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# Some New Mathematical Modelings of Junctions

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## Abstract

Most of the structures in Civil Engineering consists in assemblies of deformable bodies, thus it is of interest to dispose of efficient models of junctions between deformable solids. The classical schemes of Continuum Mechanics lead to boundary value problems involving several parameters, one being essential: the (low) thickness of the layer filled by the adhesive. For usual behaviors of the adherents and the adhesive, it is not difficult to prove existence of solutions, but their numerical approximations may be difficult due to the rather low thickness of the adhesive implying a too fine mesh. We propose a simplified but accurate mathematical modeling by a rigorous study of the asymptotic behavior of the three-dimensional adhesive when its thickness goes to zero. Depending on the stiffness of the adhesive, the limit model will replace the thin adhesive layer by either a mechanical constraint along the surface the layer shrinks toward or a material surface; the structure of the constitutive equations of the constraint or of the material surface keeping the memory of the mechanical behavior of the adhesive.

The mathematical techniques used in these studies, carried out for more than 25 years, involve variational convergences and the Trotter theory of convergence of semi-groups of operators. We will present classical results concerning standard elastic or dissipative behaviors of the adhesive and some new ones devoted to microscopic aspects, imperfectly bonded adhesive joints, loaded joints, etc. . .

*Keywords:* junctions, asymptotic analysis, variational convergence, convergence of semi-groups of operators.

## 1 Introduction

Most of the structures in Civil Engineering consists in assemblies of deformable bodies, thus it is of interest to dispose of efficient models of junctions between deformable solids. The classical schemes of Continuum Mechanics lead to boundary value problems involving several parameters, one being essential: the (low) thickness of the layer filled by the adhesive. For the usual behaviors of the adherents and the adhesive, it is not difficult to prove existence of solutions, but their numerical approximations may be difficult due to the rather low thickness of the adhesive implying a too fine mesh. Moreover, the mechanical properties of the adherents and the adhesive being very different, the involved systems may be very ill-conditioned. Hence, it is capital to propose simpler but accurate enough models. A classical way is to consider the real geometrical and mechanical data, like thickness, stiffness, etc, as parameters and to study the asymptotic behavior of the parametrized boundary value problems when these parameters go to a natural limit (0 if the quantity is small,  $+\infty$  if it is large!). This may be done by various methods: formal asymptotic expansions, singular perturbations, . . . Here, we chose the rigorous point of view of variational analysis by studying the asymptotic behavior of the minimizers of the total mechanical energy functional. We show that they converge (with respect to a topology induced by the mechanical energy) toward the solutions of a minimization problem which will be our proposal of simplified model.

Two main cases of elastic junctions have been treated in this way:

- i) the soft junctions, where the stiffness of the junction is far lower than the ones of

the adherents (it corresponds to soft adhesive bonded joints), see for instance [1] and the references therein,

- ii) the hard junctions, where the stiffness of the junction is far larger than the ones of the adherents (which may occur in some situations of welding), see for instance [2], [3] and the references therein.

Let us recall that a big difference in the nature of the asymptotic models occurs. The soft adhesive junction is replaced by a mechanical constraint between the adherents whose surface energy is a function of the relative displacement of the adherents along the interface the junction shrinks toward. On the contrary, the hard junction is replaced by a material surface perfectly stuck to the adherents, with a surface strain energy density function of the surface gradient of the displacement (here, there is no jump of displacement across the interface).

Anyway, these models are simpler than the genuine ones because surface integral functionals are involved in place of integral functionals on a thin layer. They may be accurate enough due to the rigorous convergence results: the closer the parameters to their natural limits, the sharper the models!

Here, I will describe some extensions of these two basic results which have recently been done in Montpellier in collaboration with Gérard Michaille, Oana Iosifescu and Pongpol Juntharee; all of the material is gathered in the Ph.D. thesis of Pongpol Juntharee. The first part is devoted to soft junctions and two extensions are presented. First, we consider the case when the soft adhesive bonded joint is not perfectly stuck to the adherents and after the case when the joint, perfectly bonded to the adherents, is subjected to a loading. The second part concerns hard junctions and is a first attempt to model some fracture phenomena in soldered joints. For the sake of simplicity, all the junctions considered here occupy layers of constant thickness and, like the adherents, are assumed to be elastic. Hence the starting equilibrium problems may be formulated in terms of minimization problems in some suitable function spaces and we systematically derive our asymptotic models through variational convergence methods.

## 2 Soft junctions

### 2.1 An asymptotic model for a thin, soft and imperfectly bonded elastic joint

To simplify, we confined to the case of a unique adherent lying in a domain  $\Omega$ , included in  $\{x_3 > 0\}$  with a Lipschitz-continuous boundary whose intersection  $S$  with  $\{x_3 = 0\}$  is a domain of  $\mathbb{R}^2$ . It is linked with a rigid support  $\{x_3 < -\varepsilon\}$  by an adhesive occupying the layer  $B_\varepsilon := S \times (-\varepsilon, 0)$ . The bulk energy density of the adherent is a strictly convex function of the linearized tensor of deformation  $e(u) - u$  is the displacement - with quadratic growth. The adherent is subjected to body and surface forces of densities  $f$  and  $\varphi$ , and is clamped along  $\Gamma_0 \subset \partial\Omega$ . The bulk energy density of the adhesive is a function of the linearized tensor of deformation of the type :

$$W_{\mu_S, \mu_D}(e) := \mu_S W_1(\text{tr } e) + \mu_D W_2(\text{dev}(e))$$

$$\text{tr } e := e_{11} + e_{22} + e_{33}, \quad \text{dev}(e) := e - \frac{1}{3} Id$$

which, without particular mathematical difficulties, generalizes the density associated with an isotropic linearly elastic material,  $W_1, W_2$  being strictly convex with quadratic growth (but, as for  $W$ , non necessarily quadratic!). We also assume the existence of smooth enough recession functions of order 2  $W_i^{\infty, 2}$ . The adhesive is not subjected to forces, is clamped on the rigid support and the mechanical constraint between the adhesive and the adherent is not necessarily pure adhesion but is described by a surface energy density  $h$ , which is a non negative, convex, lower semi-continuous function in  $\mathbb{R}^3$  vanishing at 0. Thus, both realistic smooth densities like  $\frac{1}{p}|\cdot|^p$  and realistic non smooth densities like indicator functions of closed convex subsets of  $\mathbb{R}^3$  may be taken into account! Assuming

the forces densities of class  $L^2$ , it is clear that an equilibrium configuration is given by the unique solution  $\bar{u}_s$  of the following problem involving the triple  $s := (\varepsilon, \mu_S, \mu_D)$ :

$$(P_s) \quad \text{Min}\{ F_s(\nu) - L(\nu); \nu \in V_s \}$$

with

$$\begin{aligned} V_s &:= \{ \nu \in L^2(\Omega_\varepsilon; \mathbb{R}^3); \nu^+ := \nu|_\Omega \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3), \nu^- := \nu|_{B_\varepsilon} \in H_{S-\varepsilon}^1(B_\varepsilon; \mathbb{R}^3) \}, \\ \Omega_\varepsilon &:= \Omega \cup B_\varepsilon \\ H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) &:= \{ \nu \in H^1(\Omega; \mathbb{R}^3); \nu = 0 \text{ on } \Gamma_0 \} \\ H_{S-\varepsilon}^1(B_\varepsilon; \mathbb{R}^3) &:= \{ \nu \in H^1(B_\varepsilon; \mathbb{R}^3); \nu = 0 \text{ on } S_{-\varepsilon} := (0, 0, -\varepsilon) + S \} \\ L(\nu) &:= \int_\Omega f(x) \cdot \nu(x) dx + \int_{\Gamma_1} \varphi(x) \cdot \nu(x) ds, \text{ the work of external loading,} \\ F_s(\nu) &:= \int_\Omega W(e(\nu^+)) dx + \int_{B_\varepsilon} W_{\mu_S, \mu_D}(e(\nu^-)) dx + \int_S h([\nu](\hat{x}, 0)) dx \\ [\nu] &:= \gamma_0(\nu^+) - \gamma_0(\nu^-), \text{ the jump of displacement across } S \end{aligned}$$

(or the relative displacement along  $S$ ) where the same symbol  $\gamma_0(w)$  denotes the trace on  $S$  of any element  $w$  of both  $H^1(B_\varepsilon; \mathbb{R}^3)$  and  $H^1(\Omega; \mathbb{R}^3)$ , of course  $\hat{x} = (x_1, x_2)$ . To get a simplified model (suitable for numerical computations), we study the asymptotic behavior of  $\bar{u}_s$  under the conditions: there exist  $\bar{s} \in \{0\} \times [0, \infty)^2$ ,  $(\bar{\mu}_S, \bar{\mu}_D) \in [0, \infty]^2$  and a positive real number  $\varepsilon_0$  such that:  $\bar{s} = \lim s$ ,  $(\bar{\mu}_S, \bar{\mu}_D) = \lim(\mu_S/\varepsilon, \mu_D/\varepsilon)$ ,  $0 = \lim(\varepsilon\mu_S, \varepsilon\mu_D)$ ,  $0 < \varepsilon < \varepsilon_0$ . It was shown in [1] that the surface energy density

$$\bar{W}_{\bar{\mu}_S, \bar{\mu}_D}(\nu) = W_{\bar{\mu}_S, \bar{\mu}_D}^{\infty, 2}(\nu) := \bar{\mu}_S W_1^{\infty, 2}(\text{tr}(\nu \otimes_s e_3)) + \bar{\mu}_D W_2^{\infty, 2}(\text{dev}(\nu \otimes_s e_3)), \quad \forall \nu \in \mathbb{R}^3$$

where

$$a \otimes_s b = \frac{1}{2}(a \otimes b + b \otimes a) \quad \forall a, b \in \mathbb{R}^3, e_3 = (0, 0, 1),$$

was the energy density associated with the mechanical constraint along  $S$  which replaced the thin soft joint perfectly bonded to the adherents. In the present case, we have shown that the imperfectly bonded joint shall be replaced by a constraint whose associated energy density is the *inf-convolution*  $g$  of  $h$  with  $\bar{W}_{\bar{\mu}_S, \bar{\mu}_D}$ :

$$g(t) := h +_e \bar{W}_{\bar{\mu}_S, \bar{\mu}_D}(t) := \inf\{ h(t') + \bar{W}_{\bar{\mu}_S, \bar{\mu}_D}(t''); t = t' + t'', t' \in \mathbb{R}^3 \}$$

That corresponds to the connecting in series of the initial mechanical constraint along  $S$  with the limit constraint of density  $\bar{W}_{\bar{\mu}_S, \bar{\mu}_D}$  ! More precisely, we establish:

*When  $s$  tends to  $\bar{s}$ , then  $\bar{u}_s|_\Omega$  converges strongly in  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  towards the unique solution  $\bar{u}$  of*

$$(P) \quad \text{Min}\{ F(\nu) - L(\nu); \nu \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \}$$

and  $F(\bar{u}) - L(\bar{u}) = \lim_{s \rightarrow \bar{s}} (F_s(\bar{u}_s) - L(\bar{u}_s))$ , where

$$F(\nu) := \begin{cases} \int_\Omega W(e(\nu)) dx + \int_S g(\gamma_0(\nu)) d\hat{x} - L(\nu), & \text{when } g(\gamma_0(\nu)) \in L^1(S), \\ +\infty, & \text{otherwise.} \end{cases}$$

This result is established by the usual strategy of variational convergence:

- i) property of compactness for all the sequences  $u_s$  with bounded energies,
- ii) upper bound for  $F_s(u_s)$ ,
- iii) lower bound for  $F_s(u_s)$ .

The point i) is obtained through estimations in function of  $\varepsilon$  of the constants involved in the inequalities of Poincaré, of Korn and of continuity of the trace operator from  $H_{\Gamma_0}^1(B_\varepsilon; \mathbb{R}^3)$  into  $L^2(S; \mathbb{R}^3)$ . The point ii) is obtained by a lifting to  $B_\varepsilon$  (similar as the one of [1]) of the field defined on  $S$  which achieves the minimum entering the definition of  $g(\gamma_0(u))$  thanks

to a capital property of Lipschitz-continuity of  $g$ , implied by the properties of  $W_i^{\infty,2}$ . Eventually, the sub-differential inequality and an integration by parts supply the point iii).

Thus, our proposal of model is simpler than the genuine one: the integral functional defined on the thin three-dimensional domain  $B_\varepsilon$  is replaced by an integral functional defined on the surface  $S$  the adhesive layer shrinks to. And, the previous convergence result shows that the closer  $s$  to  $\bar{s}$ , the more precise the model. In practice,  $\lim(\mu_D/\varepsilon)$ ,  $\lim(\mu_S/\varepsilon)$  should be replaced by the true real physical data  $\mu_D/\varepsilon$ ,  $\mu_S/\varepsilon$ . We can still improve the model by a result of corrector type by studying the asymptotic behavior of the optimal displacement in the adhesive. We have proven that it is energetically equivalent to a field, affine function in  $x_3$  whose trace on  $S$  is supplied by the minimizer involved by the definition of  $g(\gamma_0(\bar{u}))$ .

We have given various examples of realistic densities  $h$  including the one treated in [4] by means of a zoom in the third coordinate in the joint and an indirect mixed formulation with two fields (displacement and stress).

Finally, motivated by the tribological concept of the third body, a variant has been considered where the thin layer contains a far thinner and softer layer in the vicinity of the adherents.

## 2.2 Loaded adhesive joints

To be realistic, we consider a scalar problem, the unknown being, for example, the deflexion of a membrane made of three parts and subjected to a loading even in the inner part. If  $\Omega := \Sigma \times (-r, r)$ ,  $r > 0$ , where  $\Sigma$  is a bounded domain of  $\mathbb{R}^d$ ,  $d = 1, 2$ , the adhesive occupies  $B_\varepsilon := \Sigma \times (-\varepsilon/2, \varepsilon/2)$  and the adherents  $\Omega_\varepsilon := \Omega - \bar{B}_\varepsilon$ . The bulk energy densities respectively are  $\varepsilon g$  and  $f$ ,  $g$  and  $f$  are strictly convex with a quadratic growth and one assumes the existence for  $g$  of a recession function of order 2. The adherents are clamped on  $\Gamma_0 \subset \partial\Omega$  and subjected to forces whose work  $L$  is a continuous linear form on  $H_{\Gamma_0}^1(\Omega_{\varepsilon_0})$ . On the contrary, the work of the loading applied to  $B_\varepsilon$  is defined from a continuous linear form on  $V(B) := \{u \in L^2(\Omega); \frac{\partial u}{\partial x_N} \in L^2(\Omega)\}$ . If  $\tau_\varepsilon$  is the scaling operator, continuous from  $V(B_\varepsilon)$  into  $V(B)$ , defined by  $\tau_\varepsilon(u)(\hat{x}, x_N) := u(\hat{x}, x_N/\varepsilon) \forall x = (\hat{x}, x_N) \in B_\varepsilon$ , then the work is the linear form  $u \mapsto \langle S_\varepsilon, \tau_\varepsilon u \rangle$ .

Thus, the determination of equilibrium configurations leads to the problem

$$(P_\varepsilon) \quad \text{Min}\{F_\varepsilon(\nu) - L(\nu); \nu \in H_{\Gamma_0}^1(\Omega)\},$$

where

$$F_\varepsilon(\nu) := \int_{\Omega} f(\nabla \nu) dx + \varepsilon \int_{B_\varepsilon} g(\nabla \nu) dx - \langle S_\varepsilon, \tau_\varepsilon \nu \rangle$$

Clearly, the problem has a unique solution  $\bar{u}_\varepsilon$  and we aim to study its asymptotic behavior when  $\varepsilon$  tends to zero and assuming that  $S_\varepsilon$  strongly converges toward some  $S$  in the dual of  $V(B)$ . It is easy (by proceeding as in [1]) to establish that the sequences with bounded energies are relatively compact in  $L^2(\Omega)$  and in  $H_{\Gamma_0}^1(\Omega_\eta)$  weak for all positive  $\eta$ . The computation of the strong  $L^2(\Omega)$ - $\Gamma$  limit of the first two terms of  $L^2(\Omega)$  was done in [1], but as  $u \mapsto \langle S_\varepsilon, \tau_\varepsilon u \rangle$  is not a continuous perturbation on  $L^2(\Omega)$ , we expect that the limit problem will involve a mixing of the limit behavior of the strain energy of the layer and of the work of the loading acting on it.

Let

$$\begin{aligned} V_0(B) &:= \{u \in V(B); u = 0 \text{ on } \Sigma \times \{-1/2, 1/2\}\}, \\ G(u) &:= \text{Min} \left\{ \int_B g^{\infty,2} \left( \hat{0}, \frac{\partial \theta_{[u]}}{\partial x_N}(x) + [u](\hat{x}) \right) dx - \langle S, \theta \rangle; \theta \in V_0(B) \right\} - \langle S, \tilde{u} \rangle, \\ [u] &= u^+ - u^-, \tilde{u}(x) = [u](\hat{x})x_N + \frac{u^+(\hat{x}) + u^-(\hat{x})}{2}, \end{aligned}$$

we have shown :

*When  $\varepsilon$  tends to zero and  $S_\varepsilon$  to  $S$ , then  $\bar{u}_\varepsilon$  strongly converges in  $L^2(\Omega)$  toward*

the unique solution of

$$(P) \quad \text{Min} \left\{ \int_{\Omega} f(\nabla u) dx + G(u) - L(u); u \in H_{\Gamma_0}^1(\Omega - \Sigma) \right\}.$$

Hence, when  $S$  equals zero,  $G(u)$  reduces to  $\int_{\Sigma} g^{\infty,2}(\hat{0}, [u](\hat{x}))d\hat{x}$  which is nothing but the surface energy obtained in [1]. On the contrary, when  $S$  does not vanish, the functional  $G$  reads as

$$G(u) := \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta_{[u]}}{\partial x_N}(x) + [u](\hat{x})) dx - \langle S, \theta_{[u]} \rangle - \langle S, \tilde{u} \rangle,$$

where  $\theta_{[u]}$  is the minimizer involved by the first definition of  $G(u)$ , thus  $G$  generally is a non local functional, not only of the jump field  $[u]$  but also of the traces fields  $u^+$  and  $u^-$ . The appearance of an internal additional state variable  $\theta$  stems from the weak convergence of  $\tau_{\varepsilon}u_{\varepsilon}$  toward  $\theta + \tilde{u}$  in  $V(B)$  and, consequently, a lower bound of  $\varepsilon \int_{B_{\varepsilon}} g(\nabla u_{\varepsilon}) dx + \langle S_{\varepsilon}, \tau_{\varepsilon}u_{\varepsilon} \rangle$  is  $G(u)$ . To get the upper bound, we build  $u_{\varepsilon}$  in  $B_{\varepsilon}$  from the optimum  $\theta_{[u]}$  involved by  $G(u)$ .

We may give various examples of sources  $S_{\varepsilon}$  of slicing structure  $H^{N-1}|_{\Sigma} \otimes S_{\hat{x}}^{\varepsilon}$ ,  $S_{\hat{x}}^{\varepsilon} = a_{\varepsilon}(\hat{x}, \cdot)dx_N + \sum_{-\infty}^{\infty} b_{\varepsilon,n} \delta_{t_n^{\varepsilon}}(\hat{x})$ , corresponding to distributed or concentrated sources. Last, we may examine the generation by  $\nabla \tilde{u}_{\varepsilon}$  of a "gradient Young measure of concentration" that is analysed in the spirit of [5]. Moreover we express the non local  $G(u)$  in function of this measure and we get bounds for the probability measure  $\bar{\mu}_{\hat{x}}$  stemming from the desintegration of  $\bar{\mu}$ .

### 3 Hard junctions

Here, we consider the modeling of some soldered joints and revisit previous studies ([2], [3]) devoted to the asymptotic behavior of a structure made of two adherents connected by a thin and stiff adhesive layer. In [3], the adherents and the adhesive were modeled as hyperelastic through bulk energy densities with the same growth exponent  $p$  lying in  $(1, +\infty)$ , the stiffness of the adhesive being of the order of the inverse of its thickness. Here, our first attempt to account for some fracture phenomena in soldered joint is to model the adhesive as pseudo-plastic, that is to say, its behavior is described by a bulk energy density with linear growth. Hence, from the mathematical point of view, two difficulties appear: the growth of the bulk energy in the adhesive and the adherents are different and the linear growth in the adhesive will imply to work in spaces of displacement fields with free discontinuities. We use the same geometry as previously:  $\Omega := S \times (-r, r)$ ,  $r > 0$ , where  $S$  is a bounded domain of  $\mathbb{R}^2$ , the adhesive fills  $B_{\varepsilon} := S \times (-\varepsilon/2, \varepsilon/2)$  and the adherents  $\Omega_{\varepsilon} := \Omega - \bar{B}_{\varepsilon}$ . The stiffness of the material occupying the small layer being assumed to be of order  $1/\varepsilon$ , we will use the framework of small perturbations to model the adhesive. Its strain energy density reads as  $1/\varepsilon g(e(u))$ ,  $g$  being a convex function with linear growth. Concerning the adherents, there are no mathematical difficulties to assume more generally that their strain energy density  $f$  is a quasi-convex function (of the gradient of displacement  $\nabla u$ ) with a growth of order  $p \in (1, +\infty)$ . The structure made of the adhesive perfectly stuck to the two adherents is clamped on a part  $\Gamma_0$  of  $\partial\Omega$  and subjected to body and surface forces whose supports are included in  $\bar{\Omega}_{\varepsilon_0}$  and whose work is denoted by  $L(\cdot)$ . Thus the determination of equilibrium conditions leads to the problem

$$\inf \left\{ \int_{\Omega_{\varepsilon}} f(\nabla u) dx + \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon}} g(e(u)) dx - L(u); u \in A_{\varepsilon} \right\}$$

with:

$$\begin{aligned} A_{\varepsilon} &:= \{ u \in LD(\Omega; \mathbb{R}^3); u|_{\Omega_{\varepsilon}} \in W_{\Gamma_0}^{1,p}(\Omega_{\varepsilon}; \mathbb{R}^3) \}, \\ W_{\Gamma_0}^{1,p}(\Omega_{\varepsilon}; \mathbb{R}^3) &:= \{ u \in W^{1,p}(\Omega_{\varepsilon}; \mathbb{R}^3); u = 0 \text{ on } \Gamma_0 \}, \\ LD(\Omega; \mathbb{R}^3) &:= \{ u \in L^1(\Omega; \mathbb{R}^3); e(u) \in L^1(\Omega; M_S^{3 \times 3}) \}. \end{aligned}$$

Because of the linear growth of  $g$ , the problem may have no solutions but at least  $\varepsilon$ -minimizers  $\bar{u}_{\varepsilon}$ . It has at least one solution, still denoted by  $\bar{u}_{\varepsilon}$ , in

$$\bar{A}_{\varepsilon} := \{ u \in BD(\Omega; \mathbb{R}^3); u|_{\Omega_{\varepsilon}} \in W_{\Gamma_0}^{1,p}(\Omega_{\varepsilon}; \mathbb{R}^3) \},$$

where

$$BD(\Omega; \mathbb{R}^3) := \{ u \in L^1(\Omega; \mathbb{R}^3); e(u) \in M_b(\Omega; M_S^{3 \times 3}) \}.$$

As in [3] where the bulk energy of the adhesive has a superlinear growth, the adhesive, when  $\varepsilon$  tends to zero, will be replaced by a material surface whose surface energy density is a function of the surface strain denoted by  $e(\gamma_S(\hat{u}))$ , where  $\gamma_S(\hat{u})$  is the trace on  $S$  of the first two components of any element  $u$  of  $W_{\Gamma_0}^{1,p}(\Omega; \mathbb{R}^3)$  and  $e$  is the symmetrized gradient (in the sense of distributions). This density  $g_0$  stems from  $g$  by:

$$g_0(\zeta) = \text{Min}\{ g(\xi) : \xi \in M_S^{3 \times 3}, \hat{\xi} = \zeta \}, \xi \in M_S^{3 \times 3} \mapsto \hat{\xi} \in M_S^{2 \times 2}, \hat{\xi}_{\alpha\beta} = \xi_{\alpha\beta}.$$

But, whereas the traces of the limits of fields with bounded energies have surface strain tensors in  $L^p(\Omega; M_S^{2 \times 2})$  in the superlinear case, the linear growth will yield surface strain tensors not in  $L^1(\Omega; M_S^{2 \times 2})$  but in  $M_b(\Omega; M_S^{2 \times 2})$ . More precisely, if

$$A_0 := \{ u \in W_{\Gamma_0}^{1,p}(\Omega; \mathbb{R}^3); \gamma_S(\hat{u}) \in BD(S; \mathbb{R}^2) \},$$

$$BD(S; \mathbb{R}^2) = \{ u \in L^1(S; \mathbb{R}^2); e(u) \in M_b(S; M_S^{2 \times 2}) \},$$

the total strain energy functional of the asymptotic model will be :

$$F_0(u) := \begin{cases} \int_{\Omega} f(\nabla u) dx + \int_S g_0(e(\gamma_S(\hat{u}))) & \text{if } u \in A_0 \\ +\infty & \text{if } u \in L^1(\Omega; \mathbb{R}^3) - A_0 \end{cases}$$

the last term being taken in the sense of an integral of convex function of measure, by due account of the following convergence result ([6]):

*when  $\varepsilon$  tends to zero, there exist a not relabelled subsequence and  $\bar{u}$  in  $W_{\Gamma_0}^{1,p}(\Omega; \mathbb{R}^3)$  such that*

$$\bar{u}_\varepsilon \text{ weakly converges to } \bar{u} \text{ in } BD(\Omega; \mathbb{R}^3),$$

$$\bar{u}_\varepsilon \text{ weakly converges to } \bar{u} \text{ in } W_{\Gamma_0}^{1,p}(\Omega_\eta; \mathbb{R}^3) \forall \eta > 0, \gamma_S(\hat{\bar{u}}) \in BD(S; \mathbb{R}^2).$$

*Moreover,  $\bar{u}$  is solution to*

$$(P) \quad \text{Min}\{ F_0(u) - L(u); u \in L^1(\Omega; \mathbb{R}^3) \}$$

*and*

$$F_\varepsilon(\bar{u}_\varepsilon) - L(\bar{u}_\varepsilon) \rightarrow F_0(\bar{u}) - L(\bar{u}).$$

In this model, the traces on  $S$  may have discontinuities which can be interpreted in terms of macrofissures or in terms of diffuse defects or fractal cracks. Actually, due to the Sobolev embeddings, the traces on  $S$   $\gamma_S(u)$  of the displacement fields solutions to (P) being continuous when  $p > 3$ ,  $\gamma_S(\hat{u})$  as an element of  $BD(S; \mathbb{R}^2)$  does not present jumps but only fractal or diffuse singularities. It is worthwhile to note that the genuine model may involve fractures in  $B_\varepsilon$ , whereas the limit model (for  $p > 3$ ) only involve diffuse defects or fractal cracks in the material surface which replaces the adhesive ... Taking into account the geometry of the layer, one easily shows that for all sequence with bounded energies there exists  $u$  in  $W_{\Gamma_0}^{1,p}(\Omega; \mathbb{R}^3)$  and a not relabelled subsequence such that  $u_\varepsilon$  weakly converges in  $BD(\Omega; \mathbb{R}^3)$  and  $W^{1,p}(\Omega_\eta; \mathbb{R}^3)$  for all positive  $\eta$  towards  $u$ , and that  $\gamma_S(\hat{u})$  belongs to  $BD(S; \mathbb{R}^2)$  and is the weak limit, in  $BD(S; \mathbb{R}^2)$  quotiented by the set of rigid displacements of  $\mathbb{R}^2$ , of the  $x_3$ -average of  $\hat{u}_\varepsilon$  in  $B_\varepsilon$ . From that point and the very definition of  $g_0$ , one deduces that  $F_0$  is a possible lower bound with respect to the  $\Gamma$ -convergence of  $F_\varepsilon$  towards  $F_0$  for the strong topology of  $L^1(\Omega; \mathbb{R}^3)$ . To check the upper bound, one first shows that  $F_0$  is the lower semi-continuous regularization for  $L^1(\Omega; \mathbb{R}^3)$  of a functional  $\bar{F}_0$  of same expression as  $F_0$  but living on smooth fields ( $\gamma_S(u) \in C^1(\bar{S}; \mathbb{R}^3)$ ). Next is established that  $\bar{F}_0 \geq \Gamma\text{-lim sup } F_\varepsilon$  by the usual process of lifting into  $B_\varepsilon$  and one concludes by taking the l. s. c. envelope of the two members.

This result is then extended to more realistic situations of welding where the domain occupied by the global structure (adhesive + adherents) does depend on  $\varepsilon$  through suitable translations in the  $x_3$ -direction.

Next, in view to take into account materials which may undergo reversible solid/solid

phase transitions, the hypotheses of quasiconvexity and convexity for  $f$  and  $g$  respectively are dropped. A reasonable candidate for the limit functional is :

$$F_0(u) := \begin{cases} \int_{\Omega} Qf(\nabla u) dx + \int_S SQg_0(e(\gamma_S(\hat{u}))) & \text{if } u \in A_0 \\ +\infty & \text{if } u \in L^1(\Omega; \mathbb{R}^3) - A_0, \end{cases}$$

where  $Qf$  is the quasiconvex envelope of  $f$  and  $SQg_0 : M_S^{2 \times 2} \mapsto \mathbb{R}$  is the symmetric quasiconvexification of  $g_0$  defined by:

$$SQg_0(\zeta) := \inf \left\{ \frac{1}{\widehat{D}} \int_{\widehat{D}} g_0(\zeta + e(\varphi)) d\hat{x}; \varphi \in C_0^\infty(\widehat{D}; \mathbb{R}^2) \right\}.$$

We only succeed in establishing the lower bound on the subset  $\tilde{A}_0$  of  $A$  defined by:

$$\tilde{A}_0 := \{ u \in W_{\Gamma_0}^{1,p}(\Omega; \mathbb{R}^3); \gamma_S(\hat{u}) \in SBD(S; \mathbb{R}^2) \},$$

where  $SBD(S; \mathbb{R}^2)$  denotes the set of the elements  $u$  of  $BD(S; \mathbb{R}^2)$  whose Cantor part of the strain tensor  $e(u)$  vanishes, by using an additional argument of [7]. Concerning the upper bound, as previously we exhibit a functional  $\tilde{F}_0$  such that  $F_0$  is the l. s. c. regularization. The difficulty due to the differences of growth is overcome by introducing a perturbation  $\eta|\cdot|^p$  of  $g_0$ .

The last point, for numerical reasons, examines the possibilities of a regularization à la Norton-Hoff of the functional  $F_0$  involved in the limit problem. If, we recall  $g_0$  by  $h$  assumed to be positively homogeneous of degree 1 and such that

$$\exists \alpha, \beta > 0; \alpha|\xi| \leq h(\xi) \leq \beta|\xi|, \quad \forall \xi \in M_S^{2 \times 2},$$

we consider a sequence  $(h_q)_{q \in (1,p)}$  satisfying:

- i)  $h_q : M_S^{2 \times 2} \mapsto \mathbb{R}^+$  is convex and positively homogeneous of degree  $q$ ,
- ii)  $h_q \rightarrow h$  pointwise in  $M_S^{2 \times 2}$ ,
- iii)  $\exists a > 0; \forall q > 1$ , close enough to 1,  $h_q(\xi) \geq h(\xi), \forall \xi \in M_S^{2 \times 2}, |\xi| \geq a$ .

Then, we show that when  $q \rightarrow 1$ , the functional  $F_q : W_{\Gamma_0}^{1,p}(\Omega; \mathbb{R}^3) \mapsto \mathbb{R}^+ \cup \{+\infty\}$  defined by:

$$F_q(u) := \begin{cases} \int_{\Omega} f(\nabla u) dx + \int_S h_q(e(\gamma_S(\hat{u}))) & \text{if } u \in B_q \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$B_q := \{ u \in W_{\Gamma_0}^{1,p}(\Omega; \mathbb{R}^3); h_q(\gamma_S(\hat{u})) \in L^1(S) \},$$

$\Gamma$ -converges for the weak topology of  $W_{\Gamma_0}^{1,p}(\Omega; \mathbb{R}^3)$  towards

$$F_0(u) := \begin{cases} \int_{\Omega} f(\nabla u) dx + \int_S h(e(\gamma_S(\hat{u}))) & \text{if } u \in B \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$B := \{ u \in W_{\Gamma_0}^{1,p}(\Omega; \mathbb{R}^3); \gamma_S(\hat{u}) \in BD(S; \mathbb{R}^2) \}.$$

Hence, if the sequence  $(h_q)_{q \in (1,p)}$  moreover satisfies the coercivity condition

$$\exists \alpha_q > 0, \alpha_q |\xi|^q \leq h_q(\xi), \quad \forall \xi \in M_S^{2 \times 2},$$

then the problem  $\text{Min}\{F_q(u) - L(u); u \in B_q\}$  has at least one solution  $\bar{u}_q$  and there exists a not relabelled subsequence such that  $\bar{u}_q$  weakly converges in  $W_{\Gamma_0}^{1,p}(\Omega; \mathbb{R}^3)$  toward  $\bar{u}$ , solution to  $\text{Min}\{F(u) - L(u); u \in B\}$ . Because the functions  $h_q$  are convex and positively homogeneous of degree  $q$  and may be chosen differentiable, the numerical methods of convex optimization are able to easily supply approximations of  $\bar{u}_q$  and consequently of  $\bar{u}$  !...



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