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# A mixing effect induced by sources concentrated in a soft junction and the gradient concentration phenomenon

Anne-Laure Bessoud <sup>\*</sup>, Pongpol Juntharee <sup>†‡</sup>, Christian Licht <sup>†</sup>, and Gérard Michaille <sup>‡</sup>

## Abstract

We show that the variational limit of a  $\varepsilon$ -soft and thin junction problem  $(\mathcal{P}_\varepsilon)$  with sources concentrated in the junction gives rise to a surface energy mixing the internal energy and sources. The surface energy functional possesses an integral representation with respect to the Gradient Young-Concentration measures generated by sequences  $(\bar{u}_\varepsilon)_{\varepsilon>0}$  of minimizers of  $(\mathcal{P}_\varepsilon)$ .

AMS *subject classifications*: 35B40, 49Q20, 28A33

*Keywords*:  $\Gamma$ -convergence, Gradient Young measures, concentration measures, minimization problems.

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<sup>\*</sup>IMATH, EA 2134, Université du Sud Toulon-Var, Bâtiment U, BP 20132 - 83957 La Garde Cedex

<sup>†</sup>LMGC, UMR-CNRS 5508, Université Montpellier II, Case courrier 048, Place Eugène Bataillon, 34095 Montpellier Cedex 5

<sup>‡</sup>I3M, UMR-CNRS 5149, Université Montpellier II, Case courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, and MIPA, Université de Nîmes

# 1 Introduction

This paper concerns a soft thin junction subjected to concentrated sources. More precisely, let  $\Omega$  be a domain in  $\mathbb{R}^N$  and let  $B_\varepsilon := \Sigma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \subset \Omega$ ,  $\Sigma \subset \mathbb{R}^{N-1}$ , be the layer occupied by the soft thin junction (cf Figure 1). We consider the minimization problem

$$\min_{u \in W_{\Gamma_0}^{1,2}(\Omega)} \left\{ \int_{\Omega \setminus B_\varepsilon} f(\nabla u) \, dx + \varepsilon \int_{B_\varepsilon} g(\nabla u) \, dx - \langle \mathcal{S}^\varepsilon, u \rangle_\varepsilon \right\} \quad (\mathcal{P}_\varepsilon)$$

where  $W_{\Gamma_0}^{1,2}(\Omega)$  denotes the space of Sobolev functions with null trace on a part  $\Gamma_0$  of the boundary of  $\Omega$ , and the linear form  $\langle \mathcal{S}^\varepsilon, \cdot \rangle_\varepsilon$  represents the work of the source (or the loading). Let  $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})$ . A suitably rescaled  $\mathcal{S}_\varepsilon$  of  $\mathcal{S}^\varepsilon$  is assumed to strongly converge to some  $\mathcal{S}$  in the dual of the space  $V(B) := \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_N} \in L^2(\Omega) \right\}$  when  $\varepsilon$  tends to zero. A general example of such sources which are measures on  $B_\varepsilon$  is given in Section 4 of the paper. Sources of the form  $c \frac{1}{L(\varepsilon)} \mathbb{1}_{B_\varepsilon}$  where  $c$  is any constant and  $L(\varepsilon) \sim \varepsilon$ , is a trivial example of measures satisfying this condition with  $\mathcal{S} = \mathbb{1}_B$ . Note that in this paper the source (or the loading)  $\mathcal{S}_\varepsilon$  is a non  $L^2$ -continuous perturbation of the energy functional  $\int_{\Omega \setminus B_\varepsilon} f(\nabla u) \, dx + \varepsilon \int_{B_\varepsilon} g(\nabla u) \, dx$ .

Among the physical motivations of  $(\mathcal{P}_\varepsilon)$  one may mention various applications to heat conduction or electrostatic problems involving sources concentrated in the layer  $B_\varepsilon$  with conductivity or permittivity of order the size of  $B_\varepsilon$ . One may also think of membrane problems with an exterior loading concentrated in  $B_\varepsilon$  occupied by a material with stiffness of order the small size of  $B_\varepsilon$ . Such a problem with a source concentrated in the junction was considered in [3] in a one dimensional case in order to highlight and illustrate a gradient concentration phenomenon, but the authors were not able to express the variational limit problem.

This paper illustrates the same gradient concentration phenomenon with a complete description of the limit problem in the sense of  $\Gamma$ -convergence (Theorem 3.3). When the size  $\varepsilon$  of the layer goes to zero, fields  $u_\varepsilon$  of bounded energy develop a discontinuity through  $\Sigma$ . More precisely, at the variational limit, the internal energy functional of the junction  $\varepsilon \int_{B_\varepsilon} g(\nabla u) \, dx$  and the work of the loading  $\langle \mathcal{S}^\varepsilon, u \rangle_\varepsilon$  are combined into a functional of the type

$$\bar{H}(u) = \int_{\Sigma} \bar{h}(\hat{x}, u^+ - u^-, \frac{u^+ + u^-}{2}) \, d\hat{x}$$

and the limit problem reads as

$$\min_{u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)} \left\{ \int_{\Omega} f(\nabla u) \, dx + \bar{H}(u) \right\} \quad (\mathcal{P})$$

where  $u^\pm$  denote the traces on  $\Sigma$ . When regarding the various studies devoted to the asymptotic modeling of junction problems (see [2, 9, 7, 10] and references therein) the main novelty is that the density  $\bar{h}$  depends also of the mean  $\frac{u^+ + u^-}{2}$ . Furthermore we show that the sequence of minimizers of  $(\mathcal{P}_\varepsilon)$  (which converges to a minimizer  $\bar{u}$  of the limit problem  $(\mathcal{P})$ ) generates a *gradient Young-concentration measure*  $\bar{\mu}$  in the sense defined in [3]. Then we can give an *integral representation* of the internal part of  $\bar{H}$  with respect to the measure  $\bar{\mu}$  (Theorem 5.5) so that it can be localized in  $\Sigma \times \{\pm 1\}$ . Finally this provides new bounds on the measure  $\bar{\mu}$  (Corollary 5.6).

The paper is organized as follows: in Section 2 we fix notation and give a detailed description of the problem  $(\mathcal{P}_\varepsilon)$ . Section 3 is devoted to the asymptotic analysis of  $(\mathcal{P}_\varepsilon)$  in the sense of the  $\Gamma$ -convergence of the functional energy extended to  $L^2(\Omega)$  equipped with its strong topology. In Section 4 we describe a large class of suitable sources  $\mathcal{S}_\varepsilon$ . Finally Section 5 is concerned with the analysis of the gradient concentration phenomenon generated by sequences of minimizers of  $(\mathcal{P}_\varepsilon)$ . We stress the fact that one could treat the problem in  $L^p(\Omega)$ ,  $1 < p < +\infty$  in the same way without additional difficulties.

## 2 Description of the minimization problem

Let  $\varepsilon > 0$  be a small parameter intended to go to zero, more precisely taking values in a countable subset of  $(0, \varepsilon_0]$  whose 0 is the only cluster point. The reference configuration of the assembly of the two adherents and the adhesive is a cylinder  $\Omega := \Sigma \times (-r, r)$  (with  $r > \varepsilon$ ), where  $\Sigma$  is a bounded domain in  $\mathbb{R}^{N-1}$ ,  $N \geq 2$ , with Lipschitz boundary. For  $x \in \mathbb{R}^N$  we sometimes write  $x = (\hat{x}, x_N)$  where  $\hat{x} \in \mathbb{R}^{N-1}$ . In all the paper,  $C$  denotes a non negative constant which does not depend on  $\varepsilon$  and may vary from line to line. We do not relabel the various considered subsequences and the symbols  $\rightarrow$  and  $\rightharpoonup$  denote various strong convergences and weak convergences respectively. We define the following sets:

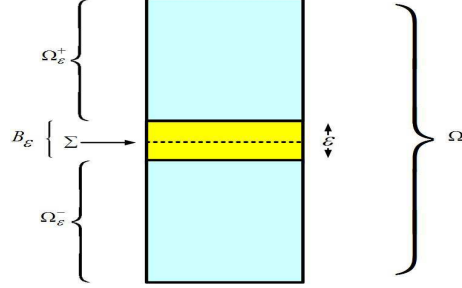


Figure 1: physical domain

- .  $B_\varepsilon := \Sigma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ ;
- .  $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})$ ;
- .  $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon}$ ;
- .  $\Gamma_0$  is a subset of the boundary  $\partial\Omega$  of  $\Omega$  such that  $\text{dist}(\overline{\Gamma_0}, \overline{\partial B_\varepsilon \cap \partial\Omega}) > 0$  for all  $\varepsilon < \varepsilon_0$ ;
- . we write  $\Omega_\varepsilon^-, \Omega_\varepsilon^+, \Omega^-, \Omega^+, B_\varepsilon^+$  and  $B_\varepsilon^-$  for the sets  $\Omega_\varepsilon \cap [x_N < 0]$  and  $\Omega_\varepsilon \cap [x_N > 0]$ ,  $\Omega \cap [x_N < 0]$ ,  $\Omega \cap [x_N > 0]$  and  $B_\varepsilon \cap [x_N > 0]$ ,  $B_\varepsilon \cap [x_N < 0]$  respectively.

We will be concerned with the following spaces:

- .  $W_{\Gamma_0}^{1,2}(\Omega_\varepsilon) := \{u \in W^{1,2}(\Omega_\varepsilon) : u = 0 \text{ on } \Gamma_0\}$ ;
- .  $W_{\Gamma_0}^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_0\}$ ;
- .  $W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma) := \{u \in W^{1,2}(\Omega \setminus \Sigma) : u = 0 \text{ on } \Gamma_0\}$ , and for every  $z \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ ,  $z^\pm$  will stand for the traces of  $z$  on  $\Sigma$  considered as a Sobolev function on  $\Omega^+$  and  $\Omega^-$  respectively.

We say that a function  $h : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$  satisfies a growth condition of order 2 if there exist  $\alpha$  and  $\beta$  in  $\mathbb{R}^+$  such that

$$\alpha |\xi|^2 \leq h(\xi) \leq \beta(1 + |\xi|^2) \text{ for all } \xi \in \mathbb{R}^N.$$

We consider two convex functions  $f, g : \mathbb{R}^N \rightarrow \mathbb{R}$  satisfying a growth condition of order 2, and we assume that there exists a positively 2-homogeneous function  $g^{\infty,2}$  satisfying

$$|g(\xi) - g^{\infty,2}(\xi)| \leq \beta(1 + |\xi|^{2-\delta}) \text{ for all } \xi \in \mathbb{R}^N, \quad (2.1)$$

for some  $\delta$ ,  $0 < \delta < 2$ . Note that  $g^{\infty,2}$  is the positively 2-homogeneous recession function of  $g$ , i.e.,

$$g^{\infty,2}(\xi) = \lim_{t \rightarrow +\infty} \frac{g(t\xi)}{t^2},$$

is convex and satisfies the same growth condition of order 2. We define the space

$$V(B_\varepsilon) := \left\{ u \in L^2(B_\varepsilon) : \frac{\partial u}{\partial x_N} \in L^2(B_\varepsilon) \right\}$$

equipped with the norm

$$\|u\|_{V(B_\varepsilon)} := \left( \int_{B_\varepsilon} |u|^2 dx + \int_{B_\varepsilon} \left| \frac{\partial u}{\partial x_N} \right|^2 dx \right)^{\frac{1}{2}}$$

and we denote the duality bracket between the topological dual space  $V'(B_\varepsilon)$  and  $V(B_\varepsilon)$  by  $\langle \cdot, \cdot \rangle_\varepsilon$ . The considered total energy functional  $F_\varepsilon : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$  is defined by

$$F_\varepsilon(u) = \begin{cases} \int_{\Omega_\varepsilon} f(\nabla u) dx + \varepsilon \int_{B_\varepsilon} g(\nabla u) dx - \langle \mathcal{S}^\varepsilon, u \rangle_\varepsilon & \text{if } u \in W_{\Gamma_0}^{1,2}(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where  $\mathcal{S}^\varepsilon$  is given in  $V'(B_\varepsilon)$ . Our aim is to describe the asymptotic behavior of the minimization problem

$$(\mathcal{P}_\varepsilon) \quad \min \{ F_\varepsilon(u) : u \in L^2(\Omega) \},$$

namely, the limit of  $\min \{ F_\varepsilon(u) : u \in L^2(\Omega) \}$  together with the limit of the minimizer  $\bar{u}_\varepsilon$ , and to identify the limit problem in the framework of  $\Gamma$ -convergence.

Let us consider the space  $V(B) := \left\{ u \in L^2(B) : \frac{\partial u}{\partial x_N} \in L^2(B) \right\}$  equipped with the norm

$$\|u\|_{V(B)} := \left( \int_B |u|^2 dx + \int_B \left| \frac{\partial u}{\partial x_N} \right|^2 dx \right)^{\frac{1}{2}},$$

and denote the duality bracket between  $V'(B)$  and  $V(B)$  by  $\langle \cdot, \cdot \rangle$ . The linear continuous operator

$$\tau_\varepsilon : V(B_\varepsilon) \rightarrow V(B)$$

is defined for every  $x = (\hat{x}, x_N) \in B$  by  $\tau_\varepsilon u(\hat{x}, x_N) := u(\hat{x}, \varepsilon x_N)$  and we denote its transposed operator by  ${}^T \tau_\varepsilon$ :

$$\langle {}^T \tau_\varepsilon \theta, u \rangle_\varepsilon = \langle \theta, \tau_\varepsilon u \rangle, \quad \forall (\theta, u) \in V'(B) \times V(B_\varepsilon).$$

We make the following assumption on the source  $\mathcal{S}^\varepsilon$ : there exists  $\mathcal{S}$  in  $V'(B)$  such that

$$\mathcal{S}_\varepsilon := ({}^T \tau_\varepsilon)^{-1} \mathcal{S}^\varepsilon \text{ strongly converges to } \mathcal{S} \text{ in } V'(B).$$

Then, in order to identify the  $\Gamma$ -limit of the functional  $F_\varepsilon$ , it will be more convenient to write the functional  $F_\varepsilon$  as

$$F_\varepsilon(u) = \begin{cases} \int_{\Omega_\varepsilon} f(\nabla u) dx + \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon u, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon u}{\partial x_N}) dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon u \rangle & \text{if } u \in W_{\Gamma_0}^{1,2}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

### 3 The variational asymptotic model

Let  $H : V(B) \rightarrow \mathbb{R}$  be the functional defined by

$$\begin{aligned} H(\theta) &:= \int_B g^{\infty,2}(\hat{\theta}, \frac{\partial \theta}{\partial x_N}) dx - \langle \mathcal{S}, \theta \rangle \\ &= H_{in}(\theta) - \langle \mathcal{S}, \theta \rangle. \end{aligned} \tag{3.1}$$

We refer the functional  $H_{in}$  as the *internal part* of  $H$ . We claim that, when  $L^2(\Omega)$  is equipped with its strong topology, the functional  $F_\varepsilon$   $\Gamma$ -converges to the functional  $F_0 : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$  given by

$$F_0(u) = \begin{cases} \int_{\Omega} f(\nabla u) \, dx + \inf_{\theta \in X(u)} H(\theta) & \text{if } u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma), \\ +\infty & \text{otherwise,} \end{cases}$$

where  $X(u) := \{\theta \in V(B) : \theta(\cdot, \pm \frac{1}{2}) = u^\pm\}$ .

Before addressing the variational convergence process, we begin by establishing some compactness properties for sequences with bounded energy. Let us introduce the  $\varepsilon$ -translate operator  $T_\varepsilon$  from  $W^{1,2}(\Omega)$  into  $W^{1,2}(\Omega \setminus \Sigma)$ . For any function  $w \in W^{1,2}(\Omega)$ ,  $\tilde{w}$  stands for its extension by reflexion on  $\Sigma \times (-2r, -r) \cup (r, 2r)$  and we define the  $\varepsilon$ -translate  $T_\varepsilon w$  of  $w$  by

$$T_\varepsilon w(\hat{x}, x_N) = \begin{cases} \tilde{w}(\hat{x}, x_N + \frac{\varepsilon}{2}) & \text{if } x \in \Omega^+; \\ \tilde{w}(\hat{x}, x_N - \frac{\varepsilon}{2}) & \text{if } x \in \Omega^-. \end{cases}$$

**Lemma 3.1** (compactness). *Let  $(u_\varepsilon)_{\varepsilon>0}$  be a sequence in  $L^2(\Omega)$  such that  $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$ . Then*

(i)

$$\int_{B_\varepsilon} |u_\varepsilon|^2 \, dx \leq C\varepsilon \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 \, dx \right); \quad (3.2)$$

(ii)

$$\sup_{\varepsilon>0} \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 \, dx \right) < +\infty; \quad (3.3)$$

(iii) *there exist  $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$  and a subsequence of  $(u_\varepsilon)_{\varepsilon>0}$  such that  $u_\varepsilon \rightharpoonup u$  in  $L^2(\Omega)$  and  $u_\varepsilon \rightharpoonup u$  in  $W_{\Gamma_0}^{1,2}(\Omega_\eta)$  for all  $\eta > 0$ ;*

(iv) *there exist  $\theta \in V(B)$  and a subsequence such that  $\tau_\varepsilon u_\varepsilon \rightharpoonup \theta$  in  $V(B)$ , i.e.*

$$\begin{aligned} \tau_\varepsilon u_\varepsilon &\rightharpoonup \theta \text{ in } L^2(B), \\ \frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N} &\rightharpoonup \frac{\partial \theta}{\partial x_N} \text{ in } L^2(B); \end{aligned}$$

*moreover,  $\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon \rightharpoonup 0$  in  $L^2(B, \mathbb{R}^{N-1})$ ;*

(v)  $\theta(\cdot, \pm \frac{1}{2}) = u^\pm$ .

*Proof. Proof of (i).* Without loss of generality, we may assume that the  $N - 1$ -dimensional Hausdorff measure of the intersection of  $\Gamma_0$  with  $[x_N > 0]$  is positive so that (i) is a mere consequence of the following Poincaré-like inequality:

$$\exists C > 0, \int_{B_\varepsilon} |\varphi|^2 \, dx \leq C\varepsilon \left( \int_{\Omega_\varepsilon^+} |\nabla \varphi|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial \varphi}{\partial x_N} \right|^2 \, dx \right) \quad \forall \varphi \in W_{\Gamma_0}^{1,2}(\Omega). \quad (3.4)$$

Indeed, because

$$\varphi(\hat{x}, x_N) = T_\varepsilon \varphi(\hat{x}, 0) + \int_{\frac{\varepsilon}{2}}^{x_N} \frac{\partial}{\partial x_N} \varphi(\hat{x}, t) \, dt \quad \forall x \in B_\varepsilon,$$

for all smooth function  $\varphi \in W_{\Gamma_0}^{1,2}(\Omega)$ , we get

$$|\varphi(\hat{x}, x_N)|^2 \leq 2 \left( |T_\varepsilon \varphi(\hat{x}, 0)|^2 + \varepsilon \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left| \frac{\partial}{\partial x_N} \varphi(\hat{x}, t) \right|^2 \, dt \right).$$

Hence, integrating on  $B_\varepsilon$  and using trace inequality and Poincaré inequality in  $\Omega^+$  give the desired inequality (3.4) for smooth  $\varphi$ , thus for all  $\varphi$  in  $W_{\Gamma_0}^{1,2}(\Omega)$  by a density argument.

*Proof of (ii).* From the coercivity conditions satisfied by  $f$  and  $g$ , estimate (3.2), and the strong convergence of  $\mathcal{S}_\varepsilon$  in  $V'(B)$ , one has

$$\begin{aligned}
\alpha \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right) &\leq C + |\langle \mathcal{S}^\varepsilon, u_\varepsilon \rangle_\varepsilon| \\
&= C + |\langle \mathcal{S}_\varepsilon, \tau_\varepsilon u_\varepsilon \rangle| \\
&\leq C + \|\mathcal{S}_\varepsilon\|_{V'(B)} \|\tau_\varepsilon u_\varepsilon\|_{V(B)} \\
&= C + \|\mathcal{S}_\varepsilon\|_{V'(B)} \left( \frac{1}{\varepsilon} \int_{B_\varepsilon} |u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{\frac{1}{2}} \\
&\leq C + C \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{1/2}.
\end{aligned}$$

Then, setting  $X_\varepsilon := \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{1/2}$ , (3.3) follows from the estimate  $\alpha X_\varepsilon^2 \leq C + CX_\varepsilon$ .

*Proof of (iii).*

*Step 1.* We claim that there exist  $z \in W^{1,2}(\Omega \setminus \Sigma)$  and a subsequence of  $(u_\varepsilon)_{\varepsilon>0}$  such that  $T_\varepsilon u_\varepsilon \rightharpoonup z$  in  $W^{1,2}(\Omega \setminus \Sigma)$  and strongly in  $L^2(\Omega \setminus \Sigma)$ . Clearly,

$$T_\varepsilon u_\varepsilon \in W^{1,2}(\Omega \setminus \Sigma) \text{ and } \frac{\partial}{\partial x_i} T_\varepsilon u_\varepsilon = T_\varepsilon \frac{\partial}{\partial x_i} u_\varepsilon \text{ for all } \varepsilon > 0. \quad (3.5)$$

Combining the Poincaré inequality, (3.3) and (3.5), we deduce

$$\sup_{\varepsilon>0} \|T_\varepsilon u_\varepsilon\|_{W^{1,2}(\Omega \setminus \Sigma)}^2 \leq C \sup_{\varepsilon>0} \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N}(x) \right|^2 dx \right) < +\infty.$$

Therefore,  $(T_\varepsilon u_\varepsilon)_{\varepsilon>0}$  is bounded in  $W^{1,2}(\Omega \setminus \Sigma)$  and the claim follows immediately.

*Step 2.* We establish that there exists  $u$  in  $L^2(\Omega)$  such that we can extract from the previous subsequence  $(u_\varepsilon)_{\varepsilon>0}$  a subsequence strongly converging to  $u$  in  $L^2(\Omega)$ . We can write

$$\int_{\Omega_\varepsilon} |u_\varepsilon(x)|^2 dx = \int_{\Omega^+ \cup \Omega^-} |T_\varepsilon u_\varepsilon(x)|^2 dx - \int_{\Sigma \times ((r-\frac{\varepsilon}{2}, r) \cup (-r, -r+\frac{\varepsilon}{2}))} |T_\varepsilon u_\varepsilon(x)|^2 dx,$$

so that

$$\|u_\varepsilon\|_{L^2(\Omega)}^2 = \int_{\Omega^+ \cup \Omega^-} |T_\varepsilon u_\varepsilon(x)|^2 dx + \int_{B_\varepsilon} |u_\varepsilon(x)|^2 dx - \int_{\Sigma \times ((r-\frac{\varepsilon}{2}, r) \cup (-r, -r+\frac{\varepsilon}{2}))} |T_\varepsilon u_\varepsilon(x)|^2 dx. \quad (3.6)$$

From step 1 and (3.2), we deduce that  $\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^2(\Omega)} < +\infty$ . Thus there exist  $u \in L^2(\Omega)$  and a not relabelled subsequence such that  $u_\varepsilon \rightharpoonup u$  in  $L^2(\Omega)$ . Let us prove that  $u = z$ . Since  $u_\varepsilon \rightharpoonup u$  in  $L^2(\Omega)$  and  $T_\varepsilon u_\varepsilon \rightharpoonup z$  in  $W^{1,2}(\Omega \setminus \Sigma)$ , we have for any  $\varphi \in C_c^\infty(\Omega)$ ,

$$\begin{aligned}
\int_{\Omega} u(x) \varphi(x) dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) \varphi(\hat{x}, x_N - \frac{\varepsilon}{2}) dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} T_\varepsilon u_\varepsilon(x) \varphi(x) dx \\
&= \int_{\Omega} z(x) \varphi(x) dx.
\end{aligned}$$

Thus  $u = z$  almost everywhere in  $\Omega$  and we deduce that  $u \in W^{1,2}(\Omega \setminus \Sigma)$ .

Moreover, from (3.2) we have that  $\int_{B_\varepsilon} |u_\varepsilon(x)|^2 dx \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . On the other hand, since  $T_\varepsilon u_\varepsilon \rightarrow z$

in  $L^2(\Omega)$ , we infer  $\int_{\Omega \cup \cup \Omega^-} |T_\varepsilon u_\varepsilon(x)|^2 dx \rightarrow \int_\Omega |z(x)|^2 dx$  and  $\int_{\Sigma \times ((r-\frac{\varepsilon}{2}, r) \cup (-r, -r+\frac{\varepsilon}{2}))} |T_\varepsilon u_\varepsilon(x)|^2 dx \rightarrow 0$ . Then we deduce that  $\|u_\varepsilon\|_{L^2(\Omega)} \rightarrow \|z\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)}$  and thus that  $(u_\varepsilon)_{\varepsilon>0}$  strongly converges to  $u$  in  $L^2(\Omega)$ .

*Step 3.* It remains to establish that for any  $\eta > 0$ , there exists a subsequence of  $(u_\varepsilon)_{\varepsilon>0}$  such that  $u_\varepsilon|_{\Omega_\eta} \rightharpoonup u|_{\Omega_\eta}$  in  $W_{\Gamma_0}^{1,2}(\Omega_\eta)$ . It will immediately result that  $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ . Let  $\eta > 0$ . Clearly, there exists  $0 < \varepsilon_1 < \eta$  such that  $\Omega_\eta \subseteq \Omega_\varepsilon$  for all  $\varepsilon \leq \varepsilon_1$ . By the Poincaré inequality we have

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{W^{1,2}(\Omega_\eta)}^2 \leq C \sup_{\varepsilon>0} \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N}(x) \right|^2 dx \right) < +\infty.$$

Thus,  $(u_\varepsilon)_{\varepsilon>0}$  is bounded in  $W_{\Gamma_0}^{1,2}(\Omega_\eta)$ , and there exist  $w \in W_{\Gamma_0}^{1,2}(\Omega_\eta)$  and a not relabelled subsequence of  $(u_\varepsilon)_{\varepsilon>0}$  satisfying  $u_\varepsilon \rightharpoonup w$  in  $L^2(\Omega_\eta)$  and  $u_\varepsilon \rightharpoonup w$  in  $W_{\Gamma_0}^{1,2}(\Omega_\eta)$ . It is easily seen that in fact  $w = u|_{\Omega_\eta}$ .

*Proof of (iv).* The weak convergence of  $\tau_\varepsilon u_\varepsilon$  to some  $\theta$  in  $V(B)$  follows from (3.2) and (3.3). Indeed

$$\begin{aligned} \sup_{\varepsilon>0} \|\tau_\varepsilon u_\varepsilon\|_{V(B)} &= \sup_{\varepsilon>0} \left( \frac{1}{\varepsilon} \int_{B_\varepsilon} |u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \sup_{\varepsilon>0} X_\varepsilon < +\infty. \end{aligned}$$

Now we deduce that  $\widehat{\nabla} \tau_\varepsilon u_\varepsilon \rightharpoonup \widehat{\nabla} \theta$  in the distributional sense so that  $\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon \rightarrow 0$  in the distributional sense. On the other hand, from the coercivity of  $g$ ,  $\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon$  weakly converges to some  $L^2(B, \mathbb{R}^{N-1})$  function. Hence,  $\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon \rightarrow 0$  in  $L^2(B, \mathbb{R}^{N-1})$ .

*Proof of (v).* Note that  $\theta(\cdot, \pm \frac{1}{2})$  is well defined. Indeed, one has

$$V(B) \subset W^{1,2}\left(\left(-\frac{1}{2}, \frac{1}{2}\right), L^2(\Sigma)\right) \subset \mathcal{C}\left(\left[-\frac{1}{2}, \frac{1}{2}\right], L^2(\Sigma)\right).$$

Clearly,  $\tau_\varepsilon u_\varepsilon(\hat{x}, \pm \frac{1}{2}) = (T_\varepsilon u_\varepsilon)^\pm(\hat{x})$  (in the sense of traces on  $\Sigma$  of  $W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ -functions) so that  $\tau_\varepsilon u_\varepsilon(\hat{x}, \pm \frac{1}{2}) \rightarrow u^\pm$  in  $L^2(\Sigma)$ . On the other hand, since

$$\tau_\varepsilon u_\varepsilon(\hat{x}, x_N) = \tau_\varepsilon u_\varepsilon(\hat{x}, \pm \frac{1}{2}) + \int_{\pm \frac{1}{2}}^{x_N} \frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N}(\hat{x}, s) ds$$

for a.e.  $x$  in  $B$ , we infer that for all  $\varphi \in \mathcal{C}_c(\Sigma)$ ,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_\Sigma \tau_\varepsilon u_\varepsilon(\hat{x}, x_N) \varphi(\hat{x}) dx = \int_\Sigma (T_\varepsilon u_\varepsilon)^\pm(\hat{x}) \varphi(\hat{x}) d\hat{x} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_\Sigma \int_{\pm \frac{1}{2}}^{x_N} \frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N}(\hat{x}, s) \varphi(\hat{x}) ds dx. \quad (3.7)$$

By passing to the limit in (3.7), we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_\Sigma \theta(\hat{x}, x_N) \varphi(\hat{x}) dx = \int_\Sigma u^\pm(\hat{x}) \varphi(\hat{x}) d\hat{x} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_\Sigma \int_{\pm \frac{1}{2}}^{x_N} \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \varphi(\hat{x}) ds dx$$

from which we deduce

$$\int_\Sigma u^\pm(\hat{x}) \varphi(\hat{x}) d\hat{x} = \int_\Sigma \theta(\hat{x}, \pm \frac{1}{2}) \varphi(\hat{x}) d\hat{x}.$$

Thus  $\theta(\cdot, \pm \frac{1}{2}) = u^\pm$  almost everywhere in  $\Sigma$ . □

**Lemma 3.2.** *For every  $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ ,  $\inf_{\theta \in X(u)} H(\theta) > -\infty$  and there exists  $\theta(u) \in X(u)$  such that  $\inf_{\theta \in X(u)} H(\theta) = H(\theta(u))$ .*



*Proof.* The proof follows from standard arguments used in the direct method of the Calculus of Variation.  $\square$

As a consequence of Lemma 3.2, in its domain  $W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ , the functional  $F_0$  may be written

$$F_0(u) = \int_{\Omega} f(\nabla u) \, dx + H(\theta(u)).$$

Theorem 3.3 is the main result of this section.

**Theorem 3.3.** *The sequence  $(F_\varepsilon)_{\varepsilon>0}$   $\Gamma$ -converges to the functional  $F_0$  when  $L^2(\Omega)$  is equipped with its strong topology.*

The proof results from the following two propositions.

**Proposition 3.4.** *For every  $u \in L^2(\Omega)$  and every  $(u_\varepsilon)_{\varepsilon>0}$  strongly converging to  $u$  in  $L^2(\Omega)$  one has*

$$F_0(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon).$$

**Proposition 3.5.** *For every  $u \in L^2(\Omega)$  there exists  $(v_\varepsilon)_{\varepsilon>0}$  strongly converging to  $u$  in  $L^2(\Omega)$  satisfying*

$$F_0(u) \geq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon).$$

*Proof of Proposition 3.4.* We may assume  $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) < +\infty$ . From Lemma 3.1  $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$  and there exists  $\theta \in X(u)$  such that  $\tau_\varepsilon u_\varepsilon \rightarrow \theta$  in  $V(B)$ . Since  $\mathcal{S}_\varepsilon \rightarrow \mathcal{S}$  in  $V'(B)$ , one has

$$\lim_{\varepsilon \rightarrow 0} \langle \mathcal{S}_\varepsilon, \tau_\varepsilon u_\varepsilon \rangle = \langle \mathcal{S}, \theta \rangle. \quad (3.8)$$

On the other hand, since from Lemma 3.1,  $u_\varepsilon \rightharpoonup u$  in  $W_{\Gamma_0}^{1,2}(\Omega_\eta)$  for all  $\eta > 0$ , one has

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla u_\varepsilon) \, dx \geq \int_{\Omega} f(\nabla u) \, dx. \quad (3.9)$$

Finally from (iv) of Lemma 3.1 and a standard lower semicontinuity argument

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx \\ & \geq \liminf_{\varepsilon \rightarrow 0} \left( \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx - \int_B g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx \right) \\ & \quad + \liminf_{\varepsilon \rightarrow 0} \int_B g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx \\ & \geq \liminf_{\varepsilon \rightarrow 0} \left( \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx - \int_B g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx \right) \\ & \quad + \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}) \, dx \\ & = \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}) \, dx \end{aligned} \quad (3.10)$$

provided that we establish

$$\lim_{\varepsilon \rightarrow 0} \left( \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx - \int_B g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx \right) = 0. \quad (3.11)$$

Since  $g^{\infty,2}$  is positively homogeneous of degree 2, and from (2.1), we have

$$\begin{aligned} & \int_B \left| \varepsilon^2 g(\widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) - g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_\varepsilon^{-1} u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \right| \, dx \\ & = \varepsilon^2 \int_B \left| g(\widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) - g^{\infty,2}(\widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \right| \, dx \\ & \leq C \varepsilon^2 \int_B \left[ 1 + \left| \widehat{\nabla} \tau_\varepsilon u_\varepsilon \right|^{2-\delta} + \left| \frac{1}{\varepsilon} \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N} \right|^{2-\delta} \right] \, dx. \end{aligned}$$

Thus, by using Hölder's inequality (take  $p = \frac{2}{2-\delta}$ ,  $q = \frac{2}{\delta}$ ) we deduce

$$\int_B \left| \varepsilon^2 g(\widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N}) - g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \right| dx \leq C\varepsilon^\delta$$

which proves (3.11). The conclusion of Proposition 3.4 follows by collecting (3.8), (3.9) and (3.10).  $\square$

*Proof of Proposition 3.5.* Let  $u \in L^2(\Omega)$ . We have to construct a sequence  $(v_\varepsilon)_{\varepsilon>0}$  strongly converging to  $u$  in  $L^2(\Omega)$  such that  $\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) \leq F_0(u)$ . If  $F_0(u) = +\infty$ , then  $u \in L^2(\Omega) \setminus W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ , and clearly, for any sequence  $(v_\varepsilon)_{\varepsilon>0}$  converging to  $u$ ,  $\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) \leq F_0(u)$  is true. Now, for the harder part, we assume  $F_0(u) < +\infty$ . Then  $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$  and

$$F_0(u) = \int_\Omega f(\nabla u(x)) dx + \inf_{\theta \in X(u)} H(\theta).$$

To complete the proof, from  $\bar{\theta} := \theta(u)$ , i.e.  $H(\bar{\theta}) = \inf_{\theta \in X(u)} H(\theta)$ , we construct a sequence  $(v_\varepsilon)_{\varepsilon>0}$  strongly converging to  $u$  in  $L^2(\Omega)$  and satisfying

$$F_0(u) \geq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon).$$

The proof is divided into four steps:

*Step 1.* Let us extend  $u$  and  $\bar{\theta}$  by 0 into  $(\mathbb{R}^{N-1} \setminus \Sigma) \times (-r, r)$  and write these extensions  $\tilde{u}$  and  $\tilde{\theta}$ . For a sequence  $\delta$  of positive numbers intended to go to 0, consider a standard sequence of molifier  $(\rho_\delta)_\delta$  and set

$$\begin{aligned} u_\delta &:= \rho_\delta * \tilde{u} \text{ defined by } \rho_\delta * \tilde{u}(\hat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\hat{x} - \hat{y}) \tilde{u}(\hat{y}, x_N) d\hat{y} \text{ for all } (\hat{x}, x_N) \in \Omega; \\ \theta_\delta &:= \rho_\delta * \tilde{\theta} \text{ defined by } \rho_\delta * \tilde{\theta}(\hat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\hat{x} - \hat{y}) \tilde{\theta}(\hat{y}, x_N) d\hat{y} \text{ for all } (\hat{x}, x_N) \in \Omega. \end{aligned}$$

Clearly,

$$\begin{cases} \theta_\delta(\hat{x}, \pm \frac{1}{2}) = u_\delta(\hat{x}, 0) \text{ for all } \hat{x} \in \Sigma, \\ u_\delta \in W^{1,2}(\Omega \setminus \Sigma), \theta_\delta \in W^{1,2}(B), \\ u_\delta \rightarrow u \text{ in } W^{1,2}(\Omega \setminus \Sigma), \theta_\delta \rightarrow \bar{\theta} \text{ in } V(B). \end{cases} \quad (3.12)$$

Next, for each  $\delta > 0$ , we define the sequence  $(v_{\delta,\varepsilon})_{\varepsilon>0}$  as follows:

$$v_{\delta,\varepsilon}(\hat{x}, x_N) = \begin{cases} u_\delta(\hat{x}, x_N \pm \frac{\varepsilon}{2}) & \text{on } \Omega_\varepsilon^\mp \\ \theta_\delta(\hat{x}, \frac{x_N}{\varepsilon}) & \text{on } B_\varepsilon. \end{cases} \quad (3.13)$$

Obviously  $v_{\delta,\varepsilon}(\hat{x}, x_N)$  belongs to  $W^{1,2}(\Omega)$  and strongly converges to  $u_\delta$  in  $L^2(\Omega)$ .

*Step 2.* We we claim that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla v_{\delta,\varepsilon})(x) dx = \int_\Omega f(\nabla u_\delta)(x) dx \quad (3.14)$$

$$\lim_{\varepsilon \rightarrow 0} \left( \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon v_{\delta,\varepsilon}, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon v_{\delta,\varepsilon}}{\partial x_N})(x) dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon v_{\delta,\varepsilon} \rangle \right) = H(\theta_\delta). \quad (3.15)$$

*Proof of (3.14):* one has

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla v_{\delta,\varepsilon})(x) dx &= \lim_{\varepsilon \rightarrow 0} \left( \int_{\Omega_\varepsilon^+} f(\nabla u_\delta)(\hat{x}, x_N - \frac{\varepsilon}{2}) dx + \int_{\Omega_\varepsilon^-} f(\nabla u_\delta)(\hat{x}, x_N + \frac{\varepsilon}{2}) dx \right) \\ &= \int_{\Omega^+} f(\nabla u_\delta)(x) dx + \int_{\Omega^-} f(\nabla u_\delta)(x) dx \\ &= \int_\Omega f(\nabla u_\delta)(x) dx. \end{aligned}$$

*Proof of (3.15):* Since  $g^{\infty,2}$  is positively homogeneous of degree 2 and  $\mathcal{S}_\varepsilon$  strongly converges to  $\mathcal{S}$  in  $V'(B)$ , one has

$$\lim_{\varepsilon \rightarrow 0} \left( \varepsilon^2 \int_B g(\widehat{\nabla} \theta_\delta, \frac{1}{\varepsilon} \frac{\partial \theta_\delta}{\partial x_N})(x) dx - \langle \mathcal{S}_\varepsilon, \theta_\delta \rangle \right) = \int_B g^{\infty,2}(\widehat{0}, \frac{\partial \theta_\delta}{\partial x_N}) dx - \langle \mathcal{S}, \theta_\delta \rangle = H(\theta_\delta).$$

*Step 3.* We establish that  $\lim_{\delta \rightarrow 0} \int_\Omega f(\nabla u_\delta) dx + H(\theta_\delta) = F_0(u)$ . Since

$$\int_\Omega f(\nabla u_\delta) dx + H(\theta_\delta) = \int_\Omega f(\nabla u_\delta) dx + \int_B g^{\infty,2}(\widehat{0}, \frac{\partial \theta_\delta}{\partial x_N}) dx - \langle \mathcal{S}, \theta_\delta \rangle,$$

the result is a straightforward consequence of (3.12).

*Step 4.* By using a standard diagonalization argument, from step 2 and step 3, there exists a mapping  $\varepsilon \mapsto \delta(\varepsilon)$  such that  $v_{\delta(\varepsilon)} \rightarrow u$  in  $L^2(\Omega)$  and

$$\lim_{\varepsilon \rightarrow 0} \left( \int_\Omega f(\nabla v_{\delta(\varepsilon)})(x) dx + \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon v_{\delta(\varepsilon)}, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon v_{\delta(\varepsilon)}}{\partial x_N})(x) dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon v_{\delta(\varepsilon)} \rangle \right) = F_0(u).$$

The sequence  $(v_\varepsilon)_{\varepsilon > 0}$  where  $v_\varepsilon := v_{\delta(\varepsilon)}$  fulfils all the conditions except the boundary condition on  $\Gamma_0$ . From assumption  $\text{dist}(\overline{\Gamma_0}, \overline{\partial B_\varepsilon} \cap \partial \Omega) > 0$ , and by using a standard slicing method due to De Giorgi in a neighborhood of  $\Gamma_0$  (see [4]), one can modify  $v_\varepsilon$  in  $\Omega_\varepsilon$  into a function  $\tilde{v}_\varepsilon$  equal to  $v_\varepsilon$  in  $B_\varepsilon$ , satisfying the boundary condition on  $\Gamma_0$ , and  $\limsup_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla v_\varepsilon) dx = \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla \tilde{v}_\varepsilon) dx$ . Still denoting by  $v_\varepsilon$  this new function, we have  $\lim_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) = F_0(u)$  and the proof is complete.  $\square$

*Remark 3.6.* In order to give an interpretation of the limit energy functional, it is worthwhile to write

$$\inf_{\theta \in X(u)} H(\theta) = \inf_{\theta \in V_0(B)} \left\{ \int_B g^{\infty,2}(\widehat{0}, \frac{\partial \theta}{\partial x_N})(x) + [u](\hat{x}) dx - \langle \mathcal{S}, \theta \rangle \right\} - \langle \mathcal{S}, \tilde{u} \rangle \quad (3.16)$$

where  $[u] = u^+ - u^-$ ,  $V_0(B) = \{\theta \in V(B) : \theta = 0 \text{ on } \Sigma \times \{\pm \frac{1}{2}\}\}$  and  $\tilde{u}(x) = x_N [u](\hat{x}) + \frac{u^+(\hat{x}) + u^-(\hat{x})}{2}$ . Therefore when the limit source  $\mathcal{S}$  vanishes on  $V(B)$ , by using Jensen's inequality,  $\inf_{\theta \in X(u)} H(\theta)$  reduces to

$$\bar{H}(u) = \int_\Sigma g^{\infty,2}(\widehat{0}, [u](\hat{x})) d\hat{x}$$

which is nothing but the surface energy of the model obtained in [9]. When the limit source is not trivial, by using the Euler equation associated with (3.16), it is easily seen that  $\bar{H}$  is a surface energy on  $\Sigma$  of the form

$$\bar{H}(u) = \int_\Sigma \bar{h}(\hat{x}, [u](\hat{x}), \frac{u^+ + u^-}{2}(\hat{x})) d\hat{x}.$$

In this case we note that the energy density depends explicitly of the mean of the traces and that the surface energy  $\bar{H}$  mixes the internal energy and the work of the loading.

## 4 Examples of measure sources $\mathcal{S}_\varepsilon$ concentrated in $B_\varepsilon$

The general form of elements of  $V'(B)$  is given for every  $\theta$  in  $V(B)$  by  $\langle \mathcal{S}, \theta \rangle = \int_B s_0 \theta dx + \int_B s_1 \frac{\partial \theta}{\partial x_N} dx$  where  $(s_0, s_1) \in L^2(B) \times L^2(B)$ . The limit sources  $\mathcal{S}$  considered in this section are generated by measures  $\mathcal{S}^\varepsilon$  in  $\mathbb{M}(B_\varepsilon)$  whose slicing structure  $\mathcal{H}^{N-1} \llcorner \Sigma \otimes \mathcal{S}_\hat{x}^\varepsilon$  is such that their slicing components  $\mathcal{S}_\hat{x}^\varepsilon$  do not present a diffuse singular part in their Lebesgue-Nikodym decomposition in  $\mathbb{M}(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ , i.e., are of the general form

$$\mathcal{S}_\hat{x}^\varepsilon = \frac{1}{\varepsilon} a_\varepsilon(\hat{x}, \varepsilon t) dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\hat{x}) \delta_{\varepsilon t_n}(\hat{x})$$

where

$$\begin{cases} a_\varepsilon \in L^2(B), b_{\varepsilon,n} \in L^2(\Sigma), \\ t_n : \Sigma \longrightarrow \left(-\frac{1}{2}, \frac{1}{2}\right) \text{ is a Borel measurable map.} \end{cases}$$

Roughly, such sources  $\mathcal{S}^\varepsilon$  are sums of a function in  $L^2(B)$  and a countable sum of surface sources, each of them being concentrated in the  $N - 1$ -dimensional surface included in  $B_\varepsilon$  whose graph is  $\varepsilon t_n$ . We make the following additional assumptions:

- (H1) there exists  $a \in L^2(B)$  such that  $a_\varepsilon \rightarrow a$  in  $L^2(B)$ ;
- (H2) there exists  $b_n \in L^2(\Sigma)$  such that  $b_{\varepsilon,n} \rightarrow b_n$  in  $L^2(\Sigma)$  when  $\varepsilon \rightarrow 0$ ;
- (H3) there exists  $c_n \in \mathbb{R}^+$  such that  $\|b_{\varepsilon,n}\|_{L^2(\Sigma)} \leq c_n$  and  $\sum_{n=-\infty}^{+\infty} c_n < +\infty$ ;

It is easy to check that the measure  $\mathcal{S}_\varepsilon = {}^T \tau_\varepsilon \mathcal{S}^\varepsilon$  of  $\mathbb{M}(B)$  is given by:  $\mathcal{S}_\varepsilon = \mathcal{H}^{N-1} \llbracket \Sigma \otimes (\mathcal{S}_\varepsilon)_{\hat{x}} \rrbracket$  where

$$(\mathcal{S}_\varepsilon)_{\hat{x}} = a_\varepsilon(\hat{x}, t) dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\hat{x}) \delta_{t_n(\hat{x})}.$$

**Proposition 4.1.** *The measure  $\mathcal{S}_\varepsilon$  strongly converges in  $V'(B)$  to the measure  $\mathcal{S}$  defined for every  $\theta \in V(B)$  by*

$$\langle \mathcal{S}, \theta \rangle = \int_B a(x) \theta(x) dx + \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x}) \theta(\hat{x}, t_n(\hat{x})) d\hat{x}.$$

Therefore, the functional  $F_\varepsilon$   $\Gamma$ -converges to the functional  $F_0 : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$  given by

$$F_0(u) = \begin{cases} \int_\Omega f(\nabla u) dx + \inf_{\theta \in X(u)} \left\{ \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}) dx - \int_B a \theta dx - \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x}) \theta(\hat{x}, t_n(\hat{x})) d\hat{x} \right\} \\ +\infty \text{ otherwise.} \end{cases} \quad \text{if } u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$$

*Proof.* The second assertion is a straightforward consequence of Theorem 3.3 provided that we establish the strong convergence of  $\mathcal{S}_\varepsilon$  to  $\mathcal{S}$  in  $V'(B)$ . For every  $\theta \in V(B)$  we have

$$\langle \mathcal{S}_\varepsilon - \mathcal{S}, \theta \rangle = \int_B (a_\varepsilon - a) \theta dx + \int_\Sigma \sum_{n=-\infty}^{+\infty} (b_{\varepsilon,n} - b_n) \theta(\hat{x}, t_n(\hat{x})) d\hat{x},$$

thus

$$|\langle \mathcal{S}_\varepsilon - \mathcal{S}, \theta \rangle| \leq \|\theta\|_{L^2(B)} \|a_\varepsilon - a\|_{L^2(B)} + \sum_{n=-\infty}^{+\infty} \left[ \|b_{\varepsilon,n} - b_n\|_{L^2(\Sigma)} \left( \int_\Sigma |\theta(\hat{x}, t_n(\hat{x}))|^2 d\hat{x} \right)^{\frac{1}{2}} \right]. \quad (4.1)$$

But it is easy to establish that there exists a non negative constant  $C$  such that

$$\left( \int_\Sigma |\theta(\hat{x}, t_n(\hat{x}))|^2 d\hat{x} \right)^{\frac{1}{2}} \leq C \|\theta\|_{V(B)}$$

so that (4.1) yields

$$\|\mathcal{S}_\varepsilon - \mathcal{S}\|_{V'(B)} \leq \|a_\varepsilon - a\|_{L^2(B)} + C \sum_{n=-\infty}^{+\infty} \|b_{\varepsilon,n} - b_n\|_{L^2(\Sigma)}.$$

The conclusion follows from assumptions (H1), (H2) and (H3).  $\square$

## 5 The gradient concentration phenomenon

We first recall the notion of gradient Young-concentration measure introduced in [3]. Let us denote the unit sphere  $\{-1, 1\}$  of  $\mathbb{R}$  by  $\mathbb{S}^0$ , and consider  $\Sigma' \subset\subset \Sigma$ ,  $B'_\varepsilon := \Sigma' \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ .

**Definition 5.1.** A pair  $(v, \mu_{\Sigma'}) \in L^2(\Omega) \times \mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^0)$  is a gradient Young-concentration measure (localized on  $\Sigma'$ ) iff there exists a sequence  $(v_\varepsilon)_{\varepsilon>0}$  in  $W_{\Gamma_0}^{1,2}(\Omega)$  satisfying

$$\begin{cases} \sup_{\varepsilon>0} \int_{\Omega \setminus B_\varepsilon} |\nabla v_\varepsilon|^2 dx < +\infty, \\ v_\varepsilon \rightarrow v \text{ in } L^2(\Omega), \\ \mu_\varepsilon := \delta_{\frac{\partial v_\varepsilon}{\partial x_N} / |\frac{\partial v_\varepsilon}{\partial x_N}|} (x) \otimes \varepsilon \mathbb{1}_{B'_\varepsilon} |\frac{\partial v_\varepsilon}{\partial x_N}|^2 dx \xrightarrow{*} \mu_{\Sigma'}. \end{cases}$$

We say that the sequence  $(v_\varepsilon)_{\varepsilon>0}$  generates the gradient Young-concentration measure  $(v, \mu_{\Sigma'})$ . We denote the set of gradient Young-concentration measures localized on  $\Sigma'$  by  $\mathcal{YC}(\Sigma')$ .

Recall that the weak convergence  $\xrightarrow{*}$  above is defined by

$$\int_{\Omega} \int_{\mathbb{S}^0} \theta(x) \varphi(\zeta) d\mu_\varepsilon = \int_{B'_\varepsilon} \varepsilon \theta(x) \tilde{\varphi} \left( \frac{\partial v_\varepsilon}{\partial x_N} \right) dx \rightarrow \int_{\Omega} \int_{\mathbb{S}^0} \theta(x) \varphi(\zeta) d\mu_{\Sigma'}$$

for all  $\theta \in \mathcal{C}(\bar{\Omega})$  and all  $\varphi \in \mathcal{C}(\mathbb{S}^0)$ , where the 2-homogeneous extension  $\tilde{\varphi} : \mathbb{R} \rightarrow \mathbb{R}$  of  $\varphi \in \mathcal{C}(\mathbb{S}^0)$  is defined for all  $\zeta \in \mathbb{R}^m$  by

$$\tilde{\varphi}(\zeta) = \begin{cases} |\zeta|^2 \varphi\left(\frac{\zeta}{|\zeta|}\right), & \text{if } \zeta \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In [3], Theorem 3.1, the gradient Young-concentration measures was characterized as follows.

**Theorem 5.2** (Characterization). A pair  $(v, \mu_{\Sigma'} = \mu_x \otimes \pi)$  belongs to  $\mathcal{YC}(\Sigma')$  if and only if  $v \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ ,  $\pi$  is concentrated on  $\bar{\Sigma}'$  and, for every  $\varphi \in \mathcal{C}(\mathbb{S}^0)$  such that  $\varphi^{**} > -\infty$ ,

$$\begin{aligned} \frac{d\pi}{d\mathcal{H}^{N-1} \llcorner \Sigma'}(x) \int_{\mathbb{S}^0} \varphi(\zeta) d\mu_x &\geq \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1} \text{ a. e. } x \in \Sigma' \\ \int_{\mathbb{S}^0} \varphi(\zeta) d\mu_x &\geq 0 \quad \text{for } \pi_s \text{ a. e. } x \in \bar{\Sigma}' \end{aligned} \tag{5.1}$$

where  $\pi = \frac{d\pi}{d\mathcal{H}^{N-1} \llcorner \Sigma'} \mathcal{H}^{N-1} \llcorner \Sigma' + \pi_s$  is the Radon-Nikodym decomposition of  $\pi$  with respect to the measure  $\mathcal{H}^{N-1} \llcorner \Sigma'$ .

*Remark 5.3.* Although from (3.3),  $\delta_{\frac{\partial v_\varepsilon}{\partial x_N} / |\frac{\partial v_\varepsilon}{\partial x_N}|} (x) \otimes \varepsilon \mathbb{1}_{B'_\varepsilon} |\frac{\partial v_\varepsilon}{\partial x_N}|^2 dx$  possesses weak cluster points in the sense of the weak convergence  $\xrightarrow{*}$  made precise above, for technical reason (proof of the sufficient conditions in Proposition 3.5 in [3]), it was not possible to state such a characterization for these cluster points because of possible concentration effects on the boundary of  $\Sigma$ . This is the reason why we deal with gradient Young-concentration measures localized on  $\Sigma' \subset\subset \Sigma$ .

Taking into account that the 2-homogeneous extension  $\tilde{\varphi} : \mathbb{R} \rightarrow \mathbb{R}$  of  $\varphi \in \mathcal{C}(\mathbb{S}^0)$  satisfying  $\varphi^{**} > -\infty$  is of the form

$$\varphi(\zeta) = \begin{cases} c\zeta^2 & \text{if } \zeta \geq 0 \\ d\zeta^2 & \text{if } \zeta \leq 0, \end{cases}$$

with  $(c, d) \in \mathbb{R}^+ \times \mathbb{R}^+$ , the above characterization theorem can be reduced to the following (cf Corollary 3.6 in [3])

**Corollary 5.4.** A measure  $(v, \mu_{\Sigma'} = (a(x)\delta_1 + b(x)\delta_{-1}) \otimes \pi)$  belongs to  $\mathcal{YC}(\Sigma')$  if and only if  $v \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ ,  $\pi$  is concentrated on  $\bar{\Sigma}'$  and

$$\frac{d\pi}{d\mathcal{H}^{N-1} \llcorner \Sigma'}(x) (a(x)c + b(x)d) \geq \varphi([v](x)) \text{ for } \mathcal{H}^{N-1} \llcorner \Sigma' \text{ a.e. } x \text{ and for all } (c, d) \in \mathbb{R}^+ \times \mathbb{R}^+$$

$$\text{where } \varphi(\zeta) = \begin{cases} c\zeta^2 & \text{if } \zeta \geq 0 \\ d\zeta^2 & \text{if } \zeta \leq 0 \end{cases}.$$

As stated in [3] Remark 2.5, every sequence  $(u_\varepsilon)_{\varepsilon>0}$  satisfying (3.3) generates a gradient Young-concentration measure. Therefore every sequence  $(\bar{u}_\varepsilon)_{\varepsilon>0}$ ,  $\bar{u}_\varepsilon \in \text{argmin } F_\varepsilon$ , generates a measure  $\bar{\mu}_{\Sigma'}$   $\in \mathcal{YC}(\Sigma')$ . Let  $\bar{u}$  be a strong limit of  $(\bar{u}_\varepsilon)_{\varepsilon>0}$  in  $L^2(\Omega)$ , then, under the condition  $g^{\infty,2}(\hat{\xi}, \xi_3) \geq g^{\infty,2}(\hat{0}, \xi_3)$ , the next theorem states that the internal term  $H_{in}(\bar{\theta})$  (cf (3.1)) where  $\bar{\theta}$  is the solution of  $\inf_{\theta \in X(\bar{u})} H(\theta)$ , possesses an integral representation with respect to the Young-concentration measure  $\bar{\mu}_{\Sigma'}$ . In some sense we localize  $H_{in}$  on  $S \times \{\pm 1\}$ . Moreover, by using Theorem 5.2 we will deduce some bounds on  $\bar{\mu}_{\Sigma'}$ .

**Theorem 5.5.** *Let  $\bar{u}_\varepsilon$  be a minimizer of  $\min \{F_\varepsilon(v) : v \in L^2(\Omega)\}$  and, for every  $\Sigma' \subset\subset \Sigma$ ,  $(\bar{u}, \bar{\mu}_{\Sigma'})$  be a gradient Young-concentration measure localized on  $\Sigma'$  generated by the sequence  $(\bar{u}_\varepsilon)_{\varepsilon>0}$ . Then the two following assertions hold:*

i)  $\bar{u}_\varepsilon \rightarrow \bar{u}$  in  $L^2(\Omega)$ ,  $F_\varepsilon(\bar{u}_\varepsilon) \rightarrow F_0(\bar{u}) = \min \{F_0(u) : u \in L^2(\Omega)\}$ ;

ii) Let  $\mathcal{F}$  be a countable family of  $\Sigma' \subset\subset \Sigma$ , then there exists  $\bar{\mu} \in \mathbb{M}(\bar{\Omega} \times \mathbb{S}^0)$ ,  $\bar{\mu} = \bar{\mu}_{\hat{x}} \otimes \bar{\pi}$  with  $\bar{\pi}$  concentrated on  $\bar{\Sigma}$  such that for all  $\Sigma' \in \mathcal{F}$ ,  $\bar{\mu}|_{\bar{\Sigma}'} \times \mathbb{S}^0 = \bar{\mu}_{\Sigma'}$ . Assume furthermore that  $g^{\infty,2}$  satisfies the condition

$$\forall \xi \in \mathbb{R}^3, \quad g^{\infty,2}(\hat{\xi}, \xi_3) \geq g^{\infty,2}(\hat{0}, \xi_3). \quad (5.2)$$

Then, every weak cluster point  $\bar{\theta}$  of the sequence  $(\tau_\varepsilon \bar{u}_\varepsilon)_{\varepsilon>0}$  in  $V(B)$  satisfies  $H(\bar{\theta}) = \inf_{\theta \in X(\bar{u})} H(\theta)$  and

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N})(\hat{x}, s) ds &= \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) d\bar{\mu}_{\hat{x}} \text{ for a.e. } \hat{x} \text{ in } \Sigma'; \\ H_{in}(\bar{\theta}) &= \int_{\Sigma} \left[ \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) d\bar{\mu}_{\hat{x}} \right] d\hat{x}. \end{aligned} \quad (5.3)$$

*Proof.* According to the variational nature of the  $\Gamma$ -convergence, for a subsequence one has

$$\begin{aligned} \bar{u}_\varepsilon &\rightarrow \bar{u} \text{ in } L^2(\Omega) \\ \lim_{\varepsilon \rightarrow 0} F_\varepsilon(\bar{u}_\varepsilon) &= F_0(\bar{u}) = \min \{F_0(v) : v \in L^2(\Omega)\} \\ &= \int_{\Omega} f(\nabla \bar{u}) dx + \inf_{\theta \in X(\bar{u})} H(\theta). \end{aligned} \quad (5.4)$$

Fix  $\Sigma' \subset\subset \Sigma$ . From (3.3), for the subsequence (possibly dependent on  $\Sigma'$ ) associated with the gradient Young-concentration measure  $(\bar{u}, \bar{\mu}_{\Sigma'})$ , there exist a subsequence and a measure  $\bar{\mu} = \bar{\mu}_{\hat{x}} \otimes \bar{\pi}$  in  $\mathbb{M}(\bar{\Omega} \times \mathbb{S}^0)$  with  $\bar{\pi}$  concentrated in  $\bar{\Sigma}$ , such that

$$\delta \frac{\partial \bar{u}_\varepsilon}{\partial x_N} / \left| \frac{\partial \bar{u}_\varepsilon}{\partial x_N} \right| (x) \otimes \varepsilon \mathbf{1}_{B_\varepsilon} \left| \frac{\partial \bar{u}_\varepsilon}{\partial x_N} \right|^2 dx \rightharpoonup \bar{\mu}.$$

Thus, from (3.11) and (5.2) we infer

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon \bar{u}_\varepsilon, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}) dx &= \lim_{\varepsilon \rightarrow 0} \int_B g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_\varepsilon \bar{u}_\varepsilon, \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}) dx \\ &\geq \lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} g^{\infty,2}(\hat{0}, \frac{\partial \bar{u}_\varepsilon}{\partial x_N}) dx \\ &= \int_{\Sigma} \left( \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) d\bar{\mu}_{\hat{x}} \right) d\bar{\pi}. \end{aligned} \quad (5.5)$$

Let  $\bar{\theta}$  be the weak limit of  $\tau_\varepsilon \bar{u}_\varepsilon$  in  $V(B)$  for the considered subsequence. Then, from (5.5) and since

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla \bar{u}_\varepsilon) dx \geq \int_{\Omega} f(\nabla \bar{u}) dx \text{ and } \lim_{\varepsilon \rightarrow 0} \langle \mathcal{S}_\varepsilon, \tau_\varepsilon \bar{u}_\varepsilon \rangle = \langle \mathcal{S}, \bar{\theta} \rangle,$$

we infer

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(\bar{u}_\varepsilon) \geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\Sigma} \left( \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} - \langle \mathcal{S}, \bar{\theta} \rangle. \quad (5.6)$$

Collecting (5.4) and (5.6) we obtain

$$\int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\theta \in X(\bar{u})} H(\theta) \geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\Sigma} \left( \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} - \langle \mathcal{S}, \bar{\theta} \rangle,$$

in particular

$$\int_{\Omega} f(\nabla \bar{u}) \, dx + H(\bar{\theta}) \geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\Sigma} \left( \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} - \langle \mathcal{S}, \bar{\theta} \rangle,$$

thus

$$\begin{aligned} \int_B g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N}) \, dx &\geq \int_{\Sigma} \left( \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} \\ &\geq \int_{\Sigma} \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \left( \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\hat{x}. \end{aligned} \quad (5.7)$$

On the other hand, by a standard lower semicontinuity argument, for every  $\varphi \in \mathcal{C}_c(\Sigma)$ ,  $\varphi \geq 0$ ,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_B \varphi(\hat{x}) g^{\infty,2}(\hat{0}, \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}) \, dx &= \int_{\Sigma} \varphi(\hat{x}) \left( \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} \\ &\geq \int_B \varphi(\hat{x}) g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N}) \, dx \end{aligned}$$

so that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N})(\hat{x}, s) \, ds \leq \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \quad \text{for a.e. } \hat{x} \in \Sigma. \quad (5.8)$$

Combining (5.7) and (5.8) we deduce

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N})(\hat{x}, s) \, ds = \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \quad \text{for a.e. } \hat{x} \in \Sigma.$$

Clearly,  $\bar{\mu}[\bar{\Sigma}' \times \mathbb{S}^0] = \bar{\mu}_{\Sigma'}$ . Now, by using a standard Cantor's diagonal process, the same equality holds for all  $\Sigma'$  of the countable family  $\mathcal{F}$ . It remains to show that  $H(\bar{\theta}) = \inf_{\theta \in X(\bar{u})} H(\theta)$ . It's enough to notice that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} F_\varepsilon(\bar{u}_\varepsilon) &= \int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\theta \in X(\bar{u})} H(\theta) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla \bar{u}_\varepsilon) \, dx + \liminf_{\varepsilon \rightarrow 0} \left( \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon \bar{u}_\varepsilon, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}) \, dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon \bar{u}_\varepsilon \rangle \right) \\ &\geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_B g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N}) \, dx - \langle \mathcal{S}, \bar{\theta} \rangle \\ &= \int_{\Omega} f(\nabla \bar{u}) \, dx + H(\bar{\theta}) \end{aligned}$$

which completes the proof.  $\square$

We define the following two constants associated with the function  $g$ :

$$c(g) := \min \left( \frac{g^{\infty,2}(\hat{0}, -1)}{g^{\infty,2}(\hat{0}, 1)}, \frac{g^{\infty,2}(\hat{0}, 1)}{g^{\infty,2}(\hat{0}, -1)} \right), \quad C(g) = \frac{1}{c(g)} = \max \left( \frac{g^{\infty,2}(\hat{0}, -1)}{g^{\infty,2}(\hat{0}, 1)}, \frac{g^{\infty,2}(\hat{0}, 1)}{g^{\infty,2}(\hat{0}, -1)} \right).$$

Recall that

$$g^{\infty,2}(\widehat{0}, \xi) = \begin{cases} g^{\infty,2}(\widehat{0}, -1) |\xi|^2 & \text{if } \xi \leq 0 \\ g^{\infty,2}(\widehat{0}, 1) |\xi|^2 & \text{if } \xi > 0 \end{cases}.$$

Moreover, from the assumption on the function  $g$ , clearly,

$$g^{\infty,2}(\widehat{0}, 1) > 0 \text{ and } g^{\infty,2}(\widehat{0}, -1) > 0.$$

We make precise the probability measure  $\bar{\mu}_{\hat{x}}$  localized on  $\Sigma' \subset \Sigma$  as follows:

$$\bar{\mu}_{\hat{x}} := p(\hat{x})\delta_1 + q(\hat{x})\delta_{-1} \quad \text{with } p(\hat{x}) + q(\hat{x}) = 1 \text{ a.e. } \hat{x} \in \Sigma'.$$

**Corollary 5.6.** *Under the assumptions of Theorem 5.5, the three following estimates hold:*

(i) for a.e.  $\hat{x}$  in  $\Sigma'$

$$c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds \leq \frac{d\bar{\pi}}{dx_N}(\hat{x}) \leq C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds, \quad (5.9)$$

$$\text{and } \frac{d\bar{\pi}}{dx_N}(\hat{x}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds \quad \text{when } g^{\infty,2}(\widehat{0}, -1) = g^{\infty,2}(\widehat{0}, 1);$$

$$(ii) \quad \frac{c(g) \left| [\bar{u}](\hat{x}) \right|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds} \leq p(\hat{x}) \leq 1 \text{ for a.e. } \hat{x} \text{ such that } [\bar{u}](\hat{x}) > 0;$$

$$(iii) \quad \frac{c(g) \left| [\bar{u}](\hat{x}) \right|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds} \leq q(\hat{x}) \leq 1 \text{ for a.e. } \hat{x} \text{ such that } [\bar{u}](\hat{x}) < 0.$$

*Proof.* Since  $\bar{\mu}_{\hat{x}} = p(\hat{x})\delta_1 + q(\hat{x})\delta_{-1}$ , we have  $\int_{\mathbb{S}^0} g^{\infty,2}(\xi) d\bar{\mu}_{\hat{x}} = p(\hat{x})g^{\infty,2}(\widehat{0}, 1) + q(\hat{x})g^{\infty,2}(\widehat{0}, -1)$  with  $p(\hat{x}) + q(\hat{x}) = 1$  a.e.  $\hat{x}$  in  $\Sigma'$  so that from (5.3), one has

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0}, \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s)) ds &= \left( \int_{\mathbb{S}^0} g^{\infty,2}(\xi) d\bar{\mu}_{\hat{x}} \right) \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \\ &= \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \left\{ p(\hat{x})g^{\infty,2}(\widehat{0}, 1) + q(\hat{x})g^{\infty,2}(\widehat{0}, -1) \right\} \text{ a.e. } \hat{x} \in \Sigma'. \end{aligned} \quad (5.10)$$

We are going to establish

$$c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds \leq \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \leq C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds.$$

From (5.10) we deduce that

$$\begin{aligned} \min \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds &\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0}, \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s)) ds \\ &= \left( \int_{\mathbb{S}^0} g^{\infty,2}(\xi) d\bar{\mu}_{\hat{x}} \right) \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \\ &= \left\{ p(\hat{x})g^{\infty,2}(\widehat{0}, 1) + q(\hat{x})g^{\infty,2}(\widehat{0}, -1) \right\} \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \\ &\leq \max \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\} \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \end{aligned} \quad (5.11)$$



and

$$\begin{aligned}
\min \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\} \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) &= \min \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\} \{p(\widehat{x}) + q(\widehat{x})\} \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \\
&\leq \left\{ p(\widehat{x})g^{\infty,2}(\widehat{0}, 1) + q(\widehat{x})g^{\infty,2}(\widehat{0}, -1) \right\} \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \\
&= \left( \int_{\mathbb{S}^0} g^{\infty,2}(\xi) d\bar{\mu}_x \right) \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0}, \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s)) ds \\
&\leq \max \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds \quad (5.12)
\end{aligned}$$

Then, from (5.11) and (5.12) we have

$$c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds = \frac{\min \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\}}{\max \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds \leq \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x})$$

and

$$\frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \leq \frac{\max \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\}}{\min \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds = C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds$$

from which we deduce,

$$c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds \leq \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \leq C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds.$$

Let us prove (ii) and (iii). According to Theorem 5.2, for every  $\varphi \in \mathcal{C}(\mathbb{S}^0)$  such that  $\varphi^{**} > -\infty$ ,

$$\frac{d\bar{\pi}}{d\mathcal{H}_{[\Sigma']}^{N-1}}(x) \int_{\mathbb{S}^0} \varphi(\zeta) d\bar{\mu}_x \geq \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1} \text{ a.e. } x \in \Sigma', \quad (5.13)$$

where  $\bar{\pi} = \frac{d\bar{\pi}}{d\mathcal{H}_{[\Sigma']}^{N-1}} \mathcal{H}_{[\Sigma']}^{N-1} + \bar{\pi}_s$  is the Radon-Nikodym decomposition of  $\bar{\pi}$  with respect to the measure  $\mathcal{H}_{[\Sigma']}^{N-1}$ . We assume that  $[\bar{u}](\widehat{x}) > 0$  and show that

$$\frac{c(g) |[\bar{u}](\widehat{x})|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds} \leq p(\widehat{x}) \leq 1.$$

Let

$$\varphi(\xi) = \begin{cases} \varphi(1) |\xi|^2 & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases}. \quad (5.14)$$

Clearly,  $\varphi^{**}([\bar{u}](\widehat{x})) = \varphi([\bar{u}](\widehat{x})) = \varphi(1) |[\bar{u}](\widehat{x})|^2$ . From the inequality (5.13), it follows that

$$\begin{aligned}
\varphi(1) |[\bar{u}](\widehat{x})|^2 &= \varphi^{**}([\bar{u}](\widehat{x})) \\
&\leq \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \left( \int_{\mathbb{S}^0} \varphi(\xi) d\bar{\mu}_x \right) \\
&\leq C(g) \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds \right) \left( \int_{\mathbb{S}^0} \varphi(\xi) d\bar{\mu}_x \right) \\
&= C(g) p(\widehat{x}) \varphi(1) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds.
\end{aligned}$$

Then, we obtain

$$\frac{|[\bar{u}](\hat{x})|^2}{C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds} = \frac{c(g) |[\bar{u}](\hat{x})|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds} \leq p(\hat{x}) \leq 1.$$

The proof of (iii) is similar. □

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