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Anne-Laure Bessoud * , Pongpol Juntharee † ‡, Christian Licht †, and Gérard Michaille ‡

Abstract

We show that the variational limit of a ε-soft and thin junction problem $(\mathcal{P}_\varepsilon)$ with sources concentrated in the junction gives rise to a surface energy mixing the internal energy and sources. The surface energy functional possesses an integral representation with respect to the Gradient Young-Concentration measures generated by sequences $(\bar{u}_\varepsilon)_{\varepsilon>0}$ of minimizers of $(\mathcal{P}_\varepsilon)$.

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1 Introduction

This paper concerns a soft thin junction subjected to concentrated sources. More precisely, let Ω be a domain in $\mathbb{R}^N$ and let $B_\varepsilon := \Sigma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \subset \Omega$, $\Sigma \subset \mathbb{R}^{N-1}$, be the layer occupied by the soft thin junction (cf Figure 1). We consider the minimization problem

$$\min_{u \in W^{1,2}_{\Gamma_0}(\Omega)} \Big\{ \int_{\Omega \setminus B_\varepsilon} f(\nabla u)\ dx + \varepsilon \int_{B_\varepsilon} g(\nabla u)\ dx - \langle \mathcal{S}^\varepsilon, u \rangle_\varepsilon \Big\} \qquad (\mathcal{P}_\varepsilon)$$

where $W^{1,2}_{\Gamma_0}(\Omega)$ denotes the space of Sobolev functions with null trace on a part Γ_0 of the boundary of Ω, and the linear form $\langle \mathcal{S}^\varepsilon, . \rangle_\varepsilon$ represents the work of the source (or the loading). Let $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})$. A suitably rescaled $\mathcal{S}_\varepsilon$ of $\mathcal{S}^\varepsilon$ is assumed to strongly converge to some $\mathcal{S}$ in the dual of the space $V(B) := \Big\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_N} \in L^2(\Omega) \Big\}$ when ε tends to zero. A general example of such sources which are measures on B_ε is given in Section 4 of the paper. Sources of the form $c \frac{1}{L(\varepsilon)} \mathbb{1}_{B_\varepsilon}$ where c is any constant and $L(\varepsilon) \sim \varepsilon$, is a trivial example of measures satisfying this condition with $\mathcal{S} = \mathbb{1}_B$. Note that in this paper the source (or the loading) $\mathcal{S}_\varepsilon$ is a non L^2-continuous perturbation of the energy functional $\int_{\Omega \setminus B_\varepsilon} f(\nabla u)\ dx + \varepsilon \int_{B_\varepsilon} g(\nabla u)\ dx$.

Among the physical motivations of $(\mathcal{P}_\varepsilon)$ one may mention various applications to heat conduction or electrostatic problems involving sources concentrated in the layer B_ε with conductivity or permittivity of order the size of B_ε. One may also think of membrane problems with an exterior loading concentrated in B_ε occupied by a material with stiffness of order the small size of B_ε. Such a problem with a source concentrated in the junction was considered in [3] in a one dimensional case in order to highlight and illustrate a gradient concentration phenomenon, but the authors were not able to express the variational limit problem.

This paper illustrates the same gradient concentration phenomenon with a complete description of the limit problem in the sense of Γ-convergence (Theorem 3.3). When the size ε of the layer goes to zero, fields u_ε of bounded energy develop a discontinuity through Σ. More precisely, at the variational limit, the internal energy functional of the junction $\varepsilon \int_{B_\varepsilon} g(\nabla u)\ dx$ and the work of the loading $\langle \mathcal{S}^\varepsilon, u \rangle_\varepsilon$ are combined into a functional of the type

$$\bar{H}(u) = \int_\Sigma \bar{h}(\hat{x}, u^+ - u^-, \frac{u^+ + u^-}{2})\ d\hat{x}$$

and the limit problem reads as

$$\min_{u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)} \Big\{ \int_\Omega f(\nabla u)\, dx + \bar{H}(u) \Big\} \qquad (\mathcal{P})$$

where $u^\pm$ denote the traces on Σ. When regarding the various studies devoted to the asymptotic modeling of junction problems (see [2, 9, 7, 10] and references therein) the main novelty is that the density $\bar{h}$ depends also of the mean $\frac{u^+ + u^-}{2}$. Furthermore we show that the sequence of minimizers of $(\mathcal{P}_\varepsilon)$ (which converges to a minimizer $\bar{u}$ of the limit problem $(\mathcal{P})$) generates a *gradient Young-concentration measure* $\bar{\mu}$ in the sense defined in [3]. Then we can give an *integral representation* of the internal part of $\bar{H}$ with respect to the measure $\bar{\mu}$ (Theorem 5.5) so that it can be localized in $\Sigma \times \{\pm 1\}$. Finally this provides new bounds on the measure $\bar{\mu}$ (Corollary 5.6).

The paper is organized as follows: in Section 2 we fix notation and give a detailed description of the problem $(\mathcal{P}_\varepsilon)$. Section 3 is devoted to the asymptotic analysis of $(\mathcal{P}_\varepsilon)$ in the sense of the Γ-convergence of the functional energy extended to $L^2(\Omega)$ equipped with its strong topology. In Section 4 we describe a large class of suitable sources $\mathcal{S}_\varepsilon$. Finally Section 5 is concerned with the analysis of the gradient concentration phenomenon generated by sequences of minimizers of $(\mathcal{P}_\varepsilon)$. We stress the fact that one could treat the problem in $L^p(\Omega)$, $1 < p < +\infty$ in the same way without additional difficulties.

2 Description of the minimization problem

Let $\varepsilon > 0$ be a small parameter intended to go to zero, more precisely taking values in a countable subset of $(0, \varepsilon_0]$ whose 0 is the only cluster point. The reference configuration of the assembly of the two adherents and the adhesive is a cylinder $\Omega := \Sigma \times (-r, r)$ (with $r > \varepsilon$), where Σ is a bounded domain in $\mathbb{R}^{N-1}$, $N \geq 2$, with Lipschitz boundary. For $x \in \mathbb{R}^N$ we sometimes write $x = (\hat{x}, x_N)$ where $\hat{x} \in \mathbb{R}^{N-1}$. In all the paper, C denotes a non negative constant which does not depend on ε and may vary from line to line. We do not relabel the various considered subsequences and the symbols $\rightarrow$ and $\rightharpoonup$ denote various strong convergences and weak convergences respectively. We define the following sets:

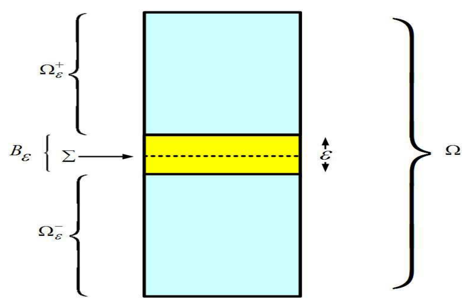

Figure 1: physical domain

- $B_\varepsilon := \Sigma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$;
- $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})$;
- $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon}$;
- Γ_0 is a subset of the boundary $\partial\Omega$ of Ω such that $\mathrm{dist}(\overline{\Gamma}_0, \overline{\partial B_\varepsilon \cap \partial\Omega}) > 0$ for all $\varepsilon < \varepsilon_0$;
- we write Ω_ε^-, Ω_ε^+, Ω^-, Ω^+, B_ε^+ and B_ε^- for the sets $\Omega_\varepsilon \cap [x_N < 0]$ and $\Omega_\varepsilon \cap [x_N > 0]$, $\Omega \cap [x_N < 0]$, $\Omega \cap [x_N > 0]$ and $B_\varepsilon \cap [x_N > 0]$, $B_\varepsilon \cap [x_N < 0]$ respectively.

We will be concerned with the following spaces:

- $W^{1,2}_{\Gamma_0}(\Omega_\varepsilon) := \{u \in W^{1,2}(\Omega_\varepsilon) : u = 0 \text{ on } \Gamma_0\}$;
- $W^{1,2}_{\Gamma_0}(\Omega) := \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_0\}$;
- $W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma) := \{u \in W^{1,2}(\Omega \setminus \Sigma) : u = 0 \text{ on } \Gamma_0\}$, and for every $z \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, $z^\pm$ will stand for the traces of z on Σ considered as a Sobolev function on Ω^+ and Ω^- respectively.

We say that a function $h : \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ satisfies a growth condition of order 2 if there exist α and β in $\mathbb{R}^+$ such that

$$\alpha |\xi|^2 \leq h(\xi) \leq \beta(1 + |\xi|^2) \text{ for all } \xi \in \mathbb{R}^N.$$

We consider two convex functions $f, g : \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfying a growth condition of order 2, and we assume that there exists a positively 2-homogeneous function $g^{\infty,2}$ satisfying

$$\left|g(\xi) - g^{\infty,2}(\xi)\right| \leq \beta(1 + |\xi|^{2-\delta}) \text{ for all } \xi \in \mathbb{R}^N, \tag{2.1}$$

for some δ, $0 < \delta < 2$. Note that $g^{\infty,2}$ is the positively 2-homogeneous recession function of g, i.e.,

$$g^{\infty,2}(\xi) = \lim_{t \to +\infty} \frac{g(t\xi)}{t^2},$$

is convex and satisfies the same growth condition of order 2. We define the space

$$V(B_\varepsilon) := \left\{ u \in L^2(B_\varepsilon) : \frac{\partial u}{\partial x_N} \in L^2(B_\varepsilon) \right\}$$

equipped with the norm

$$\|u\|_{V(B_\varepsilon)} := \Big(\int_{B_\varepsilon} |u|^2 \; dx + \int_{B_\varepsilon} \Big|\frac{\partial u}{\partial x_N}\Big|^2 \; dx \Big)^{\frac{1}{2}}$$

and we denote the duality bracket between the topological dual space $V'(B_\varepsilon)$ and $V(B_\varepsilon)$ by $\langle \; , \; \rangle_\varepsilon$. The considered total energy functional $F_\varepsilon : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f(\nabla u) \; dx + \varepsilon \int_{B_\varepsilon} g(\nabla u) \; dx - \langle \mathcal{S}^\varepsilon, u \rangle_\varepsilon \text{ if } u \in W^{1,2}_{\Gamma_0}(\Omega) \\ +\infty \text{ otherwise,} \end{cases}$$

where $\mathcal{S}^\varepsilon$ is given in $V'(B_\varepsilon)$. Our aim is to describe the asymptotic behavior of the minimization problem

$$(\mathcal{P}_\varepsilon) \qquad \min \left\{ F_\varepsilon(u) : u \in L^2(\Omega) \right\},$$

namely, the limit of $\min \left\{ F_\varepsilon(u) : u \in L^2(\Omega) \right\}$ together with the limit of the minimizer $\bar{u}_\varepsilon$, and to identify the limit problem in the framework of Γ-convergence.

Let us consider the space $V(B) := \left\{ u \in L^2(B) : \frac{\partial u}{\partial x_N} \in L^2(B) \right\}$ equipped with the norm

$$\|u\|_{V(B)} := \Big(\int_B |u|^2 \; dx + \int_B \Big|\frac{\partial u}{\partial x_N}\Big|^2 \; dx \Big)^{\frac{1}{2}},$$

and denote the duality bracket between $V'(B)$ and $V(B)$ by $\langle .,. \rangle$. The linear continuous operator

$$\tau_\varepsilon : V(B_\varepsilon) \longrightarrow V(B)$$

is defined for every $x = (\hat{x}, x_N) \in B$ by $\tau_\varepsilon u(\hat{x}, x_N) := u(\hat{x}, \varepsilon x_N)$ and we denote its transposed operator by ${}^T\tau_\varepsilon$:

$$\langle {}^T\tau_\varepsilon \theta, u \rangle_\varepsilon = \langle \theta, \tau_\varepsilon u \rangle, \; \forall (\theta, u) \in V'(B) \times V(B_\varepsilon).$$

We make the following assumption on the source $\mathcal{S}^\varepsilon$: there exists $\mathcal{S}$ in $V'(B)$ such that

$$S_\varepsilon := ({}^T\tau_\varepsilon)^{-1} \mathcal{S}^\varepsilon \text{ strongly converges to } \mathcal{S} \text{ in } V'(B).$$

Then, in order to identify the Γ-limit of the functional F_ε, it will be more convenient to write the functional F_ε as

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f(\nabla u) \; dx + \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon u, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon u}{\partial x_N}) \; dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon u \rangle \text{ if } u \in W^{1,2}_{\Gamma_0}(\Omega) \\ +\infty \text{ otherwise.} \end{cases}$$

3 The variational asymptotic model

Let $H : V(B) \longrightarrow \mathbb{R}$ be the functional defined by

$$\begin{aligned} H(\theta) \;&:=\; \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}) \; dx - \langle \mathcal{S}, \theta \rangle \\ &=\; H_{in}(\theta) - \langle \mathcal{S}, \theta \rangle. \end{aligned} \tag{3.1}$$

We refer the fiunctional H_{in} as the *internal part* of H. We claim that, when $L^2(\Omega)$ is equipped with its strong topology, the functional F_ε Γ-converges to the functional $F_0 : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$F_0(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx + \inf_{\theta \in X(u)} H(\theta) \text{ if } u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma), \\ +\infty \text{ otherwise}, \end{cases}$$

where $X(u) := \big\{\theta \in V(B) : \theta(., \pm\frac{1}{2}) = u^\pm\big\}$.

Before addressing the variational convergence process, we begin by establishing some compactness properties for sequences with bounded energy. Let us introduce the ε-translate operator T_ε from $W^{1,2}(\Omega)$ into $W^{1,2}(\Omega \setminus \Sigma)$. For any function $w \in W^{1,2}(\Omega)$, $\tilde{w}$ stands for its extension by reflexion on $\Sigma \times (-2r, -r) \cup (r, 2r)$ and we define the ε-translate $T_\varepsilon w$ of w by

$$T_\varepsilon w(\hat{x}, x_N) = \begin{cases} \widetilde{w}(\hat{x}, x_N + \frac{\varepsilon}{2}) & \text{if } x \in \Omega^+; \\ \widetilde{w}(\hat{x}, x_N - \frac{\varepsilon}{2}) & \text{if } x \in \Omega^-. \end{cases}$$

Lemma 3.1 (compactness)**.** *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^2(\Omega)$ such that $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$. Then*

(i)

$$\int_{B_\varepsilon} |u_\varepsilon|^2\, dx \le C\varepsilon\Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\, dx + \varepsilon \int_{B_\varepsilon} |\frac{\partial u_\varepsilon}{\partial x_N}|^2\, dx\Big); \tag{3.2}$$

(ii)

$$\sup_{\varepsilon>0} \Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big) < +\infty; \tag{3.3}$$

(iii) there exist $u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$ and a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $u_\varepsilon \to u$ in $L^2(\Omega)$ and $u_\varepsilon \rightharpoonup u$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$ for all $\eta > 0$;

(iv) there exist $\theta \in V(B)$ and a subsequence such that $\tau_\varepsilon u_\varepsilon \rightharpoonup \theta$ in $V(B)$, i.e.

$$\tau_\varepsilon u_\varepsilon \rightharpoonup \theta \text{ in } L^2(B), \qquad \frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N} \rightharpoonup \frac{\partial \theta}{\partial x_N} \text{ in } L^2(B);$$

moreover, $\varepsilon\widehat{\nabla}\tau_\varepsilon u_\varepsilon \rightharpoonup 0$ in $L^2(B, \mathbb{R}^{N-1})$;

(v) $\theta(., \pm\frac{1}{2}) = u^\pm$.

Proof. Proof of (i). Without loss of generality, we may assume that the $N-1$-dimensional Hausdorff measure of the intersection of Γ_0 with $[x_N > 0]$ is positive so that (i) is a mere consequence of the following Poincaré-like inequality:

$$\exists C > 0,\ \int_{B_\varepsilon} |\varphi|^2\, dx \le C\varepsilon\Big(\int_{\Omega_\varepsilon^+} |\nabla \varphi|^2\, dx + \varepsilon \int_{B_\varepsilon} \left|\frac{\partial \varphi}{\partial x_N}\right|^2\, dx\Big)\ \forall \varphi \in W^{1,2}_{\Gamma_0}(\Omega). \tag{3.4}$$

Indeed, because

$$\varphi\,(\hat{x}, x_N) = T_\varepsilon\varphi(\hat{x}, 0) + \int_{\frac{\varepsilon}{2}}^{x_N} \frac{\partial}{\partial x_N}\varphi\,(\hat{x}, t)\, dt \qquad \forall x \in B_\varepsilon,$$

for all smooth function $\varphi \in W^{1,2}_{\Gamma_0}(\Omega)$, we get

$$|\varphi(\hat{x}, x_N)|^2 \le 2\Big(|T_\varepsilon\varphi(\hat{x}, 0)|^2 + \varepsilon \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left|\frac{\partial}{\partial x_N}\varphi(\hat{x}, t)\right|^2 dt\Big).$$

Hence, integrating on B_ε and using trace inequality and Poincaré inequality in Ω^+ give the desired inequality (3.4) for smooth φ, thus for all φ in $W^{1,2}_{\Gamma_0}(\Omega)$ by a density argument.

Proof of (ii). From the coercivity conditions satisfied by f and g, estimate (3.2), and the strong convergence of $\mathcal{S}_\varepsilon$ in $V'(B)$, one has

$$\begin{aligned}
\alpha\Big(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,dx+\varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big) &\le C+|\langle\mathcal{S}^\varepsilon,u_\varepsilon\rangle_\varepsilon|\\
&= C+|\langle\mathcal{S}_\varepsilon,\tau_\varepsilon u_\varepsilon\rangle|\\
&\le C+\|\mathcal{S}_\varepsilon\|_{V'(B)}\,\|\tau_\varepsilon u_\varepsilon\|_{V(B)}\\
&= C+\|\mathcal{S}_\varepsilon\|_{V'(B)}\Big(\frac{1}{\varepsilon}\int_{B_\varepsilon}|u_\varepsilon|^2\,dx+\varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big)^{\frac12}\\
&\le C+C\Big(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,dx+\varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big)^{1/2}.
\end{aligned}$$

Then, setting $X_\varepsilon:=\Big(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,dx+\varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big)^{1/2}$, (3.3) follows from the estimate $\alpha X_\varepsilon^2\le C+CX_\varepsilon$.

Proof of (iii).

Step 1. We claim that there exist $z\in W^{1,2}(\Omega\backslash\Sigma)$ and a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $T_\varepsilon u_\varepsilon\rightharpoonup z$ in $W^{1,2}(\Omega\backslash\Sigma)$ and strongly in $L^2(\Omega\backslash\Sigma)$. Clearly,

$$T_\varepsilon u_\varepsilon\in W^{1,2}(\Omega\backslash\Sigma)\text{ and }\frac{\partial}{\partial x_i}T_\varepsilon u_\varepsilon=T_\varepsilon\frac{\partial}{\partial x_i}u_\varepsilon\ \text{ for all }\varepsilon>0. \tag{3.5}$$

Combining the Poincaré inequality, (3.3) and (3.5), we deduce

$$\sup_{\varepsilon>0}\|T_\varepsilon u_\varepsilon\|^2_{W^{1,2}(\Omega\backslash\Sigma)}\le C\sup_{\varepsilon>0}\left(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon(x)|^2\,dx+\varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}(x)\right|^2 dx\right)<+\infty.$$

Therefore, $(T_\varepsilon u_\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,2}(\Omega\backslash\Sigma)$ and the claim follows immediately.

Step 2. We establish that there exists u in $L^2(\Omega)$ such that we can extract from the previous subsequence $(u_\varepsilon)_{\varepsilon>0}$ a subsequence strongly converging to u in $L^2(\Omega)$. We can write

$$\int_{\Omega_\varepsilon}|u_\varepsilon(x)|^2\,dx=\int_{\Omega^+\cup\Omega^-}|T_\varepsilon u_\varepsilon(x)|^2\,dx-\int_{\Sigma\times\left((r-\frac{\varepsilon}{2},r)\cup(-r,-r+\frac{\varepsilon}{2})\right)}|T_\varepsilon u_\varepsilon(x)|^2\,dx,$$

so that

$$\|u_\varepsilon\|^2_{L^2(\Omega)}=\int_{\Omega^+\cup\Omega^-}|T_\varepsilon u_\varepsilon(x)|^2\,dx+\int_{B_\varepsilon}|u_\varepsilon(x)|^2\,dx-\int_{\Sigma\times\left((r-\frac{\varepsilon}{2},r)\cup(-r,-r+\frac{\varepsilon}{2})\right)}|T_\varepsilon u_\varepsilon(x)|^2\,dx. \tag{3.6}$$

From step 1 and (3.2), we deduce that $\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^2(\Omega)}<+\infty$. Thus there exist $u\in L^2(\Omega)$ and a not relabelled subsequence such that $u_\varepsilon\rightharpoonup u$ in $L^2(\Omega)$. Let us prove that $u=z$. Since $u_\varepsilon\rightharpoonup u$ in $L^2(\Omega)$ and $T_\varepsilon u_\varepsilon\rightharpoonup z$ in $W^{1,2}(\Omega\backslash\Sigma)$, we have for any $\varphi\in C_c^\infty(\Omega)$,

$$\begin{aligned}
\int_\Omega u(x)\varphi(x)dx &= \lim_{\varepsilon\to0}\int_\Omega u_\varepsilon(x)\varphi(\widehat{x},x_N-\frac{\varepsilon}{2})dx\\
&= \lim_{\varepsilon\to0}\int_\Omega T_\varepsilon u_\varepsilon(x)\varphi(x)dx\\
&= \int_\Omega z(x)\varphi(x)dx.
\end{aligned}$$

Thus $u=z$ almost everywhere in Ω and we deduce that $u\in W^{1,2}(\Omega\backslash\Sigma)$.
Moreover, from (3.2) we have that $\int_{B_\varepsilon}|u_\varepsilon(x)|^2dx\to0$ as $\varepsilon\to0$. On the other hand, since $T_\varepsilon u_\varepsilon\to z$

in $L^2(\Omega)$, we infer $\int_{\Omega^+\cup\Omega^-}|T_\varepsilon u_\varepsilon(x)|^2dx \to \int_\Omega |z(x)|^2dx$ and $\int_{\Sigma\times\left((r-\frac{\varepsilon}{2},r)\cup(-r,-r+\frac{\varepsilon}{2})\right)}|T_\varepsilon u_\varepsilon(x)|^2dx \to 0$. Then we deduce that $\|u_\varepsilon\|_{L^2(\Omega)} \to \|z\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)}$ and thus that $(u_\varepsilon)_{\varepsilon>0}$ strongly converges to u in $L^2(\Omega)$.

Step 3. It remains to establish that for any $\eta > 0$, there exists a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $u_{\varepsilon\lfloor\Omega_\eta} \rightharpoonup u_{\lfloor\Omega_\eta}$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$. It will immediately result that $u \in W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$.
Let $\eta > 0$. Clearly, there exists $0 < \varepsilon_1 < \eta$ such that $\Omega_\eta \subseteq \Omega_\varepsilon$ for all $\varepsilon \le \varepsilon_1$. By the Poincaré inequality we have

$$\sup_{\varepsilon>0}\|u_\varepsilon\|^2_{W^{1,2}(\Omega_\eta)} \le C\sup_{\varepsilon>0}\left(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon(x)|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}(x)\right|^2 dx\right) < +\infty.$$

Thus, $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$, and there exist $w \in W^{1,2}_{\Gamma_0}(\Omega_\eta)$ and a not relabelled subsequence of $(u_\varepsilon)_{\varepsilon>0}$ satisfying $u_\varepsilon \to w$ in $L^2(\Omega_\eta)$ and $u_\varepsilon \rightharpoonup w$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$. It is easily seen that in fact $w = u_{\lfloor\Omega_\eta}$.

Proof of (iv). The weak convergence of $\tau_\varepsilon u_\varepsilon$ to some θ in $V(B)$ follows from (3.2) and (3.3). Indeed

$$\begin{aligned}\sup_{\varepsilon>0}\|\tau_\varepsilon u_\varepsilon\|_{V(B)} &= \sup_{\varepsilon>0}\Big(\frac{1}{\varepsilon}\int_{B_\varepsilon}|u_\varepsilon|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big)^{\frac{1}{2}}\\ &\le C\sup_{\varepsilon>0}X_\varepsilon < +\infty.\end{aligned}$$

Now we deduce that $\widehat{\nabla}\tau_\varepsilon u_\varepsilon \rightharpoonup \widehat{\nabla}\theta$ in the distributional sense so that $\varepsilon\widehat{\nabla}\tau_\varepsilon u_\varepsilon \rightharpoonup 0$ in the distributional sense. On the other hand, from the coercivity of g, $\varepsilon\widehat{\nabla}\tau_\varepsilon u_\varepsilon$ weakly converges to some $L^2(B,\mathbb{R}^{N-1})$ function. Hence, $\varepsilon\widehat{\nabla}\tau_\varepsilon u_\varepsilon \rightharpoonup 0$ in $L^2(B,\mathbb{R}^{N-1})$.

Proof of (v). Note that $\theta(.,\pm\frac{1}{2})$ is well defined. Indeed, one has

$$V(B) \subset W^{1,2}\big((-\frac{1}{2},\frac{1}{2}),L^2(\Sigma)\big) \subset \mathcal{C}\big([-\frac{1}{2},\frac{1}{2}],L^2(\Sigma)\big).$$

Clearly, $\tau_\varepsilon u_\varepsilon(\hat{x},\pm\frac{1}{2}) = (T_\varepsilon u_\varepsilon)^\pm(\hat{x})$ (in the sense of traces on Σ of $W^{1,2}_{\Gamma_0}(\Omega\setminus\Sigma)$-functions) so that $\tau_\varepsilon u_\varepsilon(\hat{x},\pm\frac{1}{2}) \to u^\pm$ in $L^2(\Sigma)$. On the other hand, since

$$\tau_\varepsilon u_\varepsilon(\hat{x},x_N) = \tau_\varepsilon u_\varepsilon(\hat{x},\pm\frac{1}{2}) + \int_{\pm\frac{1}{2}}^{x_N}\frac{\partial\tau_\varepsilon u_\varepsilon}{\partial x_N}(\hat{x},s)\,ds$$

for a.e. x in B, we infer that for all $\varphi \in \mathcal{C}_c(\Sigma)$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\Sigma \tau_\varepsilon u_\varepsilon(\hat{x},x_N)\varphi(\hat{x})\,dx = \int_\Sigma (T_\varepsilon u_\varepsilon)^\pm(\hat{x})\varphi(\hat{x})d\hat{x} + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\Sigma\int_{\pm\frac{1}{2}}^{x_N}\frac{\partial\tau_\varepsilon u_\varepsilon}{\partial x_N}(\hat{x},s)\varphi(\hat{x})dsdx. \quad (3.7)$$

By passing to the limit in (3.7), we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\Sigma\theta(\hat{x},x_N)\varphi(\hat{x})\,dx = \int_\Sigma u^\pm(\hat{x})\varphi(\hat{x})d\hat{x} + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\Sigma\int_{\pm\frac{1}{2}}^{x_N}\frac{\partial\theta}{\partial x_N}(\hat{x},s)\varphi(\hat{x})dsdx$$

from which we deduce

$$\int_\Sigma u^\pm(\hat{x})\varphi(\hat{x})d\hat{x} = \int_\Sigma\theta(\hat{x},\pm\frac{1}{2})\varphi(\hat{x})d\hat{x}.$$

Thus $\theta(.,\pm\frac{1}{2}) = u^\pm$ almost everywhere in Σ. □

Lemma 3.2. *For every $u \in W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$, $\inf_{\theta\in X(u)} H(\theta) > -\infty$ and there exists $\theta(u) \in X(u)$ such that $\inf_{\theta\in X(u)} H(\theta) = H(\theta(u))$.*

Proof. The proof follows from standard arguments used in the direct method of the Calculus of Variation. □

As a consequence of Lemma 3.2, in its domain $W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, the functional F_0 may be written

$$F_0(u) = \int_\Omega f(\nabla u)\, dx + H(\theta(u)).$$

Theorem 3.3 is the main result of this section.

Theorem 3.3. *The sequence $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to the functional F_0 when $L^2(\Omega)$ is equipped with its strong topology.*

The proof results from the following two propositions.

Proposition 3.4. *For every $u \in L^2(\Omega)$ and every $(u_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ one has*

$$F_0(u) \le \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon).$$

Proposition 3.5. *For every $u \in L^2(\Omega)$ there exists $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ satisfying*

$$F_0(u) \ge \limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon).$$

Proof of Proposition 3.4. We may assume $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty$. From Lemma 3.1 $u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$ and there exists $\theta \in X(u)$ such that $\tau_\varepsilon u_\varepsilon \rightharpoonup \theta$ in $V(B)$. Since $\mathcal{S}_\varepsilon \to \mathcal{S}$ in $V'(B)$, one has

$$\lim_{\varepsilon\to 0} \langle \mathcal{S}_\varepsilon, \tau_\varepsilon u_\varepsilon\rangle = \langle \mathcal{S}, \theta\rangle. \tag{3.8}$$

On the other hand, since from Lemma 3.1, $u_\varepsilon \rightharpoonup u$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$ for all $\eta > 0$, one has

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)\, dx \ge \int_\Omega f(\nabla u)\, dx. \tag{3.9}$$

Finally from (iv) of Lemma 3.1 and a standard lower semicontinuity argument

$$\begin{aligned}
& \liminf_{\varepsilon\to 0} \varepsilon^2 \int_B g(\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N})\, dx \\
\ge\ & \liminf_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N})\, dx - \int_B g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N})\, dx\right) \\
& + \liminf_{\varepsilon\to 0} \int_B g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N})\, dx \\
\ge\ & \liminf_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N})\, dx - \int_B g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N})\, dx\right) \\
& + \int_B g^{\infty,2}(\hat{0}, \frac{\partial\theta}{\partial x_N})\, dx \\
=\ & \int_B g^{\infty,2}(\hat{0}, \frac{\partial\theta}{\partial x_N})\, dx
\end{aligned} \tag{3.10}$$

provided that we establish

$$\lim_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N})\, dx - \int_B g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N})\, dx\right) = 0. \tag{3.11}$$

Since $g^{\infty,2}$ is positively homogeneous of degree 2, and from (2.1), we have

$$\begin{aligned}
& \int_B \left|\varepsilon^2 g(\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) - g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_\varepsilon^{-1} u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N})\right| dx \\
=\ & \varepsilon^2 \int_B \left| g(\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) - g^{\infty,2}(\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N})\right| dx \\
\le\ & C\varepsilon^2 \int_B \left[1 + \left|\widehat{\nabla}\tau_\varepsilon u_\varepsilon\right|^{2-\delta} + \left|\frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}\right|^{2-\delta}\right] dx.
\end{aligned}$$

Thus, by using Hölder's inequality (take $p = \frac{2}{2-\delta},\ q = \frac{2}{\delta}$) we deduce

$$\int_B \left| \varepsilon^2 g(\widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N}) - g^{\infty,2}(\varepsilon \widehat{\nabla}\tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \right| dx \le C\varepsilon^\delta$$

which proves (3.11). The conclusion of Proposition 3.4 follows by collecting (3.8), (3.9) and (3.10). $\square$

Proof of Proposition 3.5. Let $u \in L^2(\Omega)$. We have to construct a sequence $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ such that $\limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \le F_0(u)$. If $F_0(u) = +\infty$, then $u \in L^2(\Omega)\backslash W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$, and clearly, for any sequence $(v_\varepsilon)_{\varepsilon>0}$ converging to u, $\limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \le F_0(u)$ is true. Now, for the harder part, we assume $F_0(u) < +\infty$. Then $u \in W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$ and

$$F_0(u) = \int_\Omega f(\nabla u(x))dx + \inf_{\theta\in X(u)} H(\theta).$$

To complete the proof, from $\overline{\theta} := \theta(u)$, i.e. $H(\overline{\theta}) = \inf_{\theta\in X(u)} H(\theta)$, we construct a sequence $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ and satisfying

$$F_0(u) \ge \limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon).$$

The proof is divided into four steps:

Step 1. Let us extend u and $\overline{\theta}$ by 0 into $(\mathbb{R}^{N-1}\setminus\Sigma)\times(-r,r)$ and write these extensions $\widetilde{u}$ and $\widetilde{\overline{\theta}}$. For a sequence δ of positive numbers intended to go to 0, consider a standard sequence of molifier $(\rho_\delta)_\delta$ and set

$$u_\delta := \rho_\delta * \widetilde{u} \text{ defined by } \rho_\delta * \widetilde{u}(\widehat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\widehat{x}-\widehat{y})\widetilde{u}(\widehat{y}, x_N)d\widehat{y} \quad \text{for all } (\widehat{x}, x_N)\in\Omega;$$
$$\theta_\delta := \rho_\delta * \widetilde{\overline{\theta}} \text{ defined by } \rho_\delta * \widetilde{\overline{\theta}}(\widehat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\widehat{x}-\widehat{y})\widetilde{\overline{\theta}}(\widehat{y}, x_N)d\widehat{y} \quad \text{for all } (\widehat{x}, x_N)\in\Omega.$$

Clearly,

$$\begin{cases} \theta_\delta(\widehat{x}, \pm\frac{1}{2}) = u_\delta(\widehat{x}, 0) \text{ for all } \widehat{x}\in\Sigma, \\ \\ u_\delta \in W^{1,2}(\Omega\setminus\Sigma), \theta_\delta \in W^{1,2}(B), \\ \\ u_\delta \to u \text{ in } W^{1,2}(\Omega\setminus\Sigma),\ \theta_\delta \to \bar{\theta} \text{ in } V(B). \end{cases} \tag{3.12}$$

Next, for each $\delta > 0$, we define the sequence $(v_{\delta,\varepsilon})_{\varepsilon>0}$ as follows:

$$v_{\delta,\varepsilon}(\widehat{x}, x_N) = \begin{cases} u_\delta(\widehat{x}, x_N \pm \frac{\varepsilon}{2}) & \text{on} \quad \Omega_\varepsilon^\mp \\ \\ \theta_\delta(\widehat{x}, \frac{x_N}{\varepsilon}) & \text{on} \quad B_\varepsilon. \end{cases} \tag{3.13}$$

Obviously $v_{\delta,\varepsilon}(\widehat{x}, x_N)$ belongs to $W^{1,2}(\Omega)$ and strongly converges to u_δ in $L^2(\Omega)$.

Step 2. We we claim that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla v_{\delta,\varepsilon})(x)dx = \int_\Omega f(\nabla u_\delta)(x)dx \tag{3.14}$$

$$\lim_{\varepsilon\to 0}\left(\varepsilon^2\int_B g(\widehat{\nabla}\tau_\varepsilon v_{\delta,\varepsilon}, \frac{1}{\varepsilon}\frac{\partial\tau_\varepsilon v_{\delta,\varepsilon}}{\partial x_N})(x)dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon v_{\delta,\varepsilon}\rangle\right) = H(\theta_\delta). \tag{3.15}$$

Proof of (3.14): one has

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla v_{\delta,\varepsilon})(x)dx &= \lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon^+} f(\nabla u_\delta)(\widehat{x}, x_N - \frac{\varepsilon}{2})dx + \int_{\Omega_\varepsilon^-} f(\nabla u_\delta(\widehat{x}, x_N + \frac{\varepsilon}{2}))dx\right) \\ &= \int_{\Omega^+} f(\nabla u_\delta)(x)dx + \int_{\Omega^-} f(\nabla u_\delta)(x)dx \\ &= \int_\Omega f(\nabla u_\delta)(x)dx. \end{aligned}$$

Proof of (3.15): Since $g^{\infty,2}$ is positively homogeneous of degree 2 and $\mathcal{S}_\varepsilon$ strongly converges to $\mathcal{S}$ in $V'(B)$, one has

$$\lim_{\varepsilon\to 0}\Big(\varepsilon^2\int_B g(\widehat{\nabla}\theta_\delta,\frac{1}{\varepsilon}\frac{\partial\theta_\delta}{\partial x_N})(x)dx-\langle\mathcal{S}_\varepsilon,\theta_\delta\rangle\Big)=\int_B g^{\infty,2}(\widehat{0},\frac{\partial\theta_\delta}{\partial x_N})dx-\langle\mathcal{S},\theta_\delta\rangle=H(\theta_\delta).$$

Step 3. We establish that $\lim_{\delta\to 0}\int_\Omega f(\nabla u_\delta)dx+H(\theta_\delta)=F_0(u)$. Since

$$\int_\Omega f(\nabla u_\delta)dx+H(\theta_\delta)=\int_\Omega f(\nabla u_\delta)dx+\int_B g^{\infty,2}(\widehat{0},\frac{\partial\theta_\delta}{\partial x_N})dx-\langle\mathcal{S},\theta_\delta\rangle,$$

the result is a straightforward consequence of (3.12).

Step 4. By using a standard diagonalization argument, from step 2 and step 3, there exists a mapping $\varepsilon\mapsto\delta(\varepsilon)$ such that $v_{\delta(\varepsilon)}\longrightarrow u$ in $L^2(\Omega)$ and

$$\lim_{\varepsilon\to 0}\Big(\int_\Omega f(\nabla v_{\delta(\varepsilon)})(x)dx+\varepsilon^2\int_B g(\widehat{\nabla}\tau_\varepsilon v_{\delta(\varepsilon)},\frac{1}{\varepsilon}\frac{\partial\tau_\varepsilon v_{\delta(\varepsilon)}}{\partial x_N})(x)dx-\big\langle\mathcal{S}_\varepsilon,\tau_\varepsilon v_{\delta(\varepsilon)}\big\rangle\Big)=F_0(u).$$

The sequence $(v_\varepsilon)_{\varepsilon>0}$ where $v_\varepsilon:=v_{\delta(\varepsilon)}$ fullfils all the conditions except the boundary condition on Γ_0. From assumption $\mathrm{dist}(\overline{\Gamma}_0,\overline{\partial B_\varepsilon\cap\partial\Omega})>0$, and by using a standard slicing method due to De Giorgi in a neighborhood of Γ_0 (see [4]), one can modify v_ε in Ω_ε into a function $\tilde{v}_\varepsilon$ equal to v_ε in B_ε, satisfying the boundary condition on Γ_0, and $\limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla v_\varepsilon)\ dx=\limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla\tilde{v}_\varepsilon)\ dx$. Still denoting by v_ε this new function, we have $\lim_{\varepsilon\to 0}F_\varepsilon(v_\varepsilon)=F_0(u)$ and the proof is complete. □

Remark 3.6. In order to give an interpretation of the limit energy functional, it is worthwhile to write

$$\inf_{\theta\in X(u)}H(\theta)=\inf_{\theta\in V_0(B)}\Big\{\int_B g^{\infty,2}(\hat{0},\frac{\partial\theta}{\partial x_N}(x)+[u](\hat{x}))\ dx-\langle\mathcal{S},\theta\rangle\Big\}-\langle\mathcal{S},\tilde{u}\rangle \tag{3.16}$$

where $[u]=u^+-u^-$, $V_0(B)=\big\{\theta\in V(B):\theta=0\text{ on }\Sigma\times\{\pm\frac{1}{2}\}\big\}$ and $\tilde{u}(x)=x_N[u](\hat{x})+\frac{u^+(\hat{x})+u^-(\hat{x})}{2}$. Thererfore when the limit source $\mathcal{S}$ vanishes on $V(B)$, by using Jensen's inequality, $\inf_{\theta\in X(u)}H(\theta)$ reduces to

$$\bar{H}(u)=\int_\Sigma g^{\infty,2}\big(\hat{0},[u](\hat{x})\big)\ d\hat{x}$$

which is nothing but the surface energy of the model obtained in [9]. When the limit source is not trivial, by using the Euler equation associated with (3.16), it is easily seen that $\bar{H}$ is a surface energy on Σ of the form

$$\bar{H}(u)=\int_\Sigma \bar{h}(\hat{x},[u](\hat{x}),\frac{u^++u^-}{2}(\hat{x}))\ d\hat{x}.$$

In this case we note that the energy density depends explicitly of the mean of the traces and that the surface energy $\bar{H}$ mixes the internal energy and the work of the loading.

4 Examples of measure sources $\mathcal{S}_\varepsilon$ concentrated in B_ε

The general form of elements of $V'(B)$ is given for every θ in $V(B)$ by $\langle\mathcal{S},\theta\rangle=\int_B s_0\theta\ dx+\int_B s_1\frac{\partial\theta}{\partial x_N}\ dx$ where $(s_0,s_1)\in L^2(B)\times L^2(B)$. The limit sources $\mathcal{S}$ considered in this section are generated by measures $\mathcal{S}^\varepsilon$ in $\mathbb{M}(B_\varepsilon)$ whose slicing structure $\mathcal{H}^{N-1}\lfloor\Sigma\otimes\mathcal{S}^\varepsilon_{\hat{x}}$ is such that their slicing components $\mathcal{S}^\varepsilon_{\hat{x}}$ do not present a diffuse singular part in their Lebesgue-Nikodym decomposition in $\mathbb{M}(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})$, i.e., are of the general form

$$\mathcal{S}^\varepsilon_{\hat{x}}=\frac{1}{\varepsilon}a_\varepsilon(\hat{x},\varepsilon t)\ dt+\sum_{n=-\infty}^{+\infty}b_{\varepsilon,n}(\hat{x})\delta_{\varepsilon t_n(\hat{x})}$$

where

$$\begin{cases} a_\varepsilon \in L^2(B),\ b_{\varepsilon,n} \in L^2(\Sigma), \\ t_n : \Sigma \longrightarrow (-\frac{1}{2}, \frac{1}{2}) \text{ is a Borel measurable map.} \end{cases}$$

Roughly, such sources $\mathcal{S}^\varepsilon$ are sums of a function in $L^2(B)$ and a countable sum of surface sources, each of them being concentrated in the $N-1$-dimensional surface included in B_ε whose graph is εt_n. We make the following additional assumptions:

(H1) there exists $a \in L^2(B)$ such that $a_\varepsilon \to a$ in $L^2(B)$;

(H2) there exists $b_n \in L^2(\Sigma)$ such that $b_{\varepsilon,n} \to b_n$ in $L^2(\Sigma)$ when $\varepsilon \to 0$;

(H3) there exists $c_n \in \mathbb{R}^+$ such that $\|b_{\varepsilon,n}\|_{L^2(\Sigma)} \le c_n$ and $\sum_{n=-\infty}^{+\infty} c_n < +\infty$;

It is easy to check that the measure $\mathcal{S}_\varepsilon =^T \tau_\varepsilon \mathcal{S}^\varepsilon$ of $\mathbb{M}(B)$ is given by: $\mathcal{S}_\varepsilon = \mathcal{H}^{N-1}\lfloor\Sigma \otimes (\mathcal{S}_\varepsilon)_{\hat{x}}$ where

$$(\mathcal{S}_\varepsilon)_{\hat{x}} = a_\varepsilon(\hat{x}, t)\ dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\hat{x})\delta_{t_n(\hat{x})}.$$

Proposition 4.1. *The measure $\mathcal{S}_\varepsilon$ strongly converges in $V'(B)$ to the measure $\mathcal{S}$ defined for every $\theta \in V(B)$ by*

$$\langle \mathcal{S}, \theta \rangle = \int_B a(x)\theta(x)\ dx + \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x})\theta(\hat{x}, t_n(\hat{x}))\ d\hat{x}.$$

Therefore, the functional F_ε Γ-converges to the functional $F_0 : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$F_0(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \inf_{\theta \in X(u)} \left\{ \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N})\ dx - \int_B a\theta\ dx - \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x})\theta(\hat{x}, t_n(\hat{x}))\ d\hat{x} \right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma) \\ +\infty \ \textit{otherwise.} \end{cases}$$

Proof. The second assertion is a straightforward consequence of Theorem 3.3 provided that we establish the strong convergence of $\mathcal{S}_\varepsilon$ to $\mathcal{S}$ in $V'(B)$. For every $\theta \in V(B)$ we have

$$\langle \mathcal{S}_\varepsilon - \mathcal{S}, \theta \rangle = \int_B (a_\varepsilon - a)\theta\ dx + \int_\Sigma \sum_{n=-\infty}^{+\infty} (b_{\varepsilon,n} - b_n)\theta(\hat{x}, t_n(\hat{x}))\ d\hat{x},$$

thus

$$|\langle \mathcal{S}_\varepsilon - \mathcal{S}, \theta \rangle| \le \|\theta\|_{L^2(B)} \|a_\varepsilon - a\|_{L^2(B)} + \sum_{n=-\infty}^{+\infty} \Big[\|b_{\varepsilon,n} - b_n\|_{L^2(\Sigma)} \Big(\int_\Sigma |\theta(\hat{x}, t_n(\hat{x})|^2\ d\hat{x} \Big)^{\frac{1}{2}} \Big]. \tag{4.1}$$

But it is easy to establish that there exists a non negative constant C such that

$$\Big(\int_\Sigma |\theta(\hat{x}, t_n(\hat{x})|^2\ d\hat{x} \Big)^{\frac{1}{2}} \le C \|\theta\|_{V(B)}$$

so that (4.1) yields

$$\|\mathcal{S}_\varepsilon - \mathcal{S}\|_{V'(B)} \le \|a_\varepsilon - a\|_{L^2(B)} + C \sum_{n=-\infty}^{+\infty} \|b_{\varepsilon,n} - b_n\|_{L^2(\Sigma)}.$$

The conclusion follows from assumptions $(H1)$, $(H2)$ and $(H3)$. $\square$

5 The gradient concentration phenomenon

We first recall the notion of gradient Young-concentration measure introduced in [3]. Let us denote the unit sphere $\{-1,1\}$ of $\mathbb{R}$ by $\mathbb{S}^0$, and consider $\Sigma' \subset\subset \Sigma$, $B'_\varepsilon := \Sigma' \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$.

Definition 5.1. *A pair $(v, \mu_{\Sigma'}) \in L^2(\Omega) \times \mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^0)$ is a gradient Young-concentration measure (localized on Σ') iff there exists a sequence $(v_\varepsilon)_{\varepsilon>0}$ in $W^{1,2}_{\Gamma_0}(\Omega)$ satisfying*

$$\begin{cases} \sup\limits_{\varepsilon>0} \displaystyle\int_{\Omega\setminus B_\varepsilon} |\nabla v_\varepsilon|^2 \, dx < +\infty, \\ v_\varepsilon \to v \text{ in } L^2(\Omega), \\ \mu_\varepsilon := \delta_{\frac{\partial v_\varepsilon}{\partial x_N}/\left|\frac{\partial v_\varepsilon}{\partial x_N}\right|(x)} \otimes \varepsilon \mathbb{1}_{B'_\varepsilon} |\frac{\partial v_\varepsilon}{\partial x_N}|^2 \, dx \overset{*}{\rightharpoonup} \mu_{\Sigma'}. \end{cases}$$

We say that the sequence $(v_\varepsilon)_{\varepsilon>0}$ generates the gradient Young-concentration measure $(v, \mu_{\Sigma'})$. We denote the set of gradient Young-concentration measures localized on Σ' by $\mathcal{YC}(\Sigma')$.

Recall that the weak convergence $\overset{*}{\rightharpoonup}$ above is defined by

$$\int_{\bar{\Omega}} \int_{\mathbb{S}^0} \theta(x)\varphi(\zeta) d\mu_\varepsilon = \int_{B'_\varepsilon} \varepsilon\theta(x)\tilde{\varphi}(\frac{\partial v_\varepsilon}{\partial x_N}) \, dx \to \int_{\bar{\Omega}} \int_{\mathbb{S}^0} \theta(x)\varphi(\zeta) \, d\mu_{\Sigma'}$$

for all $\theta \in \mathcal{C}(\bar{\Omega})$ and all $\varphi \in \mathcal{C}(\mathbb{S}^0)$, where the 2-homogeneous extension $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of $\varphi \in \mathcal{C}(\mathbb{S}^0)$ is defined for all $\zeta \in \mathbb{R}^m$ by

$$\tilde{\varphi}(\zeta) = \begin{cases} |\zeta|^2 \varphi(\frac{\zeta}{|\zeta|}), & \text{if } \zeta \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In [3], Theorem 3.1, the gradient Young-concentration measures was characterized as follows.

Theorem 5.2 (Characterization)**.** *A pair $(v, \mu_{\Sigma'} = \mu_x \otimes \pi)$ belongs to $\mathcal{YC}(\Sigma')$ if and only if $v \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, π is concentrated on $\bar{\Sigma}'$ and, for every $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$,*

$$\begin{aligned} &\frac{d\pi}{d\mathcal{H}^{N-1}\lfloor\Sigma'}(x) \int_{\mathbb{S}^0} \varphi(\zeta) \, d\mu_x \geq \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1} \text{ a. e. } x \in \Sigma' \\ &\int_{\mathbb{S}^0} \varphi(\zeta) \, d\mu_x \geq 0 \quad \text{for } \pi_s \text{ a. e. } x \in \bar{\Sigma}' \end{aligned} \tag{5.1}$$

where $\pi = \frac{d\pi}{d\mathcal{H}^{N-1}\lfloor\Sigma'}\mathcal{H}^{N-1}\lfloor\Sigma' + \pi_s$ is the Radon-Nikodym decomposition of π with respect to the measure $\mathcal{H}^{N-1}\lfloor\Sigma'$.

Remark 5.3. Although from (3.3), $\delta_{\frac{\partial v_\varepsilon}{\partial x_N}/\left|\frac{\partial v_\varepsilon}{\partial x_N}\right|(x)} \otimes \varepsilon \mathbb{1}_{B_\varepsilon} |\frac{\partial v_\varepsilon}{\partial x_N}|^2 \, dx$ possesses weak cluster points in the sense of the weak convergence $\overset{*}{\rightharpoonup}$ made precise above, for technical reason (proof of the sufficient conditions in Proposition 3.5 in [3]), it was not possible to state such a characterization for these cluster points because of possible concentration effects on the boundary of Σ. This is the reason why we deal with gradient Young-concentration measures localized on $\Sigma' \subset\subset \Sigma$.

Taking into account that the 2-homogeneous extension $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of $\varphi \in \mathcal{C}(\mathbb{S}^0)$ satisfying $\varphi^{**} > -\infty$ is of the form

$$\varphi(\zeta) = \begin{cases} c\zeta^2 \text{ if } \zeta \geq 0 \\ d\zeta^2 \text{ if } \zeta \leq 0, \end{cases}$$

with $(c,d) \in \mathbb{R}^+ \times \mathbb{R}^+$, the above characterization theorem can be reduced to the following (cf Corollary 3.6 in [3])

Corollary 5.4. *A measure $(v, \mu_{\Sigma'} = (a(x)\delta_1 + b(x)\delta_{-1}) \otimes \pi)$ belongs to $\mathcal{YC}(\Sigma')$ if and only if $v \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, π is concentrated on $\bar{\Sigma}'$ and*

$$\frac{d\pi}{d\mathcal{H}^{N-1}\lfloor\Sigma'}(x)\big(a(x)c + b(x)d\big) \geq \varphi([v](x)) \text{ for } \mathcal{H}^{N-1}\lfloor\Sigma' \text{ a.e. } x \text{ and for all } (c,d) \in \mathbb{R}^+ \times \mathbb{R}^+$$

where $\varphi(\zeta) = \begin{cases} c\zeta^2 \text{ if } \zeta \geq 0 \\ d\zeta^2 \text{ if } \zeta \leq 0 \end{cases}$.

As stated in [3] Remark 2.5, every sequence $(u_\varepsilon)_{\varepsilon>0}$ satisfying (3.3) generates a gradient Young-concentration measure. Therefore every sequence $(\bar{u}_\varepsilon)_{\varepsilon>0}$, $\bar{u}_\varepsilon \in \operatorname{argmin} F_\varepsilon$, generates a measure $\bar{\mu}_{\Sigma'} \in \mathcal{YC}(\Sigma')$. Let $\bar{u}$ be a strong limit of $(\bar{u}_\varepsilon)_{\varepsilon>0}$ in $L^2(\Omega)$, then, under the condition $g^{\infty,2}(\hat{\xi},\xi_3) \geq g^{\infty,2}(\hat{0},\xi_3)$, the next theorem states that the internal term $H_{in}(\bar{\theta})$ (cf (3.1)) where $\bar{\theta}$ is the solution of $\inf_{\theta\in X(\bar{u})} H(\theta)$, possesses an integral representation with respect to the Young-concentration measure $\bar{\mu}_{\Sigma'}$. In some sense we localize H_{in} on $S \times \{\pm 1\}$. Moreover, by using Theorem 5.2 we will deduce some bounds on $\bar{\mu}_{\Sigma'}$.

Theorem 5.5. *Let $\bar{u}_\varepsilon$ be a minimizer of $\min\{F_\varepsilon(v) : v \in L^2(\Omega)\}$ and, for every $\Sigma' \subset\subset \Sigma$, $(\bar{u}, \bar{\mu}_{\Sigma'})$ be a gradient Young-concentration measure localized on Σ' generated by the sequence $(\bar{u}_\varepsilon)_{\varepsilon>0}$. Then the two following assertions hold:*

i) $\bar{u}_\varepsilon \to \bar{u}$ *in* $L^2(\Omega)$, $F_\varepsilon(\bar{u}_\varepsilon) \to F_0(\bar{u}) = \min\{F_0(u) : u \in L^2(\Omega)\}$;

ii) Let $\mathcal{F}$ be a countable familly of $\Sigma' \subset\subset \Sigma$, then there exists $\bar{\mu} \in \mathbb{M}(\bar{\Omega} \times \mathbb{S}^0)$, $\bar{\mu} = \bar{\mu}_{\hat{x}} \otimes \bar{\pi}$ with $\bar{\pi}$ concentrated on $\bar{\Sigma}$ such that for all $\Sigma' \in \mathcal{F}$, $\bar{\mu}\lfloor\bar{\Sigma}' \times \mathbb{S}^0 = \bar{\mu}_{\Sigma'}$. Assume furthermore that $g^{\infty,2}$ satisfies the condition

$$\forall \xi \in \mathbb{R}^3,\ g^{\infty,2}(\hat{\xi},\xi_3) \geq g^{\infty,2}(\hat{0},\xi_3). \tag{5.2}$$

Then, every weak cluster point $\bar{\theta}$ of the sequence $(\tau_\varepsilon \bar{u}_\varepsilon)_{\varepsilon>0}$ in $V(B)$ satisfies $H(\bar{\theta}) = \inf_{\theta\in X(\bar{u})} H(\theta)$ and

$$\begin{aligned} &\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\hat{0}, \frac{\partial\bar{\theta}}{\partial x_N})(\hat{x},s)\ ds = \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi_3)\ d\bar{\mu}_{\hat{x}} \textit{ for a.e. } \hat{x} \textit{ in } \Sigma'; \\ &H_{in}(\bar{\theta}) = \int_\Sigma \Big[\frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi_3)\ d\bar{\mu}_{\hat{x}}\Big]\ d\hat{x}. \end{aligned} \tag{5.3}$$

Proof. According to the variational nature of the Γ-convergence, for a subsequence one has

$$\begin{aligned} &\bar{u}_\varepsilon \to \bar{u} \text{ in } L^2(\Omega) \\ &\lim_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon) = F_0(\bar{u}) = \min\{F_0(v) : v \in L^2(\Omega)\} \\ &\qquad = \int_\Omega f(\nabla\bar{u})\ dx + \inf_{\theta\in X(\bar{u})} H(\theta). \end{aligned} \tag{5.4}$$

Fix $\Sigma' \subset\subset \Sigma$. From (3.3), for the subsequence (possibly dependent on Σ') associated with the gradient Young-concentration measure $(\bar{u}, \bar{\mu}_{\Sigma'})$, there exist a subsequence and a measure $\bar{\mu} = \bar{\mu}_{\hat{x}} \otimes \bar{\pi}$ in $\mathbb{M}(\bar{\Omega} \times \mathbb{S}^0)$ with $\bar{\pi}$ concentrated in $\bar{\Sigma}$, such that

$$\delta_{\frac{\partial\bar{u}_\varepsilon}{\partial x_N}/\left|\frac{\partial\bar{u}_\varepsilon}{\partial x_N}\right|\ (x)} \otimes \varepsilon \mathbb{1}_{B_\varepsilon} \left|\frac{\partial\bar{u}_\varepsilon}{\partial x_N}\right|^2\ dx \rightharpoonup \bar{\mu}.$$

Thus, from (3.11) and (5.2) we infer

$$\begin{aligned} \lim_{\varepsilon\to 0} \varepsilon^2 \int_B g\big(\widehat{\nabla}\tau_\varepsilon\bar{u}_\varepsilon, \frac{1}{\varepsilon}\frac{\partial\tau_\varepsilon\bar{u}_\varepsilon}{\partial x_N}\big)\ dx &= \lim_{\varepsilon\to 0} \int_B g^{\infty,2}\big(\varepsilon\widehat{\nabla}\tau_\varepsilon\bar{u}_\varepsilon, \frac{\partial\tau_\varepsilon\bar{u}_\varepsilon}{\partial x_N}\big)\ dx \\ &\geq \lim_{\varepsilon\to 0} \int_{B_\varepsilon} g^{\infty,2}\big(\hat{0}, \frac{\partial\bar{u}_\varepsilon}{\partial x_N}\big)\ dx \\ &= \int_{\bar{\Sigma}} \Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi_3)\ d\bar{\mu}_{\hat{x}}\Big)\ d\bar{\pi}. \end{aligned} \tag{5.5}$$

Let $\bar{\theta}$ be the weak limit of $\tau_\varepsilon\bar{u}_\varepsilon$ in $V(B)$ for the considered subsequence. Then, from (5.5) and since

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f(\nabla\bar{u}_\varepsilon)\ dx \geq \int_\Omega f(\nabla\bar{u})\ dx \text{ and } \lim_{\varepsilon\to 0} \langle\mathcal{S}_\varepsilon, \tau_\varepsilon\bar{u}_\varepsilon\rangle = \langle\mathcal{S}, \bar{\theta}\rangle,$$

we infer

$$\lim_{\varepsilon\to 0} F_\varepsilon(\bar u_\varepsilon) \geq \int_\Omega f(\nabla \bar u)\ dx + \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi - \langle \mathcal{S},\bar\theta\rangle. \tag{5.6}$$

Collecting (5.4) and (5.6) we obtain

$$\int_\Omega f(\nabla\bar u)\ dx + \inf_{\theta\in X(\bar u)} H(\theta) \geq \int_\Omega f(\nabla \bar u)\ dx + \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi - \langle\mathcal{S},\bar\theta\rangle,$$

in particular

$$\int_\Omega f(\nabla\bar u)\ dx + H(\bar\theta) \geq \int_\Omega f(\nabla \bar u)\ dx + \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi - \langle\mathcal{S},\bar\theta\rangle,$$

thus

$$\begin{aligned}\int_B g^{\infty,2}\big(\hat 0,\frac{\partial\bar\theta}{\partial x_N}\big)\ dx &\geq \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi\\ &\geq \int_{\bar\Sigma}\frac{d\bar\pi}{d\hat x}(\hat x)\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x}\Big)\ d\hat x.\end{aligned} \tag{5.7}$$

On the other hand, by a standard lower semicontinuity argument, for every $\varphi\in\mathcal{C}_c(\Sigma)$, $\varphi\geq 0$,

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_B \varphi(\hat x) g^{\infty,2}\big(\hat 0,\frac{\partial\tau_\varepsilon\bar u_\varepsilon}{\partial x_N}\big)\ dx &= \int_\Sigma\varphi(\hat x)\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi\\ &\geq \int_B \varphi(\hat x) g^{\infty,2}\big(\hat 0,\frac{\partial\bar\theta}{\partial x_N}\big)\ dx\end{aligned}$$

so that

$$\int_{-\frac12}^{\frac12} g^{\infty,2}\big(\hat 0,\frac{\partial\bar\theta}{\partial x_N}\big)(\hat x,s)\ ds \leq \frac{d\bar\pi}{d\hat x}(\hat x)\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x}\quad\text{for a.e. } \hat x\in\Sigma. \tag{5.8}$$

Combining (5.7) and (5.8) we deduce

$$\int_{-\frac12}^{\frac12} g^{\infty,2}\big(\hat 0,\frac{\partial\bar\theta}{\partial x_N}\big)(\hat x,s)\ ds = \frac{d\bar\pi}{d\hat x}(\hat x)\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x}\quad\text{for a.e. } \hat x\in\Sigma.$$

Clearly, $\bar\mu\lfloor\bar\Sigma'\times\mathbb{S}^0 = \bar\mu_{\Sigma'}$. Now, by using a standard Cantor's diagonal process, the same equality holds for all Σ' of the countable familly $\mathcal{F}$. It remains to show that $H(\bar\theta) = \inf_{\theta\in X(\bar u)} H(\theta)$. It's enough to notice that

$$\begin{aligned}\lim_{\varepsilon\to 0} F_\varepsilon(\bar u_\varepsilon) &= \int_\Omega f(\nabla\bar u)\ dx + \inf_{\theta\in X(\bar u)} H(\theta)\\ &\geq \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla\bar u_\varepsilon)\ dx + \liminf_{\varepsilon\to 0}\Big(\varepsilon^2\int_B g\big(\widehat\nabla\tau_\varepsilon\bar u_\varepsilon,\frac1\varepsilon\frac{\partial\tau_\varepsilon\bar u_\varepsilon}{\partial x_N}\big)\ dx - \langle\mathcal{S}_\varepsilon,\tau_\varepsilon\bar u_\varepsilon\rangle\Big)\\ &\geq \int_\Omega f(\nabla\bar u)\ dx + \int_B g^{\infty,2}\big(\hat 0,\frac{\partial\bar\theta}{\partial x_N}\big)\ dx - \langle\mathcal{S},\bar\theta\rangle\\ &= \int_\Omega f(\nabla\bar u)\ dx + H(\bar\theta)\end{aligned}$$

which completes the proof. □

We define the following two constants associated with the function g:

$$c(g) := \min\Big(\frac{g^{\infty,2}(\widehat 0,-1)}{g^{\infty,2}(\widehat 0,1)},\frac{g^{\infty,2}(\widehat 0,1)}{g^{\infty,2}(\widehat 0,-1)}\Big),\ C(g) = \frac{1}{c(g)} = \max\Big(\frac{g^{\infty,2}(\widehat 0,-1)}{g^{\infty,2}(\widehat 0,1)},\frac{g^{\infty,2}(\widehat 0,1)}{g^{\infty,2}(\widehat 0,-1)}\Big).$$

Recall that

$$g^{\infty,2}(\widehat{0},\xi) = \begin{cases} g^{\infty,2}(\widehat{0},-1)\,|\xi|^2 & \text{if} \quad \xi \le 0 \\ g^{\infty,2}(\widehat{0},1)\,|\xi|^2 & \text{if} \quad \xi > 0 \end{cases}.$$

Moreover, from the assumption on the function g, clearly,

$$g^{\infty,2}(\widehat{0},1) > 0 \text{ and } g^{\infty,2}(\widehat{0},-1) > 0.$$

We make precise the probability measure $\bar{\mu}_{\hat{x}}$ localized on $\Sigma' \subset\subset \Sigma$ as follows:

$$\bar{\mu}_{\hat{x}} := p(\hat{x})\delta_1 + q(\hat{x})\delta_{-1} \quad \text{with} \quad p(\hat{x}) + q(\hat{x}) = 1 \text{ a.e. } \hat{x} \in \Sigma'.$$

Corollary 5.6. *Under the assumptions of Theorem 5.5, the three following estimates hold:*

(i) for a.e. $\hat{x}$ in Σ'

$$c(g)\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds \le \frac{d\overline{\pi}}{dx_N}(\hat{x}) \le C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds, \tag{5.9}$$

and $\dfrac{d\overline{\pi}}{dx_N}(\hat{x}) = \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds$ *when* $g^{\infty,2}(\widehat{0},-1) = g^{\infty,2}(\widehat{0},1)$*;*

(ii) $\dfrac{c(g)\ |[\bar{u}](\hat{x})|^2}{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds} \le p(\hat{x}) \le 1$ *for a.e. $\hat{x}$ such that $[\bar{u}](\hat{x}) > 0$;*

(iii) $\dfrac{c(g)\ |[\bar{u}](\hat{x})|^2}{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds} \le q(\hat{x}) \le 1$ *for a.e. $\hat{x}$ such that $[\bar{u}](\hat{x}) < 0$.*

Proof. Since $\bar{\mu}_{\hat{x}} = p(\widehat{x})\delta_1 + q(\widehat{x})\delta_{-1}$, we have $\int_{\mathbb{S}^0} g^{\infty,2}(\xi)d\overline{\mu}_{\hat{x}} = p(\widehat{x})g^{\infty,2}(\widehat{0},1) + q(\widehat{x})g^{\infty,2}(\widehat{0},-1)$ with $p(\widehat{x}) + q(\widehat{x}) = 1$ a.e. $\widehat{x}$ in Σ' so that from (5.3), one has

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0},\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s))ds &= \left(\int_{\mathbb{S}^0} g^{\infty,2}(\xi)d\overline{\mu}_{\hat{x}}\right)\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ &= \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\left\{p(\widehat{x})g^{\infty,2}(\widehat{0},1) + q(\widehat{x})g^{\infty,2}(\widehat{0},-1)\right\} \text{ a.e. } \widehat{x} \in \Sigma'. \end{aligned} \tag{5.10}$$

We are going to establish

$$c(g)\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \le \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \le C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds.$$

From (5.10) we deduce that

$$\begin{aligned}\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds &\le \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0},\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s))ds \\ &= \left(\int_{\mathbb{S}^0} g^{\infty,2}(\xi)d\overline{\mu}_x\right)\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ &= \left\{p(\widehat{x})g^{\infty,2}(\widehat{0},1) + q(\widehat{x})g^{\infty,2}(\widehat{0},-1)\right\}\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ &\le \max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \end{aligned} \tag{5.11}$$

and

$$\begin{aligned}
\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\frac{d\pi}{d\widehat{x}}(\widehat{x}) &= \min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\{p(\widehat{x})+q(\widehat{x})\}\frac{d\pi}{d\widehat{x}}(\widehat{x}) \\
&\leq \left\{p(\widehat{x})g^{\infty,2}(\widehat{0},1)+q(\widehat{x})g^{\infty,2}(\widehat{0},-1)\right\}\frac{d\pi}{d\widehat{x}}(\widehat{x}) \\
&= \left(\int_{\mathbb{S}^0} g^{\infty,2}(\xi)d\overline{\mu}_x\right)\frac{d\pi}{d\widehat{x}}(\widehat{x}) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0},\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s))ds \\
&\leq \max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds
\end{aligned} \tag{5.12}$$

Then, from (5.11) and (5.12) we have

$$c(g)\int_{\frac{-1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds = \frac{\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}{\max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \leq \frac{d\pi}{d\widehat{x}}(\widehat{x})$$

and

$$\frac{d\pi}{d\widehat{x}}(\widehat{x}) \leq \frac{\max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}{\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds = C(g)\int_{\frac{-1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds$$

from which we deduce,

$$c(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \leq \frac{d\pi}{d\widehat{x}}(\widehat{x}) \leq C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds.$$

Let us prove (ii) and (iii). According to Theorem 5.2, for every $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$,

$$\frac{d\overline{\pi}}{d\mathcal{H}^{N-1}_{\lfloor\Sigma'}}(x)\int_{\mathbb{S}^0}\varphi(\zeta)\ d\overline{\mu}_x \geq \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1}\text{a.e. } x\in\Sigma', \tag{5.13}$$

where $\overline{\pi} = \dfrac{d\overline{\pi}}{d\mathcal{H}^{N-1}_{\lfloor\Sigma'}}\mathcal{H}^{N-1}_{\lfloor\Sigma'} + \overline{\pi}_s$ is the Radon-Nikodym decomposition of $\overline{\pi}$ with respect to the measure $\mathcal{H}^{N-1}_{\lfloor\Sigma'}$. We assume that $[\overline{u}](\hat{x}) > 0$ and show that

$$\frac{c(g)\ \left|[\overline{u}](\hat{x})\right|^2}{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds} \leq p(\hat{x}) \leq 1.$$

Let

$$\varphi(\xi) = \begin{cases} \varphi(1)\left|\xi\right|^2 & \text{if} \quad \xi \geq 0 \\ 0 & \text{if} \quad \xi < 0 \end{cases}. \tag{5.14}$$

Clearly, $\varphi^{**}([\overline{u}](\widehat{x})) = \varphi([\overline{u}](\widehat{x})) = \varphi(1)\ \left|[u](\widehat{x})\right|^2$. From the inequality (5.13), it follows that

$$\begin{aligned}
\varphi(1)\ \left|[\overline{u}](\widehat{x})\right|^2 &= \varphi^{**}([\overline{u}](\widehat{x})) \\
&\leq \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\left(\int_{\mathbb{S}^0}\varphi(\xi)d\overline{\mu}_x\right) \\
&\leq C(g)\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds\Big)\Big(\int_{\mathbb{S}^0}\varphi(\xi)d\overline{\mu}_x\Big) \\
&= C(g)p(\widehat{x})\varphi(1)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds.
\end{aligned}$$

Then, we obtain

$$\frac{|[\bar{u}](\hat{x})|^2}{C(g)\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2\,ds} = \frac{c(g)\;|[\bar{u}](\hat{x})|^2}{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2\,ds} \leq p(\hat{x}) \leq 1.$$

The proof of (iii) is similar. □

6 References

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