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Nonparametric regression on hidden Φ -mixing variables: identifiability and consistency of a pseudo-likelihood based estimation procedure

Thierry Dumont* and Sylvain Le Corff†

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Abstract

This paper outlines a new nonparametric estimation procedure for unobserved Φ -mixing processes. It is assumed that the only information on the stationary hidden states $(X_k)_{k \geq 0}$ is given by the process $(Y_k)_{k \geq 0}$, where Y_k is a noisy observation of $f_*(X_k)$. The paper introduces a maximum pseudo-likelihood procedure to estimate the function f_* and the distribution $\nu_{*,b}$ of (X_0, \dots, X_{b-1}) using blocks of observations of length b . The identifiability of the model is studied in the particular cases $b = 1$ and $b = 2$ and the consistency of the estimators of f_* and of $\nu_{*,b}$ as the number of observations grows to infinity is established.

1 Introduction

The model considered in this paper is made of a bivariate stochastic process $\{(X_k, Y_k)\}_{k \geq 0}$ where only a sample of the observation sequence $\{Y_k\}_{k \geq 0}$ is available. These observations are given by

$$Y_k = f_*(X_k) + \epsilon_k,$$

where f_* is a function defined on a space \mathbb{X} and taking values in \mathbb{R}^ℓ . The measurement noise $(\epsilon_k)_{k \geq 0}$ is an independent and identically distributed (i.i.d.) sequence of Gaussian random vectors of \mathbb{R}^ℓ . This paper proposes a new method to estimate the function f_* and the distribution of the hidden states using only the observations $(Y_k)_{k \geq 0}$. Note that the setting introduced here encompasses the case of hidden Markov models in which the state sequence $(X_k)_{k \geq 0}$ is a Markov chain, the observations $(Y_k)_{k \geq 0}$ are independent conditionally on $(X_k)_{k \geq 0}$ and where the conditional distribution of Y_k given the state sequence depends only on X_k . These hidden models can be applied in a large variety of disciplines such as financial econometrics [15], biology [3] or speech recognition [11].

It is clear that the model considered in this paper is not identifiable with no additional assumptions and that the statistical inference of f_* may not be possible. For instance, if $\tilde{X}_k = \sigma(X_k)$ where $\sigma : \mathbb{X} \rightarrow \mathbb{X}$ is a bijective function, then $Y_k = f_* \circ \sigma^{-1}(\tilde{X}_k) + \epsilon_k$. Therefore, there exist a function \tilde{f} and a process $(\tilde{X}_k)_{k \geq 0}$ on \mathbb{X} fully characterizing the distribution

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of the observation process $(Y_k)_{k \geq 0}$ and it is not possible to define a consistent estimator of f_\star using the observations $(Y_k)_{k \geq 0}$ only. It is then natural to study the identifiability with some assumptions on the model, which is done in Proposition 4.1 and Proposition 4.3 below. To our best knowledge, these are the first results about the identifiability of the nonparametric regression models on hidden variables studied in this paper. It is assumed that the state-space \mathbb{X} is a compact subset of \mathbb{R}^m and that f_\star is a C^1 diffeomorphism. The C^1 regularity hypothesis on the target function f_\star allows to perform the estimation procedure in a Sobolev setting such as in classical regression frameworks. The invertibility of f_\star is a somehow strong assumption. Nevertheless, in the case $\ell \geq 2m + 1$, this assumption is satisfied for a dense class of functions in C^1 . Proposition 4.1 establishes that if \tilde{X}_0 has a distribution with probability density ν and if $\tilde{f} : \mathbb{X} \rightarrow \mathbb{R}^\ell$ is such that $\tilde{f}(\tilde{X}_0)$ and $f_\star(X_0)$ have the same distribution then: (a) $\tilde{f} = f_\star \circ \phi$ with $\phi : \mathbb{X} \rightarrow \mathbb{X}$ a bijective function and (b) ν is obtained by a transformation of the distribution density of X_0 involving ϕ . When $m = 1$ and X_0 is uniformly distributed on \mathbb{X} , ϕ is an isometric transformation of \mathbb{X} and the model is almost identifiable. Proposition 4.3 states a similar result on f_\star and on the distribution of (X_0, X_1) when $(\tilde{f}(\tilde{X}_0), \tilde{f}(\tilde{X}_1))$ and $(f_\star(X_0), f_\star(X_1))$ have the same distribution. As a striking consequence, Corollary 4.4 shows that if the density of the distribution of X_1 conditionally on $X_0 = x$ is of the form $q_\star(x, x') = c_\star(x)\rho_\star(\|x - x'\|)$, then q_\star and the full distribution of (X_0, X_1) are identifiable. In addition $f = f_\star \circ \phi$ with $\phi : \mathbb{X} \rightarrow \mathbb{X}$ an isometric function.

This paper also proposes a new method to estimate the function f_\star and the distribution $\nu_{b,\star}$ of the hidden states (X_0, \dots, X_{b-1}) for a fixed parameter b using only the observations $(Y_k)_{k \geq 0}$. Note that this nonparametric estimation problem differs from classical regression settings since the variables $(X_k)_{k \geq 0}$ are not observed. In *errors-in-variables* models, the random variables $(X_k)_{k \geq 0}$ are i.i.d. and observed through a sequence $(Z_k)_{k \geq 0}$, i.e. $Z_k = X_k + \eta_k$ and $Y_k = f_\star(X_k) + \epsilon_k$, where the variables $(\eta_k)_{k \geq 0}$ are i.i.d with known distribution. Many solutions have been proposed to solve this problem, see [9] and [10] for a ratio of deconvolution kernel estimators, [12] for B-splines estimators and [4] for a procedure based on the minimization of a penalized contrast. In the case where the hidden state is a Markov chain, [14] and [13] considered the following observation model $Y_k = X_k + \epsilon_k$, where the random variables $\{\epsilon_k\}_{k \geq 0}$ are i.i.d. with known distribution. [14] (resp. [13]) proposed an estimate of the transition density (resp. the stationary density and the transition density) of the Markov chain $(X_k)_{k \geq 0}$ based on the minimization of a penalized L_2 contrast. However, there does not exist any result on the nonparametric estimation problem studied in this paper with unobserved states $(X_k)_{k \geq 0}$.

As mentioned in the discussion on the identifiability of the model, it is assumed that f_\star is a C^1 diffeomorphism. It is therefore rational to introduce a likelihood based procedure which penalizes the complexity of the estimated function f . The observations are decomposed into non-overlapping blocks $(Y_{kb}, \dots, Y_{(k+1)b-1})$ and the pseudo-loglikelihood of the observations (Y_0, \dots, Y_{nb-1}) considered in this paper is given by the sum of the loglikelihood of $(Y_{kb}, \dots, Y_{(k+1)b-1})$ for $k \in \{0, \dots, n-1\}$. The estimator $(\hat{f}_n, \hat{\nu}_n)$ of $(f_\star, \nu_{b,\star})$ is defined as a maximizer of the penalized version of the pseudo-likelihood of the observations (Y_0, \dots, Y_{nb-1}) . This estimator of f_\star can be used to define an estimator \hat{p}_n of the density of the distribution of (Y_0, \dots, Y_{b-1}) . Theorem 3.1 states that the Hellinger distance between \hat{p}_n and the true distribution of a block of observations vanishes as the number of observations grows to infinity. More precisely, this Hellinger distance converges at a rate which can be chosen as close as possible to $n^{-1/4}$. To establish this result, the penalization function

needs only to be lower bounded by a power of the supremum norm. The rate obtained in Theorem 3.1 is slower than the rate of convergence given in [20, Theorem 10.6] which uses a penalty term directly based on the regularity of the density (and not on the regularity of f as it is considered in this paper). We believe that the rate of convergence of Theorem 3.1 could be improved for particular choices of function space and complexity function used in the penalized pseudo-likelihood. The consistency of $(\widehat{f}_n, \widehat{\nu}_n)$ follows as a consequence together with some continuity properties, see Corollary 3.3. The rate of convergence of \widehat{f}_n to f_\star remains an open problem and seems to be very challenging.

It is also proven that the results presented in this paper hold in the special case where the function f_\star belongs to a Sobolev space provided that the order s of the Sobolev space and the order p of the associated L_p space satisfy $s > m/p + 1$ to allow the use of classical embedding into the space of continuously differentiable functions on \mathbb{X} . Proposition 5.1 establishes the consistency of the estimator of f_\star when the penalization function is based on the Sobolev norm.

The proof of the convergence of the Hellinger distance between \widehat{p}_n and the true distribution of a block of observations relies on a concentration inequality for the empirical process of the observations. This result is obtained by an extension of the concentration inequality for Φ -mixing processes given in [19, Theorem 3]. The inequality of [19, Theorem 3] holds for empirical processes based on uniformly bounded functions which is not the case in the model presented here but a similar control can be proven under the assumptions of this paper. Then, the control of the expectation of the supremum of the empirical process is given by a direct application of the maximal inequality for dependent processes of [7].

The theoretical results given in the paper are supported by numerical experiments: in the case where $m = 1$, an Expectation-Maximization (see [6]) based algorithm is outlined to compute $\widehat{\nu}_n$ and \widehat{f}_n .

The model and the estimators are presented in Section 2. The consistency results are displayed in Section 3. The identifiability in the cases $b = 1$ and $b = 2$ is addressed in Section 4. The application to a Sobolev class of function is detailed in Section 5 and the algorithm and numerical experiments are displayed in Section 6. Section 7 gathers important proofs on the identifiability and consistency needed to state the main results.

2 Model and definitions

Let $(\Omega, \mathcal{E}, \mathbb{P})$ be a probability space and $(\mathbb{X}, \mathcal{X})$ be a general state space equipped with a measure μ . Let $(X_k)_{k \geq 0}$ be a stationary process defined on Ω and taking values in \mathbb{X} . This process is only partially observed through the sequence $(Y_k)_{k \geq 0}$ which takes values in \mathbb{R}^ℓ , $\ell \geq 1$. For any $k \geq 1$ the sequence (x_1, \dots, x_k) is denoted by $x_{1:k}$. The observations $(Y_k)_{k \geq 0}$ are given by

$$Y_k \stackrel{\text{def}}{=} f_\star(X_k) + \epsilon_k, \quad (1)$$

where $f_\star : \mathbb{X} \rightarrow \mathbb{R}^\ell$ is a measurable function and $(\epsilon_k)_{k \geq 0}$ are i.i.d. with density φ with respect to the Lebesgue measure λ of \mathbb{R}^ℓ , given, for any $y_{1:\ell} \in \mathbb{R}^\ell$, by:

$$\varphi(y_{1:\ell}) \stackrel{\text{def}}{=} (2\pi)^{-\ell/2} \exp\left(-\frac{1}{2} \sum_{j=1}^{\ell} y_j^2\right). \quad (2)$$

The covariance matrix of the observation noise is assumed to be the identity matrix. This does not imply any loss of generality since with a nonsingular covariance matrix Σ , the observation Y_k can just be replaced by $\Sigma^{-1}Y_k$.

One of the objective of this paper is the estimation of the target function $f_\star \in \mathcal{F}$ where \mathcal{F} is a set of functions from \mathbb{X} to \mathbb{R}^ℓ . The results presented in Section 3 and Section 4 are applied in Section 5 when \mathcal{F} is a Sobolev space.

Let b be a positive integer. For any sequence $(x_k)_{k \geq 0}$, define $\mathbf{x}_k \stackrel{\text{def}}{=} (x_{kb}, \dots, x_{(k+1)b-1})$ and for any function $f : \mathbb{X} \rightarrow \mathbb{R}^\ell$, define $\mathbf{f} : \mathbb{X}^b \rightarrow \mathbb{R}^{b\ell}$ by

$$\mathbf{x} = (x_0, \dots, x_{b-1}) \mapsto \mathbf{f}(\mathbf{x}) \stackrel{\text{def}}{=} (f(x_0), \dots, f(x_{b-1})) .$$

The distribution of \mathbf{X}_0 is assumed to have a density $\nu_{b,\star}$ with respect to the measure $\mu^{\otimes b}$ on \mathbb{X}^b which is assumed to lie in a set of probability densities \mathcal{D}_b . For all $f \in \mathcal{F}$ and $\nu \in \mathcal{D}_b$, let $p_{f,\nu}$ be defined, for all $\mathbf{y} \in \mathbb{R}^{b\ell}$, by

$$p_{f,\nu}(\mathbf{y}) \stackrel{\text{def}}{=} \int \nu(\mathbf{x}) \prod_{k=0}^{b-1} \varphi(y_k - f(x_k)) \mu^{\otimes b}(\mathrm{d}\mathbf{x}) . \quad (3)$$

Note that $p_\star \stackrel{\text{def}}{=} p_{f_\star, \nu_{b,\star}}$ is the probability density of \mathbf{Y}_0 defined in (1). The function

$$y_{0:(n+1)b-1} \mapsto \sum_{k=0}^{n-1} \ln p_{f,\nu}(\mathbf{y}_k)$$

is referred to as the pseudo log-likelihood of the observations up to time $nb - 1$.

The estimation procedure introduced in this paper is based on the M-estimation presented in [21] and [20]. Consider a function $I : \mathcal{F} \rightarrow \mathbb{R}^+$ which characterizes the complexity of functions in \mathcal{F} and let ρ_n and λ_n be some positive numbers. Define the following ρ_n -Maximum Pseudo-Likelihood Estimator (ρ_n -MPLE) of $(f_\star, \nu_{b,\star})$:

$$\left(\widehat{f}_n, \widehat{\nu}_n \right) \stackrel{\text{def}}{=} \operatorname{argmax}_{f \in \mathcal{F}, \nu \in \mathcal{D}^b}^{\rho_n} \left\{ \sum_{k=0}^{n-1} \ln p_{f,\nu}(\mathbf{Y}_k) - \lambda_n I(f) \right\} , \quad (4)$$

where $\operatorname{argmax}_{f \in \mathcal{F}, \nu \in \mathcal{D}^b}^{\rho_n}$ is one of the couples (f', ν') such that

$$\sum_{k=0}^{n-1} \ln p_{f',\nu'}(\mathbf{Y}_k) - \lambda_n I(f') \geq \sup_{f \in \mathcal{F}, \nu \in \mathcal{D}^b} \left\{ \sum_{k=0}^{n-1} \ln p_{f,\nu}(\mathbf{Y}_k) - \lambda_n I(f) \right\} - \rho_n .$$

The consistency of the estimators is established using a control for empirical processes associated with mixing sequences. The Φ -mixing coefficient between two σ -fields $\mathcal{U}, \mathcal{V} \subset \mathcal{E}$ is defined in [5] by

$$\Phi(\mathcal{U}, \mathcal{V}) \stackrel{\text{def}}{=} \sup_{\substack{U \in \mathcal{U}, V \in \mathcal{V}, \\ \mathbb{P}(U) > 0}} \left| \frac{\mathbb{P}(U \cap V)}{\mathbb{P}(U)} - \mathbb{P}(V) \right| .$$

The process $(X_k)_{k \geq 0}$ is said to be Φ -mixing when $\lim_{i \rightarrow \infty} \Phi_i^X = 0$ where, for all $i \geq 1$,

$$\Phi_i^X \stackrel{\text{def}}{=} \Phi(\sigma(X_k ; k \leq 0), \sigma(X_k ; k \geq i)) , \quad (5)$$

$\sigma(X_k ; k \in C)$ being the σ -field generated by $(X_k)_{k \in C}$ for any $C \subset \mathbb{Z}$. As in [19], the required concentration inequality for the empirical process is established under the following assumption on the Φ -mixing coefficients of $(X_k)_{k \geq 0}$.

H1 The stationary process $(X_k)_{k \geq 0}$ satisfies

$$\Phi \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} (\Phi_i^X)^{1/2} < \infty, \quad (6)$$

where Φ_i^X is given by (5).

Remark 2.1. Assume that $(X_k)_{k \geq 0}$ is a Markov chain with transition kernel Q and stationary distribution π and that there exist $\epsilon > 0$ and a measure ϑ on \mathbb{X} such that, for all $x \in \mathbb{X}$ and all $A \in \mathcal{E}$,

$$Q(x, A) \geq \epsilon \vartheta(A).$$

Then, by [17, Theorem 16.2.4], there exists $\rho \in (0, 1)$ such that, for all $x \in \mathbb{X}$ and all $A \in \mathcal{E}$,

$$|Q^n(x, A) - \nu(A)| \leq \rho^n.$$

Therefore, for all $n, i > 0$ and $A, B \in \mathcal{E}$ such that $\pi(A) > 0$,

$$\begin{aligned} & |\mathbb{P}(X_{n+i} \in B | X_n \in A) - \mathbb{P}(X_{n+i} \in B)| \\ & \leq |\mathbb{P}(X_{n+i} \in B | X_n \in A) - \pi(B)| + |\mathbb{P}(X_{n+i} \in B | X_n \in A) - \pi(B)|, \\ & \leq \frac{1}{\pi(A)} \left| \int_A (Q^i(x, B) - \pi(B)) \pi(dx) \right| + \left| \int_{\mathbb{X}} (Q^i(x, B) - \pi(B)) \pi(dx) \right|, \\ & \leq 2\rho^i. \end{aligned}$$

The Φ -mixing coefficients associated with $(X_k)_{k \geq 0}$ decrease geometrically and Assumption **H1** is satisfied.

3 General convergence results

Denote by \hat{p}_n the estimator of p_* defined by

$$\hat{p}_n \stackrel{\text{def}}{=} p_{\hat{f}_n, \hat{\nu}_n}. \quad (7)$$

The first step to prove the consistency of the estimators is to establish the convergence of \hat{p}_n to p_* using a suitable metric. This is done in Theorem 3.1 where the only assumption related to the penalization procedure is that the complexity function I is lower bounded by a power of the supremum norm. Consider the following assumptions.

H2 There exist $C > 0$ and $\nu > 0$ such that for all $f \in \mathcal{F}$,

$$\|f\|_{\infty} \leq CI(f)^{\nu}, \quad (8)$$

where, for any $f \in \mathcal{F}$, $\|f\|_{\infty} \stackrel{\text{def}}{=} \max_{1 \leq j \leq \ell} \sup_{x \in \mathbb{X}} |f^j(x)|$.

H3 There exist $0 < \nu_- < \nu_+ < +\infty$ such that, for all $\nu \in \mathcal{D}_b$ $\nu_- \leq \nu \leq \nu_+$.

The convergence of \widehat{p}_n to p_\star is established using the Hellinger metric defined, for any probability densities p_1 and p_2 on $\mathbb{R}^{b\ell}$, by

$$h(p_1, p_2) \stackrel{\text{def}}{=} \left[\frac{1}{2} \int \left(p_1^{1/2}(y) - p_2^{1/2}(y) \right)^2 dy \right]^{1/2}. \quad (9)$$

Theorem 3.1 provides a rate of convergence of \widehat{p}_n to p_\star and a bound for the complexity $I(\widehat{f}_n)$ of the estimator \widehat{f}_n . For any sequence of real random variables $(Z_n)_{n \geq 0}$ and any sequence of positive numbers $(\alpha_n)_{n \geq 0}$, we write $Z_n = O_{\mathbb{P}}(\alpha_n)$ if

$$\lim_{T \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{P}(|Z_n| > T\alpha_n) = 0.$$

and $Z_n = o_{\mathbb{P}}(\alpha_n)$ if for all $T > 0$,

$$\lim_{n \rightarrow +\infty} \mathbb{P}(|Z_n| > T\alpha_n) = 0.$$

Theorem 3.1. *Assume H1-3 hold for some v such that $b\ell v < 1$. Assume also that λ_n and ρ_n satisfy*

$$\lambda_n n^{-1} \xrightarrow{n \rightarrow +\infty} 0, \quad \lambda_n n^{-1/2} \xrightarrow{n \rightarrow +\infty} +\infty \quad \text{and} \quad \rho_n = O_{\mathbb{P}}\left(\frac{\lambda_n}{n}\right). \quad (10)$$

Then,

$$h^2(\widehat{p}_n, p_\star) = O_{\mathbb{P}}\left(\frac{\lambda_n}{n}\right) \quad \text{and} \quad I(\widehat{f}_n) = O_{\mathbb{P}}(1). \quad (11)$$

Condition 10 implies that the rate of convergence of the Hellinger distance between \widehat{p}_n and the true density p_\star is slower than $n^{-1/4}$. This rate is slower than the rate of convergence given in [20, Theorem 10.6]. [20, Theorem 10.6] performs a localization of the empirical process concentration which allows to use a penalty term that depends on the regularity of the density. The rate of convergence may then depend on the regularity of the true density p_\star such that more regularity implies a faster rate of convergence. The concentration result on the empirical process obtained in Proposition 3.2 is established globally on the class of functions $\{p_{f,\nu}; f \in \mathcal{F}, \nu \in \mathcal{D}^b\}$ and no localization is performed. Then, the rate $n^{-1/4}$ corresponds to the "worse case" rate. Such a localization of the empirical process concentration is a difficult problem in our setting since the regularity targets the class of functions \mathcal{F} rather than the class of densities $\{p_{f,\nu}; f \in \mathcal{F}, \nu \in \mathcal{D}^b\}$ such as in [20]. Measuring the concentration of the empirical process when f is close to f_\star remains an open problem even for particular cases. We believe that the rate of convergence of Theorem 3.1 could be improved for particular choices of \mathcal{F} and complexity function I if the Hellinger distance $h(p_{f,\nu}, p_\star)$ could be controlled by a distance between f and f_\star .

The proof of Theorem 3.1 relies on a *basic inequality* which provides a simultaneous control of the Hellinger risk $h^2(\widehat{p}_n, p_\star)$ and of the complexity of the estimator $I(\widehat{f}_n)$. Define for any density function p on $\mathbb{Y}^{b\ell}$,

$$g_p \stackrel{\text{def}}{=} \frac{1}{2} \ln \frac{p + p_\star}{2p_\star}. \quad (12)$$

Let \mathbb{P}_n be the empirical distribution based on the observations $\{\mathbf{Y}_k\}_{k=0}^{n-1}$, i.e., for any measurable set A of $\mathbb{R}^{b\ell}$,

$$\mathbb{P}_n(A) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_A(\mathbf{Y}_k).$$

By (4) and (7), the basic inequality of [20, Lemma 10.5], states that:

$$h^2(\widehat{p}_n, p_\star) + 4\lambda_n n^{-1} I(\widehat{f}_n) \leq 16 \int g_{\widehat{p}_n} d(\mathbb{P}_n - \mathbb{P}_\star) + 4\lambda_n n^{-1} I(f_\star) + \rho_n. \quad (13)$$

Therefore, a control of $\int g_{\widehat{p}_n} d(\mathbb{P}_n - \mathbb{P}_\star)$ in the right hand side of (13) will simultaneously provide a bound on the growth of $h^2(\widehat{p}_n, p_\star)$ and of $I(\widehat{f}_n)$. This control is given in Proposition 3.2.

Proposition 3.2. *Assume H1-3 hold. There exists a positive constant c such that, for any $\eta > 0$, there exist A and N such that for any $n \geq N$ and any $x > 0$,*

$$\mathbb{P} \left[\sup_{f \in \mathcal{F}, \nu \in \mathcal{D}_b} \frac{|\int g_{p_{f,\nu}} d(\mathbb{P}_n - \mathbb{P}_\star)|}{1 \vee I(f)^\gamma} \geq c\Phi \left(\sqrt{\frac{x}{n}} + \frac{x}{n} \right) + \frac{A}{\sqrt{n}} \right] \leq \frac{2e^{-\alpha x}}{1 - e^{-\alpha x}},$$

where

$$\gamma \stackrel{\text{def}}{=} b\ell v + \eta \quad \text{and} \quad \alpha \stackrel{\text{def}}{=} \frac{\log(2)(\gamma - v)}{2^{2\gamma}} = \frac{\log(2)((b\ell - 1)v + \eta)}{2^{2(b\ell v + \eta)}}.$$

Proposition 3.2 is proved in Section. 7.1.

Proof of Theorem 3.1. Since $v^{-1} > b\ell$, $\eta > 0$ in Proposition 3.2 can be chosen such that $\gamma = b\ell v + \eta = 1$. For this choice of η , Proposition 3.2 implies that

$$\frac{\int g_{\widehat{p}_n} d(\mathbb{P}_n - \mathbb{P}_\star)}{1 \vee I(\widehat{f}_n)} = O_{\mathbb{P}}(n^{-\frac{1}{2}}).$$

Combined with (13), this yields

$$h^2(\widehat{p}_n, p_\star) + 4\lambda_n n^{-1} I(\widehat{f}_n) \leq (1 \vee I(\widehat{f}_n)) O_{\mathbb{P}}(n^{-\frac{1}{2}}) + 4\lambda_n n^{-1} I(f_\star) + \rho_n. \quad (14)$$

Then, (14) directly implies that

$$4 I(\widehat{f}_n) \leq (1 \vee I(\widehat{f}_n)) O_{\mathbb{P}}(n^{\frac{1}{2}} \lambda_n^{-1}) + 4I(f_\star) + \rho_n n \lambda_n^{-1},$$

which, together with (10), gives

$$I(\widehat{f}_n) = O_{\mathbb{P}}(1).$$

Combining this result with (14) again leads to

$$h^2(\widehat{p}_n, p_\star) + O_{\mathbb{P}}(\lambda_n n^{-1}) \leq O_{\mathbb{P}}(n^{-\frac{1}{2}}) + 4\lambda_n n^{-1} I(f_\star) + \rho_n.$$

This concludes the proof of Theorem 3.1. \square

Theorem 3.1 shows that $h^2(\widehat{p}_n, p_*)$ vanishes as $n \rightarrow +\infty$. However, this does not imply the convergence of $(\widehat{f}_n, \widehat{\nu}_n)$ to $(f_*, \nu_{b,*})$. The convergence of the estimators $(\widehat{f}_n, \widehat{\nu}_n)$ is addressed in the case where the set \mathcal{D}_b may be written as

$$\mathcal{D}_b = \{\nu_a; a \in \mathcal{A}\}, \quad (15)$$

where \mathcal{A} is a parameter set not necessarily of finite dimension. The ρ_n -MPLE is then given by:

$$(\widehat{f}_n, \widehat{a}_n) \stackrel{\text{def}}{=} \operatorname{argmax}_{f \in \mathcal{F}, a \in \mathcal{A}}^{\rho_n} \left\{ \sum_{k=0}^{n-1} \ln p_{f, \nu_a}(\mathbf{Y}_k) - \lambda_n I(f) \right\}.$$

Assume that \mathcal{A} is equipped with a distance $d_{\mathcal{A}}$ such that \mathcal{A} is compact with respect to the topology defined by $d_{\mathcal{A}}$. Assume also that \mathcal{F} is equipped with a metric $d_{\mathcal{F}}$ such that $\mathcal{F}_M \stackrel{\text{def}}{=} \{f \in \mathcal{F}; I(f) \leq M\}$ is compact for all $M > 0$ with respect to the topology defined by $d_{\mathcal{F}}$. Let d be the product distance on $\mathcal{F} \times \mathcal{A}$. Assume that the function $(f, a) \mapsto h^2(p_{f, \nu_a}, p_*)$ is continuous with respect to the topology on $\mathcal{F} \times \mathcal{A}$ induced by d . Corollary 3.3 establishes the convergence of $(\widehat{f}_n, \widehat{a}_n)$ to the set \mathcal{E}_* defined as:

$$\mathcal{E}_* \stackrel{\text{def}}{=} \{(f, a) \in \mathcal{F} \times \mathcal{A}; h(p_{f, \nu_a}, p_{f_*, \nu_{a_*}}) = 0\}. \quad (16)$$

Corollary 3.3. *Assume H1-3 hold for some v such that $vbl < 1$. Assume also that λ_n and ρ_n satisfy*

$$\lambda_n n^{-1} \xrightarrow{n \rightarrow +\infty} 0, \quad \lambda_n n^{-1/2} \xrightarrow{n \rightarrow +\infty} +\infty \quad \text{and} \quad \rho_n = O\left(\frac{\lambda_n}{n}\right).$$

Then,

$$d((\widehat{f}_n, \widehat{a}_n), \mathcal{E}_*) = o_{\mathbb{P}}(1).$$

Corollary 3.3 is a direct consequence of Theorem 3.1 and of the properties of $d_{\mathcal{A}}$ and $d_{\mathcal{F}}$ and is therefore omitted. The few assumptions on the model allow only to establish the convergence of the estimators $(\widehat{f}_n, \widehat{a}_n)$ to the set \mathcal{E}_* in Corollary 3.3.

4 Identifiability when \mathbb{X} is a subset of \mathbb{R}^m

The aim of this section is to characterize the set \mathcal{E}_* given by (16) when $b = 1$ and when $b = 2$ (the characterization of \mathcal{E}_* when $b > 2$ follows the same lines) with some additional assumptions on the model, on \mathcal{F} and on \mathcal{D}_b . In the sequel, ν_* must satisfy $0 < \nu_- \leq \nu_* \leq \nu_+$ for some constants ν_- and ν_+ .

It is assumed that \mathbb{X} is a subset of \mathbb{R}^m for some $m \geq 1$ and that μ is the Lebesgue measure. For any subset A of \mathbb{R}^m , $\overset{\circ}{A}$ stands for the interior of A and \overline{A} for the closure of A . Consider the following assumptions on the state-space \mathbb{X} .

- H4** a) \mathbb{X} is non empty, compact and $\overline{\overset{\circ}{\mathbb{X}}} = \mathbb{X}$,
b) \mathbb{X} is arcwise and simply connected.

The compactness implies that \mathbb{X} is closed and that continuous functions on \mathbb{X} are bounded. By the last assumption of H4a), the interior of \mathbb{X} is not empty and any element in \mathbb{X} is the limit of elements of the interior of \mathbb{X} . Finally, \mathbb{X} is arcwise and simply

connected to ensure topological properties used in the proofs of the identifiability results below. A function f is said to be \mathcal{C}^1 (resp. a \mathcal{C}^1 -diffeomorphism) if f is the restriction to \mathbb{X} of a \mathcal{C}^1 function (resp. a \mathcal{C}^1 diffeomorphism) defined on an open neighborhood of \mathbb{X} in \mathbb{R}^m .

H5 f_\star is a \mathcal{C}^1 -diffeomorphism from \mathbb{X} to $f_\star(\mathbb{X})$.

H5 assumes that f_\star needs to be invertible. Nevertheless, this restrictive assumption is satisfied for almost every functions when $\ell \geq 2m+1$. Indeed, Whitney's embedding theorem ([22]) states that any continuous function from \mathbb{X} to \mathbb{R}^ℓ can be approximated by a smooth embedding.

In the case $b = 1$, Proposition 4.1 discusses the identifiability when \mathcal{F} is a subset of \mathcal{C}^1 . For any $\phi : \mathbb{X} \rightarrow \mathbb{X}$, let J_ϕ be the determinant of the Jacobian matrix of ϕ .

Proposition 4.1 (b=1). *Assume that H4 and H5 hold. Let $f \in \mathcal{C}^1$ and let ν be a probability density with respect to μ such that $0 < \nu_- \leq \nu \leq \nu_+$. Then, $h(p_{f,\nu}, p_{f_\star,\nu_\star}) = 0$ if and only if f_\star and f have the same image in \mathbb{R}^ℓ , $\phi = f_\star^{-1} \circ f$ is bijective and, for μ almost every $x \in \mathbb{X}$,*

$$\nu(x) = |J_\phi(x)|\nu_\star(\phi(x))$$

The proof of Proposition 4.1 is given in Section 7.2. When $\mathcal{F} \subset \mathcal{C}^1$, Proposition 4.1 and H3 implies that the set \mathcal{E}_\star defined in (16) is given by

$$(f, a) \in \mathcal{E}_\star \Leftrightarrow \text{There exists a bijective function } \phi \in \mathcal{C}^1(\mathbb{X}, \mathbb{X}) \text{ such that} \\ f = f_\star \circ \phi \text{ and } \nu_a = |J_\phi| \cdot \nu_\star \circ \phi \text{ } \mu \text{ almost everywhere in } \mathbb{X} .$$

Remark 4.2. Proposition 4.1 states that the candidates (f, ν) to characterize the distribution of Y_0 are necessarily related to (f_\star, ν_\star) through a state-space transformation denoted by ϕ . In the particular case where $\mathbb{X} = [0, 1]$ ($m = 1$) and $\nu_\star = 1$, Proposition 4.1 implies a sharper result. Assuming that $\nu = \nu_\star$, Proposition 4.1 implies the existence of a \mathcal{C}^1 and bijective function ϕ satisfying $f = f_\star \circ \phi$ and $|J_\phi| = 1$. Therefore, $\phi : x \mapsto x$ or $\phi : x \mapsto 1 - x$ which are the two possible isometric transformations of $[0, 1]$.

When ν_\star is unknown and continuous we can define the uniform random variable on $[0, 1]$: $\tilde{X}_0 = F(X_0)$ where F is the \mathcal{C}^1 and strictly increasing cumulative distribution function of X_0 . The observation Y_0 can be written $Y_0 = \tilde{f}_\star(\tilde{X}_0) + \epsilon_0$ where $\tilde{f}_\star = f_\star \circ F^{-1}$ satisfies the same hypothesis as f_\star . Thus, if the true density function ν_\star is continuous, the function $\tilde{f}_\star = f_\star \circ F^{-1}$ can be identified up to an isometric transformation of $[0, 1]$ from the distribution of Y_0 only. This cannot be extended to the case $m > 1$ where $|J_\phi| = 1$ does not necessarily imply that ϕ is isometric but only that ϕ preserves the volumes.

Proposition 4.3 discuss the identifiability when $b = 2$. In this case, $\nu_{2,\star}$ can be written $\nu_{2,\star}(x, x') = \nu_\star(x)q_\star(x, x')$ where q_\star is a transition density with stationary probability density ν_\star . For any transition density q on \mathbb{X}^2 satisfying

$$\text{for all } x, x' \in \mathbb{X}, 0 < q_- \leq q(x, x') \leq q_+, \quad (17)$$

there exists a stationary density ν associated with q satisfying, for all $x \in \mathbb{X}$, $q_- \leq \nu(x) \leq q_+$. Denote by ν_q this density.

Proposition 4.3 (b=2). *Assume that H4 and H5 hold. Let $f \in \mathcal{C}^1$ and q be a transition density satisfying (17). Let $\nu_2(x, x') = \nu_q(x)q(x, x')$. Then, $h(p_{f,\nu_2}, p_{f_\star,\nu_{\star,2}}) = 0$ if and only*

if f_\star and f have the same image in \mathbb{R}^ℓ , $\phi = f_\star^{-1} \circ f$ is bijective and $\mu \otimes \mu$ almost everywhere in \mathbb{X}^2 ,

$$q(x, x') = |J_\phi(x')| q_\star(\phi(x), \phi(x')) . \quad (18)$$

Proposition 4.3 is proved in Section 7.3.

Corollary 4.4. *Consider the same assumptions as in Proposition 4.3. Assume in addition that q_\star and q are of the form:*

$$q_\star(x, x') = c_\star(x) \rho_\star(\|x - x'\|) , \quad q(x, x') = c(x) \rho(\|x - x'\|) ,$$

where ρ and ρ_\star are two continuous functions defined on \mathbb{R}_+ . Assume in addition that ρ_\star is one-to-one. Then, $h(p_{f, \nu_2}, p_{f_\star, \nu_{\star, 2}}) = 0$ if and only if f_\star and f have the same image in \mathbb{R}^ℓ , $\phi = f_\star^{-1} \circ f$ is an isometry on \mathbb{X} and $q = q_\star$.

The proof of Corollary 4.4 is given in Section 7.3. When $\mathcal{F} \subset \mathcal{C}^1$ and for any a in \mathcal{A} , $\nu_a \in \mathcal{D}_2$ is of the form

$$\nu_a(x, x') = \nu_{q_a}(x) q_a(x, x') \text{ with } q_a(x, x') = c_a(x) \rho_a(\|x - x'\|) ,$$

where $\rho_- \leq \rho_a \leq \rho_+$, Corollary 4.4 implies that the set \mathcal{E}_\star defined in (16) is given by

$$(f, a) \in \mathcal{E}_\star \Leftrightarrow f = f_\star \circ \phi \text{ with } \phi \text{ an isometry and } q_a = q_\star$$

Finally, if the only isometry of \mathbb{X} is the identity function, and if there exists a unique a_\star in \mathcal{A} such that $q_{a_\star} = q_\star$, then $\mathcal{E}_\star = \{(f_\star, a_\star)\}$ and the model is fully identifiable.

5 Application when \mathcal{F} is a Sobolev class of functions

In this section, \mathbb{X} is a subset of \mathbb{R}^m , $m \geq 1$ and the results of Section 3 and Section 4 are applied to a specific class of functions \mathcal{F} with an example of complexity function I satisfying H2. Let $p \geq 1$, define

$$L^p \stackrel{\text{def}}{=} \left\{ f : \mathbb{X} \rightarrow \mathbb{R}^\ell ; \|f\|_{L^p}^p = \int_{\mathbb{X}} \|f(x)\|^p \mu(dx) < \infty \right\} .$$

For any $f : \mathbb{X} \rightarrow \mathbb{R}^\ell$ and any $j \in \{1, \dots, \ell\}$, the j^{th} component of f is denoted by f_j . For any vector $\alpha \stackrel{\text{def}}{=} \{\alpha_i\}_{i=1}^m$ of non-negative integers, we write $|\alpha| \stackrel{\text{def}}{=} \sum_{i=1}^m \alpha_i$ and $D^\alpha f : \mathbb{X} \rightarrow \mathbb{R}^\ell$ for the vector of partial derivatives of order α of f in the sense of distributions. Let $s \in \mathbb{N}$ and $W^{s,p}$ be the Sobolev space on \mathbb{X} with parameters s and p , *i.e.*,

$$W^{s,p} \stackrel{\text{def}}{=} \{f \in L^p; D^\alpha f \in L^p, \alpha \in \mathbb{N}^m \text{ and } |\alpha| \leq s\} . \quad (19)$$

$W^{s,p}$ is equipped with the norm $\|\cdot\|_{W^{s,p}}$ defined, for any $f \in W^{s,p}$, by

$$\|f\|_{W^{s,p}} \stackrel{\text{def}}{=} \left(\sum_{0 \leq |\alpha| \leq s} \|D^\alpha f\|_{L^p}^p \right)^{1/p} . \quad (20)$$

The results of Section 3 and Section 4 can be applied to the class $\mathcal{F} = W^{s,p}$ under the following assumption.

H6 \mathbb{X} has a locally Lipschitz boundary.

H6 means that all x on the boundary of \mathbb{X} has a neighbourhood whose intersection with the boundary of \mathbb{X} is the graph of a Lipschitz function. For any $j \in \{1, \dots, \ell\}$ and $f \in W^{s,p}$, f_j belongs to $W^{s,p}(\mathbb{X}, \mathbb{R})$, the Sobolev space of real-valued functions with parameters s and p . Let $k \geq 0$, by [1, Theorem 6.3], if $s > m/p + k$ and if **H4a**) and **H6** holds, $W^{s,p}(\mathbb{X}, \mathbb{R})$ is compactly embedded into $(\mathcal{C}^k(\mathbb{X}, \mathbb{R}), \|\cdot\|_{\mathcal{C}^k})$. Arguing component by component, $W^{s,p}$ is compactly embedded into $\mathcal{C}^k \stackrel{\text{def}}{=} \mathcal{C}^k(\mathbb{X}, \mathbb{R}^\ell)$. Moreover, the identity function $id : W^{s,p} \rightarrow \mathcal{C}^k$ being linear and continuous, there exists a positive coefficient κ such that, for any $f \in W^{s,p}$,

$$\|f\|_{\mathcal{C}^k} \leq \kappa \|f\|_{W^{s,p}} . \quad (21)$$

Then, if $s > m/p + k$, for any $f \in \mathcal{F} = W^{s,p}$,

$$\|f\|_\infty \leq \kappa \|f\|_{W^{s,p}} . \quad (22)$$

If the complexity function is defined by $I(f) = \|f\|_{W^{s,p}}^{1/v}$ with $vbl < 1$, then **H2** holds and Theorem 3.1 can be applied. Moreover, by [1, Theorem 6.3], the subspace \mathcal{F}_M , $M \geq 1$ are quasi-compact in \mathcal{C}^k . Let $d_{\mathcal{A}}$ be a metric on the space \mathcal{A} introduced in (15) such that \mathcal{A} is compact and that, for $\mu \otimes \mu$ almost every $(x, x') \in \mathbb{X}^2$, $a \mapsto \nu_a(x, x')$ is continuous. By applications of the dominated convergence theorem, this implies the continuity of $(f, a) \mapsto h(p_{f, \nu_a}, p_\star)$. Define

$$\mathcal{F}_\star \stackrel{\text{def}}{=} \{f \in W^{s,p}; \exists a \in \mathcal{A} \text{ such that } (f, a) \in \mathcal{E}_\star\} .$$

Then, Proposition 5.1 is a direct application of Corollary 3.3.

Proposition 5.1 ($\mathcal{F} = \mathbf{W}^{s,p}$, $s > m/p + k$, $k \geq 0$). *Assume that **H1**, **H3**, **H4a**) and **H6** hold. Assume also that $I(f) = \|f\|_{W^{s,p}}^{1/v}$ for some v such that $vbl < 1$ and that λ_n and ρ_n satisfy*

$$\lambda_n n^{-1} \xrightarrow{n \rightarrow +\infty} 0, \quad \lambda_n n^{-1/2} \xrightarrow{n \rightarrow +\infty} +\infty \text{ and } \rho_n = O\left(\frac{\lambda_n}{n}\right) .$$

Then,

$$d_{\mathcal{C}^k}(\widehat{f}_n, \mathcal{F}_\star) = o_{\mathbb{P}}(1) .$$

Moreover, as shown in Section 7.2 the assumption $\overline{\mathbb{X}} = \mathbb{X}$, together with the continuity of the functions in \mathcal{F} provided by (21) imply that for any f in \mathcal{F}_\star , $f(\mathbb{X}) = f_\star(\mathbb{X})$. Define the Hausdorff distance $d_{\mathcal{H}}(A, B)$ between two compact subsets A and B of \mathbb{R}^ℓ as

$$d_{\mathcal{H}}(A, B) \stackrel{\text{def}}{=} \max\left(\sup_{a \in A} \inf_{b \in B} \|a - b\|_{\mathbb{R}^\ell}, \sup_{b \in B} \inf_{a \in A} \|a - b\|_{\mathbb{R}^\ell}\right)$$

Proposition 5.1 implies Corollary 5.2.

Corollary 5.2 ($\mathcal{F} = \mathbf{W}^{s,p}$, $s > m/p$). *Assume that **H1**, **H3**, **H4a**) and **H6** hold. Assume also that $I(f) = \|f\|_{W^{s,p}}^{1/v}$ for some v such that $vbl < 1$ and that λ_n and ρ_n satisfy*

$$\lambda_n n^{-1} \xrightarrow{n \rightarrow +\infty} 0, \quad \lambda_n n^{-1/2} \xrightarrow{n \rightarrow +\infty} +\infty \text{ and } \rho_n = O\left(\frac{\lambda_n}{n}\right) .$$

Then,

$$d_{\mathcal{H}}(\widehat{f}_n(\mathbb{X}), f_\star(\mathbb{X})) = o_{\mathbb{P}}(1) .$$

Corollary 5.2 establishes the consistency of the estimator $\widehat{f}_n(\mathbb{X})$ of the image of f_\star in \mathbb{R}^ℓ . This result is particularly interesting since $f_\star(\mathbb{X})$ is a manifold of dimension $m \leq \ell$ in \mathbb{R}^ℓ . Thus, the proposed estimation procedure allows to approximate such manifolds, possibly of low dimensions, that are only observed with additive noise in \mathbb{R}^ℓ . Moreover, this result holds under relatively low regularity assumptions on the manifold.

Proposition 5.3 below states the consistency of the estimators $(\widehat{f}_n, \widehat{a}_n)$ in the case $b = 2$ and $\mathcal{F} = W^{s,p}$. Assume that for any a in \mathcal{A} , $\nu_a \in \mathcal{D}_2$ is of the form

$$\nu_a(x, x') = \nu_{q_a}(x)q_a(x, x') \text{ with } q_a(x, x') = c_a(x)\rho_a(\|x - x'\|),$$

where $\rho_- \leq \rho_a \leq \rho_+$. It is also assumed that there exists a unique $a_\star \in \mathcal{A}$ such that $\nu_\star = \nu_{a_\star}$ and that ρ_{a_\star} is one-to-one. Proposition 5.3 is a direct application of Corollary 3.3 and of Proposition 4.3 and is stated without proof.

Proposition 5.3 ($\mathcal{F} = \mathbf{W}^{s,p}$, $s > \mathbf{m}/\mathbf{p} + \mathbf{k}$, $\mathbf{k} \geq 1$, $\mathbf{b} = 2$). *Assume that H1 and H3-6 hold. Assume also that $I(f) = \|f\|_{W^{s,p}}^{1/v}$ for some v such that $2v\ell < 1$ and that λ_n and ρ_n satisfy*

$$\lambda_n n^{-1} \xrightarrow{n \rightarrow +\infty} 0, \lambda_n n^{-1/2} \xrightarrow{n \rightarrow +\infty} +\infty \text{ and } \rho_n = O\left(\frac{\lambda_n}{n}\right).$$

Then,

$$\mathcal{F}_\star = \{f_\star \circ \phi; \phi \text{ is an isometry of } \mathbb{X}\},$$

and

$$d_{C^k}(\widehat{f}_n, \mathcal{F}_\star) = o_{\mathbb{P}}(1) \text{ and } d_{\mathcal{A}}(\widehat{a}_n, a_\star) = o_{\mathbb{P}}(1),$$

6 Numerical experiments

6.1 Proposed Expectation Maximization algorithm

This section introduces a practical algorithm to compute the estimators defined in (4) in the cases $b = 1$ and $b = 2$. It is assumed that there exists $a_\star > 0$ such that for all $x, x' \in [0, 1]$,

$$\nu_{\star,2}(x, x') = \nu_{a_\star}(x)q_{a_\star}(x, x'),$$

where

$$q_{a_\star}(x, x') \stackrel{\text{def}}{=} C_{a_\star}(x)\exp\left(-\frac{|x' - x|}{a_\star}\right).$$

The unique stationary density associated with q_{a_\star} is given, for all $x \in [0, 1]$, by

$$\nu_{a_\star}(x) \propto C_{a_\star}^{-1}(x) = \int_{[0,1]} \exp\left(-\frac{|x' - x|}{a_\star}\right) dx'. \quad (23)$$

For practical considerations, we choose $\mathcal{F} = W^{s,2}$ and $I(f) = \|f\|_{W^{s,2}}^{1/v}$.

We introduce an Expectation-Maximization (EM) based procedure which iteratively produces a sequence of estimates $\widehat{a}^p, \widehat{f}^p$, $p \geq 0$, see [6]. Assume the current parameter

estimates are given by \widehat{a}^p and \widehat{f}^p . The estimates \widehat{a}^{p+1} and \widehat{f}^{p+1} are defined as one of the maximizer of the function Q :

$$(a, f) \mapsto Q((a, f), (\widehat{a}^p, \widehat{f}^p)) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} \mathbb{E}_{\widehat{a}^p, \widehat{f}^p} [\ln p_{f,a}(X_{2k:2k+1}, Y_{2k:2k+1}) | Y_{2k:2k+1}] - \lambda_n \|f\|_{W^{s,2}}^{1/v},$$

where $\mathbb{E}_{\widehat{a}^p, \widehat{f}^p}[\cdot]$ denotes the conditional expectation under the model parameterized by \widehat{a}^p and \widehat{f}^p and where

$$p_{f,a}(x, x', y, y') = \nu_a(x) q_a(x, x') \varphi(y - f(x)) \varphi(y - f(x')).$$

The intermediate quantity $Q((a, f), (\widehat{a}^p, \widehat{f}^p))$ can be written:

$$Q((a, f), (\widehat{a}^p, \widehat{f}^p)) = Q_p^1(a) + Q_p^2(f),$$

where

$$Q_p^1(a) = \sum_{k=0}^{n-1} \mathbb{E}_{\widehat{a}^p, \widehat{f}^p} [\ln \{\nu_a(X_{2k}) q_a(X_{2k}, X_{2k+1})\} | Y_{2k:2k+1}],$$

$$Q_p^2(f) = \sum_{k=0}^{n-1} \mathbb{E}_{\widehat{a}^p, \widehat{f}^p} [\ln \{\varphi(Y_{2k} - f(X_{2k})) \varphi(Y_{2k+1} - f(X_{2k+1}))\} | Y_{2k:2k+1}] - \lambda_n \|f\|_{W^{s,2}}^{1/v}.$$

Therefore \widehat{a}^{p+1} is obtained by maximizing the function $a \mapsto Q_p^1(a)$ and \widehat{f}^{p+1} by maximizing the function $f \mapsto Q_p^2(f)$.

Remark 6.1. In the case $b = 1$, the functions Q_p^1 and Q_p^2 are given by:

$$Q_p^1(a) = \sum_{k=0}^{n-1} \mathbb{E}_{\widehat{a}^p, \widehat{f}^p} [\ln \{\nu_a(X_k)\} | Y_k],$$

$$Q_p^2(f) = \sum_{k=0}^{n-1} \mathbb{E}_{\widehat{a}^p, \widehat{f}^p} [\ln \{\varphi(Y_k - f(X_k))\} | Y_k] - \lambda_n \|f\|_{W^{s,2}}^{1/v}.$$

Lemma 6.2 proves that the penalized pseudo-likelihood increases at each iteration of this EM based algorithm.

Lemma 6.2. *The sequences \widehat{a}^p and \widehat{f}^p satisfy*

$$\sum_{k=0}^{n-1} \ln p_{\widehat{f}^{p+1}, \widehat{a}^{p+1}}(\mathbf{Y}_k) - \lambda_n I(\widehat{f}^{p+1}) \geq \sum_{k=0}^{n-1} \ln p_{\widehat{f}^p, \widehat{a}^p}(\mathbf{Y}_k) - \lambda_n I(\widehat{f}^p).$$

Proof. The proof follows the same lines as the one for the usual EM algorithm. For all

$0 \leq k \leq n-1$, all $f \in \mathcal{F}$ and all $a \in \mathcal{A}$

$$\begin{aligned}
\ln \left[p_{f,a}(\mathbf{Y}_k) e^{-\lambda_n I(f)/n} \right] &= \ln \left[\int p_{f,a}(\mathbf{x}, \mathbf{Y}_k) e^{-\lambda_n I(f)/n} \mu^{\otimes 2}(\mathrm{d}\mathbf{x}) \right], \\
&= \ln \left[\int p_{f,a}(\mathbf{x}, \mathbf{Y}_k) e^{-\lambda_n I(f)/n} \frac{p_{\hat{f}^p, \hat{a}^p}(\mathbf{x} | \mathbf{Y}_k)}{p_{\hat{f}^p, \hat{a}^p}(\mathbf{x} | \mathbf{Y}_k)} \mu^{\otimes 2}(\mathrm{d}\mathbf{x}) \right], \\
&= \ln \left[\int p_{\hat{f}^p, \hat{a}^p}(\mathbf{x} | \mathbf{Y}_k) \frac{p_{f,a}(\mathbf{x}, \mathbf{Y}_k) e^{-\lambda_n I(f)/n}}{p_{\hat{f}^p, \hat{a}^p}(\mathbf{x} | \mathbf{Y}_k)} \mu^{\otimes 2}(\mathrm{d}\mathbf{x}) \right], \\
&\geq \int p_{\hat{f}^p, \hat{a}^p}(\mathbf{x} | \mathbf{Y}_k) \ln \left[\frac{p_{f,a}(\mathbf{x}, \mathbf{Y}_k) e^{-\lambda_n I(f)/n}}{p_{\hat{f}^p, \hat{a}^p}(\mathbf{x} | \mathbf{Y}_k)} \right] \mu^{\otimes 2}(\mathrm{d}\mathbf{x}),
\end{aligned}$$

where the last inequality comes from the concavity of $x \mapsto \log x$. Then,

$$\begin{aligned}
&\ln \left[p_{f,a}(\mathbf{Y}_k) e^{-\lambda_n I(f)/n} \right] - \ln \left[p_{\hat{f}^p, \hat{a}^p}(\mathbf{Y}_k) e^{-\lambda_n I(\hat{f}^p)/n} \right] \\
&\geq \mathbb{E}_{\hat{a}^p, \hat{f}^p} \left[\ln p_{f,a}(\mathbf{X}_k, \mathbf{Y}_k) - \ln p_{\hat{f}^p, \hat{a}^p}(\mathbf{X}_k, \mathbf{Y}_k) \middle| \mathbf{Y}_k \right] - \frac{\lambda_n}{n} \left(I(f) - I(\hat{f}^p) \right).
\end{aligned}$$

The proof is concluded by definition of \hat{a}^{p+1} and \hat{f}^{p+1} . \square

6.2 Numerical approximations

The approximations are presented in the case $b = 2$, the case $b = 1$ follows the same lines (see Remark 6.1).

Computation of \hat{a}^{p+1}

By definition of ν_a and q_a , the new estimate is set as a minimizer of the function

$$a \mapsto \log(a + a^2(\exp(-1/a) - 1)) + \frac{1}{na} \sum_{k=0}^{n-1} \mathbb{E}_{\hat{a}^p, \hat{f}^p} [\|X_{2k} - X_{2k+1}\| | Y_{2k:2k+1}].$$

In the following, the expectation in the right hand side of the last equation is estimated using a standard Monte Carlo algorithm (with an importance sampling mechanism) based on N_a simulations. For all $0 \leq k \leq n-1$, $(X_{2k}^i, X_{2k+1}^i)_{i=1}^{N_a}$ are independently sampled uniformly in $[0, 1] \times [0, 1]$ and associated with the importance weights:

$$\omega_k^i \propto \nu_{\hat{a}^p}(X_{2k}^i) q_{\hat{a}^p}(X_{2k}^i, X_{2k+1}^i) \varphi(Y_{2k} - \hat{f}^p(X_{2k}^i)) \varphi(Y_{2k+1} - \hat{f}^p(X_{2k+1}^i)). \quad (24)$$

Then, \hat{a}^{p+1} is obtained as a minimizer of the function

$$a \mapsto \log(a + a^2(\exp(-1/a) - 1)) + \frac{1}{na} \sum_{k=0}^{n-1} \sum_{i=1}^{N_a} \omega_k^i \|X_{2k}^i - X_{2k+1}^i\|.$$

Computation of \hat{f}^{p+1}

\hat{f}^{p+1} is obtained by maximizing the function $f \mapsto Q_p^2(f)$ which is equivalent to minimize the function:

$$f \mapsto \frac{1}{2} \sum_{k=0}^{n-1} \mathbb{E}_{\hat{a}^p, \hat{f}^p} [\|Y_{2k} - f(X_{2k})\|^2 + \|Y_{2k+1} - f(X_{2k+1})\|^2 | \mathbf{Y}_k] + \lambda_n \|f\|_{W^{s,2}}^{1/v}. \quad (25)$$

Each expectation in (25) is approximated using Monte Carlo simulations. For all $k \in \{0, \dots, n-1\}$, N_f random variables $(X_{2k}^i, X_{2k+1}^i)_{i=1}^{N_f}$ are sampled uniformly in $[0, 1] \times [0, 1]$ and associated with the random weights $(\omega_k^i)_{i=1}^{N_f}$ given by (24). Then, N_f couples of states $(\tilde{X}_{2k}^i, \tilde{X}_{2k+1}^i)_{i=1}^{N_f}$ are drawn from the point mass distribution $\sum_{i=1}^{N_f} \omega_k^i \delta_{(X_{2k}^i, X_{2k+1}^i)}$. \hat{f}^{p+1} is then obtained by minimizing the function:

$$f \mapsto \frac{1}{2N_f} \sum_{k=0}^{n-1} \sum_{i=1}^{N_f} \left(\|Y_{2k} - f(\tilde{X}_{2k}^i)\|^2 + \|Y_{2k+1} - f(\tilde{X}_{2k+1}^i)\|^2 \right) + \lambda_n \|f\|_{W^{s,2}}^{1/v}.$$

For practical considerations, the new estimate \hat{f}^{p+1} is computed using cubic smoothing splines which are designed to minimize

$$f \mapsto \frac{1}{2N_f} \sum_{k=0}^{n-1} \sum_{i=1}^{N_f} \left(\|Y_{2k} - f(\tilde{X}_{2k}^i)\|^2 + \|Y_{2k+1} - f(\tilde{X}_{2k+1}^i)\|^2 \right) + \lambda_n \|f''\|_{L^2}^2.$$

Proposition 5.3 relies on the assumption that $I(f) = \|f\|_{W^{s,p}}^{1/v}$ with $2\nu\ell < 1$. However, choosing $I(f) = \|f''\|_{L^2}^2$ allows to define an algorithm easy to implement with a good convergence behavior and where the minimization procedure can be directly implemented using Matlab or R built-in routines.

6.3 Experimental results

The proposed EM based algorithm is applied with $a_\star = 1$ and (in this case $\ell = 2$ and $m = 1$):

$$\begin{aligned} f_\star &: [0, 1] \rightarrow \mathbb{R}^2 \\ x &\mapsto (\cos(\pi x), \sin(\pi x)). \end{aligned}$$

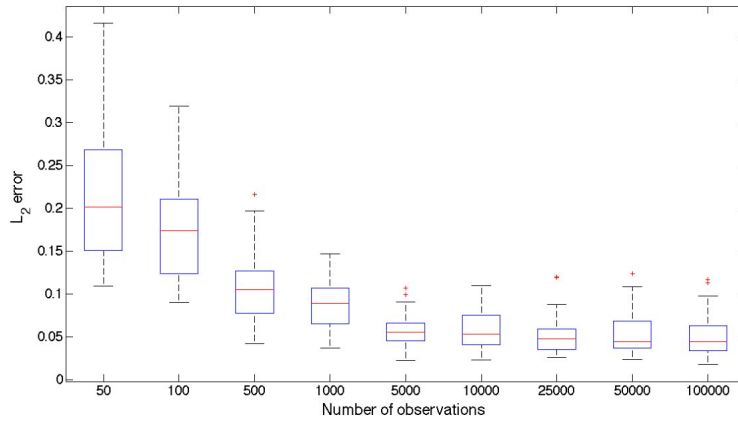
The algorithm is run to estimate f_\star , $f_\star([0, 1])$ and a_\star with initial estimates given by $\hat{a}^0 = 4$ and

$$\begin{aligned} \hat{f}^0 &: [0, 1] \rightarrow \mathbb{R}^2 \\ x &\mapsto (x, 0). \end{aligned}$$

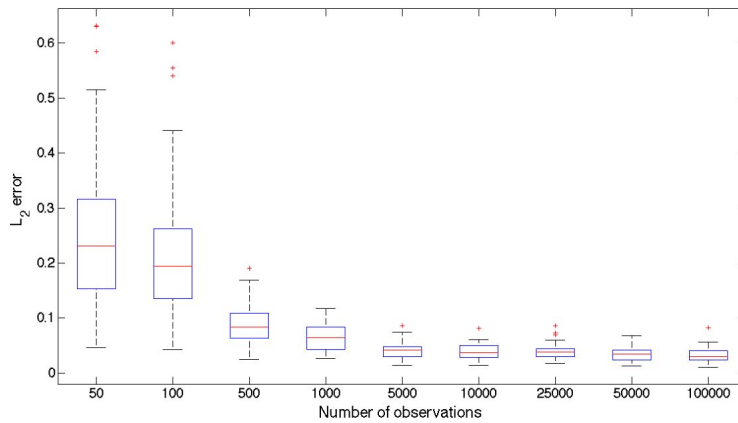
Case $b = 1$

In this section, according to Remark 4.2, $\nu_\star = 1$ is assumed to be known. The estimation is performed with $N_f = 100$ and the penalization constant λ is chosen such that $\lambda \propto \log(n)n^{1/2}$ in order to satisfy the assumption of Section 5. Figure 1 displays the L_2 error of the estimation of f_\star after 100 iterations of the algorithm as a function of the number of observations. The L_2 estimation error decreases quickly for small values of n (lower than

5000) and then goes on decreasing at a lower rate as n increases. It can be seen that even with a great number of observations, a small bias still remains for both functions (with a mean a bit lower than 0.05). Indeed, there is always small errors in the estimation of f_* around $x = 0$ and $x = 1$.



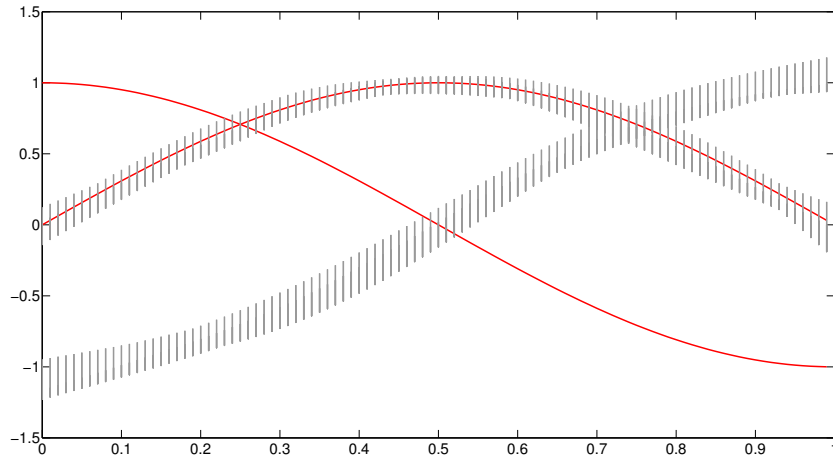
(a) f_1 .



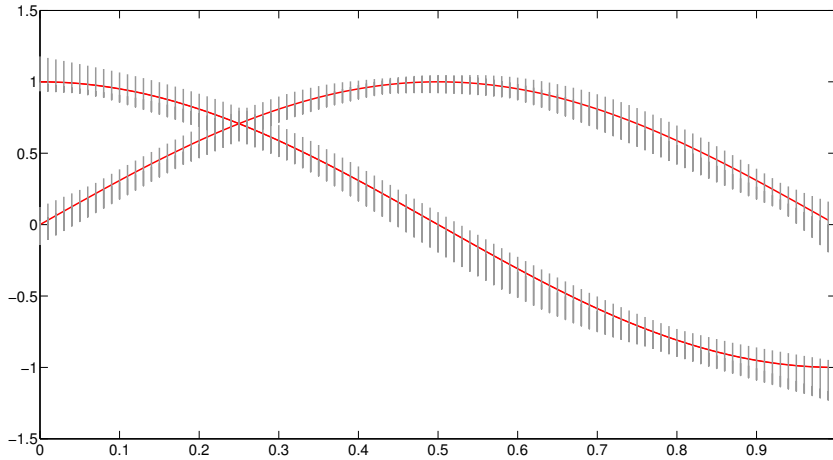
(b) f_2 .

Figure 1: L_2 error after 100 iterations over 100 Monte Carlo runs.

Figure 2 shows the estimates after 100 iterations when $n = 25,000$. It can be seen that the second component of f_* is estimated with accuracy while the first component of f_* is recovered up to the isometry $x \mapsto 1 - x$ (the isometry is used in Figure 1 to compute the L_2 error). This simulation illustrates the identifiability results obtained in Section 4.



(a) With no isometry for f_1 .



(b) With the isometry $x \mapsto 1 - x$ for f_1 .

Figure 2: True functions (bold line) and estimates after 100 iterations (vertical lines) over 100 Monte Carlo runs ($n = 25.000$).

Case $b = 2$

In this section, a_* and f_* are estimated. The Monte Carlo approximations are computed using $N_a = N_f = 200$ and 20.000 observations (*i.e.* $n = 10.000$) are sampled.

Figure 3 displays the estimation a_* as a function of the number of iterations of the algorithm over 50 independent Monte Carlo runs. The estimates converge to the true value of a_* after a few number of iterations (about 25).

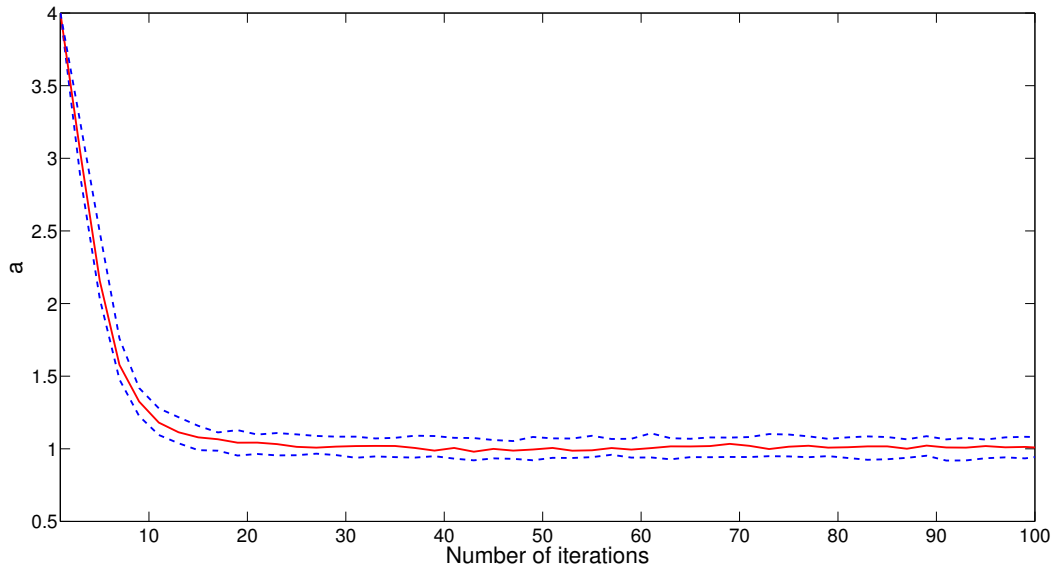


Figure 3: Estimation of a_* as a function of the number of iterations. The true value is $a_* = 1$. Median (bold line) and upper and lower quartiles (dotted line) over 50 Monte Carlo runs.

Figure 4 illustrates Corollary 5.2. It displays the estimation of $f_*([0, 1])$ after 100 iterations for several Monte Carlo runs. It shows that despite the variability of the estimation, the image is well estimated with few observations.

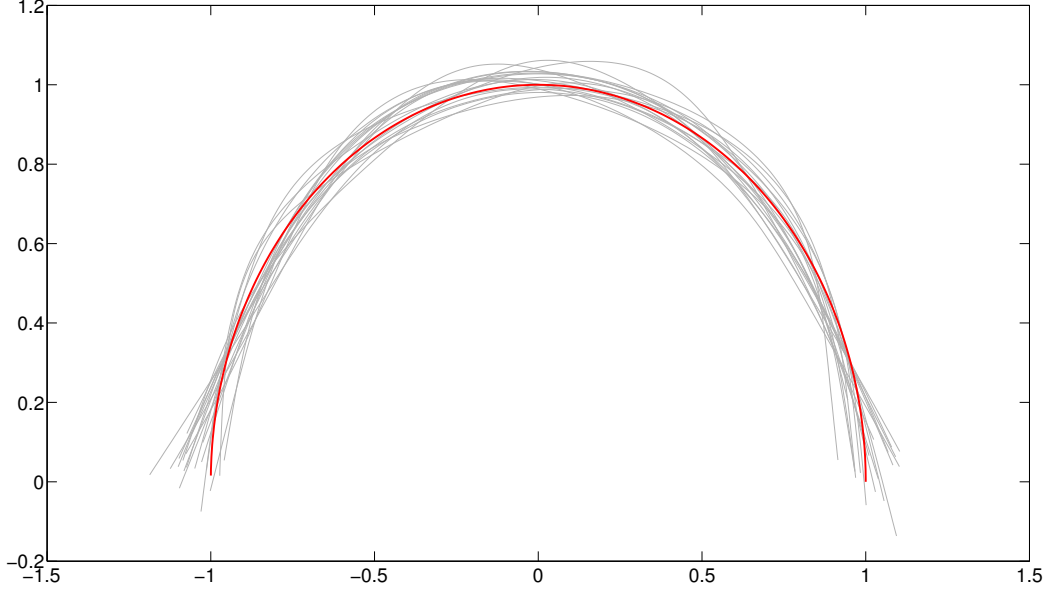


Figure 4: True image $f_*([0, 1])$ (red) and estimations after 100 iterations of the algorithm over 100 Monte Carlo runs (grey).

7 Proofs

7.1 Proof of Proposition 3.2

The proof relies on the application of Proposition A.1 and Proposition A.2 to obtain first a concentration inequality for the class of functions \mathcal{G}_M , where $M \geq 1$, defined as:

$$\mathcal{G}_M \stackrel{\text{def}}{=} \{g_{p_f, \nu} ; \nu \in \mathcal{D}_b, f \in \mathcal{F} \text{ and } I(f) \leq M\} ,$$

where $p_{f, \nu}$ is defined by (3) and $g_{p_f, \nu}$ by (12). For any $p > 0$, denote by $L_p(\mathbb{P}_*)$ the set of functions $g : \mathbb{R}^{b\ell} \rightarrow \mathbb{R}$ such that $\mathbb{E}_* [|g|^p] < +\infty$. For any $\kappa > 0$ and any set \mathcal{G} of functions from $\mathbb{R}^{b\ell}$ to \mathbb{R} , let $N(\kappa, \mathcal{G}, \|\cdot\|_{L_p(\mathbb{P}_*)})$ be the smallest integer N such that there exists a set of functions $\{(g_i^L, g_i^U)\}_{i=1}^N$ for which:

- a) $\|g_i^U - g_i^L\|_{L_p(\mathbb{P}_*)} \leq \kappa$ for all $i \in \{1, \dots, N\}$;
- b) for any g in \mathcal{G} , there exists $i \in \{1, \dots, N\}$ such that

$$g_i^L \leq g \leq g_i^U .$$

$N(\delta, \mathcal{G}, \|\cdot\|_{L_p(\mathbb{P}_*)})$ is the κ -number with bracketing of \mathcal{G} , and $H(\kappa, \mathcal{G}, \|\cdot\|_{L_p(\mathbb{P}_*)}) \stackrel{\text{def}}{=} \ln N(\kappa, \mathcal{G}, \|\cdot\|_{L_p(\mathbb{P}_*)})$ is the κ -entropy with bracketing of \mathcal{G} . For any g , define

$$S_n(g) \stackrel{\text{def}}{=} n \int g d(\mathbb{P}_n - \mathbb{P}_*) . \tag{26}$$

Application of Proposition A.1 Proposition A.1 is applied to the class of functions $\overline{\mathcal{G}}_M$ defined as

$$\overline{\mathcal{G}}_M \stackrel{\text{def}}{=} \{\mathbf{g} - \mathbb{E}_\star[\mathbf{g}]; g \in \mathcal{G}_M\} .$$

Since $(\boldsymbol{\epsilon}_k)_{k \geq 0}$ is i.i.d. and $(\mathbf{X}_k)_{k \geq 0}$ is Φ -mixing, $(\mathbf{Y}_k)_{k \geq 0}$ is also Φ -mixing with mixing coefficients $(\phi_i^{\mathbf{Y}})_{i \geq 0}$ satisfying, for all $i \geq 1$,

$$\phi_i^{\mathbf{Y}} \leq \phi_i^{\mathbf{X}} = \phi_{(i-1)b+1}^{\mathbf{X}} .$$

Therefore $\Phi^{\mathbf{Y}} = \sum_{i \geq 1} (\phi_i^{\mathbf{Y}})^{1/2} < \infty$. By H2, there exists $C > 0$ such that for any $i \geq 0$, and any $g \in \mathcal{G}_M$,

$$\begin{aligned} |\mathbf{g}(\mathbf{Y}_i)| &\leq CM^v (1 + \|\mathbf{Y}_i\|) \leq CM^v (1 + \|\mathbf{f}_\star(\mathbf{X}_i)\| + \|\boldsymbol{\epsilon}_i\|) , \\ &\leq CM^v (1 + \|f_\star\|_\infty + \|\boldsymbol{\epsilon}_i\|) , \\ &\leq CM^v (1 + \|\boldsymbol{\epsilon}_i\|) . \end{aligned}$$

Define $\mathbf{U}_i \stackrel{\text{def}}{=} CM^v (1 + \|\boldsymbol{\epsilon}_i\|)$. Then $(\mathbf{U}_i)_{i \geq 0}$ is i.i.d., $\mathbf{g}(\mathbf{Y}_i) \leq \mathbf{U}_i + \mathbb{E}[\mathbf{U}_0]$ and there exist positive constants ν and c such that

$$\mathbb{E}[(\mathbf{U}_i + \mathbb{E}[\mathbf{U}_0])^{2k}] \leq k! \nu c^{k-1} ,$$

where $\nu = CM^{2v}$ and $c = CM^{2v}$. Then, by Proposition A.1, there exists a positive constant c such that for any positive x ,

$$\mathbb{P} \left[\sup_{g \in \mathcal{G}_M} |S_n(g)| \geq \mathbb{E} \left[\sup_{g \in \mathcal{G}_M} |S_n(g)| \right] + c \Phi^{\mathbf{Y}} (\sqrt{nx} + x) M^v \right] \leq e^{-x} . \quad (27)$$

Application of Proposition A.2 Proposition A.2 is used to control the inner expectation in (27). Let $r > 1$. By [16, Lemma 7.26] and since the Hellinger distance is bounded by 1, there exists a constant δ such that for any $g = g_{p_f, \nu} \in \mathcal{G}_M$.

$$\|g\|_{\mathbb{L}_{2r}(\mathbb{P}_\star)}^{2r} \leq Ch^2(p_f, \nu, p_\star) \leq \delta .$$

By Lemma B.1, for any $p' \geq 1$, and any $s' > bl/p'$, provided that $d > s' + bl(1 - \frac{1}{p'})$, there exists a constant C such that, for all $u > 0$,

$$H(u, \|\cdot\|_{\mathbb{L}_{2r}(\mathbb{P}_\star)}, \mathcal{G}_M) \leq C \left(\frac{M^{v(s'+d+\frac{bl}{p'})}}{u^{2r}} \right)^{bl/s'} . \quad (28)$$

For any $p' \geq 1$, and any $s' > bl/p'$, provided that $d > s' + bl(1 - \frac{1}{p'})$, there exists a constant C such that

$$\begin{aligned} \varphi(\delta) &\stackrel{\text{def}}{=} \int_0^\delta H^{1/2}(u, \|\cdot\|_{\mathbb{L}_{2r}(\mathbb{P}_\star)}, \mathcal{G}_M) du , \\ &\leq CM^{(s'+d+bl/p')\frac{blv}{2s'}} \int_0^\delta u^{-rb/s'} du . \end{aligned}$$

Choosing $d \leq s' + bl(1 - \frac{1}{p'}) + 2$, if $s' \rightarrow +\infty$ then $u \mapsto u^{-rb/s'}$ and the last integral is finite, and $(s' + d + bl/p')\frac{blv}{2s'}$ tends to blv , so that for any $\eta > 0$ there exists a positive constant C such that

$$\varphi(\delta) \leq CM^{blv+\eta} .$$

Finally, by Proposition A.2 for any $\eta > 0$, there exists a constant A such that for n large enough

$$\mathbb{E} \left[\sup_{g \in \mathcal{G}_M} |S_n(g)| \right] \leq A\sqrt{n}M^{blv+\eta}.$$

Then, by (27), this yields

$$\mathbb{P} \left[\sup_{g \in \mathcal{G}_M} |S_n(g)| \geq c\Phi^{\mathbf{Y}}(\sqrt{nx} + x)M^v + A\sqrt{n}M^{blv+\eta} \right] \leq e^{-x}. \quad (29)$$

Proposition 3.2 is then proved using a peeling argument. By (26) and (29), for any $M \geq 1$, any $n \geq N$ and any $x > 0$, if $\gamma = blv + \eta$,

$$\mathbb{P} \left[\sup_{g \in \mathcal{G}_M} \frac{|\int g d(\mathbb{P}_n - \mathbb{P}_\star)|}{M^\gamma} \geq c\Phi^{\mathbf{Y}} \left(\sqrt{\frac{x}{n}} + \frac{x}{n} \right) + \frac{A}{\sqrt{n}} \right] \leq e^{-M^{\gamma-v}x}. \quad (30)$$

We can write

$$\mathbb{P} \left[\sup_{f \in \mathcal{F}, \nu \in \mathcal{D}_b} \frac{|\int g_{p_{f,\nu}} d(\mathbb{P}_n - \mathbb{P}_\star)|}{1 \vee I(f)^\gamma} \geq c\Phi^{\mathbf{Y}} \left(\sqrt{\frac{x}{n}} + \frac{x}{n} \right) + \frac{(2^\gamma \vee 1)A}{\sqrt{n}} \right] \leq P_1 + \sum_{k=0}^{+\infty} T_k,$$

where

$$P_1 \stackrel{\text{def}}{=} \mathbb{P} \left[\sup_{\substack{f \in \mathcal{F}; I(f) \leq 1, \\ \nu \in \mathcal{D}_b}} \frac{|\int g_{p_{f,\nu}} d(\mathbb{P}_n - \mathbb{P}_\star)|}{1 \vee I(f)^\gamma} \geq c\Phi^{\mathbf{Y}} \left(\sqrt{\frac{x}{n}} + \frac{x}{n} \right) + \frac{(2^\gamma \vee 1)A}{\sqrt{n}} \right],$$

$$T_k \stackrel{\text{def}}{=} \mathbb{P} \left[\sup_{\substack{f \in \mathcal{F}; 2^k < I(f) \leq 2^{k+1}, \\ \nu \in \mathcal{D}_b}} \frac{|\int g_{p_{f,\nu}} d(\mathbb{P}_n - \mathbb{P}_\star)|}{1 \vee I(f)^\gamma} \geq c\Phi^{\mathbf{Y}} \left(\sqrt{\frac{x}{n}} + \frac{x}{n} \right) + \frac{(2^\gamma \vee 1)A}{\sqrt{n}} \right].$$

By (30),

$$\begin{aligned} P_1 &\leq \mathbb{P} \left[\sup_{g \in \mathcal{G}_1} \left| \int g d(\mathbb{P}_n - \mathbb{P}_\star) \right| \geq c\Phi^{\mathbf{Y}} \left(\sqrt{\frac{x}{n}} + \frac{x}{n} \right) + \frac{(2^\gamma \vee 1)A}{\sqrt{n}} \right], \\ &\leq \mathbb{P} \left[\sup_{g \in \mathcal{G}_1} \left| \int g d(\mathbb{P}_n - \mathbb{P}_\star) \right| \geq c\Phi^{\mathbf{Y}} \left(\sqrt{\frac{x}{n}} + \frac{\sqrt{cx}}{n} \right) + \frac{A}{\sqrt{n}} \right], \\ &\leq e^{-x} \end{aligned}$$

and for all $k \geq 0$,

$$\begin{aligned} T_k &\leq \mathbb{P} \left[\sup_{g \in \mathcal{G}_{2^{k+1}}} \frac{|\int g d(\mathbb{P}_n - \mathbb{P}_\star)|}{2^{\gamma(k+1)}} \geq \frac{c}{2^\gamma} \Phi^{\mathbf{Y}} \left(\sqrt{\frac{x}{n}} + \frac{x}{n} \right) + \frac{(2^\gamma \vee 1)A}{2^\gamma \sqrt{n}} \right], \\ &\leq \mathbb{P} \left[\sup_{g \in \mathcal{G}_{2^{k+1}}} \frac{|\int g d(\mathbb{P}_n - \mathbb{P}_\star)|}{2^{\gamma(k+1)}} \geq c\Phi^{\mathbf{Y}} \left(\sqrt{\frac{x}{2^{2\gamma}n}} + \frac{x}{2^{2\gamma}n} \right) + \frac{A}{\sqrt{n}} \right], \\ &\leq e^{-2^{(\gamma-v)(k+1)}x/2^{2\gamma}}. \end{aligned}$$

Using (30),

$$\begin{aligned}
\mathbb{P} \left[\sup_{f \in \mathcal{F}, \nu \in \mathcal{D}_b} \frac{|f g_{p_{f,\nu}} d(\mathbb{P}_n - \mathbb{P}_\star)|}{1 \vee I(f)^\gamma} \geq c \Phi^{\mathbf{Y}} \left(\sqrt{\frac{x}{n}} + \frac{x}{n} \right) + \frac{(2^\gamma \vee 1)A}{\sqrt{n}} \right] \\
\leq e^{-x} + \sum_{k=0}^{\infty} e^{-2^{(\gamma-\nu)(k+1)}x/2^{2\gamma}} \\
\leq e^{-x} + \sum_{k=0}^{\infty} e^{-(k+1)x \log(2)(\gamma-\nu)/2^{2\gamma}} \\
\leq e^{-x} + \frac{e^{-\alpha x}}{1 - e^{-\alpha x}},
\end{aligned}$$

which concludes the proof of Proposition 3.2.

7.2 Proof of Proposition 4.1

Assume that $h(p_{f,\nu}, p_{f_\star, \nu_\star}) = 0$ (the proof of the converse proposition is straightforward). Let X'_0 be a random variable on \mathbb{X} with distribution $\nu(x)\mu(dx)$. Since ϵ_0 is a Gaussian random variable, $h(p_{f,\nu}, p_{f_\star, \nu_\star}) = 0$ implies that $f(X'_0)$ has the same distribution as $f_\star(X_0)$.

Proof that f and f_\star have the same image in \mathbb{R}^ℓ . Let $y \in f(\mathbb{X})$ and $n \geq 1$. Using $\nu \geq \nu_-$, the continuity of f and $\overline{\mathbb{X}} = \mathbb{X}$, $f(X'_0)$ has the same distribution as $f_\star(X_0)$ implies that,

$$\begin{aligned}
\mathbb{P} \{ X_0 \in f_\star^{-1}(B(y, n^{-1})) \} &= \mathbb{P} \{ X'_0 \in f^{-1}(B(y, n^{-1})) \}, \\
&\geq \nu_- \mu \{ f^{-1}(B(y, n^{-1})) \} > 0,
\end{aligned}$$

as $f^{-1}(B(y, n^{-1}))$ is a nonempty open subset of \mathbb{X} . Therefore $f_\star^{-1}(B(y, n^{-1}))$ is nonempty and for all $n \geq 1$, there exists $x_n \in \mathbb{X}$ such that $\|y - f_\star(x_n)\| < n^{-1}$. For all $n \geq 1$, $f_\star(x_n)$ is in the compact set $f_\star(\mathbb{X})$ which implies that $y \in f_\star(\mathbb{X})$. The proof of the converse inclusion follows the same lines.

Proof that ϕ is bijective. Since $f(X'_0)$ has the same distribution as $f_\star(X_0)$, X_0 has the same distribution as $\phi(X'_0)$ where $\phi \stackrel{\text{def}}{=} f_\star^{-1} \circ f$. To prove that $|J_\phi| > 0$, we use the following result due to [8, Theorem 2, p.99].

Lemma 7.1. *If $\phi : \mathbb{X} \rightarrow \mathbb{X}$ is Lipschitz then, for any integrable function g ,*

$$\int_{\mathbb{X}} g(x) |J_\phi(x)| \mu(dx) = \int_{\mathbb{X}} \sum_{x \in \phi^{-1}(\{y\})} g(x) \mu(dy).$$

Define $A \stackrel{\text{def}}{=} \{x \in \mathbb{X}; \forall x' \in \phi^{-1}(\{x\}), |J_\phi(x')| > 0\}$. Let h_1 be a bounded measurable

real function on \mathbb{X} and define $h \stackrel{\text{def}}{=} \mathbb{1}_A h_1$. By Lemma 7.1,

$$\begin{aligned} \mathbb{E}[h \circ \phi(X'_0)] &= \int_{\mathbb{X}} h_1(\phi(x')) \mathbb{1}_A(\phi(x')) \nu(x') \mu(dx') , \\ &= \int_{\mathbb{X}} h_1(\phi(x')) \mathbb{1}_A(\phi(x')) \frac{\nu(x')}{|J_\phi(x')|} |J_\phi(x')| \mu(dx') , \\ &= \int_{\mathbb{X}} h_1(x) \mathbb{1}_A(x) \sum_{x' \in \phi^{-1}(\{x\})} \frac{\nu(x')}{|J_\phi(x')|} \mu(dx) . \end{aligned}$$

As X_0 has the same distribution as $\phi(X'_0)$,

$$\int_{\mathbb{X}} h_1(x) \mathbb{1}_A(x) \nu_\star(x) \mu(dx) = \int_{\mathbb{X}} h_1(x) \mathbb{1}_A(x) \sum_{x' \in \phi^{-1}(\{x\})} \frac{\nu(x')}{|J_\phi(x')|} \mu(dx) .$$

Applying Lemma 7.1 with $g \stackrel{\text{def}}{=} \mathbb{1}_{|J_\phi|=0}$ implies that $\mathbb{1}_A = 1$ μ -a.s. in \mathbb{X} and, μ -a.s.,

$$\nu_\star(x) = \sum_{x' \in \phi^{-1}(\{x\})} \frac{\nu(x')}{|J_\phi(x')|} . \quad (31)$$

Therefore, for μ almost every $x \in \mathbb{X}$ and for all $x' \in \phi^{-1}(\{x\})$,

$$|J_\phi(x')| \geq \frac{\nu_-}{\nu_+} .$$

By continuity of J_ϕ and using that $\overline{\mathbb{X}} = \mathbb{X}$, $|J_\phi(x)| > 0$ for all $x \in \mathbb{X}$. Therefore, ϕ is locally invertible and, since \mathbb{X} is compact, simply connected and arcwise connected, ϕ is bijective by [2, Theorem 1.8, p.47]. Then (31) ensures that for μ almost every $x \in \mathbb{X}$,

$$\nu_\star(\phi(x)) = \frac{\nu(x)}{|J_\phi(x)|} ,$$

which concludes the proof of Proposition 4.1.

7.3 Proof of Proposition 4.3 and Corollary 4.4

Proof of Proposition 4.3 The proof of (18) follows the same lines as the proof of Proposition 4.1. Let (X'_0, X'_1) be a random variable on \mathbb{X}^2 with probability density $\nu(x)q(x, x')$ on \mathbb{X}^2 . $h(p_{f, \nu^2}, p_{f_\star, \nu_\star^2}) = 0$ implies that $h(p_{f, \nu}, p_{f_\star, \nu_\star}) = 0$ and, by Proposition 4.1, $f(\mathbb{X}) = f_\star(\mathbb{X})$ and $\phi = f_\star^{-1} \circ f$ is bijective. Moreover, since (ϵ_0, ϵ_1) has a Gaussian distribution, $h(p_{f, \nu^2}, p_{f_\star, \nu_\star^2}) = 0$ implies that $(\phi(X'_0), \phi(X'_1))$ has the same distribution as (X_0, X_1) so that for any x in \mathbb{X} and any bounded measurable function f on \mathbb{X} ,

$$\mathbb{E}[\phi(X'_1) | X'_0 = \phi^{-1}(x)] = \mathbb{E}[X_1 | X_0 = x] .$$

Following the proof of Proposition 4.1, this gives (18).

Proof of Corollary 4.4 Assume now that

$$\begin{aligned} q_\star(x, x') &= c_\star(x)\rho_\star(\|x - x'\|) , \\ q(x, x') &= c(x)\rho(\|x - x'\|) . \end{aligned}$$

We may assume, using an eventual modification of c_\star and c that $\rho(0) = \rho_\star(0) = 1$. By (18),

$$c(x)\rho(\|x - x'\|) = |J_\phi(x')|c_\star(\phi(x))\rho_\star(\|\phi(x) - \phi(x')\|) . \quad (32)$$

Applying (32) with $x = x'$ implies $|J_\phi(x)| = c(x)/c_\star(\phi(x))$. Therefore,

$$\frac{|J_\phi(x)|}{|J_\phi(x')|} = \frac{\rho_\star(\|\phi(x) - \phi(x')\|)}{\rho(\|x - x'\|)} = \frac{\rho_\star(\|\phi(x') - \phi(x)\|)}{\rho(\|x' - x\|)} = \frac{|J_\phi(x')|}{|J_\phi(x)|}$$

and then, for all $x \in \mathbb{X}$, $|J_\phi(x)| = 1$.

Now (32) implies that for any x and x' in \mathbb{X} ,

$$\rho(\|x - x'\|) = \rho_\star(\|\phi(x) - \phi(x')\|) . \quad (33)$$

Let $x_0 \in \overset{\circ}{\mathbb{X}}$, $y_0 = \phi(x_0)$ and $d_0, d'_0 > 0$ be such that $B(x_0, d_0) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^m, \|x_0 - x\| < d_0\} \subset \mathbb{X}$ and $\phi(B(x_0, d_0)) \subset B(y_0, d'_0)$.

Let $d < d_0$ and denote by $S(x_0, d)$ the set $S(x_0, d) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^m, \|x_0 - x\| = d\}$. As ρ_\star is one-to-one, write $F = \rho_\star^{-1} \circ \rho$. (33) implies that $\phi(S(x_0, d)) \subset S(y_0, F(d))$. Furthermore, using the compactness and the connectivity of $S(x_0, d)$, $\phi(S(x_0, d)) = S(y_0, F(d))$ which, together with the continuity of ϕ , guarantees that $\phi(B(x_0, d)) = B(y_0, F(d))$. Finally, because ϕ preserves the volumes, for any $d < d_0$, $F(d) = d$ and for any $x \in \mathbb{X}$ and any $x' \in B(x, d_0)$, $\|x - x'\| = \|\phi(x) - \phi(x')\|$. The proof is concluded using the connectivity of \mathbb{X} .

A Concentration results for the empirical process of unbounded functions

Proposition A.1 provides a concentration inequality on the empirical process over a class of functions \mathcal{G} for which $|g(Z_i)|$ can be bounded uniformly in $g \in \mathcal{G}$ by an independent process U_i with bounded moments. This unusual condition is more general than the settings considered in [19, Theorem 3] which considers a uniformly bounded class of functions.

Proposition A.1. *Let $(Z_n)_{n \geq 0}$ be a Φ -mixing process taking values in a set \mathcal{Z} . Assume that the Φ -mixing coefficients associated with $(Z_n)_{n \geq 0}$ satisfy:*

$$\Phi \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \phi_i^{1/2} < \infty .$$

Let \mathcal{G} be some countable class of real valued measurable functions defined on \mathcal{Z} . Assume that there exists a sequence of independent random variables $(U_i)_{i \geq 0}$ such that:

- for any g in \mathcal{G} ,

$$|g(Z_i)| \leq U_i \text{ a.s. ;} \quad (34)$$

- there exists some positive numbers ν and c such that, for any $k \geq 1$:

$$\sum_{i=0}^{n-1} \mathbb{E} [U_i^{2k}] \leq k! n \nu c^{k-1}. \quad (35)$$

Then, for any positive x ,

$$\mathbb{P} [S_n \geq 2\Phi (2\sqrt{n\nu x} + \sqrt{cx})] \leq e^{-x},$$

where

$$S_n = \sup_{g \in \mathcal{G}} \left| \sum_{i=0}^{n-1} g(Z_i) \right| - \mathbb{E} \left[\sup_{g \in \mathcal{G}} \left| \sum_{i=0}^{n-1} g(Z_i) \right| \right].$$

Proof. For any real valued random variable and for any real random variable X , define $\psi_X(\lambda) \stackrel{\text{def}}{=} \ln(\mathbb{E}[\exp(\lambda X)])$, Following the proof of [19, Theorem 3] together with the discussion about the dependence structure in [19, Section 2], we have

$$\exp\left(\psi_{S_n}\left(\frac{\lambda}{4}\right)\right) \leq \mathbb{E} \left[\exp \left[\lambda^2 \frac{\Phi^2}{4} V^2 \right] \right]^{\frac{1}{2}} \exp \left[\lambda^2 \frac{\Phi^2}{8} \mathbb{E} [V^2] \right], \quad (36)$$

where $V^2 \stackrel{\text{def}}{=} \sum_{i=1}^n U_i^2$. Using (34) and by independence of the $(U_i)_{i \geq 0}$,

$$\begin{aligned} \exp\left(\psi_{S_n}\left(\frac{\lambda}{4}\right)\right) &\leq \mathbb{E} \left[\exp \left[\lambda^2 \frac{\Phi^2}{4} \sum_{i=1}^n U_i^2 \right] \right]^{\frac{1}{2}} \exp \left[\lambda^2 \frac{\Phi^2}{8} \sum_{i=1}^n \mathbb{E}[U_i^2] \right], \\ &\leq \prod_{i=1}^n \mathbb{E} \left[\exp \left[\lambda^2 \frac{\Phi^2}{4} U_i^2 \right] \right]^{\frac{1}{2}} \exp \left[\lambda^2 \frac{\Phi^2}{8} \sum_{i=1}^n \mathbb{E}[U_i^2] \right]. \end{aligned}$$

Thus,

$$\psi_{S_n}(\lambda/4) \leq \frac{1}{2} \sum_{i=1}^n \ln \left\{ \mathbb{E} \left[\exp \left(\lambda^2 \frac{\Phi^2}{4} U_i^2 \right) \right] \right\} + \lambda^2 \frac{\Phi^2}{8} \sum_{i=1}^n \mathbb{E} [U_i^2].$$

Since for any $u > 0$, $\ln(u) \leq u - 1$, this yields

$$\psi_{S_n}(\lambda/4) \leq \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k!} \left[\lambda^2 \frac{\Phi^2}{4} \right]^k \sum_{i=1}^n \mathbb{E} [U_i^{2k}] + \lambda^2 \frac{\Phi^2}{8} \sum_{i=1}^n \mathbb{E} [U_i^2].$$

Then, by (35),

$$\psi_{S_n}(\lambda/4) \leq n\nu \left[\lambda^2 \frac{\Phi^2}{4} \right] \frac{1}{2} \sum_{k=0}^{\infty} \left[\lambda^2 \frac{\Phi^2}{4} c \right]^k + \left[\lambda^2 \frac{\Phi^2}{8} \nu \right].$$

If $0 < \lambda^2 \Phi^2 c / 4 < 1$,

$$\begin{aligned} \psi_{S_n}(\lambda/4) &\leq n\nu \lambda^2 \frac{\Phi^2}{8} \frac{1}{1 - \lambda^2 \frac{\Phi^2}{4} c} + n\nu \lambda^2 \frac{\Phi^2}{8}, \\ &\leq n\nu \lambda^2 \frac{\Phi^2}{4} \frac{1}{1 - \lambda^2 \frac{\Phi^2}{4} c}. \end{aligned}$$

Define $\nu' \stackrel{\text{def}}{=} 8n\nu\Phi^2$ and $c' \stackrel{\text{def}}{=} 2\Phi\sqrt{c}$. Therefore,

$$\psi_{S_n}(\lambda/4) \leq \frac{\nu'(\lambda/4)^2}{2(1 - c'(\lambda/4))}. \quad (37)$$

Hence, for all $0 < \lambda < 1/c'$,

$$\psi_{S_n}(\lambda) \leq \frac{\nu'\lambda^2}{2(1 - c'\lambda)}. \quad (38)$$

By the Bernstein type inequality (38), [16, Lemma 2.3] gives, for any measurable set A with $\mathbb{P}(A) > 0$,

$$\mathbb{E}_A[S_n] \leq \sqrt{2\nu' \ln\left(\frac{1}{\mathbb{P}(A)}\right)} + c' \ln\left(\frac{1}{\mathbb{P}(A)}\right).$$

Hence, by [16, Lemma 2.4], for any positive x ,

$$\mathbb{P}\left[S_n \geq \sqrt{2\nu'x} + c'x\right] \leq e^{-x}.$$

□

Proposition A.2 below provides a control on the expectation of the empirical process. It introduces a β -mixing condition (see [5]) which is weaker than the Φ -mixing condition considered in Proposition A.1. The β -mixing coefficient between two σ -fields $\mathcal{U}, \mathcal{V} \subset \mathcal{E}$ is defined in [5] by

$$\beta(\mathcal{U}, \mathcal{V}) \stackrel{\text{def}}{=} \frac{1}{2} \sup \sum_{(i,j) \in I \times J} |\mathbb{P}(U_i \cap V_j) - \mathbb{P}(U_i)\mathbb{P}(V_j)|,$$

where the supremum is taken over all finite partitions $(U_i)_{i \in I}$ and $(V_j)_{j \in J}$ respectively \mathcal{U} and \mathcal{V} measurable. The corresponding mixing coefficients $(\beta_i)_{i \geq 0}$ associated with a process $(X_k)_{k \geq 0}$ satisfy $\beta_i < \phi_i$ for all $i \geq 1$.

Proposition A.2. *Let $(Z_i)_{i \geq 0}$ be a stationary process taking values in a Polish space \mathcal{Z} . Denote by \mathbb{P}_\star the distribution of Z_0 and by \mathbb{E}_\star the expectation under \mathbb{P}_\star . Assume that $(Z_i)_{i \geq 0}$ is β -mixing with β coefficients $(\beta_i)_{i \geq 0}$ satisfying*

$$\sum_{i=1}^{\infty} \beta_i < \infty.$$

Let \mathcal{G} be a countable class of functions on \mathcal{Z} . Assume that there exist $r > 1$ and $\sigma > 0$ such that for any $g \in \mathcal{G}$,

$$\|g\|_{\mathbb{L}_{2r}(\mathbb{P}_\star)} \stackrel{\text{def}}{=} \mathbb{E}_\star[g^{2r}]^{1/2r} \leq \delta.$$

Assume also that the bracketing function satisfies

$$\int_0^1 \sqrt{H(u, \|\cdot\|_{\mathbb{L}_{2r}(\mathbb{P}_\star)}, \mathcal{G})} du < \infty.$$

Then,

$$\varphi(\delta) := \int_0^\delta \sqrt{H(u, \|\cdot\|_{\mathbb{L}_{2r}(\mathbb{P}_\star)}, \mathcal{G})} du$$

is finite and there exists a constant A such that for n big enough

$$\mathbb{E} \left[\sup_{g \in \mathcal{G}} |S_n(g)| \right] \leq \sqrt{n} A \varphi(\delta), \quad (39)$$

where, for all $g \in \mathcal{G}$, $S_n(g) = \sum_{i=0}^{n-1} g(Z_i) - n \mathbb{E}_* [g(Z_0)]$.

Proof. This is a direct application of the remark following [7, Theorem 3]. \square

B Entropy of the class \mathcal{G}_M

Lemma B.1. *For any $p' \geq 1$, and any $s' > bl/p'$, provided that $d > s' + bl(1 - \frac{1}{p'})$, there exists a constant C such that*

$$\forall u > 0, H(u, \|\cdot\|_{\mathbb{L}_{2r}(\mathbb{P}_*)}, \mathcal{G}_M) \leq C \left(\frac{M^{v(s' + d + \frac{bl}{p'})}}{u^{2r}} \right)^{bl/s'}. \quad (40)$$

Proof. By [16, Lemma 7.26], for any densities of probability p_2 and p_1 on \mathbb{R}^{bl} ,

$$\|g_{p_2} - g_{p_1}\|_{\mathbb{L}_{2r}(\mathbb{P}_*)}^{2r} \leq C \|\sqrt{p_2} - \sqrt{p_1}\|_{\mathbb{L}_2(\mathbb{R}^{bl})}^2.$$

Since $\|\sqrt{p_2} - \sqrt{p_1}\|_{\mathbb{L}_2(\mathbb{R}^{bl})}^2 \leq \|p_2 - p_1\|_{\mathbb{L}_1(\mathbb{R}^{bl})}$, this yields, for any $u > 0$,

$$H(u, \|\cdot\|_{\mathbb{L}_{2r}(\mathbb{P}_*)}, \mathcal{G}_M) \leq H\left(\frac{u^{2r}}{C}, \|\cdot\|_{\mathbb{L}_1(\mathbb{R}^{bl})}, \mathcal{P}_M\right), \quad (41)$$

where $\mathcal{P}_M \stackrel{\text{def}}{=} \{p_{f,\nu}; \nu \in \mathcal{D}_b, f \in \mathcal{F} \text{ and } I(f) \leq M\}$. Thus, it remains to bound the entropy with bracketing of the class of functions \mathcal{P}_M associated with $\|\cdot\|_{\mathbb{L}_1(\mathbb{R}^{bl})}$ to control the entropy with bracketing of the class of functions \mathcal{G}_M associated with the $\|\cdot\|_{\mathbb{L}_{2r}(\mathbb{P}_*)}$.

By [18] the entropy of \mathcal{P}_M is bounded by proving that it is included in some weighted Sobolev Space. Define the polynomial weighting function $\langle \mathbf{y} \rangle^d \stackrel{\text{def}}{=} (1 + \|\mathbf{y}\|^2)^{d/2}$ parametrized by $d \in \mathbb{R}$ where $\mathbf{y} \in \mathbb{R}^{bl}$. Furthermore, define for $p' \geq 1$, and $s' > bl/p'$ the weighted Sobolev space

$$W^{s',p'}(\mathbb{Y}^b, \langle \mathbf{y} \rangle^d) \stackrel{\text{def}}{=} \left\{ f; f \cdot \langle \mathbf{y} \rangle^d \in W^{s',p'}(\mathbb{Y}^b, \mathbb{R}) \right\}.$$

Lemma B.2 ensures that, for any $p' \geq 1$, $s' > bl/p'$ any even integer d , the renormalized classes of functions $\mathcal{P}_M / M^{v(s' + d + \frac{bl}{p'})}$, $M \geq 1$ belong to the same bounded subspace of $W^{s',p'}(\mathbb{R}^{bl}, \langle \mathbf{y} \rangle^d)$. By [18, Corollary 4], for any $p' \geq 1$, and any $s' > bl/p'$, provided that $d > s' + bl(1 - \frac{1}{p'})$, there exists a constant C such that

$$\forall \epsilon > 0, H\left(\epsilon, \|\cdot\|_{\mathbb{L}_1(\mathbb{R}^{bl})}, \mathcal{P}_M / M^{v(s' + d + \frac{bl}{p'})}\right) \leq C \epsilon^{-bl/s'}.$$

The proof is concluded by (41). \square

Lemma B.2. *Assume that H2 holds for some $v > 0$. Then, for any $p' \geq 1$, $s' > bl/p'$ and any even and positive number d , there exists a positive constant C such that for any $f \in \mathcal{F}$ and any $\nu \in \mathcal{D}_b$,*

$$\|p_{f,\nu} \cdot \langle \mathbf{y} \rangle^d\|_{W^{s',p'}(\mathbb{R}^{bl}, \mathbb{R})} \leq C \kappa(v, f)^{s' + d + \frac{bl}{p'}},$$

where $\kappa(v, f) \stackrel{\text{def}}{=} 1 \vee I(f)^v$.

Proof. Let f be a function in \mathcal{F} , for any $\nu \in \mathcal{D}^b$,

$$\|p_{f,\nu} \cdot \langle \mathbf{y} \rangle^d\|_{W^{s',p'}(\mathbb{R}^{b\ell},\mathbb{R})}^{p'} = \sum_{|\alpha| \leq s'} \|D^\alpha (p_{f,\nu} \cdot \langle \mathbf{y} \rangle^d)\|_{L_{p'}}^{p'}.$$

Applying the general Leibniz rule component by component, for any $\alpha \in \mathbb{N}^{b\ell}$,

$$D^\alpha (p_{f,\nu} \cdot \langle \mathbf{y} \rangle^d) = \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} D^{\alpha'} (\langle \mathbf{y} \rangle^d) D^{\alpha - \alpha'} (p_{f,\nu}), \quad (42)$$

where $\binom{\alpha}{\alpha'} \stackrel{\text{def}}{=} \prod_{j=1}^{b\ell} \binom{\alpha_j}{\alpha'_j}$. Then, Lemma B.2 requires to control $\|D^{\alpha^{(1)}} (\langle \mathbf{y} \rangle^d) D^{\alpha^{(2)}} (p_{f,\nu})\|_{L_{p'}}$ for any given $\alpha^{(1)}$ and $\alpha^{(2)}$ in $\mathbb{N}^{b\ell}$. For any α in $\mathbb{N}^{b\ell}$, there exists a polynomial function P_α with degree lower than $|\alpha|$ such that, for any $\mathbf{y} \in \mathbb{R}^{b\ell}$,

$$D^\alpha p_{f,\nu}(\mathbf{y}) = \int_{\mathbf{x} \in \mathbb{X}^b} P_\alpha(\mathbf{f}(\mathbf{x}) - \mathbf{y}) \exp\left\{-\frac{1}{2}\|\mathbf{f}(\mathbf{x}) - \mathbf{y}\|^2\right\} q_{a,b}(\mathbf{x}) \mu^{\otimes b}(\mathrm{d}\mathbf{x}). \quad (43)$$

Moreover, since d is an even number, for any $\alpha \in \mathbb{N}^{b\ell}$ such that $|\alpha| \leq d$, $D^\alpha \langle \mathbf{y} \rangle^d$ is a polynomial function denoted by $P_{d,\alpha}$ with degree lower than $d - |\alpha|$. In the case where $|\alpha| > d$, $D^\alpha \langle \mathbf{y} \rangle^d = 0$.

By H2, there exists constant $C > 0$ such that, for any $\mathbf{x} \in \mathbb{X}^b$, $\|\mathbf{f}(\mathbf{x})\| \leq CI(f)^v \leq C\kappa(v, f)$. Since $P_{\alpha^{(2)}}$ and $P_{d,\alpha^{(1)}}$ are both polynomial functions, there exist a constant C depending on $\alpha^{(1)}, \alpha^{(2)}$ and d such that, for any $\mathbf{y} \in \mathbb{R}^{b\ell}$ and any $\mathbf{x} \in \mathbb{X}^b$,

$$|P_{d,\alpha^{(1)}}(\mathbf{y}) P_{\alpha^{(2)}}(\mathbf{f}(\mathbf{x}) - \mathbf{y})| \leq \mathbf{1}_{|\alpha^{(1)}| \leq d} \left[C(1 + \|\mathbf{y}\|)^{d - |\alpha^{(1)}|} \times (\kappa(v, f) + \|\mathbf{y}\|)^{|\alpha^{(2)}|} \right].$$

Define the following subset of $\mathbb{R}^{b\ell}$

$$A_f \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbb{R}^{b\ell}; \|\mathbf{y}\| \leq \kappa(v, f)\}.$$

$\|\mathbf{f}(\mathbf{x}) - \mathbf{y}\|$ can be lower bounded by 0 when \mathbf{y} belongs to A_f and by $|\kappa(v, f) - \|\mathbf{y}\||$ when \mathbf{y} belongs to A_f^c . Therefore, uniformly in $\mathbf{x} \in \mathbb{X}^b$,

$$\exp\left\{-\frac{1}{2}\|\mathbf{f}(\mathbf{x}) - \mathbf{y}\|^2\right\} \leq \mathbf{1}_{A_f}(\mathbf{y}) + \mathbf{1}_{A_f^c}(\mathbf{y}) e^{-\frac{1}{2}(\kappa(v, f) - \|\mathbf{y}\|)^2}.$$

Thus, there exists a constant $C > 0$ which does not depend on a , such that, for any $p' \geq 1$,

$$\|D^{\alpha^{(1)}} (\langle \mathbf{y} \rangle^d) D^{\alpha^{(2)}} (p_{f,\nu})\|_{L_{p'}}^{p'} \leq \mathbf{1}_{|\alpha^{(1)}| \leq d} \left[C\kappa(v, f)^{p'|\alpha^{(2)}|} (I_1 + I_2) \right],$$

where,

$$\begin{aligned} I_1 &\stackrel{\text{def}}{=} \int_{A_f} (1 + \|\mathbf{y}\|)^{p'(d - |\alpha^{(1)}|)} \left(1 + \frac{\|\mathbf{y}\|}{\kappa(v, f)}\right)^{p'|\alpha^{(2)}|} \lambda^{\otimes b}(\mathrm{d}\mathbf{y}), \\ I_2 &\stackrel{\text{def}}{=} \int_{A_f^c} (1 + \|\mathbf{y}\|)^{p'(d - |\alpha^{(1)}|)} \left(1 + \frac{\|\mathbf{y}\|}{\kappa(v, f)}\right)^{p'|\alpha^{(2)}|} e^{-\frac{p'}{2}(\kappa(v, f) - \|\mathbf{y}\|)^2} \lambda^{\otimes b}(\mathrm{d}\mathbf{y}). \end{aligned}$$

By the change of variables $\mathbf{y}' = (\kappa(v, f))^{-1}\mathbf{y}$ in I_1 and I_2 , there exists a constant C such that

$$\|D^{\alpha^{(1)}}(\langle \mathbf{y} \rangle^d) D^{\alpha^{(2)}}(p_{f,\nu})\|_{L_{p'}}^{p'} \leq C \kappa(v, f)^{p'(|\alpha^{(2)}| - |\alpha^{(1)}| + d) + b\ell}. \quad (44)$$

Using (44) in (42) with $\alpha^{(1)} = \alpha'$ and $\alpha^{(2)} = \alpha - \alpha'$ for any $|\alpha| \leq s'$ and $\alpha' \leq \alpha$ concludes the proof of Lemma B.2. \square

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