamma, \rho_\Gamma) - D(n_\lambda, n_\lambda), J = \iint \frac{|\Gamma(x,y)|^2}{|x-y|} dx dy - D(n_\lambda, n_\lambda)$ . We prove

**Lemma 2.3.** There holds

$$\begin{aligned} \text{Tr}_{\mathcal{P}^0}(\mathcal{D}^0 N') &= m(\alpha) + \frac{g'_1(0)^2}{2\lambda^2 m} \int |\nabla\psi_1|^2 dx + o(\lambda^{-2}), \\ \frac{\alpha}{2}I &= -\frac{\alpha(2\alpha_r(0) - \alpha_r(0)^2)}{2\lambda} D(n_1, n_1) + o\left(\frac{\alpha\alpha_r(0)}{\lambda}\right), \\ \alpha J &= o\left(\frac{\alpha\alpha_r(0)}{\lambda}\right), \\ \text{Tr}_{\mathcal{P}^0}(D\gamma) &= \frac{\alpha(\alpha_r(0) - \alpha_r(0)^2)}{2\lambda} D(n_1, n_1) + o\left(\frac{\alpha\alpha_r(0)}{\lambda}\right) \end{aligned}$$

such that *in fine* we get

$$\mathcal{E}(\Gamma) = m(\alpha) + \frac{\alpha\alpha_r(0)}{2\lambda} E_{\text{CP}} + o\left(\frac{\alpha\alpha_r(0)}{\lambda}\right).$$

Lemma 2.3. is proved in section 4.1 and Theorem 1. follows immediatly.

A corollary is then

**Proposition 2.** For each  $q \neq 0, 1$  there holds  $E(1) < E(1-q) + E(q)$ .

Theorem 1.[7] assures that *there exists a minimizer of  $E(1)$ .*

We study such a minimizer taking the form  $Q = \gamma + |\psi\rangle\langle\psi|$  with  $D_Q\psi = \mu\psi$ .

We write  $v_\gamma = \rho_\gamma \star |\cdot|^{-1}$  and  $R_\gamma(x, y) = \frac{\gamma(x,y)}{|x-y|}$ : as  $(|\psi|^2 \star |\cdot|^{-1} - \frac{\psi(x)\psi(y)^*}{|x-y|})\psi = 0$  we have

$$(\mathcal{D}^0 + \alpha(v_\gamma - R_\gamma))\psi = \mu\psi. \quad (15)$$

A natural question arises: does it have a form similar to the previous trial test, in particular does its energy have the same asymptotic expansion at order 1 ?

**Theorem 3.** *There exist three constants  $\alpha_1, L_1, \Lambda_1 > 0$  such that for  $\alpha \leq \alpha_1, L \leq L_1, \Lambda \geq \Lambda_1$  in the regime  $\alpha(\log(\Lambda))^3 \gtrsim 1$  there holds*

$$E(1) = m(\alpha) + \frac{(\alpha\alpha_r(0))^2 m(\alpha)}{2(g'_1(0))^2} E_{\text{CP}} + o((\alpha\alpha_r(0))^2). \quad (16)$$

As it can be guessed we will follow the same path as the one for Theorem 1. We will first prove that

**Lemma 2.4.**  $\|\psi\|_{H^{3/2}} = O(1)$ ,

enabling us to apply the fixed-point method with  $n = |\psi|^2$  and  $N = |\psi\rangle\langle\psi|$  and by so constructing the minimizer as a fixed point. Using the estimates that we deduce from the fixed-point method and equation (15) we then prove that

**Lemma 2.5.**  $\langle |\nabla|^2 \psi, \psi \rangle = O((\alpha\alpha_r(0))^2)$ .

It implies that a minimizer of  $E(1)$  has the same estimates of the trial test concerning the quantities  $D(n, n), D(\rho_\gamma, \rho_\gamma)$  and  $\iint \frac{|\gamma(x, y)|^2}{|x-y|} dx dy$ : they are respectively  $O(L\alpha), O(L^2(L\alpha))$  and  $O((L\alpha)^2)$ .

Following [7], we apply a scaling transform to the minimizer with a scale  $\alpha\alpha_r(0)$ : we get  $\underline{\psi} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \in H^1(\mathbf{C}^4)$ . The previous results will give

**Lemma 2.6.**  $\|\underline{\psi}\|_{H^{3/2}} = O(1), \|\chi\|_{H^1} = O(L\alpha)$ .

This last lemma enables us to estimate  $E(1)$  and to obtain the result of Theorem 3. Thus

**Theorem 4.** writing  $C_0^2 := \frac{2g_1'(0)^2}{(\alpha\alpha_r(0))^2 m(\alpha)}$  there holds in the regime (9)

$$\liminf_{\alpha, \Lambda^{-1} \rightarrow 0} C_0^2(E(1) - m(\alpha)) = \limsup_{\alpha, \Lambda^{-1} \rightarrow 0} C_0^2(E(1) - m(\alpha)) = E_{\text{CP}}. \quad (17)$$

If  $L = \alpha \log(\Lambda) \leq \min(L_0, L_1)$  is fixed then  $\lim_{\alpha \rightarrow 0} \Lambda^{-1} = 0$  so (17) holds with  $\alpha \rightarrow 0$ .

*Remark 2.7.* This answers an open question stated in [8].

## 3 Preliminary results

### 3.1 The fixed point method

*Notation 3.1.* For a compact operator  $Q$  we will write  $R_Q$  or  $R(Q)$  the operator whose kernel is  $\frac{Q(x, y)}{|x-y|}$  and  $\varphi_Q$  the function  $\rho_Q \star |\cdot|^{-1}$ . In general we take the notation of [5].

As shown in [5] we can use the Cauchy expansion to write (at least formally)

$$\chi_{(-\infty, 0)}(\mathcal{D}^0 + \alpha(\varphi_Q - R_Q)) - \chi_{(-\infty, 0)}(\mathcal{D}^0) = \sum_{k=1}^{\infty} \alpha^k Q_k, \quad (18a)$$

$$Q_k = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} d\eta \frac{1}{\mathcal{D}^0 + i\eta} \left( (R_Q - \varphi_Q) \frac{1}{\mathcal{D}^0 + i\eta} \right)^k. \quad (18b)$$

We also expand  $(R - \varphi)^k$ :  $Q_k := \sum_{j=0}^k Q_{j, j-k}$  like in [5] (the first number denotes the number of  $(R)$ 's). This equation is about the vacuum without external field: to consider an electron (represented by  $N := |\psi\rangle\langle\psi|$ ) we have to add its field  $n := |\psi|^2$  together with the operator  $\frac{N(x, y)}{|x-y|}$  in the exchange term and get:  $\rho' = \rho + n, Q' = Q + N, \varphi'_Q = \varphi_{Q'}$  and so the equation

$$\chi_{(-\infty, 0)}(\mathcal{D}^0 + \alpha(\varphi'_Q - R'_Q)) - \chi_{(-\infty, 0)}(\mathcal{D}^0) = F_Q(Q', \rho') = \sum_{k=1}^{\infty} \alpha^k Q_k(Q', \rho'). \quad (19)$$

There holds  $\widehat{\rho}_{0,1}(p) = -\widehat{\rho}'(p)B_\Lambda(p)$  so taking the density  $\rho$  of that equation we obtain  $\rho_{Q'} = F_\rho(Q', \rho')$ , with

$$\widehat{F}_\rho(p) = \frac{1}{1 + \alpha B_\Lambda(p)} \left( \alpha(\widehat{\rho}_{1,0}(p) + \widehat{n}(p)) + \sum_{k \geq 2} \alpha^k \widehat{\rho}_k(p) \right). \quad (20)$$

Of course this is not enough: we must precise the domain of the function

$$F := F_Q \times F_\rho. \quad (21)$$

We will first consider the Banach space  $\mathcal{X} = \mathcal{Q} \times \mathfrak{C}$  defined by the norms

$$\|Q\|_{\mathcal{Q}}^2 = \iint \tilde{E}(p-q)^2 \tilde{E}(p+q) |\widehat{Q}(p,q)|^2 dpdq, \quad \|\rho\|_{\mathfrak{C}}^2 = \int \frac{\tilde{E}(k)^2}{|k|^2} |\widehat{\rho}(k)|^2 dk,$$

and  $\|(Q, \rho)\|_{\mathcal{X}} = 2C_1^{3/2}(2\sqrt{2}\|\rho\|_{\mathfrak{C}} + C_R\sqrt{2}\|Q\|_{\mathcal{Q}})$  where  $C_R$  is defined in [5] and  $|g_0(p)| \leq C_1$ ,  $|g_1(p)| \leq C_1|p|$ . Like in [5] we can show that we can apply the Banach fixed point theorem in  $\mathcal{X} \cap B(0, R_\Lambda)$  where  $R_\Lambda$  is  $O(\sqrt{\log(\Lambda)})$  when  $\sqrt{L\alpha} \leq \varepsilon$ : in our regime where  $\alpha \log(\Lambda) \leq L_0$  the condition on  $\alpha$  holds for  $\alpha$  sufficiently small.

We will denote by  $\nu$  the Lipschitz constant of  $F$  in  $B(0, R_\Lambda)$ :  $\nu = O(\sqrt{L\alpha})$ . Indeed we can show that  $\|dF\|_{L(\mathcal{X})} \lesssim \sqrt{L\alpha}$ .

We also introduce the norms

$$\begin{aligned} \|Q\|_S^2 &= \iint \frac{|Q(x,y)|^2}{|x-y|} dx dy, \quad \|Q\|_F^2 = \text{Tr}(|\mathcal{D}^0|Q^*Q), \\ \|Q\|_E^2 &= \iint \max(\tilde{E}(p), (\tilde{E}(p-q))^2, \tilde{E}(p-q)\tilde{E}(p+q)) |\widehat{Q}(p,q)|^2 dpdq, \end{aligned}$$

and  $\sqrt{\frac{2}{\pi}}\|\cdot\|_S \leq \|\cdot\|_F \leq \|\cdot\|_E \leq \|\cdot\|_{\mathcal{Q}}$ .

*Remark 3.2.* By looking closely at the estimates of [5] we realize that we can take another choice of norms for  $F$  and so another choice of Banach space on which applying the Banach fixed point theorem. Indeed let us take a radial function  $f: \mathbf{R}^3 \rightarrow [1, +\infty)$ : as long as there exists a constant  $C > 0$  such that

$$f(p-q) \leq C(f(p-p_1) + f(p_1-q)), \quad \alpha \left( \int_{r=0}^{\Lambda} \frac{dr}{f(r)^2} \right)^{1/2} =: \theta = o(1),$$

we can apply the theorem with the norms

$$*\|Q\|_{\mathcal{Q}}^2 = \iint f(p-q)^2 \tilde{E}(p+q) |\widehat{Q}(p,q)|^2 dpdq, \quad *\|\rho\|_{\mathfrak{C}}^2 = \int \frac{f(k)^2}{|k|^2} |\widehat{\rho}(k)|^2 dk,$$

even if it means changing the weights of the norms and restricting to  $\theta \ll 1$ . With  $f(p) = \tilde{E}(p)^a$ ,  $\frac{1}{2} \leq a \leq 1$  we have  $\theta = O(\sqrt{L\alpha}) \xrightarrow{\alpha \rightarrow 0} 0$  so there will be no problem in our regime.

### 3.2 Some inequalities

Let us recall Hardy's, Kato's and Kato-Seiler-Simon's inequalities we will use throughout this paper: for  $\phi \in L^2(\mathbf{R}^3)$ ,  $f, g \in \mathcal{B}(\mathbf{R}^3, \mathbf{C}^4)$  (Borelian functions) there hold:

$$\int \frac{|\phi(x)|^2}{|x|^2} dx \leq 4 \langle |\nabla|^2 \phi, \phi \rangle, \quad (22a)$$

$$\int \frac{|\phi(x)|^2}{|x|} dx \leq \frac{\pi}{2} \langle |\nabla| \phi, \phi \rangle, \quad (22b)$$

$$\|f(x)g(i\nabla)\|_{\mathfrak{S}_p} \leq (2\pi)^{-\frac{3}{p}} \|f\|_{L^p} \|g\|_{L^p}, \quad 2 \leq p < \infty. \quad (22c)$$

In particular (22b) and (22c) give

**Lemma 3.3.** Let  $Q \in \mathcal{Q}_\Lambda$  and  $\rho \in \mathcal{C}$ , then we have  $(\varphi = \rho \star |\cdot|^{-1})$

$$\begin{aligned} \| |R_Q| \mathcal{D}^0 |^{-\frac{1}{2}} \|_{\mathcal{B}}, \| |\mathcal{D}^0|^{-\frac{1}{2}} R_Q \|_{\mathcal{B}} &\lesssim \|Q\|_S, \\ \| |\varphi| \mathcal{D}^0 |^{-\frac{1}{2}} \|_{\mathfrak{S}_6}, \| |\mathcal{D}^0|^{-\frac{1}{2}} \varphi \|_{\mathfrak{S}_6} &\lesssim (\log(\Lambda))^{\frac{1}{6}} \|\rho\|_{\mathcal{C}}, \\ \| |\varphi| \mathcal{D}^0 |^{-t} \|_{\mathfrak{S}_6}, \| |\mathcal{D}^0|^{-t} \varphi \|_{\mathfrak{S}_6} &\leq K_t \|\rho\|_{\mathcal{C}}, \quad t > 1/2. \end{aligned}$$



Let us consider  $R = R_Q$  with  $Q \in \mathcal{Q}_\Lambda$ . [5] introduces the norm  $\|R\|_R^2 = \iint \frac{\tilde{E}(p-q)^2}{\tilde{E}(p+q)} |\hat{R}(p, q)|^2 dpdq$  and the proof of Lemma 8.[5] enables us to say that

**Lemma 3.4.** Let  $t \geq 0$ .

$$\| |\mathcal{D}^0|^{-1/2} R |\mathcal{D}^0|^{-1/2} \|_{\mathfrak{S}_2}^2 \lesssim \iint \tilde{E}(p+q) |\hat{Q}(p, q)|^2 dpdq, \quad (23a)$$

$$\iint \frac{\tilde{E}(p-q)^t}{\tilde{E}(q)^2} |\hat{R}(p, q)|^2 dpdq \lesssim \iint \tilde{E}(p-q)^t \tilde{E}(p+q) |\hat{Q}(p, q)|^2 dpdq, \quad (23b)$$

$$\iint \frac{|\hat{R}(p, q)|^2}{\tilde{E}(q)} dpdq \lesssim \iint \tilde{E}(p-q) \tilde{E}(p+q) |\hat{Q}(p, q)|^2 dpdq. \quad (23c)$$

(23a) is straightforward for  $\tilde{E}(p)^{-1} \tilde{E}(q)^{-1} \lesssim \tilde{E}(p+q)^{-1}$  and (23c) is due to the fact that  $\tilde{E}(q)^{-1} \lesssim \frac{\tilde{E}(p-q)}{\tilde{E}(p+q)}$ . Following the proof of 8.[5] we have

$$\iint \frac{\tilde{E}(p-q)^t}{\tilde{E}(q)^2} |\hat{R}(p, q)|^2 dpdq \leq 8 \iint \tilde{E}(2v)^t \tilde{E}(2l) h(l, v) |\hat{Q}(l+v, l-v)|^2 dpdq,$$

$$h(l, v) \leq \tilde{E}(2l)^1 (2\pi^2)^{-2} \iint dudl' (\tilde{E}(u-v)^2 \tilde{E}(2l')^{1+1} |l-u|^2 |l'-u|^2)^{-1} \lesssim 1.$$

## 4 Proofs

### 4.1 Proof of Lemma 2.3.

We apply the Banach theorem with initial data  $(N, n) \in \mathcal{X}$ : we note the iterations

$$\gamma'_j = \gamma_j + N, \quad \vec{p}'_j = \vec{p}_j + n \quad (24)$$

with  $\gamma_{j+1} = \chi_{(-\infty, 0)}(\mathcal{D}^0 + \alpha(\varphi'_{\vec{p}'_j} - R'(\gamma_j))) - \mathcal{P}^0_-$  (so  $\gamma_0 = 0$ ). All the estimates we need about  $\gamma$  etc. are in Appendix B, in particular we will use (68):  $\|\gamma\|_E \lesssim L\alpha$  where we recall  $\|\cdot\|_{\mathfrak{S}_2} \leq \|\cdot\|_E$  and we define

$$\tau := \alpha\alpha_r(0). \quad (25)$$

*Remark 4.1.* Here  $\lambda^{-1}$  and  $\tau$  are of the same order  $L\alpha$  but the use of  $\tau$  means we estimate a quantity depending on  $\gamma$  while the use of  $\lambda^{-1}$  means we estimate a quantity depending on  $\psi_\lambda$ .

A direct calculation shows that  $\|\mathcal{P}^0_- |\mathcal{D}^0| \psi_\lambda\|_{L^2} = O(\lambda^{-1})$  and  $\| |\mathcal{D}^0| \psi_\lambda \|_{L^2} = O(1)$ . We will often use

$$\|\pi\psi_\lambda\|_{L^2} \leq \|\gamma\psi_\lambda\|_{L^2} + \|\mathcal{P}^0_- \psi_\lambda\|_{L^2} \lesssim (\tau + \lambda^{-1}). \quad (26)$$

*Notation 4.2.* Let us note  $\phi_\lambda := \frac{(1-\pi)\psi_\lambda}{\| (1-\pi)\psi_\lambda \|_{L^2}}$  and  $N' = |\phi_\lambda\rangle\langle\phi_\lambda|$ .

• Looking at the kernel of  $H =: [|\mathcal{D}^0|, \gamma] = [|\mathcal{D}^0|, \pi]$ ,  $\|H\|_{\mathfrak{S}_2} \leq \|\gamma\|_E$  is immediate.

$$4.1.1 \quad J = \iint (|\gamma(x, y)|^2 + 2\Re(\gamma(x, y), \Phi_\lambda(x, y))) |x-y|^{-1} dx dy.$$

(6) and (68) show that  $\|\gamma\|_S^2 = O(\tau^2)$ . By Cauchy-Schwarz inequality and (22a) ( $G = |f\rangle\langle g|$ )

$$|\langle\gamma, G\rangle_S| \leq \min(\|\gamma\|_S \|G\|_S, 2\|\gamma\|_{\mathfrak{S}_2} \|\nabla|f\rangle\langle g|\|_{L^2}).$$

Now thanks to (22b) and (72):  $\iint |\pi\psi_\lambda(x)|^2 |x-y|^{-1} |\pi\psi_\lambda(y)|^2 dx dy \lesssim \|\pi\psi_\lambda\|_{L^2}^2 \langle |\mathcal{D}^0| \pi\psi_\lambda, \pi\psi_\lambda \rangle$  and  $\langle |\mathcal{D}^0| \pi\psi_\lambda, \pi\psi_\lambda \rangle \leq \|\pi\psi_\lambda\|_{L^2} (\|H\|_B + \|\pi|\mathcal{D}^0| \psi_\lambda\|_{L^2})$ , so we obtain  $(\tau + \lambda^{-1})^4$ .

In the same way

$\iint |\psi_\lambda(x)|^2 |x-y|^{-1} |\pi\psi_\lambda(y)|^2 dx dy \lesssim \|\pi\psi_\lambda\|_{L^2}^2 \langle |\nabla| \psi_\lambda, \psi_\lambda \rangle \lesssim \frac{(\tau + \lambda^{-1})}{\lambda}$  and finally:  $\iint |\psi_\lambda(x)|^2 |x-y|^{-2} |\psi_\lambda(y)|^2 dx dy \leq 4\|\psi_\lambda\|_{L^2} \langle |\nabla|^2 \psi_\lambda, \psi_\lambda \rangle \leq 4\lambda^{-2}$ . Thus  $J = O(\tau^2 + \lambda^{-2}) = O((L\alpha)^2)$ .

**4.1.2**  $I = D(\rho_\gamma, \rho_\gamma) + 2D(\rho_\gamma, |\phi_\lambda|^2)$ .

According to the self-consistent equation satisfied by  $\rho_\gamma$ , we write

$$\widehat{\rho}_\gamma(p) = -\alpha_r(p)\widehat{n}(p) + (1 - \alpha_r(p))\widehat{\rho}_{1,0}(p) + (1 - \alpha_r(p)) \sum_{k=2}^{\infty} \alpha^k \widehat{\rho}_k(p) \quad (27)$$

where we recall that  $\alpha_r(p) = \frac{\alpha B_\Lambda(p)}{1 + \alpha B_\Lambda(p)}$ . Thus

$$\begin{aligned} D(\rho_\gamma, \rho_\gamma) &= 4\pi \int_p \left( \alpha_r(p)^2 |\widehat{n}(p)|^2 + (1 - \alpha_r(p))^2 |\alpha \widehat{\rho}_{1,0}(p)|^2 + (1 - \alpha_r(p))^2 \left| \sum \right|^2 \right. \\ &\quad \left. + 2\Re \left( \alpha_r(p)(1 - \alpha_r(p)) \overline{\widehat{n}(p)} \left( \alpha \widehat{\rho}_{1,0}(p) + \sum \right) + (1 - \alpha_r(p))^2 \overline{\alpha \widehat{\rho}_{1,0}(p)} \sum \right) \right) \frac{dp}{|p|^2}, \end{aligned}$$

and by Cauchy-Schwarz inequality, we just look at  $\int \frac{|\widehat{\rho}(p)|^2}{|p|^2} dp$  with  $\rho = n, \rho_{1,0}, \sum$ . By Proposition 9. in a neighbourhood of 0 *independent* of  $\alpha, \Lambda$  in the regime (9), for  $\varepsilon = \frac{1}{6}$ , there holds ( $|k| = x < r_\varepsilon$ ):

$$\frac{|B_\Lambda(x) - B_\Lambda(0)|}{x} \lesssim (\Lambda^{-1} + x^{1/2}) =: z(x). \quad (28)$$

Then

$$\int_p \frac{\alpha_r(p)^2 |\widehat{n}_\lambda(p)|^2}{|p|^2} dp = \frac{1}{\lambda} \int_p \frac{\alpha_r(\frac{p}{\lambda})^2 |\widehat{n}_1(p)|^2}{|p|^2} dp,$$

For  $\lambda \geq r_\varepsilon^{-4}$  and  $p \in B(0, \lambda^{3/4})$ :  $|B_\Lambda(p/\lambda) - B_\Lambda(0)| \leq \frac{|p|}{\lambda} (z(\lambda^{-1/4}) + K\Lambda^{-1})$ . As  $f_1 : t \in \mathbf{R}^+ \rightarrow \frac{t}{1+t}$  and  $f_2 = f_1^2$  have bounded derivatives (by 1 and 2 respectively), for  $p$  with  $B_\Lambda(p) \neq B_\Lambda(0)$ ,

$$|\alpha_r(p) - \alpha_r(0)| \leq \alpha |B_\Lambda(p) - B_\Lambda(0)|, \quad |\alpha_r(p)^2 - \alpha_r(0)^2| \leq 2\alpha |B_\Lambda(p) - B_\Lambda(0)| \text{ so}$$

$$\begin{aligned} \int_{|p| \leq \lambda^{3/4}} |f_i(\alpha B_\Lambda(p)) - f_i(\alpha B_\Lambda(0))| \frac{|\widehat{n}_\lambda(p)|^2 dp}{|p|^2} &\leq 2\alpha \frac{z(\lambda^{-1/4}) + K\Lambda^{-1}}{\lambda} \int \frac{|\widehat{n}_1(p)|^2 dp}{|p|} \\ &\lesssim \alpha \frac{z(\lambda^{-1/4}) + \Lambda^{-1}}{\lambda} \|n_1\|_c \|\psi_1\|_{L^4}. \end{aligned}$$

As  $f_1(t), f_2(t) \leq t^2$  then

$$\int_{|p| > \lambda^{3/4}} \alpha_r(p)^i \frac{|\widehat{n}_1(p)|^2}{|p|^2} dp \lesssim \lambda^{-3/2} L^i \int |\widehat{n}_1(p)|^2 dp \lesssim \lambda^{-3/2} L^i \|\psi_1\|_{H^1}^2 = O(L^i \lambda^{-3/2})$$

and

**Lemma 4.3.**  $\int_p \alpha_r(p)^i \frac{|\widehat{n}_\lambda(p)|^2}{|p|^2} dp = \alpha_r(0)^i \frac{D(n_1, n_1)}{\lambda} + \underset{\lambda \rightarrow \infty}{o}(L^i \lambda^{-1})$ .

Furthermore  $\int_p \alpha^2 (1 - \alpha_r(p))^2 \frac{|\widehat{\rho}_{1,0}(p)|^2}{|p|^2} dp \lesssim \alpha^2 \|\rho_{1,0}\|_c^2$  where

$$\widehat{\rho}_{1,0}(p) = \frac{2^{-1}}{(2\pi)^{3/2}} \int_{|l+k/2|, |l-k/2| < \Lambda} \text{Tr}_{\mathbf{C}^4}(\widehat{R}_{\gamma'}(l+k/2, l-k/2) M(l-k/2, l+k/2)) dl, \quad (29)$$

writing  $R(\gamma') = \sum_{k \geq 1} (R_{\gamma_{k+1}} - R_{\gamma_k}) + R_{\gamma_1} + R_N$  we propagate by linearity in (29): thanks to (65b) and (70a) there holds

$$\alpha^2 \|\rho_{1,0}\|_c^2 \lesssim \alpha^2 (\lambda^{-3} \|\psi_\lambda\|_{H^1}^4 + L\sqrt{L\alpha} D(n_1, n_1) + O(\alpha\sqrt{L\alpha})).$$

Then  $\|\sum\|_c \leq \alpha^2$  is immediate with the estimates of [5].

Now  $|\phi_\lambda|^2(x) = \frac{1}{1 - \|\pi\psi_\lambda\|_{L^2}^2} (|\psi_\lambda(x)|^2 + |\pi\psi_\lambda(x)|^2 - 2\langle \pi\psi_\lambda(x), \psi_\lambda(x) \rangle)$ . For the two last terms, we use Cauchy-Schwarz inequality to get thanks to (22b)  $K \frac{(\tau + \lambda^{-1})^2}{\lambda}$  (cf 4.1.1 and  $|\langle \pi\psi_\lambda(x), \psi_\lambda(x) \rangle| \leq |\pi\psi_\lambda(x)| |\psi_\lambda(x)|$  etc.)

Then  $D(\rho_\gamma, n_\lambda) = -4\pi \int \alpha_r(p) |\widehat{n_\lambda}(p)|^2 \frac{dp}{|p|^2} + D(\alpha\rho_{1,0}, n_\lambda) + D(\sum_{k \geq 2} \alpha^k \rho_k, n_\lambda)$ .  
 With the same method as for  $D(\rho_\gamma, \rho_\gamma)$  and Cauchy-Schwarz inequality:

$$D(\rho_\gamma, |\phi_\lambda|^2) = -\alpha_r(0)D(n_\lambda, n_\lambda) + \underset{\lambda \rightarrow \infty}{o} \left( \frac{L}{\lambda} \right).$$

Since  $\frac{1}{1 - \|\pi\psi_\lambda\|_{L^2}^2} = 1 + O((\tau + \lambda^{-1})^2)$ , we finally obtain:

$$I = -\frac{2\alpha_r(0) + \alpha_r(0)^2}{\lambda} D(n_1, n_1) + \underset{\lambda \rightarrow \infty}{o} \left( \frac{L}{\lambda} \right) \quad (30)$$

#### 4.1.3 $\text{Tr}_{\mathcal{P}^0}(\mathcal{D}^0 N') = \text{Tr}(DN')$ = $\langle \mathcal{D}^0 \phi_\lambda, \phi_\lambda \rangle$ .

We emphasize that  $\psi_\lambda$  has no lower part as a spinor.

As in **a**) there holds  $|\langle \mathcal{D}^0 \pi\psi_\lambda, \pi\psi_\lambda \rangle| \leq \|\pi\psi_\lambda\|_{L^2} \left\| \mathcal{D}^0 |\pi\psi_\lambda| \right\|_{L^2}$  and thanks to (72),  
 $\left\| \mathcal{D}^0 |\pi\psi_\lambda| \right\|_{L^2} \lesssim K(\tau + \lambda^{-1})$  so  $|\langle \mathcal{D}^0 \pi\psi_\lambda, \pi\psi_\lambda \rangle| = o((L\alpha)^2)$ .

Hence  $\langle \mathcal{D}^0 \phi_\lambda, \phi_\lambda \rangle = \frac{\langle \mathcal{D}^0 \psi_\lambda, \psi_\lambda \rangle}{1 - \|\pi\psi_\lambda\|_{L^2}^2} + \langle \mathcal{D}^0 |\mathcal{P}^0 \psi_\lambda, \psi_\lambda \rangle + o((L\alpha)^2)$ . Indeed  
 $\langle \mathcal{D}^0 \psi_\lambda, \pi\psi_\lambda \rangle = \langle \pi \mathcal{D}^0 \psi_\lambda, \pi\psi_\lambda \rangle$  etc.

*Notation 4.4.* We will write  $\langle g_0 \psi, \psi \rangle$  for  $(2\pi)^{-3} \int g_0(p) |\widehat{\psi}(p)|^2 dp$  etc.

As  $g'_0(0) = 0$  and  $\|g''_0\|_\infty \lesssim \alpha$  and the  $(g'_1)_{\alpha, \Lambda}$ 's are uniformly continuous in a neighbourhood of 0 (cf Proposition 5. in Appendix A)

$$\begin{aligned} \frac{\langle \mathcal{D}^0 \psi_\lambda, \psi_\lambda \rangle}{1 - \|\pi\psi_\lambda\|_{L^2}^2} &= \langle g_0 \psi_\lambda, \psi_\lambda \rangle (1 + \langle \mathcal{P}^0 \psi_\lambda, \psi_\lambda \rangle) + o((L\alpha)^2) \\ &= g_0(0) + \frac{g_0(0)}{4} \langle \frac{g_1^2}{g_0^2} \psi_\lambda, \psi_\lambda \rangle + o((L\alpha)^2) \\ &= g_0(0) + \frac{g_1^2(0)}{4g_0(0)\lambda^2} \langle |\nabla|^2 \psi_1, \psi_1 \rangle + o((L\alpha)^2). \end{aligned}$$

Furthermore  $\langle |\mathcal{D}^0| \mathcal{P}^0 \psi_\lambda, \psi_\lambda \rangle = \frac{1}{2} \langle (|\mathcal{D}^0| - g_0) \psi_\lambda, \psi_\lambda \rangle = \frac{1}{4g_0(0)} \langle g_1^2 \psi_\lambda, \psi_\lambda \rangle + o(\lambda^{-2})$ .  
 Finally

$$\text{Tr}_{\mathcal{P}^0}(\mathcal{D}^0 N') = \langle \mathcal{D}^0 \phi_\lambda, \phi_\lambda \rangle = g_0(0) + \frac{g_1^2(0)}{2\lambda^2 g_0(0)} \langle |\nabla|^2 \psi_1, \psi_1 \rangle + o((L\alpha)^2) \quad (31)$$

#### 4.1.4 $\text{Tr}_{\mathcal{P}^0}(D\gamma)$ .

*Notation 4.5.* Let us write  $B = R'_\gamma - \varphi'_\gamma = R(\gamma + N) - (\rho_\gamma + n) \star |\cdot|^{-1}$ .

*Remark 4.6.* Let us recall Lemma 1.[5]: if  $P, \Pi$  are two projectors such that  $P - \Pi \in \mathfrak{S}_2$  then

$$Q \in \mathfrak{S}_1^P \iff Q \in \mathfrak{S}_1^\Pi \text{ and then } \text{Tr}_P(Q) = \text{Tr}_\Pi(Q).$$

Here we will take  $P = \mathcal{P}^0$  and  $\Pi := \chi_{(-\infty, 0)}(\mathcal{D}^0 + \alpha B)$ : formally (cf [9])

$$\text{Tr}_{\mathcal{P}^0}((\mathcal{D}^0 + \alpha B)\gamma) \stackrel{?}{=} \text{Tr}(|\mathcal{D}^0|\gamma^2) + \alpha \text{Tr}_{\mathcal{P}^0}(B\gamma) \quad (32a)$$

$$\text{Tr}_{\mathcal{P}^0}((\mathcal{D}^0 + \alpha B)\gamma) \stackrel{?}{=} -\text{Tr}(|\mathcal{D}^0 + \alpha B|\gamma^2) \stackrel{?}{=} -\text{Tr}(|\mathcal{D}^0|\gamma^2) + o(\text{Tr}(|\mathcal{D}^0|\gamma^2)) \quad (32b)$$

so we would like to show that  $\text{Tr}(|\mathcal{D}^0|\gamma^2) = -\frac{\alpha}{2} \text{Tr}_{\mathcal{P}^0}(B\gamma) + o(\tau^2)$ .

Two problems arise: are  $B\gamma, BQ_k(\gamma)$  in  $\mathfrak{S}_1^{\mathcal{P}^0}$  and how can we evaluate  $|\mathcal{D}^0 + \alpha B| - |\mathcal{D}^0|$ ? We will deal with the last question in Appendix C and prove

#### Lemma 4.7.

$$\text{Tr}(|\mathcal{D}^0 + \alpha B|\gamma^2) = \text{Tr}(|\mathcal{D}^0|\gamma^2) + O(\alpha\tau^2).$$

Supposing those facts are true we get  $\text{Tr}(|\mathcal{D}^0|\gamma^2) = -\frac{\alpha}{2} \text{Tr}_{\mathcal{P}^0}(B\gamma) + O(\alpha\tau^2)$ .  
 We use (23c):

$$\|R'_\gamma \gamma\|_{\mathfrak{S}_1} \leq \|R(\gamma) |\mathcal{D}^0|^{-1/2}\|_{\mathfrak{S}_2} \| |\mathcal{D}^0|^{1/2} \gamma \|_{\mathfrak{S}_2} + \|R(N)\|_{\mathfrak{S}_2} \|\gamma\|_{\mathfrak{S}_2} \lesssim (\tau + \lambda^{-1})\tau.$$

Then let us prove  $\text{Tr}_{\mathcal{P}^0}(\varphi'_\gamma) = D(\rho_\gamma + n_\lambda, \rho_\gamma)$ . In fact if  $Q \in \mathfrak{S}_1^{\mathcal{P}^0}$  and if  $\int \text{Tr}(\widehat{Q}(p, p))dp$  exists then it is equal to  $\text{Tr}_{\mathcal{P}^0}(Q)$  for  $\mathcal{P}^0 = f(i\nabla)$ : in Fourier space  $\text{Tr}_{\mathbb{C}^4}(\widehat{\mathcal{P}}^0(p)\widehat{Q}(p, p)\widehat{\mathcal{P}}^0(p)) = 0$ .

$$\begin{aligned} (2\pi)^{-3/2} \iint_{|p|, |q| < \Lambda} \widehat{\varphi}'_\gamma(p - q)(\text{Tr}(\widehat{\gamma}(p, q)))^* dpdq &= (2\pi)^{-3/2} \iint_{|u + \frac{k}{2}|, |u - \frac{k}{2}| < \Lambda} \widehat{\varphi}'_\gamma(k)(\text{Tr}(\widehat{\gamma}(u + k/2, u - k/2)))^* dudk \\ &= \int_k \widehat{\varphi}'_\gamma(k)\widehat{\rho}_\gamma(k)^* dk = 4\pi \int_k \frac{\widehat{\rho}'_\gamma(k)\widehat{\rho}_\gamma(k)^*}{|k|^2} dk = D(\rho_\gamma, \rho'_\gamma). \end{aligned}$$

Like in the calculation of  $I$  there holds

$$D(\rho_\gamma, \rho_\gamma + n_\lambda) = \frac{\alpha_r(0)^2 - \alpha_r(0)}{\lambda} D(n_1, n_1) + o\left(\frac{L}{\lambda}\right),$$

so

$$\text{Tr}(|\mathcal{D}^0|\gamma^2) = \alpha \frac{\alpha_r(0)^2 - \alpha_r(0)}{2\lambda} D(n_1, n_1) + o\left(\frac{L}{\lambda}\right). \quad (33)$$

*Remark 4.8.* The calculation above is correct if  $\widehat{\gamma}(p, q) \in \mathcal{C}^0(B(0, \Lambda)^2)$ :

$$A_1 = \iint_{|u \pm \frac{k}{2}| < \Lambda} \frac{|\widehat{\rho}(k)|}{|k|^2} |\widehat{\gamma}(u + \frac{k}{2}, u - \frac{k}{2})| dudk \lesssim \Lambda^{3/2} \|\rho\|_{\mathcal{C}} (\Lambda^{3/2} \|\widehat{\gamma}\|_{L^\infty} + \|\gamma\|_{\mathfrak{S}_2}).$$

We conclude by continuity of  $Q \in \mathfrak{S}_1^{\mathcal{P}^0} \mapsto \rho_Q \in \mathcal{C}$  shown in [7] and of  $Q \in \mathfrak{S}_1^{\mathcal{P}^0} \mapsto \text{Tr}_{\mathcal{P}^0}(\varphi'_\gamma Q)$  and the density of  $\mathcal{C}^0(B(0, \Lambda)^2)$  in  $\mathcal{F}(\mathfrak{S}_1^{\mathcal{P}^0}(\mathfrak{H}_\Lambda))$ .

Indeed, using the notations of [5] and [4]:  $\gamma^{e_1 e_2} = \mathcal{P}^0_{e_1} \gamma \mathcal{P}^0_{e_2}$ , there holds (cf 3.3.):

$$(\varphi'_\gamma Q)^{-} = (\mathcal{P}^0_-[\varphi'_\gamma, \mathcal{P}^0_+]|D^0|^{-1/2})|D^0|^{1/2} Q^{+-} + (\varphi'_\gamma |D^0|^{-1/2})^{-} |D^0|^{1/2} Q^{-} \in \mathfrak{S}_1(\mathfrak{H}_\Lambda) \quad (34)$$

and so  $|\text{Tr}_{\mathcal{P}^0}(\varphi'_\gamma Q)| \leq \|\rho'_\gamma\|_{\mathcal{C}} \Lambda^{1/2} (\log(\Lambda))^{1/6} \|Q\|_{\mathfrak{S}_{1, \mathcal{P}^0}}$  with

$$\|Q\|_{\mathfrak{S}_{1, \mathcal{P}^0}} := \|Q^{-}\|_{\mathfrak{S}_1} + \|Q^{+-}\|_{\mathfrak{S}_1} + \|Q^{-+}\|_{\mathfrak{S}_2} + \|Q^{+-}\|_{\mathfrak{S}_2}. \quad (35)$$

$BQ_k(\gamma) \in \mathfrak{S}_1^{\mathcal{P}^0}(\mathfrak{H}_\Lambda)$ . We recall:  $|D^0|^{-1/2} \varphi'_\gamma \in \mathfrak{S}_2(\mathfrak{H}_\Lambda)$ ,  $|D^0| \in \mathcal{B}(\mathfrak{H}_\Lambda)$ ,  $|D^0|^{-1/2} R'_\gamma \in \mathfrak{S}_2(\mathfrak{H}_\Lambda)$ , and thanks to [5]  $\gamma^{++}, \gamma^{--} \in \mathfrak{S}_1(\mathfrak{H}_\Lambda)$ .

These facts enable us to show: for  $k \geq 4$ ,  $Q_k(\gamma) \in \mathfrak{S}_1(\mathfrak{H}_\Lambda)$  since  $Q_k^{+ \dots +} = Q_k^{- \dots -} = 0$  (by the residuum formula in the formula of the kernel in Fourier space as shown in [5]).

We can adapt Lemma of [4] and prove in the same way:

**Lemma 4.9.** For  $0 \leq t < 1/2$  there holds with  $A = \|\rho'_\gamma\|_{\mathcal{C}} + \|\gamma'\|_S$

$$\begin{aligned} \||D^0|^{1/2+t} Q_2(\gamma)\|_{\mathfrak{S}_{3/2}} &\leq K_t A^2, \quad \||D^0| Q_3(\gamma)\|_{\mathfrak{S}_{6/5}} \leq K_t A^3, \\ \||D^0|^t \widetilde{Q}_4(\gamma) |D^0|^t\|_{\mathfrak{S}_1} &\leq K_t \left( A^4 + \alpha A^5 + \alpha^2 \left( \frac{\|\rho'_\gamma\|_{\mathcal{C}}^6}{\text{dist}(0, \sigma(\mathcal{D}^0 + \alpha B))} + A^6 \right) \right). \end{aligned}$$

where  $\gamma = \sum_{j=1}^{k-1} \alpha^j Q_j + \alpha^k \widetilde{Q}_k$ .

Using the same method as in [4] with  $D(x) := \mathcal{D}^0 + xB(\|\rho'_\gamma\|_{\mathcal{C}} + \|\gamma\|_S)^{-1}$  (there exists  $0 < x_0 \in \mathbf{R}^+$  with  $|D(x)| \geq \frac{1}{2}$ ,  $-x_0 < x < x_0$ ) we obtain

**Lemma 4.10.** Let  $0 \leq t < 1/2$ , then there exists  $K_t > 0$  such that

$$\begin{aligned} \||D^0|^t Q_2^{\pm\pm}(\gamma) |D^0|^t\|_{\mathfrak{S}_1} &\leq K_t \\ \||D^0|^t Q_3^{\pm\pm}(\gamma) |D^0|^t\|_{\mathfrak{S}_1} &\leq K_t \end{aligned}$$

Therefore  $\gamma, \tilde{Q}_4(\gamma), Q_3(\gamma), Q_2(\gamma) \in \mathfrak{S}_1^{\mathcal{P}^0}(\mathfrak{H}_\Lambda)$  and so  $Q_1(\gamma) \in \mathfrak{S}_1^{\mathcal{P}^0}(\mathfrak{H}_\Lambda)$ .  
Then as in (34):  $\mathcal{P}^0(BQ_k)\mathcal{P}^0_- = \underbrace{\mathcal{P}^0[B, \mathcal{P}^0_+]}_{\in \mathfrak{S}_2(\mathfrak{H}_\Lambda)} \underbrace{|\mathcal{D}^0|^{-1/2} |\mathcal{D}^0|^{1/2} Q_k^{+-}}_{\in \mathfrak{S}_2(\mathfrak{H}_\Lambda)} + \underbrace{\mathcal{P}^0_- B |\mathcal{D}^0|^{-1}}_{\in \mathfrak{S}_6(\mathfrak{H}_\Lambda)} \underbrace{|\mathcal{D}^0| Q_k^{--}}_{\in \mathfrak{S}_1(\mathfrak{H}_\Lambda)}$ ,

so  $BQ_k(\gamma) \in \mathfrak{S}_1^{\mathcal{P}^0}(\mathfrak{H}_\Lambda)$ .

*Remark 4.11.* As  $\Lambda \rightarrow +\infty$  there holds  $\langle |\mathcal{D}^0|^2 \psi_1, \psi_1 \rangle - D(n_1, n_1) = E_{\text{CP}} + o(1)$ .  
In fact  $\psi_1 = (\phi_1, 0)^T$  where  $\phi_1 = P_\Lambda \phi'_1 / \|P_\Lambda \phi'_1\|_{L^2}$  and  $\phi'_1$  is the minimizer of  
Choquard-Pekar energy.  $P_\Lambda$  is the projector onto  $\mathfrak{H}_\Lambda$  and by so  $\phi_1^{(\Lambda)} \xrightarrow[\Lambda \rightarrow +\infty]{H^1} \phi'_1$ .

Then writing  $n' = |\phi'_1|^2$  there holds by (22b)

$$\begin{aligned} \left| \|n_1\|_c - \|n'\|_c \right| &\leq \|n_1 - n'\|_c \leq (\langle |\nabla| \psi_1, \psi_1 \rangle + \langle |\nabla| \phi'_1, \phi'_1 \rangle) \left| \|\psi_1\|_{L^2}^2 - \|\phi'_1\|_{L^2}^2 \right| \\ &\leq \langle |\nabla| \phi'_1, \phi'_1 \rangle \left| \|\psi_1\|_{L^2}^2 - \|\phi'_1\|_{L^2}^2 \right| \xrightarrow[\Lambda \rightarrow \infty]{} 0. \end{aligned}$$

## 4.2 Proof of Proposition 2.

Let us prove now the binding inequalities for  $0 < q < 1$ . According to Lieb's principle ([7]) for each  $q$  we can take minimizing sequences for  $E(q)$  of the form

$$Q_{(k)} = P_{(k)} - \mathcal{P}^0_- + q|\psi_k\rangle\langle\psi_k|, \quad Q_{(k)} \in \mathcal{Q}_\Lambda, P_k^2 = P_k, P_k \psi_k = 0, \text{Tr}_{\mathcal{P}^0}(P_k - \mathcal{P}^0_-) = 0, k \in \mathbf{N} \quad (36)$$

and we note as before  $\gamma_k = P_k - \mathcal{P}^0_-$ ,  $n_k = |\psi_k|^2$ ,  $N_k = |\psi_k\rangle\langle\psi_k|$ . We will forget to emphasize the dependence in  $k$ .

Writing  $I_\gamma(N) = \alpha \Re \left( D(\rho_\gamma, n) - \iint \frac{\text{Tr}_{\mathbf{C}^4}(N(x,y)^* \gamma(x,y))}{|x-y|} dx dy \right)$ ;  $\mathcal{E}(Q)$  can be written:

$$\mathcal{E}(Q) = \mathcal{E}(\gamma) + q\langle \mathcal{D}^0 \psi, \psi \rangle + qI_\gamma(N) = (1-q)\mathcal{E}(\gamma) + q\mathcal{E}(\gamma + N).$$

Taking the lim inf, we obtain

$$E(q) = \liminf_{k \rightarrow \infty} ((1-q)\mathcal{E}(\gamma) + q\mathcal{E}(\gamma + N)) \geq (1-q) \liminf_{k \rightarrow \infty} \mathcal{E}(\gamma) + qE(1).$$

Either  $x = \liminf_{k \rightarrow \infty} \mathcal{E}(\gamma) > 0$  and  $E(q) > qE(1)$  or  $x = 0$ . What happens in the second case? Up to the extraction of a subsequence we can assume that  $\liminf \mathcal{E}(\gamma)$  is a limit. Thanks to (6) it implies  $\text{Tr}(|\mathcal{D}^0| \gamma^2) + D(\rho_\gamma, \rho_\gamma) \xrightarrow[k \rightarrow \infty]{} 0$ .

As  $P_k \psi_k = 0$  we obtain  $\|\mathcal{P}^0_+ \psi\|^2 = \|\psi\|^2 - \|\mathcal{P}^0_- \psi\|^2 = 1 - \|\gamma \psi\|^2 \rightarrow 1$  and  $\langle \mathcal{D}^0 \psi, \psi \rangle = \langle |\mathcal{D}^0| \psi^+, \psi^+ \rangle + \langle |\mathcal{D}^0| \gamma \psi, \psi^- \rangle$  where  $\psi^\varepsilon = \mathcal{P}^0_\varepsilon \psi$ . As  $\mathcal{D}^0$  is bounded on  $\mathfrak{H}_\Lambda$  and  $\|\psi\|_2 = 1$ ,  $\liminf_{k \rightarrow \infty} \langle \mathcal{D}^0 \psi, \psi \rangle \geq m(\alpha)$ , so by Cauchy-Schwartz inequality  $I_\gamma(N) \rightarrow 0$  and

$$\liminf_{k \rightarrow \infty} \mathcal{E}(Q_k) = E(q) \geq \liminf_{k \rightarrow \infty} \mathcal{E}(\gamma) + q \liminf_{k \rightarrow \infty} I_\gamma(N) + q \liminf_{k \rightarrow \infty} \langle \mathcal{D}^0 \psi, \psi \rangle \geq qm(\alpha).$$

It implies  $E(q) = qm(\alpha)$ , but we can use the method of Section 4.1. to prove that  $E(q) < qm(\alpha)$  for sufficiently small  $\alpha$  and  $L$  in regard with  $q$ :

$$Q + \mathcal{P}^0_- = \chi_{(-\infty, 0)} \left( \mathcal{D}^0 + \alpha (\varphi_\gamma + qm \star |\cdot|^{-1 - \frac{\gamma(x,y) + qN(x,y)}{|x-y|}}) \right) + q \frac{|(1-\pi)\psi_\lambda\rangle\langle(1-\pi)\psi_\lambda|}{1 - \|\pi\psi_\lambda\|_{L^2}^2}.$$

If we assume that  $E(q) = qm(\alpha)$  once  $E(1) < m(\alpha)$  has been proven, we also obtain  $E(q) > qE(1)$ . We thus get  $E(q) + E(1-q) > qE(1) + (1-q)E(1) = E(1)$ .

There remains the case  $q > 1$ . However it has been proven in [7] that for each integer  $N$ ,  $E$  is concave on  $[N, N+1]$ . Thus thanks to (6) there holds

$$E(q) \geq q(1 - \alpha \frac{\pi}{4})m(\alpha)$$

and it suffices that  $E(2) > E(1)$  to obtain *in fine*  $E(q) > E(1)$  for  $q > 1$ . For  $\alpha < \frac{2}{\pi}$  it is therefore true and as  $E(q) > 0$  for  $q \neq 0$  we obtain the binding inequalities for  $q > 1$  and hence for all  $q$ .

### 4.3 Proof of Theorem 3.

*Notation 4.12.*

- Let  $Q = \gamma + |\psi\rangle\langle\psi|$  be the minimizer written with the notation of Section 2.
- As before  $N = |\psi\rangle\langle\psi|, n = |\psi|^2$ .
- We have  $|\psi\rangle\langle\psi| = \chi_{(0,\mu]}(D_Q)$  with  $D_Q := \mathcal{D}^0 + \alpha(R'_\gamma - \varphi'_\gamma)$ .
- $\mu$  is chosen such that  $D_Q\psi = |D_Q|\psi = \mu\psi$ :  $\mu \leq m(\alpha)$ .
- We note  $C_0^2 := \frac{2g'_1(0)^2}{(\alpha\alpha_r(0))^2 m(\alpha)}$  and  $c := \frac{(g'_1(0))^2}{\alpha\alpha_r(0)m(\alpha)}$ .
- As  $(R(N) - \varphi_{|\psi|^2})\psi = 0$ , there holds

$$(\mathcal{D}^0 + \alpha(R(\gamma) - \varphi_\gamma))\psi = \mu\psi \quad (37)$$

- We note  $v_\gamma := \varphi_\gamma, b_\gamma := v_\gamma - R_\gamma, d := \mathcal{D}^0$ . We remark:

$$\langle v_\gamma\psi, \psi \rangle = D(\rho_\gamma, n), \quad |\langle R_\gamma\psi, \psi \rangle| \leq \|\gamma\|_S \|n\|_C. \quad (38)$$

- We mean by  $\langle g_0\psi, \psi \rangle$ :  $(2\pi)^{-3} \int_p g_0(p) |\widehat{\psi}(p)|^2 dp$  etc.

*Remark 4.13.* Throughout this section we will prove estimates more and more precise of the norms of  $\psi, n, \gamma, \rho_\gamma$ .

#### 4.3.1 $\|\psi\|_{H^{3/2}} = O(1)$ .

First let us prove that we can construct  $Q$  as a fixed point with the norm of [5]: it suffices to prove that  $\|n\|_C, \|N\|_Q = O(1)$  and as  $\|N\|_Q \lesssim \|\psi\|_{H^{3/2}}^2$  we will first prove Lemma 2.4.

Thanks to 38 and (22b) there holds

$$\langle \mathcal{D}^0\psi, \psi \rangle = \langle D_Q\psi, \psi \rangle - \alpha \langle b_\gamma\psi, \psi \rangle = \langle |D_Q|\psi, \psi \rangle + O(\alpha\sqrt{\langle |\mathcal{D}^0|\psi, \psi \rangle} (\|\gamma\|_S + \|\rho_\gamma\|_C)).$$

Thanks to C.3. and the fact that  $\|N\|_S^2 - \|n\|_C^2 = 0$  there holds by Cauchy-Schwartz inequality and (22b):

$$\begin{aligned} \mathcal{E}(Q) &= \mathcal{E}(\gamma) + \langle \mathcal{D}^0\psi, \psi \rangle + \alpha \Re(D(\rho, n) - \langle \gamma, N \rangle_S) \\ &\geq (1 - K\alpha) \text{Tr}(|\mathcal{D}^0|\gamma^2) + \frac{\alpha}{2} D(\rho_\gamma, \rho_\gamma) + (1 - C_2\alpha) \langle |\mathcal{D}^0|\psi, \psi \rangle - \alpha\sqrt{\langle |\mathcal{D}^0|\psi, \psi \rangle} (\|\gamma\|_F + \|\rho_\gamma\|_C), \end{aligned}$$

as  $\mathcal{E}(Q) \leq m(\alpha)$  we thus have

$$\text{Tr}(|\mathcal{D}^0|\gamma^2) + \alpha D(\rho_\gamma, \rho_\gamma) + \langle |\mathcal{D}^0|\psi, \psi \rangle = O(1). \quad (39)$$

Thanks to (37) we have

$$\langle \mathcal{D}^0\psi, \mathcal{D}^0\psi \rangle = \mu^2 \|\psi\|_{L^2}^2 - 2\alpha\mu \Re \langle b_\gamma\psi, \psi \rangle + \alpha^2 \|b_\gamma\psi\|_{L^2}^2. \quad (40)$$

Then as

$$|\langle b_\gamma\psi, f \rangle| \leq \sqrt{\frac{\pi}{2}} \|\nabla^{1/2}\psi\|_{L^2} (\|\rho_\gamma\|_C + \|\gamma\|_S) \|f\|_{L^2},$$

by duality

$$\|b_\gamma\psi\|_{L^2} \leq \sqrt{\frac{\pi}{2}} \|\nabla^{1/2}\psi\|_{L^2} (\|\rho_\gamma\|_C + \|\gamma\|_S). \quad (41)$$

Furthermore

$$\begin{aligned} \alpha \langle v_\gamma\psi, \psi \rangle &= \alpha D(\rho_\gamma, n) = O(\sqrt{\alpha \|\nabla\psi\|_{L^2}}), & \alpha^2 |\langle v_\gamma\psi, v_\gamma\psi \rangle| &\lesssim \alpha(\alpha \|\rho_\gamma\|_C^2) \langle |\nabla|\psi, \psi \rangle, \\ |\langle R_\gamma\psi, \psi \rangle| &\leq \|\gamma\|_S \|n\|_C = O(1), & |\langle R_\gamma\psi, R_\gamma\psi \rangle| &\lesssim \|\gamma\|_S^2 \langle |\nabla|\psi, \psi \rangle \end{aligned}$$

while there holds thanks to Proposition 5.

$$|\langle (g_0^2 - g_0(0)^2)\psi, \psi \rangle| \leq K\alpha \langle |\nabla|^2 \psi, \psi \rangle.$$

Thus  $\|\psi\|_{H^1} = O(1)$  and

$$\langle g_1^2 \psi, \psi \rangle \lesssim \alpha^{2/3} \quad (42)$$

In particular there holds  $\|\psi\|_{L^4} \lesssim \|\widehat{\psi}\|_{L^{4/3}} \lesssim \|\psi\|_{H^1}$  and  $\|n\|_{L^2} = O(1)$ . Moreover there holds  $D(n, n) \leq \frac{\pi}{2} \langle |\nabla| \psi, \psi \rangle$  such that  $\|n\|_{\mathfrak{C}} = O(1)$ .

Then by (37) we have  $|d|^2 \psi = \mu d\psi - \alpha db_\gamma \psi$  such that

$$\langle |d|^3 \psi, \psi \rangle = \mu \langle |d| d\psi, \psi \rangle + \alpha \langle |d|^{1/2} (R_\gamma - v_\gamma) |d|^{-3/2} |d|^{3/2} \psi, |d|^{1/2} \psi \rangle.$$

Then thanks to (23b) and C.2., writing

$$|d|^{1/2} b_\gamma |d|^{-3/2} = [|d|^{1/2}, b_\gamma] |d|^{-3/2} + b_\gamma |d|^{-1}$$

we get  $\| |d|^{1/2} b_\gamma |d|^{-3/2} \|_{\mathfrak{B}} \lesssim (\|\gamma\|_S + \|\rho_\gamma\|_C) + \sqrt{\iint \widetilde{E}(p-q) \widetilde{E}(p+q) |\widehat{\gamma}(p, q)|^2 dpdq}$ .

Thanks to Remark 3.2 and the fact that

$$\int \frac{\widetilde{E}(p)}{|p|^2} |\widehat{n}(p)|^2 dp, \quad \iint \widetilde{E}(p-q) \widetilde{E}(p+q) |\widehat{N}(p, q)|^2 dpdq \lesssim 1,$$

we can apply the fixed point method *with the choice of norms*

$$\star \|q_0\|_{\mathfrak{Q}}^2 := \iint \widetilde{E}(p-q) \widetilde{E}(p+q) |\widehat{q}_0(p, q)|^2 dpdq, \text{ etc.}$$

and construct this minimizer as a fixed point with these norms. In the same way as in Appendix B we get that  $\star \|\gamma\|_{\mathfrak{Q}} \lesssim 1$ : we obtain  $\langle |d|^3 \psi, \psi \rangle = O(1)$ . Therefore we can apply the fixed point method with the norm of [5]:  $\|\cdot\|_{\mathfrak{Q}}, \|\cdot\|_{\mathfrak{C}}$  etc.

#### 4.3.2 $\langle |\nabla|^2 \psi, \psi \rangle = O((\alpha \alpha_r(0))^2)$ .

We note  $x = (\langle g_1^2 \psi, \psi \rangle)^{1/4}$ . Thanks to (69), (70a) and B.3 (that gives  $\|\rho_{1,0}(N)\|_C, \|n\|_{L^2} \lesssim (\alpha^{1/3})^{3/2} = \alpha^{1/2}$  with (42)), there holds

$$\|\rho_\gamma\|_C \lesssim \alpha^{3/2} + (L\alpha)^{3/2} + Lx + \alpha \sqrt{L\alpha} x^2 \quad (43a)$$

$$\|\gamma\|_S \lesssim L\alpha + \sqrt{L\alpha} x + \alpha x^2. \quad (43b)$$

Thus going back to (40) we have (cf 5. for  $\|g_0''\|_\infty$ )

$$\begin{aligned} \langle d^2 \psi, \psi \rangle &= x^4 + m(\alpha)^2 + \langle 2g_0 g_0'' |\nabla|^2 \psi, \psi \rangle = x^4 + m(\alpha) + O(\alpha x^4) \\ \alpha \langle b_\gamma \psi, \psi \rangle + \alpha^2 \|b_\gamma \psi\|_{L^2}^2 &\leq K_1 \alpha^{5/2} x + K_2 L \alpha x^2 + K_3 \alpha^2 x^3 + K_4 (L\alpha)^2 x^4 + K_6 \alpha^4 x^6 \\ \mu^2 \|\psi\|_{L^2}^2 &\leq m(\alpha)^2. \end{aligned}$$

As  $x = O(\alpha^{1/6})$ ,  $\alpha^4 x^6 = O(\alpha^5) = O((L\alpha)^2)$ , therefore

$$x^4 \leq k_0 c^{-2} + k_1 \alpha^{5/2} x + k_2 (L\alpha) x^2 + k_3 \alpha^2 x^3. \quad (44)$$

Finally

$$x \leq k_0^{1/4} c^{-1/2} + k_1^{1/3} \alpha^{5/6} + k_2^{1/2} (L\alpha)^{1/2} + k_3 \alpha^2 \lesssim (L\alpha)^{1/2} + \alpha^{5/6}, \quad (45)$$

and there holds  $x^4 \leq K(L\alpha)^2 = O(c^{-2})$  provided that

$$\alpha^{5/3} = O(L\alpha) \Leftrightarrow \alpha \log(\Lambda)^3 \geq K > 0.$$

Thus the same estimates as for the test function hold for the minimizer:

$$\begin{aligned} \|\gamma\|_{\mathfrak{Q}} &\lesssim \alpha, & \|\rho_\gamma\|_{\mathfrak{C}} &\lesssim L\sqrt{L\alpha}, \\ \|\gamma\|_{\mathfrak{E}} &\lesssim L\alpha, & \|\rho_\gamma\|_C &\lesssim L\sqrt{L\alpha} \end{aligned}$$

where for  $\|\rho_\gamma\|_C, \|\rho_\gamma\|_{\mathfrak{C}}$  we use now  $\|\rho_{1,0}(N)\|_C^2 \lesssim c^{-3}$  by B.3.

### 4.3.3 The spinor $\psi$ .

*Remark 4.14.* We follow now the path of [7] and [3].

We consider the problem associated with  $E_{c=1,\alpha,\Lambda}$ . As in [7] we note

$$U_c^* : \begin{array}{l} \mathfrak{H}_\Lambda \rightarrow \mathfrak{H}_{c\Lambda} \\ \phi \mapsto c^{3/2}\phi(c(\cdot)), \end{array}$$

and so  $U_c\phi(x) = c^{-3/2}\phi(x/c)$ .

There holds a scaling correspondence between  $(1, \alpha, \Lambda)$  and  $(c, c\alpha, c\Lambda)$  :

$$E_{c,c\alpha,c\Lambda}(U_c^*QU_c) = c^2E_{1,\alpha,\Lambda}(Q).$$

To distinguish the objects of  $(c, c\alpha, c\Lambda)$  we underline them:

$$\begin{array}{l} \underline{\psi}(x) = U_c^*\psi(x) = c^{3/2}\psi(cx), \\ \underline{\gamma}(x, y) = U_c^*\gamma U_c(x, y) = c^3\gamma(cx, cy), \\ \underline{\rho}_\gamma(x) = c^3\rho_\gamma(cx), \underline{v} = \rho_\gamma \star |\cdot|^{-1}, \\ \underline{R}(x, y) = \underline{\gamma}(x, y)|x - y|^{-1}, \end{array} \quad \left\{ \begin{array}{l} \underline{\mathcal{D}}^0 = c^2U_c^*\mathcal{D}^0U_c = \underline{m}c^2\beta + cT, \\ \underline{m} = g_0(-i\nabla/c), \\ T = cg_1(-i\nabla/c)\alpha \cdot \frac{-i\nabla}{|\nabla|}, \\ D = cg_1(-i\nabla/c)\sigma \cdot \frac{-i\nabla}{|\nabla|}. \end{array} \right.$$

There holds  $|\nabla| \leq |D| \leq C_1|\nabla|$  and

$$\left\{ \begin{array}{l} \|\underline{\gamma}\|_S = \sqrt{c}|\gamma|_S, \\ \|\underline{\rho}_\gamma\|_c = \sqrt{c}|\rho_\gamma|_c \end{array} \right. \quad \text{so} \quad \left\{ \begin{array}{l} \||D^0|^{-1/2}\underline{R}\|_{\mathfrak{B}} \lesssim \|\gamma\|_S = \sqrt{c}|\gamma|_S \text{ etc.} \\ \||D^0|^{-1}\underline{v}\|_{\mathfrak{S}_6} \lesssim \|\rho_\gamma\|_c = \sqrt{c}|\rho_\gamma|_c \text{ etc.} \end{array} \right.$$

We have shown  $\langle g_1^2\psi, \psi \rangle = O((L\alpha)^2)$ , so choosing  $c := \frac{g_1'(0)^2}{\alpha\alpha_\gamma(0)}$ ,  $\underline{\psi}$  has finite  $H^1$  norm.

*Remark 4.15.* Here the constant of scaling  $c$  corresponds to  $\lambda$  of the test function.

Moreover thanks to (37) it satisfies

$$\underline{m}c^2\beta\underline{\psi} + cT\underline{\psi} + \alpha c(\underline{v} - \underline{R})\underline{\psi} = \mu c^2\underline{\psi}. \quad (46)$$

Considering the upper part  $\varphi$  and the lower part  $\chi$  of  $\psi$ :

$$\underline{m}c^2\underline{\varphi} + cD\underline{\chi} + \alpha c\underline{v}\underline{\varphi} - \alpha c(\underline{R}\underline{\psi})_1 = \mu c^2\underline{\varphi} \quad (47a)$$

$$-\underline{m}c^2\underline{\chi} + cD\underline{\varphi} + \alpha c\underline{v}\underline{\chi} - \alpha c(\underline{R}\underline{\psi})_2 = \mu c^2\underline{\chi} \quad (47b)$$

From (47b) we obtain

$$\underline{\chi} = \frac{D\underline{\varphi}}{\underline{m}c + \mu c} + \frac{\alpha}{\underline{m}c + \mu c}((\underline{R}\underline{\psi})_2 - \underline{v}\underline{\chi}).$$

We take the  $L^2$ -norm:

$$\|\underline{\chi}\|_{L^2} \lesssim \frac{\|\underline{\psi}\|_{H^1}}{c} + \frac{\alpha}{\sqrt{c}}(\|\rho_\gamma\|_c + \|\gamma\|_S) \lesssim \frac{1}{c} + \frac{\alpha L\sqrt{L\alpha}}{\sqrt{c}} + \frac{\alpha L\alpha}{\sqrt{c}} \lesssim \frac{1}{c}.$$

In particular the lower part  $\chi$  tends to 0 in  $L^2(\mathfrak{H})$  at speed  $c^{-1}$ .

As  $T$  exchanges upper and lower spinors, by Cauchy-Schwarz inequality:

$$\begin{aligned} \langle \mathcal{D}^0\psi, \psi \rangle &= \langle g_0\varphi, \varphi \rangle - \langle g_0\chi, \chi \rangle + 2\Re(\langle g_1\sigma \cdot \frac{-i\nabla}{|\nabla|}\varphi, \chi \rangle) \\ &= m(\alpha)\|\varphi\|_2^2 + O(c^{-2}) \\ &= m(\alpha) + O(c^{-2}). \end{aligned}$$

It enables us to estimate

$$\mu = m(\alpha) + O(c^{-2}) \text{ and } E(1) = \mathcal{E}(\gamma') = m(\alpha) + O(c^{-2}). \quad (48)$$



From (47a) we obtain

$$D\underline{\chi} = \frac{(\mu c^2 - \underline{m}c^2)\underline{\varphi}}{c} + \alpha[(R\underline{\psi})_1 - \underline{V}\underline{\varphi}].$$

As  $\mu = m(\alpha) + O(c^{-2})$ , its  $L^2$ -norm has the following upper bound:

$$\|D\underline{\chi}\|_{L^2} \lesssim \alpha + \alpha\sqrt{c}(L\alpha + L\sqrt{L\alpha}) \lesssim \alpha,$$

writing  $Y^2 = \langle g_1^3\psi, \psi \rangle$ , we get the middle estimates

$$\|\underline{\chi}\|_{H^1} \lesssim \alpha \quad (49a)$$

$$\|\underline{\chi}\|_{H^1} \lesssim (\alpha Y + c^{-1}) + L\alpha. \quad (49b)$$

Indeed writing  $\mu = m(\alpha) + \delta m$ ,  $c^2 \times \frac{\delta m}{c} \underline{\varphi}$  has  $L^2$ -norm lesser than  $Kc^{-1}$ . Then:

$$\left| g_0(p/c) - g_0(0) \right| = \begin{cases} \left| \int_0^1 g'_0(tp/c) dt \frac{|p|}{c} \right| & \leq K\alpha \frac{|p|}{c} \\ \left| \int_0^1 g''_0(tp/c)(1-t) dt \frac{|p|^2}{c^2} \right| & \leq K\alpha \frac{|p|^2}{c^2} \end{cases}.$$

In particular

$$\langle g_1\chi, \chi \rangle \leq \sqrt{\langle \chi, \chi \rangle \langle g_1^2\chi, \chi \rangle} = O(c^{-1} \times (\alpha Y + c^{-1})c^{-1}) = O(\alpha Y c^{-2} + c^{-3}) \quad (50)$$

and there also holds the middle estimate:  $\|\chi\|_{H^1} \lesssim c^{-1} + \alpha c^{-1}$ .

#### 4.3.4 $\|U_c^*\psi\|_{H^{3/2}} = O(1)$ .

As before:

$$|d|^{1/2}R_\gamma\psi = [|d|^{1/2}, R_\gamma]|d|^{-1}|d|\psi + R_\gamma|d|^{1/2}\psi,$$

and thanks to (23b)

$$\| [|d|^{1/2}, R_\gamma]|d|^{-1} \|_{\mathfrak{S}_2}^2 \lesssim \iint \tilde{E}(p-q) \tilde{E}(p+q) |\hat{\gamma}(p,q)|^2 dpdq \lesssim c^{-2}.$$

Thanks to (50) there holds (with  $Y^2 = \langle g_1^3\psi, \psi \rangle$ ):

$$\left| \mu \langle g_1\alpha \cdot \frac{-i\nabla}{|i\nabla|} \psi, |\nabla|\psi \rangle \right| \lesssim \| |\nabla|^{3/2}\varphi \|_{L^2} \| |\nabla|^{1/2}\chi \|_{L^2} = O(Yc^{-3/2} + Y^{3/2}\sqrt{\alpha}c^{-1}).$$

Then:

$$\begin{aligned} \langle g_0^2\psi, \psi \rangle &= m(\alpha)^2 + 2(2\pi)^{-3} \int_p \left( \int_{t=0}^1 (1-t)g_0(tp)g''_0(tp)dt \right) |p|^3 |\hat{\psi}(p)|^2 dp \\ &= m(\alpha)^2 + K\alpha Y^2, \\ \langle g_0\beta\psi, |\nabla|\psi \rangle &= (2\pi)^{-3} \left( \int_p g_0(p)|p|\hat{\psi}(p)|^2 dp - 2 \int_p g_0(p)|p|\hat{\chi}(p)|^2 dp \right) \\ &= \langle g_0\psi, |\nabla|\psi \rangle + O(\alpha Y c^{-2} + c^{-3}) \\ &= m(\alpha)\langle |\nabla|\psi, \psi \rangle + O(\alpha Y^2 + \alpha Y c^{-2} + c^{-3}), \\ \langle \mathcal{D}^0\psi, |\nabla|\psi \rangle &= \langle g_0\beta\psi, |\nabla|\psi \rangle + 2\Re(\langle g_1\sigma \cdot \frac{-i\nabla}{|i\nabla|}\varphi, |\nabla|\chi \rangle) \\ &= m(\alpha)\langle |\nabla|\psi, \psi \rangle + O(\alpha Y^2 + \alpha Y c^{-2} + c^{-3} + Yc^{-3/2} + Y^{3/2}\sqrt{\alpha}c^{-1}), \\ \mu \langle \mathcal{D}^0\psi, |\nabla|\psi \rangle &= m(\alpha)^2 \langle |\nabla|\psi, \psi \rangle + O(\alpha Y^2 + Y^{3/2}\sqrt{\alpha}c^{-1} + Yc^{-3/2} + c^{-3}). \end{aligned}$$

We write down  $S = g_1(-i\nabla)\sigma \cdot \frac{-i\nabla}{|i\nabla|}$ .

With the same method as in C.2:

$$\left\| [|\nabla|, v]|d|^{-3/2} \right\|_{\mathcal{B}}, \left\| [|\nabla|^{1/2}, v]|d|^{-1} \right\|_{\mathcal{B}}, \left\| v|d|^{-1/2} \right\|_{\mathcal{B}} \lesssim \|\rho_\gamma\|_c \sqrt{\log(\Lambda)}.$$

$$\begin{aligned}
|\langle R_\gamma \psi, S|\nabla|\psi\rangle| &\leq |(\|\nabla\|^{1/2}, R_\gamma)|d|^{-1}|d|\psi, S|\nabla\|^{1/2}\psi\rangle| + |\langle R_\gamma|\nabla\|^{1/2}\psi, S|\nabla\|^{1/2}\psi\rangle| \\
&\lesssim Y\|\gamma\|_S(1 + \underbrace{\langle|\nabla\|d|\psi, \psi\rangle}_{\text{from } \|\|R_\gamma|\nabla\|^{1/2}\psi\|}), \\
|\langle Sv_\gamma\varphi, |\nabla|\chi\rangle| &\leq 3C_1|\langle|\nabla|v_\gamma\varphi, |\nabla|\chi\rangle| \\
&\leq 3C_1|\langle|\nabla|, v_\gamma\rangle|d|^{-3/2}|d|^{3/2}\varphi, |\nabla|\chi\rangle| + 3C_1|\langle v_\gamma|d|^{-1/2}|d|^{1/2}|\nabla|\varphi, |\nabla|\chi\rangle| \\
&\lesssim \alpha c^{-1}\sqrt{\log(\Lambda)}\|\rho_\gamma\|_c Y = KL(L\alpha)^2 Y, \\
|\langle Sv_\gamma\chi, |\nabla|\varphi\rangle| &\lesssim |\langle|\nabla\|^{1/2}v_\gamma\chi, |\nabla\|^{3/2}\varphi\rangle| \\
&\lesssim |\langle|\nabla\|^{1/2}, v_\gamma\rangle|d|^{-1}|d|\chi, |\nabla\|^{3/2}\varphi\rangle| + |\langle v_\gamma|d|^{-1/2}|d|^{1/2}|\nabla\|^{1/2}\chi, |\nabla\|^{3/2}\varphi\rangle| \\
&\lesssim Y\sqrt{\log(\Lambda)}\|\rho_\gamma\|_c \times \|\|d|\chi\|\|_{L^2} \lesssim Y\sqrt{\log(\Lambda)}L\sqrt{L\alpha}(\alpha c^{-1} + c^{-1}), \\
|\langle v_\gamma\varphi, |\nabla|\varphi\rangle| &= \left| \iint \frac{(|\nabla|\varphi)^*(x)\varphi(x)\rho(y)}{|x-y|} dx dy \right| \leq Y^2\|\rho_\gamma\|_c \text{ etc.} \\
|\langle R_\gamma\varphi, |\nabla|\varphi\rangle| &= \left| \iint \frac{(|\nabla|\varphi)^*(x)\gamma(x,y)\varphi(y)}{|x-y|} dx dy \right| \leq Y^2\|\gamma\|_S \text{ etc.}
\end{aligned}$$

Therefore :  $Y^2(1 - K\alpha) \leq K_0 c^{-3} + K_1(L\alpha^2)Y + K_3\sqrt{\alpha}c^{-1}Y^{3/2}$ .

As  $L\alpha^2 = \underset{\Lambda \rightarrow +\infty}{o}((L\alpha)^{3/2})$ , we deduce  $\langle|\nabla\|^3\psi, \psi\rangle = O(c^{-3})$  and so

$$\|\psi\|_{H^{3/2}} = O(1).$$

We now improve (49a) as written before:

$$\begin{aligned}
g_0(p/c) - g_0(0) &= \int_0^1 g'_0(tp/c) \frac{|p|}{c} dt = \int_0^1 (1-t)g''_0(tp/c) \frac{|p|^2}{c^2} dt \\
|g_0(p/c) - g_0(0)|^2 &= \left| \int_0^1 g'_0(tp/c) dt \int_0^1 (1-u)g''_0(up/c) du \right| \frac{|p|^3}{c^3},
\end{aligned}$$

and therefore

$$\|(m(\alpha) - \underline{m})c\underline{\psi}\|_{L^2} \leq K\sqrt{\frac{\|g'_0\|_\infty \|g''_0\|_\infty}{c}} = K\alpha\sqrt{L\alpha} = o(c^{-1}). \quad (51)$$

So

$$\|\underline{\chi}\|_{H^1} = O(c^{-1}) \text{ and } \|\|\nabla|\chi\|\|_{H^1} = O(c^{-2}). \quad (52)$$

### 4.3.5 Estimation of $E(1)$ .

Thanks to (47b)

$$\chi = \frac{S\varphi}{g_0 + \mu} + \alpha \frac{(R_\gamma\psi)_2 - v_\gamma\chi}{g_0 + \mu} = \frac{S\varphi}{g_0 + \mu} + \delta\chi,$$

where the remainder  $\delta\chi$  has  $L^2$ -norm lesser than  $K\alpha L\sqrt{L\alpha} = o(c^{-1})$ . Thus as  $\|g_1\psi\|_{L^2} = O(c^{-1})$ , thanks to 5. we have the following asymptotic expansion:

$$\begin{aligned}
E(1) + \frac{\alpha\alpha_r(0)}{2c}D(\underline{n}, \underline{n}) &= \langle g_0\varphi, \varphi\rangle - \langle g_0\frac{S}{g_0+\mu}\varphi, \frac{S}{g_0+\mu}\varphi\rangle + 2\Re\langle \frac{S}{g_0+\mu}\varphi, S\varphi\rangle + o(c^{-2}) \\
&= m(\alpha)(1 - 2\langle \frac{g_1^2}{(g_0+\mu)^2}\varphi, \varphi\rangle) + 2\langle \frac{g_1^2}{g_0+\mu}\varphi, \varphi\rangle + o(c^{-2}) \\
&= m(\alpha) - \langle \frac{g_1^2}{2m(\alpha)}\varphi, \varphi\rangle + \langle \frac{g_1^2}{m(\alpha)}\varphi, \varphi\rangle + o(c^{-2}) \\
&= m(\alpha) + \frac{1}{2m(\alpha)}\langle g_1^2\varphi, \varphi\rangle + o(c^{-2}) \\
&= m(\alpha) + \frac{1}{2m(\alpha)}\langle g_1^2\psi, \psi\rangle + o(c^{-2}),
\end{aligned}$$

where to deal with  $g_0$  we use  $\langle|\nabla\|^3\varphi, \varphi\rangle = O(c^{-3})$  and  $|g'_0| = O(\alpha)$  and treat the  $((g_0 + \mu)^{-1})$ 's one after the other. For the last line we use the fact that  $\langle|\nabla\|^2\chi, \chi\rangle = O(c^{-3})$ .

Writing with  $\underline{\psi}$ :

$$C_0^2(E(1)-m(\alpha)) = \frac{1}{(g_1'(0))^2(2\pi)^3} \int c^2 g_1 \left(\frac{p}{c}\right)^2 |\widehat{\underline{\psi}}(p)|^2 dp - \iint \frac{|\psi(x)|^2 |\psi(y)|^2}{|x-y|} dx dy + o(1). \quad (53)$$

We recall (cf 6.) the  $(g_1')_{\alpha,\Lambda}$ 's are *uniformly* continuous in a neighbourhood of 0; cutting in Fourier space at level  $|p| = \sqrt{c}$  there holds

$$\begin{aligned} \int_{|p| \leq \sqrt{c}} c^2 g_1(p/c)^2 |\widehat{\underline{\psi}}(p)|^2 dp &= \int_{|p| \leq \sqrt{c}} g_1'(0)^2 |p|^2 |\widehat{\underline{\psi}}(p)|^2 dp + \int_{|p| \leq \sqrt{c}} \left( \int_{t=0}^1 (g_1'(tp/c) - g_1'(0)) dt \right)^2 |p|^2 |\widehat{\underline{\psi}}(p)|^2 dp \\ &\quad + 2g_1'(0) \int_{|p| \leq \sqrt{c}} \left( \int_{t=0}^1 (g_1'(tp/c) - g_1'(0)) dt \right) |p|^2 |\widehat{\underline{\psi}}(p)|^2 dp \\ &= \int_{|p| \leq \sqrt{c}} g_1'(0)^2 |p|^2 |\widehat{\underline{\psi}}(p)|^2 dp + O \left( \sup_{|q| \leq c^{-1/2}} \{ |g_1'(q) - g_1'(0)| \} \| |\nabla| \underline{\psi} \|^2 \right) \\ &= \int_{|p| \leq \sqrt{c}} g_1'(0)^2 |p|^2 |\widehat{\underline{\psi}}(p)|^2 dp + o_{c \rightarrow +\infty}(1). \end{aligned}$$

Moreover:

$$\begin{aligned} \int_{|p| \geq \sqrt{c}} c^2 g_1(p/c)^2 |\widehat{\underline{\psi}}(p)|^2 dp &\lesssim \int_{|p| \geq \sqrt{c}} \frac{|p|^3}{|p|} |\widehat{\underline{\psi}}(p)|^2 dp \\ &\lesssim \frac{1}{\sqrt{c}} \langle |\nabla|^3 \underline{\psi}, \underline{\psi} \rangle \lesssim c^{-1/2} \xrightarrow{c \rightarrow +\infty} 0. \end{aligned}$$

Thus

$$\frac{1}{(g_1'(0))^2} \langle c^2 g_1^2(\cdot/c) \underline{\psi}, \underline{\psi} \rangle - D(\underline{n}, \underline{n}) = \langle |\nabla|^2 \underline{\psi}, \underline{\psi} \rangle - D(\underline{n}, \underline{n}) + o(1),$$

By unicity of the asymptotic expansion and by definition of  $E_{\text{CP}}$  we thus have

$$E(1) = m(\alpha) + C_0^{-2} E_{\text{CP}} + o((\alpha \alpha_r(0))^2). \quad (54)$$

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# Appendices

## A The operator $\mathcal{D}^0$

### A.1 The functions $g_0$ and $g_1$

As established in [8],  $\mathcal{D}^0$  is solution of the following equation in the Fourier space

$$\widehat{\mathcal{D}}^0 = \widehat{D}^0 + \frac{\alpha}{4\pi^2} \frac{\widehat{\mathcal{D}}^0}{|\mathcal{D}^0|} \star \frac{1}{|\cdot|^2} \quad \text{in } \mathcal{B}(B(0, \Lambda), \text{End}(\mathbf{C}^4)) \quad (55)$$

and by a bootstrap argument  $\widehat{\mathcal{D}}^0 \in \cap_{m \geq 1} H^m(\overline{B(0, \Lambda)})$ . With the notation of 1.1. it shows that  $g_0, \mathbf{g}_1$  are smooth while  $g_1(p) = \mathbf{g}_1(p) \cdot \omega_p$  is *a priori*  $\mathcal{C}^\infty(B(0, \Lambda) \setminus \{0\})$  and there holds

$$g_0(|p|) = 1 + \frac{\alpha}{4\pi^2} \int_{|r| < \Lambda} dr \frac{1}{|p-r|^2} \frac{g_0(|r|)}{\sqrt{g_1(|r|)^2 + g_0(|r|)^2}}, \quad (56a)$$

$$g_1(|p|) = |p| + \frac{\alpha}{4\pi^2} \int_{|r| < \Lambda} dr \frac{\omega_p \cdot \omega_r}{|p-r|^2} \frac{g_1(|r|)}{\sqrt{g_1(|r|)^2 + g_0(|r|)^2}}. \quad (56b)$$

*Remark A.1.* We recall here that  $C_1 > 0$  is a constant such that  $g_1(r) \leq C_1 r$  and  $|g_0|_\infty \leq C_1$ .

Let us show first that

**Proposition 5.**  $g_1 \in \mathcal{C}^1([0, \Lambda], \mathbf{R})$  and  $g'_0(0) = 0$ .

Moreover writing  $\|d^2 g_1\|_\star = \sup_{0 < |p| \leq \Lambda} \| |p| d^2 g_1(p) \|$  we have

$$\begin{cases} \|g'_0\|_\infty = O(\alpha) \\ \|g'_1\|_\infty = O(1) \end{cases} \quad \text{and} \quad \begin{cases} \|g''_0\|_\infty = O(\alpha) \\ \|d^2 g_1\|_\star = O(1) \end{cases} .$$

In fact it suffices to differentiate (55) to get  $g'_0(p), g'_1(p)$  and then taking the norm to obtain the first part; then we differentiate once more to get the second part.

### A.1.1 Proof of 5: part 1.

We can define  $dg_1(p)$  for  $p \neq 0$ . First we have

$$dg_0(p)h = \frac{\alpha}{4\pi^2} \int \frac{dq}{|p-q|^2} \left( \frac{dg_0(q)h}{\tilde{E}(q)} - \frac{g_0(q)dg_0(q)h + g_1(q)dg_1(q)h}{\tilde{E}(q)^2} \frac{g_0(q)}{\tilde{E}(q)} \right).$$

We remark that for  $p \neq 0$  we have:

$$\begin{cases} dg_1(p)h = g'_1(|p|)\langle \omega_p, h \rangle \\ \langle dg_1(p) \cdot \omega_p, \omega_p \rangle = g'_1(|p|). \end{cases}$$

Then

$$d\mathbf{g}_1(p) \cdot h = h + \frac{\alpha}{4\pi^2} \int \frac{dq}{|p-q|^2} \left( \frac{d\mathbf{g}_1(q) \cdot h}{\tilde{E}(q)} - \frac{g_0(q)dg_0(q)h + g_1(q)dg_1(q)h}{\tilde{E}(q)^2} \frac{\mathbf{g}_1(q)}{\tilde{E}(q)} \right),$$

so that for any  $\omega \in \mathbf{S}^2$ :

$$\begin{aligned} g'_1(x) &= 1 + \frac{\alpha}{4\pi^2} \int_{|q| \leq \Lambda} \frac{dq}{|x\omega - q|^2} \left( \left( \frac{g_1(q)}{|q|} (1 - \langle \omega, \omega_q \rangle^2) + g'_1(q) \langle \omega_q, \omega \rangle^2 (1 - \frac{g_1^2(q)}{\tilde{E}(q)^2}) \right) \frac{1}{\tilde{E}(q)} \right. \\ &\quad \left. - \frac{g_1(q)}{\tilde{E}(q)} \frac{\langle \omega, \omega_q \rangle^2}{\tilde{E}(q)} \frac{g_0(q)g'_0(q)}{\tilde{E}(q)} \right). \end{aligned}$$

The regularity of  $g_1$  (as a function of  $\mathbf{R}^+$ ) will come from the continuous extension to  $x = 0$  of the formula above.

We have

$$|g'_0(|p|)| \leq \frac{\alpha}{4\pi^2} \int \frac{dq}{|p-q|^2} \left( \frac{|g'_0|_\infty}{\tilde{E}(q)} + |g_0|_\infty \frac{|g'_0|_\infty + |g'_1|_\infty}{\tilde{E}(q)^2} \right) \quad (57a)$$

$$|g'_1(|p|)| \leq 1 + \frac{\alpha}{4\pi^2} \int \frac{dq}{|p-q|^2} \left( \frac{|g'_1|_\infty}{\tilde{E}(q)} + \frac{|g'_0|_\infty + |g'_1|_\infty}{\tilde{E}(q)} \right). \quad (57b)$$

Thus

$$\begin{cases} |g'_0|_\infty \leq K_1 \alpha \log(\Lambda) |g'_0|_\infty + K_2 \alpha |g'_1|_\infty \\ |g'_1|_\infty \leq 1 + K_3 \alpha \log(\Lambda) (|g'_0|_\infty + |g'_1|_\infty) \end{cases}$$

and  $|g'_0|_\infty \lesssim \alpha$ ,  $|g'_1|_\infty \leq 1 + K\alpha \log(\Lambda)$ .

*Remark A.2.* In particular we get  $g'_1(0) = 1 + O(L) > 0$  for  $L$  sufficiently small.

Since  $g_0 \in \mathcal{C}^\infty(B(0, \Lambda), \mathbf{R})$  and radial, necessarily

$$dg_0(0) = 0 \text{ and } g'_0(0) = dg_0(0)\omega = 0, \forall \omega \in \mathbf{S}^2.$$

### A.1.2 Proof of 5: part 2.

Let us now calculate  $d^2\mathcal{D}^0$ . We note  $h_* = \frac{g_*}{E(\cdot)}$  and  $j = \tilde{E}(\cdot)^{-1}$ : the coefficient of  $\beta$  in  $d^2\mathcal{D}^0(p)h^2$  is

$$d^2g_0(p)h^2 = \frac{\alpha}{4\pi^2} \int_q \frac{dq}{|p-q|^2} d^2h_0(q)h^2,$$

where

$$\begin{aligned} d^2h_0(q)h^2 &= \frac{d^2g_0(p) \cdot h^2}{\tilde{E}(q)} - \frac{2}{\tilde{E}(q)^3} dg_0(q)h [g_0(q)dg_0(q)h + g_1(q)dg_1(q)h] \\ &\quad - \frac{g_0(q)}{\tilde{E}(q)^3} [(dg_0(q)h)^2 + g_0(q)d^2g_0(q)h^2 + (dg_1(q)h)^2 + g_1(q)d^2g_1(q)h^2] \\ &\quad + 3 \frac{g_0(q)}{\tilde{E}(q)^5} [g_0(q)dg_0(q)h + g_1(q)dg_1(q)h]^2. \end{aligned}$$

Furthermore, there holds

$$d^2 \mathbf{g}_1(p) h^2 = \frac{\alpha}{4\pi^2} \int \frac{dq}{|p-q|^2} \left( \frac{d^2 \mathbf{g}_1(q) h^2}{\tilde{E}(q)} + 2d\mathbf{g}_1(q) h dj(q) h + \mathbf{g}_1(q) d^2 j(q) h^2 \right)$$

and taking the scalar product with  $\omega_p$  we get

$$\begin{aligned} |p| |d^2 g_1(p)| &\leq C_1 + \frac{\alpha}{4\pi^2} \int \frac{|p|dq}{|p-q|^2 |q| E(q)} \|d^2 g_1\|_* + \frac{\alpha}{4\pi^2} \int \frac{|p|dq}{|p-q|^2 E(q)^2} \|d^2 g_1\|_* \\ &\quad + \frac{\alpha}{4\pi^2} \int_q \frac{|p|dq}{|p-q|^2} \left( \frac{1}{E(q)^2} (|dg_0|^2 + |dg_1|^2) + \frac{g_0(q)}{E(q)^2} |d^2 g_0| + \frac{3}{E(q)^2} (|dg_0| + |dg_1|)^2 \right. \\ &\quad \left. + 2(|dg_1| + C_1) \frac{|dg_0| + |dg_1|}{E(q)^2} + \frac{1}{E(q)} \frac{2|dg_1| + 4C_1}{|q|} \right), \end{aligned}$$

as there holds  $\langle |p| d^2 \mathbf{g}_1(p) h^2, \omega_p \rangle = |p| d^2 g_1^p \cdot h^2 + \frac{g_1(p)}{|p|} (\langle \omega_p, h \rangle^2 - \|h\|^2)$ .

Analogously there holds

$$\begin{aligned} |d^2 g_0(p)| &\leq \frac{\alpha}{4\pi^2} \left( \int \frac{C_1 dq}{E(q)^2 |p-q|^2} \|d^2 g_1\|_* \right. \\ &\quad \left. \int_q \frac{dq}{|p-q|^2} \left( \frac{|d^2 g_0|}{E(q)} + 2 \frac{|dg_0| (|dg_0| + |dg_1|)}{E(q)^2} + \frac{g_0(q)}{E(q)} \frac{|dg_0|^2 + |dg_1|^2}{E(q)^2} \right. \right. \\ &\quad \left. \left. + \frac{g_0(q)^2 |d^2 g_0|}{E(q)^2 E(q)} + 3 \frac{g_0(q)}{E(q)} \frac{(|dg_0| + |dg_1|)^2}{E(q)^2} \right) \right). \end{aligned}$$

As  $\frac{|p|}{|p-q|^2 |q|} \leq 2 \max(\frac{1}{|p-q||q|}, \frac{1}{|p-q|^2})$ , there holds

$$\int_{|q| \leq \Lambda} \frac{dq |p|}{|p-q|^2 |q| E(q)} \leq 2 \left( \int_{|q| \leq \Lambda} \frac{dq}{|p-q||q| E(q)} + \int_{|q| \leq \Lambda} \frac{dq}{|p-q| E(q)} \right),$$

we recall then that the convolution of radial nonnegative functions is radial nonnegative. Hence we obtain

$$\begin{cases} \|g_0''\|_\infty \leq K\alpha \\ \|d^2 g_1\|_* \leq C_1 + K\alpha \log(\Lambda) \end{cases}$$

### A.1.3 Variations of $d\mathbf{g}_1$ .

Then we can show that for  $p, q \in \mathbf{R}^3 \cap B(0, \Lambda)$

$$\int_{|l| < \Lambda} \left| |p-l|^{-1} - |q-l|^{-1} \right| \frac{dl}{\tilde{E}(l)} \leq 8\pi |p-q| \int_{r=-\Lambda}^\Lambda \frac{dr}{\sqrt{1+r^2}} \lesssim \log(\Lambda) |p-q| \quad (58)$$

so

**Proposition 6.** The function

$$d\mathbf{g}_1(p) = \text{id} + \frac{\alpha}{4\pi^2} \int_{|r| < \Lambda} \frac{dr}{|p-r| \tilde{E}(r)} \left( d\mathbf{g}_1(r) - \mathbf{g}_1(r) \frac{g_0(r) dg_0(r) + g_1(r) dg_1(r)}{\tilde{E}(r)^2} \right)$$

is in  $C^0(B(0, \Lambda), L(\mathbf{R}^3, \mathbf{C}^4))$  and

$$|d\mathbf{g}_1(p) - d\mathbf{g}_1(q)| \leq KL |p-q|.$$

In particular the same holds for  $g_1(t) = \langle \mathbf{g}_1(t\omega), \omega \rangle$  and  $KLt$ .

In fact it suffices to split  $B(0, \Lambda)$  in two domains:

We write  $F_p = \mathbf{R}^3 \cap \{r : |p-r| \leq |q-r|\}$ ,  $F_q = \mathbf{R}^3 \cap \{r : |q-r| \leq |p-r|\}$ .

In  $F_p \cap B(0, \Lambda)$  we take spherical coordinates centered in  $p$ , in  $F_q \cap B(0, \Lambda)$  centered in  $q$ . There holds

$$\left| |p-r|^{-1} - |q-r|^{-1} \right| \leq \begin{cases} \frac{|p-q|}{|p-r|^2} & \text{for } r \in F_p, \\ \frac{|p-q|}{|q-r|^2} & \text{for } r \in F_q. \end{cases}$$

Proposition 5. enables us to prove

**Lemma A.3.** Let  $p, q \in B(0, \Lambda)$  and  $k = p - q$ . There holds

$$\frac{\tilde{E}(p)\tilde{E}(q) - \langle \mathbf{g}(p), \mathbf{g}(q) \rangle}{\tilde{E}(p)\tilde{E}(q)(\tilde{E}(p) + \tilde{E}(q))} \leq \min(2, \frac{2K|k|^2}{E(p)}, \frac{2K|k|^2}{E(q)}).$$

where we can choose  $K \leq 2$  for  $\alpha \log(\Lambda)$  sufficiently small.

In fact we can write for  $a, b, t = b - a \in \mathbf{R}^3$ :  $|a||b| - \langle a, b \rangle = \frac{a^2 t^2 - (t, a)^2}{|a||b| + \langle a, b \rangle}$ . If  $\langle a, b \rangle > -\frac{|a||b|}{2}$  then  $A = \frac{|a||b| - \langle a, b \rangle}{|a||b|} \leq \frac{2a^2 t^2}{a^2 b^2}$ , by symmetry there is also  $A \leq \frac{2b^2 t^2}{a^2 b^2}$ . Else  $\langle a, b \rangle \leq -\frac{|a||b|}{2}$ , then  $\frac{1}{|a||b|(|a||b| + \langle a, b \rangle)} \geq 2(a^2 b^2)^{-1}$  and

$$\begin{aligned} 2\frac{t^2}{b^2} &\geq 2\frac{a^2 + b^2 + |a||b|}{b^2} \geq 2 \\ 2\frac{t^2}{a^2} &\geq 2\frac{a^2 + b^2 + |a||b|}{a^2} \geq 2. \end{aligned}$$

*Remark A.4.* This last estimate assures us that we can apply the fixed point method with  $\mathcal{D}^0$  instead of with  $D^0$ . Indeed all the estimates of [5] remains the same *modulo* multiplicative constants: here because of A.3 we must add 2;  $C_1$  also appear.

## A.2 The function $B_\Lambda$

We recall that

$$B_\Lambda(k) = \frac{1}{\pi^2 |k|^2} \int_{|p=l-\frac{k}{2}, |q=l+\frac{k}{2}| < \Lambda} \frac{\tilde{E}(p)\tilde{E}(q) - \langle \mathbf{g}(p), \mathbf{g}(q) \rangle}{\tilde{E}(p)\tilde{E}(q)(\tilde{E}(p) + \tilde{E}(q))} dl \geq 0.$$

This formula holds only for  $k \neq 0$ : our first purpose is to extend it continuously to 0. Thanks to A.3. we can say that  $B_\Lambda(k) \leq K \log(\Lambda)$ .

*Notation A.5.* Throughout this part,  $p = l + \frac{k}{2}, q = l - \frac{k}{2}$ .

Let us write  $I = \pi^2 |k|^2 B_\Lambda(k)$ , its integrand  $f(l)$  and  $x = |k|$ . Let  $0 < \varepsilon < \frac{2}{3}$  and  $s = \frac{1}{3} + \varepsilon$ . We look at  $x < 1$  and split the domain in three:

$$B = \{l : |l| \leq x^s\}, A = \{l : x^s < |l| < \Lambda - \frac{x}{2}\},$$

$$C = \{l : |l - \frac{k}{2}|, |l + \frac{k}{2}| < \Lambda\} \setminus \{l : |l| < \Lambda - \frac{x}{2}\} \subset \{l : \Lambda - \frac{x}{2} < |l| < \Lambda\} = C'.$$

With A.3. we obtain the following behaviour *independent* of  $\alpha, \Lambda$  in the regime (9)

$$|I_B| \leq Kx^{2+3s} = Kx^{3+3\varepsilon} = \underset{x \rightarrow 0}{o}(x^3), \quad |I_C| \leq Kx^2 \log\left(\frac{\Lambda}{\Lambda - \frac{x}{2}}\right) \underset{x \rightarrow 0}{\sim} \frac{Kx^3}{\Lambda}. \quad (59)$$

There remains  $I_A$ : we rewrite  $f(l)$  as

$$f(l) = \frac{|\mathbf{g}(p) \wedge \mathbf{g}(q)|^2}{\tilde{E}(p)\tilde{E}(q)(\tilde{E}(p) + \tilde{E}(q))(\tilde{E}(p)\tilde{E}(q) + \mathbf{g}(p) \cdot \mathbf{g}(q))} \quad (60)$$

where  $|\mathbf{g}(p) \wedge \mathbf{g}(q)|^2 = \sum_i |\Delta_{0i}|^2 + \sum_{i,j} |\Delta_{ij}|^2$ ,

$$\Delta_{0i} = \begin{vmatrix} g_0(p) & g_0(q) \\ (\mathbf{g}_1(p))_i & (\mathbf{g}_1(q))_i \end{vmatrix} = \begin{vmatrix} \delta g_0 & g_0(q) \\ (\delta \mathbf{g}_1)_i & (\mathbf{g}_1(q))_i \end{vmatrix} \quad (61a)$$

$$\Delta_{ij} = \begin{vmatrix} (\mathbf{g}_1(p))_i & (\mathbf{g}_1(q))_i \\ (\mathbf{g}_1(p))_j & (\mathbf{g}_1(q))_j \end{vmatrix} = \begin{vmatrix} (\delta \mathbf{g}_1)_i & (\mathbf{g}_1(q))_i \\ (\delta \mathbf{g}_1)_j & (\mathbf{g}_1(q))_j \end{vmatrix} \quad (61b)$$

$$\delta g_* = g_*(p) - g_*(q).$$

If we take  $k$  along a *fixed* ray:  $k = x\omega$  we have

$$\begin{aligned} \frac{1}{x} \delta g_0(k, l) &= \int_{t=0}^1 dg_0(l + (t-1/2)k) \cdot \omega dt \xrightarrow{x \rightarrow 0} g'_0(|l|)\omega_l \cdot \omega \\ \frac{1}{x} \delta \mathbf{g}_1(k, l) &= \int_{t=0}^1 d\mathbf{g}_1(l + (t-1/2)k) \cdot \omega dt \xrightarrow{x \rightarrow 0} d\mathbf{g}_1(l) \cdot \omega, \end{aligned}$$

We write  $\mathbf{g}_l^\omega = \begin{pmatrix} g_0'(|l|)\omega_l \cdot \omega \\ d\mathbf{g}_1(l) \cdot \omega \end{pmatrix}$  and  $\tilde{E}_l^\omega = |\mathbf{g}_l^\omega|$ .

In fact, as  $A, g_0, g_1$  are radial symmetric so is  $I_A(k)$  and for  $\omega \in \mathbf{S}^2$  fixed and  $p' = l + \frac{x\omega}{2}, q' = l - \frac{x\omega}{2}$  there holds

$$I_A(k = x\omega_k) = \frac{1}{\pi^2 x^2} \int_{x^s < |l| < \Lambda - \frac{x}{2}} \frac{\tilde{E}(p') \tilde{E}(q') - \langle \mathbf{g}(p'), \mathbf{g}(q') \rangle}{\tilde{E}(p') \tilde{E}(q') (\tilde{E}(p') + \tilde{E}(q'))} dl,$$

$f'(l) = \frac{f(l)}{x^2} \chi_{l \in A}$  is also symmetric. By Proposition 5.:

$|f'(l)| \leq K \frac{1}{(1+|l|^2)^{3/2}} \chi_{|l| \leq \Lambda - x/2}$  and by dominated convergence we have

**Proposition 7.**

$$B_\Lambda(k) \xrightarrow{k \rightarrow 0} \frac{1}{\pi^2} \int_{|l| \leq \Lambda} \frac{|\mathbf{g}_l^\omega \wedge \mathbf{g}_l|^2}{4\tilde{E}(l)^5} dl =: B_\Lambda(0). \quad (62)$$

As there holds by symmetry

$$\int_{\mathbf{n} \in \mathbf{S}^2} \langle \mathbf{n}, \omega \rangle^2 d\mathbf{n} = \frac{4}{3}\pi, \quad \int_{\mathbf{n} \in \mathbf{S}^2} |d\mathbf{g}_1(|l|\mathbf{n}) \cdot \omega|^2 d\mathbf{n} = \frac{4}{3}\pi \left( (g_1')^2(l) + 2\frac{g_1(l)^2}{|l|^2} \right) \quad (63)$$

we have

$$B_\Lambda(0) = \frac{1}{3\pi} \left( \int_{u=0}^{\Lambda} u^2 \frac{((g_0')^2(u) + (g_1')^2(u) + 2\frac{g_1(u)^2}{|u|^2})(g_0^2(u) + g_1^2(u))}{(g_0(u)^2 + g_1(u)^2)^{5/2}} du - \int_{u=0}^{\Lambda} u^2 \frac{(g_0 g_0'(u) + g_1 g_1'(u))^2}{(g_0(u)^2 + g_1(u)^2)^{5/2}} du \right),$$

and

$$B_\Lambda(0) = \frac{1}{3\pi} \left( \int_{u=0}^{\Lambda} u^2 \frac{(g_0')^2(u) + (g_1')^2(u) + 2\frac{g_1(u)^2}{|u|^2}}{(g_0(u)^2 + g_1(u)^2)^{3/2}} du - \int_{u=0}^{\Lambda} u^2 \frac{(g_0 g_0'(u) + g_1 g_1'(u))^2}{(g_0(u)^2 + g_1(u)^2)^{5/2}} du \right).$$

Finally thanks to Proposition 5. and remark A.2:

**Proposition 8.**

$$B_\Lambda(0) = \frac{2}{3\pi} \log(\Lambda) + O(L \log(\Lambda) + 1).$$

Let us look at the variations  $|k|^{-1} |B_\Lambda(k) - B_\Lambda(0)|$ . Let  $f_0$  be the integrand in Proposition 7.: we have  $|\int_B f_0| \leq K x^{3s} = O(x^{1+3\varepsilon})$  and

$|\int_C f_0| \leq K \log(\frac{\Lambda}{\Lambda-x/2}) = O(\frac{x}{\Lambda})$ . There remains the integration over  $A$ .

For  $|l| \geq x^s$ :  $\frac{x}{|l|} = O(x^{2/3-\varepsilon})$  so we can expand the integrand of  $I_A(x)$  at order 1. Indeed:

$$\tilde{E}(p)^{-1} = \tilde{E}(l)^{-1} \left\{ 1 + \frac{\tilde{E}(p) - \tilde{E}(l)}{\tilde{E}(l)} \right\}^{-1} = \tilde{E}(l)^{-1} \left\{ 1 + \frac{\tilde{E}(l) - \tilde{E}(p)}{\tilde{E}(l)} + O\left(\frac{x^2}{\tilde{E}(l)^2}\right) \right\} \text{ etc.}$$

where as  $\tilde{E}(l) \geq 1$  the  $O(\cdot)$  is independent of  $l$ .

Writing  $h(l, k) = \tilde{E}(p) \tilde{E}(q) - \mathbf{g}(p) \cdot \mathbf{g}(q)$  there holds

$$I_A(x) = \frac{1}{x^2} \int_A \frac{h(l, k)}{2\tilde{E}(l)^3} dl + \frac{1}{x^2} \int_A \frac{h(l, k)}{2\tilde{E}(l)^3} \left( \frac{2\tilde{E}(l) - \tilde{E}(p) - \tilde{E}(q)}{\tilde{E}(l)} + \frac{2\tilde{E}(l) - \tilde{E}(p) - \tilde{E}(q)}{2\tilde{E}(l)} + O\left(\frac{x^2}{\tilde{E}(l)^2}\right) \right).$$

By Taylor formula (at order 2):

$$|2\tilde{E}(l) - (\tilde{E}(p) + \tilde{E}(q))| \leq \int_t \int_u dt du K x^{1+2/3-\varepsilon} = K x^{1+2/3-\varepsilon}.$$



Thanks to Proposition 6. we have by Taylor formula at order 1:

$$\left| \frac{\delta \mathbf{g}}{x} - \mathbf{g}_i^\omega \right| \lesssim Lx,$$

so *in fine*

**Proposition 9.** There exists  $0 < r_\varepsilon \in \mathbf{R}^+$ , independent of  $\alpha, \Lambda$  in the regime (9) such that for  $|k| < r_\varepsilon$ :

$$|k|^{-1} |B_\Lambda(k) - B_\Lambda(0)| \leq K(\Lambda^{-1} + L^2|k| + |k|^{3\varepsilon} + |k|^{2/3-\varepsilon}).$$

Choosing  $\varepsilon := 6^{-1}$  there holds:

$$|k|^{-1} |B_\Lambda(k) - B_\Lambda(0)| \leq K(\Lambda^{-1} + |k|^{1/2}).$$

## B Estimates in the fixed point method

*Remark B.1.* We note  $\|Q\|_{\mathcal{Q}}^2 = \iint \tilde{E}(p+q) |\widehat{Q}(p,q)|^2 dpdq$  and by the proofs of Lemmas 8.[5] and 11.[5] there hold

$$\|\rho_{1,0}(Q)\|_{\mathcal{C}} \leq K\sqrt{\log(\Lambda)} \|Q\|_{\mathcal{Q}} \text{ and } \|Q_{0,1}\|_{\mathcal{Q}} \leq K\sqrt{\log(\Lambda)} \|\rho\|_{\mathcal{C}}. \quad (64)$$

### B.1 Preliminary estimates.

Let us form the test function of Lemma 2.3. or construct the minimizer as a fixed point: let us decompose  $\Gamma = \gamma + |\psi\rangle\langle\psi|$  where we know  $\|\psi\|_{H^{3/2}} = O(1)$  and  $\langle |\nabla|^2 \psi, \psi \rangle \lesssim (\alpha\alpha_r(0))^2$ .

*Notation B.2.* We write as before  $N = |\psi\rangle\langle\psi|$  and  $n = |\psi|^2$ . We also write the Choquard-Pekar minimizer  $\psi_{\text{CP}}$ . We take the notation of Section 4.1 for the iterations of the fixed point method.

- As  $\|(N, n)\|_{\mathcal{X}} = O(1)$ , for  $L, \alpha$  sufficiently small,  $\|(\gamma_k, \overline{\rho}_k)\|_{\mathcal{X}} = O(1)$  due to the fact that  $\|(\gamma_1, \overline{\rho}_1)\|_{\mathcal{X}} \lesssim (L + \alpha^2)\|(N, n)\|_{\mathcal{X}}$  and the function  $F$  is a contraction of constant  $\nu = O(\sqrt{L\alpha})$ .
- By (22a)  $\|R(N)\|_{\mathfrak{S}_2} \lesssim \|\nabla\psi\|_{L^2} \lesssim L\alpha$ . Then by 11.[5]:

$$\|Q_{1,0}(N)\|_E \lesssim \|R(N)\|_{\mathfrak{S}_2} \lesssim L\alpha.$$

- Moreover  $|\widehat{Q_{0,1}(N)}(p, q)|^2 = \frac{(4\pi)^2}{2^5 \pi^3} \frac{|\widehat{n}(p-q)|^2}{|p-q|^4} |M(p, q)|^2$  in the notation of [5] so  $\|Q_{0,1}(N)\|_{\mathcal{Q}} \leq \sqrt{\log(\Lambda)} \|n\|_{\mathcal{C}}$ . We recall:  $\|Q_{0,1}(\rho)\|_{\mathcal{Q}} \leq \sqrt{\log(\Lambda)} \|\rho\|_{\mathcal{C}}$ .
- Now  $\rho_{1,0}$ : there holds

$$\begin{aligned} \int_k \frac{f(k)^2}{|k|^2} |\widehat{\rho_{1,0}}(k)|^2 dk &\lesssim \int_k f(k)^2 dk \left( \iint_{|l| < \Lambda, r} \frac{|\widehat{\psi}(l-k/2)| |\widehat{\psi}(l+k/2)|}{r^2 E(l+r)^2} dl dr \right)^2 \\ &\lesssim K \int_k f(k)^2 dk \left( \int_{|l| < \Lambda} |\widehat{\psi}(l-k/2)| |\widehat{\psi}(l+k/2)| dl \right)^2, \end{aligned}$$

and by Young inequality

$$\|\rho_{1,0}(N_\lambda)\|_{\mathcal{C}} \lesssim \lambda^{-3/2} \|\psi_{\text{CP}}\|_{W^{1,4/3}} \quad (65a)$$

$$\|\rho_{1,0}(N)\|_{\mathcal{C}} \lesssim \lambda^{-3/2} \|\widehat{\psi}\|_{L^{4/3}}^2 \lesssim \lambda^{-3/2} \|\psi\|_{H^1}^2 \quad (65b)$$

*Remark B.3.* In fact, if we know that  $\psi_A := A^{3/2}\psi(Ax) = O(1)$  in  $H^1$  then

$$\|\rho_{1,0}(N)\|_c \lesssim A^{-3/2} \|\psi_A\|_{L^4}^2 \lesssim A^{-3/2} \|\psi_A\|_{H^1}$$

and

$$\|n\|_{L^2} = A^{-3/2} \|\psi_A\|_{L^4}^2 \lesssim A^{-3/2} \|\psi_A\|_{H^1}.$$

By remark B.1. we have

$$\begin{aligned} \|\rho_{1,0}(\gamma_1)\|_c &\lesssim \alpha \|\rho_{1,0}(Q_1(N, n))\|_c + \sum_{k \geq 2} \alpha^k \|\rho_{1,0}(Q_k(N, n))\|_c \\ &\lesssim (\alpha(L\alpha)^{3/2} + L\sqrt{L\alpha}) + \sqrt{\log(\Lambda)} \sum_{k \geq 2} \alpha^k \|Q_k\|_{\mathcal{Q}} \lesssim (\alpha(L\alpha) + L\sqrt{L\alpha} + \alpha\sqrt{L\alpha}). \end{aligned}$$

Writing  $\delta\gamma = (\gamma_2 - \gamma_1)$  we also have  $\|\rho_{1,0}(\delta\gamma)\|_c \lesssim \sqrt{\log(\Lambda)^*} \|\delta\gamma\|_{\mathcal{Q}}$ . We note  $x = \langle |\nabla|^2 \psi, \psi \rangle^{1/4}$ .

$\delta\gamma = \sum_{k \geq 1} \alpha^k (Q'_k(\gamma_1, \bar{\rho}_1) - Q_k(N, n)) = \alpha Q_1(\gamma_1, \bar{\rho}_1) + \sum_{k \geq 2} \alpha^k (Q'_k(\gamma_1, \bar{\rho}_1) - Q'_k(N, n))$  and thanks to (23a) for  $Q_{1,0}$ , Lemmas 11, 13, 15 of [5] for  $Q_{0,1}, Q_k$  we get

$$\alpha \|\rho_{1,0}(\delta\gamma)\|_c \lesssim (\alpha^2 \sqrt{L\alpha} x^2 + L^2 \alpha x + L\alpha^2). \quad (66)$$

Indeed

$$\begin{aligned} * \|Q_{1,0}(\gamma_1)\|_{\mathcal{Q}} &\lesssim * \|\gamma_1\|_{\mathcal{Q}} \\ &\lesssim \sqrt{L\alpha} x + \alpha x^2 + \alpha^2 \\ * \|Q_{0,1}(\bar{\rho}_1)\|_{\mathcal{Q}} &\lesssim \sqrt{\log(\Lambda)} \|\bar{\rho}_1\|_c \\ &\lesssim L\sqrt{\log(\Lambda)} x + \sqrt{L\alpha}. \end{aligned}$$

## B.2 Estimates of $\|\gamma\|_{\mathcal{Q}}, \|\gamma\|_E, \|\gamma\|_F, \|\rho_\gamma\|_{\mathcal{E}}, \|\rho_\gamma\|_c$ .

We write:  $\gamma = \sum_{k \geq 1} (\gamma_{k+1} - \gamma_k) + \gamma_1$  and  $\gamma_1 = \sum_{k \geq 1} \alpha^k Q'_k$ , taking the norm we obtain

$$\|\gamma\| \leq \sum_{k \geq 1} \nu^k \|F(N, n) - (N, n)\|_x + \|\gamma_1\| = \nu \|(\gamma_1, \bar{\rho}_1)\|_x + \|\gamma_1\|. \quad (67)$$

Underlining the terms with the biggest estimates:

$$\|\bar{\rho}_1\|_{\mathcal{E}} \lesssim \|\check{W} \star n\|_{\mathcal{E}} + \alpha \|\rho_{1,0}(N)\|_{\mathcal{E}} + \sum_{k=2}^{\infty} \alpha^k \|(N, n)\|_x^k \lesssim \sqrt{L\alpha} + Lx, \text{ and finally}$$

Depending on taking  $\|\cdot\|_E$  or  $\|\cdot\|_{\mathcal{Q}}$  we have

$$\begin{aligned} \|\gamma_1\|_{\mathcal{Q}} &\lesssim \alpha (\|Q_{0,1}(n)\|_{\mathcal{Q}} + \|Q_{1,0}(N)\|_{\mathcal{Q}}) + \sum_{k=2}^{\infty} \alpha^k \|(N, n)\|_x^k \lesssim \sqrt{L\alpha} \|n\|_{\mathcal{E}} + \alpha, \\ \|\gamma_1\|_E &\lesssim \alpha (\|Q_{0,1}(n)\|_{\mathcal{Q}} + \|Q_{1,0}(N)\|_E) + \sum_{k=2}^{\infty} \alpha^k \|(N, n)\|_x^k \lesssim \sqrt{L\alpha} \|n\|_{\mathcal{E}} + L\alpha. \end{aligned}$$

$$\|\gamma\|_{\mathcal{Q}} \lesssim \alpha, \quad \|\gamma\|_E \lesssim L\alpha. \quad (68)$$

Emphasizing the dependance of  $x = \|\nabla \psi\|_{L^2}^{1/2}$  (for the proof of Lemma 2.5)

$$\begin{aligned} \|\gamma\|_F &\leq \sum_{k \geq 2} \|\gamma_{k+1} - \gamma_k\|_{\mathcal{Q}} + \|\gamma_2 - \gamma_1\|_F + \|\gamma_1\|_F \lesssim L\alpha + \|\gamma_2 - \gamma_1\|_F + \|\gamma_1\|_F \\ \|\gamma_1\|_F &\lesssim \alpha (\|Q_{1,0}(N)\|_F + \|Q_{0,1}(n)\|_F) + \alpha^2 \lesssim \alpha x^2 + (\sqrt{L\alpha}) + \alpha^2 \\ \|\gamma_2 - \gamma_1\|_F &\lesssim \alpha (\|Q_{1,0}(\gamma_1)\|_F + \|Q_{0,1}(\bar{\rho}_1)\|_F) + \alpha^2 \lesssim \alpha \|\gamma_1\|_{\mathcal{Q}} + \sqrt{L\alpha} \|\bar{\rho}_1\|_c + \alpha^2 \\ &\lesssim \alpha (* \|\gamma_1\|_{\mathcal{Q}}) + \sqrt{L\alpha} (Lx + \sqrt{L\alpha}) + \alpha^2 \lesssim \alpha (\sqrt{L\alpha} x + \alpha) + L\sqrt{L\alpha} x + L\alpha \\ &\lesssim L\sqrt{L\alpha} x + L\alpha. \end{aligned}$$

So

$$\|\gamma\|_F \lesssim \alpha x^2 + \sqrt{L\alpha}x + L\alpha. \quad (69)$$

Analogously

$$\begin{aligned} \|\rho_\gamma\|_c &\lesssim \|\check{\alpha}_r \star n\|_c + \alpha(\|\rho_{1,0}(\delta\gamma)\|_c + \|\rho_{1,0}(\gamma_1)\|_c + \|\rho_{1,0}(N)\|_c + \sum_{k \geq 2} \|\rho_{1,0}(\gamma_{k+1} - \gamma_k)\|_c) + \alpha^2 \\ &\lesssim Lx + \alpha(L^2x + \alpha\sqrt{L\alpha}x^2 + L\alpha) + (Lx + \sqrt{L\alpha}x^2 + \alpha\sqrt{L\alpha}) + \|\rho_{1,0}(N)\|_c + (L\alpha)^{3/2} \end{aligned}$$

$$\|\rho_\gamma\|_c \lesssim \alpha\|\rho_{1,0}(N)\|_c + (L\alpha)^{3/2} + Lx + \alpha\sqrt{L\alpha}x^2 \quad (70a)$$

where  $(L\alpha)^{3/2}$  comes from  $\alpha\|\rho_{1,0}(\Sigma)\|_c \leq K\sqrt{L\alpha}\|\Sigma\|_{\mathcal{Q}}$ , the other terms are negligible and  $x = \langle |\nabla|^2\psi, \psi \rangle^{1/4}$ .

With the test function we get:  $\|\rho_\gamma\|_{\mathcal{E}}, \|\rho_\gamma\|_c \lesssim Lx\|\psi_{\text{CP}}\|_{\mathcal{E}}$ , it is easier for there is a simple estimate of  $\rho_{1,0}(N)$ .

### B.3 Estimates of $\|\gamma S\psi_\lambda\|_{L^2}$ , $S = \text{id}$ , $|\mathcal{D}^0|$ .

We write

$\|\gamma S\psi_\lambda\|_{L^2} \leq \|\gamma\|_{\mathcal{B}}\|S\psi_\lambda\|_{L^2}$ . Looking at the expression of  $Q_k(\gamma'\rho'_\gamma)$  (cf [5]) it is straightforward that

$$\begin{aligned} \|Q_{1,0}(\gamma')\|_{\mathcal{E}_2} &\lesssim \|R(N)\|_{\mathcal{E}_2} + \|\frac{\hat{R}(p,q)}{E(p)+E(q)}\|_{L^2} \\ &\lesssim \|R(N)\|_{\mathcal{E}_2} + \|\gamma\|_{\mathcal{Q}} \lesssim L\alpha, \\ \|Q_{0,1}(\rho'_\gamma)\|_{\mathcal{E}_2} &\lesssim \|\rho'_\gamma\|_c \lesssim \sqrt{L\alpha}. \end{aligned}$$

So

$$\|\gamma\|_{\mathcal{B}} \leq \|\gamma\|_{\mathcal{E}_2} \lesssim \alpha(\|Q_{1,0}(\gamma')\|_{\mathcal{E}_2} + \|Q_{0,1}(\rho'_\gamma)\|_{\mathcal{E}_2}) + \alpha^2 \lesssim \alpha\sqrt{L\alpha}. \quad (71)$$

Thus

$$\|\gamma|\mathcal{D}^0|\psi_\lambda\|_{L^2} \leq K\alpha\sqrt{L\alpha} = o(L\alpha). \quad (72)$$

## C The operator $|\mathcal{D}^0 + \alpha B| - |\mathcal{D}^0|$

### C.1 $\text{Tr}(|\mathcal{D}^0 + \alpha B| - |\mathcal{D}^0|)$ .

We use the following formula: for  $x > 0$

$$\sqrt{x} = \frac{1}{\pi} \int_0^{+\infty} \frac{x}{x+u} \frac{du}{\sqrt{u}} \quad (73)$$

by applying it to  $|\mathcal{D}^0|^2$  and  $|\mathcal{D}^0 + \alpha B|^2$ . Indeed we can write  $|\mathcal{D}^0 + \alpha B|^2 = |\mathcal{D}^0|(1 + \alpha T)|\mathcal{D}^0|$  where  $T = G + \alpha W = O(1)$  in  $\mathcal{B}(\mathfrak{H}_\Lambda)$  thanks to 3.3.

Similarly we write

$$|\mathcal{D}^0 + \alpha B|^2 + u = \sqrt{|\mathcal{D}^0|^2 + u}(1 + \alpha(G_u + \alpha W_u))\sqrt{|\mathcal{D}^0|^2 + u}, T_u = G_u + \alpha W_u.$$

$$\begin{aligned} |\mathcal{D}^0 + \alpha B| &= \frac{1}{\pi} \int_u \frac{|\mathcal{D}^0|(1 + \alpha G + \alpha^2 W)}{\sqrt{|\mathcal{D}^0|^2 + u}} \frac{1}{1 + \alpha G_u + \alpha^2 W_u} \frac{1}{\sqrt{|\mathcal{D}^0|^2 + u}} \frac{du}{\sqrt{u}} \\ &= |\mathcal{D}^0| + \frac{\alpha}{\pi} \int_u \left( |\mathcal{D}^0| G \frac{|\mathcal{D}^0|}{\sqrt{|\mathcal{D}^0|^2 + u}} \frac{1}{\sqrt{|\mathcal{D}^0|^2 + u}} - |\mathcal{D}^0| \frac{|\mathcal{D}^0|}{\sqrt{|\mathcal{D}^0|^2 + u}} G_u \frac{1}{\sqrt{|\mathcal{D}^0|^2 + u}} \right) \frac{du}{\sqrt{u}} \\ &\quad + \int_u (\dots) \end{aligned}$$

where the last integral is a bounded operator,  $O(\alpha^2)$  as power series in  $\alpha$ :

$$\begin{aligned} \int_u (\dots) &= \frac{\alpha^2}{\pi} \int_u \left( |\mathcal{D}^0| W |\mathcal{D}^0| \frac{1}{|\mathcal{D}^0 + \alpha B|^2} - |\mathcal{D}^0 + \alpha B|^2 \frac{1}{\sqrt{|\mathcal{D}^0|^2 + u}} W_u \frac{1}{\sqrt{|\mathcal{D}^0|^2 + u}} \right) \frac{du}{\sqrt{u}} \\ &\quad + \frac{\alpha^2}{\pi} \int_u |\mathcal{D}^0| (1 + \alpha T) \frac{|\mathcal{D}^0|}{\sqrt{|\mathcal{D}^0|^2 + u}} \frac{T_u^2}{1 + \alpha T_u} \frac{1}{\sqrt{|\mathcal{D}^0|^2 + u}} \frac{du}{\sqrt{u}}. \end{aligned}$$

*Notation C.1.* To simplify we will write

$$\begin{aligned} d &= \mathcal{D}^0, & q &= \mathcal{D}^0 + \alpha B, & q^2 &= d^2 + \alpha y, & d_u &= \sqrt{|\mathcal{D}^0|^2 + u} \\ g &= G, G_u, & w &= W, W_u, & m &= g, v & b &= B. \end{aligned}$$

Let  $q_1, q_2 \in \mathcal{Q}$ :

$$\left| \text{Tr} \left( |d| g \frac{|d|}{d_u^2} q_1 q_2 \right) \right| = \left| \text{Tr} \left( g \frac{|d|}{d_u^2} q_1 q_2 |d| \right) \right| \leq \|g|d|^{1/4}\|_{\mathcal{B}} \left\| \frac{|d|^{3/4}}{d_u^2} \right\|_{\mathcal{B}} \|q_1 |d|^{1/2}\|_{\mathfrak{S}_2} \| |d|^{-1/2} q_2 |d| \|_{\mathfrak{S}_2}, \quad (74)$$

as  $\tilde{E}(q)^2 \tilde{E}(p)^{-1} \leq 2C_1^2 \tilde{E}(p - q) \tilde{E}(p + q)$ ,  $\| |d|^{-1/2} q_2 |d| \|_{\mathfrak{S}_2} \leq K \|q_2\|_E$  and  $\|q_1 |d|^{1/2}\|_{\mathfrak{S}_2} \leq \|q_1\|_E$  is immediate.

Then  $G = b|d|^{-1} + |d|^{-1}b$  and  $T = |d|^{-1}b^2|d|^{-1}$  such that

$$\begin{aligned} g|d|^{1/4} &= b|d|^{-3/4} + |d|^{-1}[b, |d|^{1/4}] + |d|^{-3/4}b \\ T|d|^{1/4} &= |d|^{-1}b|d|^{-3/4} \end{aligned}$$

We treat the commutator in C.2, for the others Lemma 3.3 gives

$$\|b|d|^{-3/4}\|_{\mathcal{B}} \leq K(\|\rho'_\gamma\|_c + \|\gamma'\|_{\mathcal{Q}}).$$

For a bounded borelian function  $f$ ,  $\|f(\mathcal{D}^0)\|_{\mathcal{B}} = \sup_{x \in \sigma(d)} |f(x)|$ ; the function  $x > 0 \rightarrow \frac{x^s}{x^2+u}$ ,  $s \leq 1$  reaches its maximum at the point  $x_0 = \sqrt{\frac{su}{2-s}}$  where  $f(x_0) = \frac{u^{s/2}}{u(1+\frac{s}{2-s})} \frac{s}{2-s} \leq u^{s/2-1}$ . For  $x_0 < 1$  the norm is  $f(1)$  and for  $x_0 > \tilde{E}(\Lambda)$  it is  $f(\tilde{E}(\Lambda))$  such that

$$\int \left\| \left\| \frac{|d|^s}{d_u^2} \right\|_{\mathcal{B}} \frac{du}{\sqrt{u}} \right\| \leq \begin{cases} K_s & s < 1 \\ K \log(\Lambda) & s = 1 \end{cases}$$

Similarly the trace  $\text{Tr}\{|d|T|d|d_u^{-1}v d_u^{-1}q_1 q_2\}$  is equal to the trace of

$$T \frac{|d|^{3/4}}{d_u} (|d|^{1/4}v) d_u^{-1} (q_1 |d|^{1/2}) (|d|^{-1/2} q_2 |d|)$$

and  $\text{Tr}\{|d|m \frac{|d|}{d_u} m \frac{1}{d_u} q_1 q_2\}$  to the trace of:

$$\begin{aligned} &|d|^{-\frac{1}{2}} m |d|^{\frac{3}{4}} \frac{|d|^{\frac{1}{4}}}{d_u} m \frac{|d|^{\frac{1}{2}}}{d_u} (|d|^{-\frac{1}{2}} q_1 |d|) (|d|^{-1} q_2 |d|^{\frac{1}{2}}) \quad (75a) \\ &\text{and} \begin{cases} |d|^{-\frac{1}{2}} G |d|^{\frac{3}{4}} = |d|^{-\frac{1}{2}} b |d|^{-\frac{1}{4}} + |d|^{-\frac{3}{2}} [b, |d|^{\frac{1}{2}}] + |d|^{-1} b \\ |d|^{-\frac{1}{2}} T |d|^{\frac{3}{4}} = |d|^{-\frac{3}{2}} b |d|^{\frac{1}{2}} |d|^{-\frac{1}{2}} b |d|^{-\frac{1}{4}} \end{cases} \end{aligned}$$

Thanks to Lemma 3.3.,

$|d|^{-\frac{1}{2}} b |d|^{-\frac{1}{2}} \in \mathcal{B}(\mathfrak{H}_\Lambda)$  and there holds

**Lemma C.2.**  $\left\| \left\| |d|^{-\frac{3}{2}} [b, |d|^{\frac{1}{2}}] \right\|_{\mathcal{B}} \right\|, \left\| \left\| |d|^{-1} [b, |d|^{\frac{1}{4}}] \right\|_{\mathcal{B}} \right\| \lesssim (\|\gamma'\|_{\mathcal{Q}} + \|\rho'\|_c).$

The estimation with  $R(\gamma')$  comes from (23b): indeed we have

$$|\widetilde{E}(p)^s - \widetilde{E}(q)^s| \leq K \frac{|p-q|}{\widetilde{E}(p)^{1-s} + \widetilde{E}(q)^{1-s}}, \quad s = \frac{1}{2}, \frac{1}{4} \text{ etc.}$$

Then with  $f \in \mathfrak{H}_\Lambda$  there holds with  $\Phi = |d|^{-\frac{3}{2}} [\varphi'_a, |d|^{\frac{1}{2}}]$

$$\int_p |\widehat{\Phi}f(p)|^2 dp \leq K \iint \frac{dpdq}{\widetilde{E}(p)^3} \frac{|\widetilde{E}(p) - \widetilde{E}(q)|^2}{|p-q|^4} \frac{|\widehat{\rho}'_\gamma(p-q)|^2}{\widetilde{E}(p) + \widetilde{E}(q)} \int |f(q)|^2 dq,$$

and we do the same for the last term.

Let us now deal with  $\text{Tr}(|\mathcal{D}^0 + \alpha B|\gamma^2)$ . Using (73) we have the trace of an integral. We can change the order of summation for the integrand (which is the operator  $\frac{q^2}{q^2+u} \frac{\gamma^2}{\sqrt{u}}$ ) is non-negative. Doing so, we then expand the operator  $\frac{q^2}{q^2+u}$  into the six operators we have written previously. Estimating the absolute value of the traces we obtain:

$$\text{Tr}(|\mathcal{D}^0 + \alpha B|\gamma^2) = \text{Tr}(|\mathcal{D}^0|\gamma^2) + O(\alpha(\|\gamma'\|_{\mathcal{Q}} + \|\rho'_\gamma\|_c)\|\gamma\|_E^2) = \text{Tr}(|\mathcal{D}^0|\gamma^2) + O(\alpha(L\alpha)^2) \quad (76)$$

where  $(\|\gamma'\|_{\mathcal{Q}} + \|\rho'_\gamma\|_c)$  comes from the estimates of the bounded operator  $\|g|d|^{1/4}\|_{\mathcal{B}}$  etc.

The integration over  $u$  gives a constant  $K$  while  $\|\gamma\|_E \lesssim L\alpha$ .

## C.2 $\langle |\mathcal{D}^0 + \alpha B|\phi, \phi \rangle, \phi \in H^{1/2}$ .

We want to prove

**Lemma C.3.** There exists  $C_2 > 0$  such that

$$\langle |\mathcal{D}^0 + \alpha B|\phi, \phi \rangle \geq (1 - C_2\alpha)\langle |\mathcal{D}^0|\phi, \phi \rangle. \quad (77)$$

Indeed we go back to (73):

$$\begin{aligned} q(q^2 + u)^{-1}q - d(d^2 + u)^{-1}d &= q(d^2 + u)^{-1}q - d(d^2 + u)^{-1}d + q((q^2 + u)^{-1} - (d^2 + u)^{-1})q \\ &= \alpha b \frac{d}{d^2+u} + \alpha \frac{d}{d^2+u} b + \alpha^2 b(d^2 + u)^{-1}b - \alpha q(q^2 + u)^{-1}y(d^2 + u)^{-1}q \\ &= \alpha(x_1 + x_2 + \alpha x_3 - x_4). \end{aligned}$$

And then we do the same as before. For instance let us treat  $x_4$ :

writing  $s = |d|^{1/2}$ ,  $r_u = (d^2 + u)^{1/2}$  and  $\phi^+, \phi^-$  according to  $\chi_{(0,\infty)}(|\mathcal{D}^0 + \alpha B|)$ , there holds:

$$\begin{aligned} \langle x_4\phi^+, \phi^+ \rangle &= \langle s^{-1}|q|s^{-1} \frac{sr_u^{-1}}{r_u} (q^2 + u)^{-1} r_u r_u^{-1} y r_u^{-3/4} \frac{r_u^{-5/4}}{s} s^{-1}|q|s^{-1} s\phi^+, s\phi^+ \rangle \\ |\langle x_4\phi^+, \phi^+ \rangle| &\lesssim \|sr_u^{-1}\|_{\mathcal{B}} \|r_u^{-5/4}s\|_{\mathcal{B}} \|r_u^{-1}y r_u^{-3/4}\|_{\mathcal{B}} \langle |d|\phi^+, \phi^+ \rangle. \end{aligned}$$

Then  $|d|^{-1}y|d|^{-3/4} = b|d|^{-3/4} + |d|^{-1}b|d|^{1/4} + \alpha|d|^{-1}bb|d|^{-3/4}$  and we finish as before.

We do the same for  $(\phi^-, \phi^+)$  etc.

Then integrating over  $u$  we get

$$\left| \langle |\mathcal{D}^0 + \alpha B|\phi, \phi \rangle - \langle |\mathcal{D}^0|\phi, \phi \rangle \right| \leq K\alpha(\sqrt{\text{Tr}(|\mathcal{D}^0|(\gamma')^2)} + \|\rho'_\gamma\|_c)\langle |\mathcal{D}^0|\phi, \phi \rangle, \quad (78)$$

assuming that  $\|r_u^{-1}R_{\gamma'}r_u^{1/4}\|_{\mathcal{B}} \lesssim \sqrt{\text{Tr}(|\mathcal{D}^0|(\gamma')^2)}$ .

Indeed, we write  $r_u^{-1}R_{\gamma'}r_u^{1/4} = r_u^{-3/4}R_{\gamma'} + r_u^{-1}[R_{\gamma'}, r_u^{1/4}]$ ; then in Fourier space there holds

$$(r_u^{-1}\widehat{[R_{\gamma'}, r_u^{1/4}]})(p, q) = \text{Cst} \times \frac{1}{\sqrt{\widetilde{E}(p)^2 + u}} ((\widetilde{E}(p)^2 + u)^{1/4} - (\widetilde{E}(q)^2 + u)^{1/4}) \widehat{R}(p, q)$$

and so

$$|(r_u^{-1}[\widehat{R_{\gamma'}}, r_u^{1/4}])(p, q)| \lesssim \frac{|p - q|^{1/4}}{\sqrt{\widetilde{E}(p)^2 + u}} |\widehat{R}(p, q)|.$$

Following the methods of [5], for  $1/2 < \theta < 3/2$ :

$$\begin{aligned} \iint \frac{|p - q|^{1/2}}{\widetilde{E}(p)^2 + u} |\widehat{R}(p, q)|^2 dp dq &\lesssim \iiint \frac{|k|^{1/2}}{\widetilde{E}(v + k/2)^2 + u} \frac{\widetilde{E}(2l)^{1/2+\theta}}{\widetilde{E}(2l')^{1/2+\theta}} \frac{|\widehat{\gamma}'(l + k, l - k)|^2}{|l - v|^2 |l' - v|^2} dk dl dv dl' \\ &\lesssim \iint |k|^{1/2} \widetilde{E}(2l)^{1/2} |\widehat{\gamma}'(l + k, l - k)|^2 \times K_\theta(l, k) dl dk, \end{aligned}$$

where

$$K_\theta(l, k) = \widetilde{E}(2l)^\theta \iint \frac{dv dl'}{|l - v|^2 |l' - v|^2 \widetilde{E}(2l')^{1/2+\theta} (\widetilde{E}(v + k/2)^2 + u)}.$$

First, there holds

$$\int \frac{dl'}{|l' - v|^2 \widetilde{E}(2l')^{1/2+\theta}} \lesssim \frac{1}{|v|^{\theta-1/2}}.$$

At last we have to prove that

$$\int \frac{\widetilde{E}(2l)^\theta dv}{|l - v|^2 |v|^{\theta-1/2} (\widetilde{E}(v + k/2)^2 + u)} \lesssim 1.$$

- If  $|l| \geq 1$  we use the inequality  $\frac{1}{u + \widetilde{E}(v + k/2)^2} \leq \frac{1}{|v + k/2|^{1/2+1}}$  and if  $|l| < 1$  we use the inequality  $\frac{1}{u + \widetilde{E}(v + k/2)^2} \leq \frac{1}{|v + k/2|^{3/2-\theta}}$ .
- If  $|k| \geq 2|l|$  we make the change of variables  $v' = \frac{v}{|k|}$  and if  $|k| < 2|l|$  we make another one:  $v' = \frac{v}{|l|}$ .