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A mixing effect induced by sources concentrated in a soft junction and the gradient concentration phenomenon

Anne-Laure Bessoud *, Pongpol Juntharee ^{† ‡}, Christian Licht [†], and Gérard Michaille [‡]

Abstract

We show that the variational limit of a ε -soft and thin junction problem ($\mathcal{P}_{\varepsilon}$) with sources concentrated in the junction gives rise to a surface energy mixing the internal energy and sources. The surface energy functional possesses an integral representation with respect to the Gradient Young-Concentration measures generated by sequences $(\bar{u}_{\varepsilon})_{\varepsilon>0}$ of minimizers of ($\mathcal{P}_{\varepsilon}$).

AMS subject classifications: 35B40, 49Q20, 28A33

Keywords: F-convergence, Gradient Young measures, concentration measures, minimization problems.

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1 Introduction

This paper concerns a soft thin junction subjected to concentrated sources. More precisely, let Ω be a domain in \mathbb{R}^N and let $B_{\varepsilon} := \Sigma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \subset \Omega$, $\Sigma \subset \mathbb{R}^{N-1}$, be the layer occupied by the soft thin junction (cf Figure 1). We consider the minimization problem

$$\min_{u \in W_{\Gamma_0}^{1,2}(\Omega)} \left\{ \int_{\Omega \setminus B_{\varepsilon}} f(\nabla u) \, dx + \varepsilon \int_{B_{\varepsilon}} g(\nabla u) \, dx - \langle \mathcal{S}^{\varepsilon}, u \rangle_{\varepsilon} \right\} \tag{$\mathcal{P}_{\varepsilon}$}$$

where $W_{\Gamma_0}^{1,2}(\Omega)$ denotes the space of Sobolev functions with null trace on a part Γ_0 of the boundary of Ω , and the linear form $\langle S^{\varepsilon}, . \rangle_{\varepsilon}$ represents the work of the source (or the loading). Let $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})$. A suitably rescaled S_{ε} of S^{ε} is assumed to strongly converge to some S in the dual of the space $V(B) := \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_N} \in L^2(\Omega) \right\}$ when ε tends to zero. A general example of such sources which are measures on B_{ε} is given in Section 4 of the paper. Sources of the form $c\frac{1}{L(\varepsilon)}\mathbbm{1}_{B_{\varepsilon}}$ where c is any constant and $L(\varepsilon) \sim \varepsilon$, is a trivial example of measures satisfying this condition with $S = \mathbbm{1}_B$. Note that in this paper the source (or the loading) S_{ε} is a non L^2 -continuous perturbation of the energy functional $\int_{\Omega \setminus B_{\varepsilon}} f(\nabla u) \, dx + \varepsilon \int_{B_{\varepsilon}} g(\nabla u) \, dx$.

Among the physical motivations of $(\mathcal{P}_{\varepsilon})$ one may mention various applications to heat conduction or electrostatic problems involving sources concentrated in the layer B_{ε} with conductivity or permittivity of order the size of B_{ε} . One may also think of membrane problems with an exterior loading concentrated in B_{ε} occupied by a material with stiffness of order the small size of B_{ε} . Such a problem with a source concentrated in the junction was considered in [3] in a one dimensional case in order to highlight and illustrate a gradient concentration phenomenon, but the authors were not able to express the variational limit problem.

This paper illustrates the same gradient concentration phenomenon with a complete description of the limit problem in the sense of Γ -convergence (Theorem 3.3). When the size ε of the layer goes to zero, fields u_{ε} of bounded energy develop a discontinuity through Σ . More precisely, at the variational limit, the internal energy functional of the junction $\varepsilon \int_{B_{\varepsilon}} g(\nabla u) dx$ and the work of the loading $\langle S^{\varepsilon}, u \rangle_{\varepsilon}$ are combined into a functional of the type

$$\bar{H}(u) = \int_{\Sigma} \bar{h}(\hat{x}, u^{+} - u^{-}, \frac{u^{+} + u^{-}}{2}) d\hat{x}$$

and the limit problem reads as

$$\min_{u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)} \left\{ \int_{\Omega} f(\nabla u) \, dx + \bar{H}(u) \right\} \tag{P}$$

where u^{\pm} denote the traces on Σ . When regarding the various studies devoted to the asymptotic modeling of junction problems (see [2, 9, 7, 10] and references therein) the main novelty is that the density \bar{h} depends also of the mean $\frac{u^++u^-}{2}$. Furthermore we show that the sequence of minimizers of $(\mathcal{P}_{\varepsilon})$ (which converges to a minimizer \bar{u} of the limit problem (\mathcal{P})) generates a gradient Young-concentration measure $\bar{\mu}$ in the sense defined in [3]. Then we can give an integral representation of the internal part of \bar{H} with respect to the measure $\bar{\mu}$ (Theorem 5.5) so that it can be localized in $\Sigma \times \{\pm 1\}$. Finally this provides new bounds on the measure $\bar{\mu}$ (Corollary 5.6).

The paper is organized as follows: in Section 2 we fix notation and give a detailed description of the problem $(\mathcal{P}_{\varepsilon})$. Section 3 is devoted to the asymptotic analysis of $(\mathcal{P}_{\varepsilon})$ in the sense of the Γ -convergence of the functional energy extended to $L^2(\Omega)$ equipped with its strong topology. In Section 4 we describe a large class of suitable sources S_{ε} . Finally Section 5 is concerned with the analysis of the gradient concentration phenomenon generated by sequences of minimizers of $(\mathcal{P}_{\varepsilon})$. We stress the fact that one could treat the problem in $L^p(\Omega)$, 1 in the same way without additional difficulties.

2 Description of the minimization problem

Let $\varepsilon > 0$ be a small parameter intended to go to zero, more precisely taking values in a countable subset of $(0, \varepsilon_0]$ whose 0 is the only cluster point. The reference configuration of the assembly of the two adherents and the adhesive is a cylinder $\Omega := \Sigma \times (-r, r)$ (with $r > \varepsilon$), where Σ is a bounded domain in \mathbb{R}^{N-1} , $N \ge 2$, with Lipschitz boundary. For $x \in \mathbb{R}^N$ we sometimes write $x = (\hat{x}, x_N)$ where $\hat{x} \in \mathbb{R}^{N-1}$. In all the paper, C denotes a non negative constant which does not depend on ε and may vary from line to line. We do not relabel the various considered subsequences and the symbols \rightarrow and \rightharpoonup denote various strong convergences and weak convergences respectively. We define the following sets:



Figure 1: physical domain

- $B_{\varepsilon} := \Sigma \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right);$
- . $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2});$
- $\Omega_{\varepsilon} = \Omega \setminus \overline{B_{\varepsilon}};$
- . Γ_0 is a subset of the boundary $\partial\Omega$ of Ω such that $\operatorname{dist}(\overline{\Gamma}_0, \overline{\partial B_{\varepsilon} \cap \partial\Omega}) > 0$ for all $\varepsilon < \varepsilon_0$;
- . we write Ω_{ε}^{-} , Ω_{ε}^{+} , Ω^{-} , Ω^{+} , B_{ε}^{+} and B_{ε}^{-} for the sets $\Omega_{\varepsilon} \cap [x_{N} < 0]$ and $\Omega_{\varepsilon} \cap [x_{N} > 0]$, $\Omega \cap [x_{N} < 0]$, $\Omega \cap [x_{N} < 0]$ respectively.

We will be concerned with the following spaces:

- $. \ W^{1,2}_{\Gamma_0}(\Omega_{\varepsilon}) := \left\{ u \in W^{1,2}(\Omega_{\varepsilon}) : u = 0 \text{ on } \Gamma_0 \right\};$
- . $W^{1,2}_{\Gamma_0}(\Omega) := \left\{ u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_0 \right\};$
- . $W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma) := \left\{ u \in W^{1,2}(\Omega \setminus \Sigma) : u = 0 \text{ on } \Gamma_0 \right\}$, and for every $z \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, z^{\pm} will stand for the traces of z on Σ considered as a Sobolev function on Ω^+ and Ω^- respectively.

We say that a function $h : \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ satisfies a growth condition of order 2 if there exist α and β in \mathbb{R}^+ such that

$$\alpha |\xi|^2 \le h(\xi) \le \beta(1+|\xi|^2)$$
 for all $\xi \in \mathbb{R}^N$.

We consider two convex functions $f, g : \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfying a growth condition of order 2, and we assume that there exists a positively 2-homogeneous function $g^{\infty,2}$ satisfying

$$\left|g(\xi) - g^{\infty,2}(\xi)\right| \le \beta(1 + \left|\xi\right|^{2-\delta}) \text{ for all } \xi \in \mathbb{R}^N,$$
(2.1)

for some δ , $0 < \delta < 2$. Note that $g^{\infty,2}$ is the positively 2-homogeneous recession function of g, i.e.,

$$g^{\infty,2}(\xi) = \lim_{t \to +\infty} \frac{g(t\xi)}{t^2},$$

is convex and satisfies the same growth condition of order 2. We define the space

$$V(B_{\varepsilon}) := \left\{ u \in L^2(B_{\varepsilon}) : \frac{\partial u}{\partial x_N} \in L^2(B_{\varepsilon}) \right\}$$

equipped with the norm

$$|u||_{V(B_{\varepsilon})} := \left(\int_{B_{\varepsilon}} |u|^2 \ dx + \int_{B_{\varepsilon}} \left|\frac{\partial u}{\partial x_N}\right|^2 \ dx\right)^{\frac{1}{2}}$$

and we denote the duality bracket between the topological dual space $V'(B_{\varepsilon})$ and $V(B_{\varepsilon})$ by $\langle , \rangle_{\varepsilon}$. The considered total energy functional $F_{\varepsilon}: L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$F_{\varepsilon}(u) = \begin{cases} \int_{\Omega_{\varepsilon}} f(\nabla u) \, dx + \varepsilon \int_{B_{\varepsilon}} g(\nabla u) \, dx - \langle \mathcal{S}^{\varepsilon}, u \rangle_{\varepsilon} \text{ if } u \in W^{1,2}_{\Gamma_0}(\Omega) \\ +\infty \text{ otherwise,} \end{cases}$$

where $\mathcal{S}^{\varepsilon}$ is given in $V'(B_{\varepsilon})$. Our aim is to describe the asymptotic behavior of the minimization problem

$$(\mathcal{P}_{\varepsilon})$$
 min $\{F_{\varepsilon}(u) : u \in L^{2}(\Omega)\},\$

namely, the limit of min $\{F_{\varepsilon}(u) : u \in L^2(\Omega)\}$ together with the limit of the minimizer \bar{u}_{ε} , and to identify the limit problem in the framework of Γ -convergence.

Let us consider the space $V(B) := \left\{ u \in L^2(B) : \frac{\partial u}{\partial x_N} \in L^2(B) \right\}$ equipped with the norm

$$||u||_{V(B)} := \left(\int_{B} |u|^2 dx + \int_{B} \left|\frac{\partial u}{\partial x_N}\right|^2 dx\right)^{\frac{1}{2}},$$

and denote the duality bracket between V'(B) and V(B) by $\langle ., . \rangle$. The linear continuous operator

$$\tau_{\varepsilon}: V(B_{\varepsilon}) \longrightarrow V(B)$$

is defined for every $x = (\hat{x}, x_N) \in B$ by $\tau_{\varepsilon} u(\hat{x}, x_N) := u(\hat{x}, \varepsilon x_N)$ and we denote its transposed operator by ${}^T \tau_{\varepsilon}$:

$$\langle {}^T \tau_{\varepsilon} \theta, u \rangle_{\varepsilon} = \langle \theta, \tau_{\varepsilon} u \rangle, \ \forall (\theta, u) \in V'(B) \times V(B_{\varepsilon}).$$

We make the following assumption on the source $\mathcal{S}^{\varepsilon}$: there exists \mathcal{S} in V'(B) such that

$$S_{\varepsilon} := ({}^{T}\tau_{\varepsilon})^{-1} \mathcal{S}^{\varepsilon}$$
 strongly converges to \mathcal{S} in $V'(B)$.

Then, in order to identify the Γ -limit of the functional F_{ε} , it will be more convenient to write the functional F_{ε} as

$$F_{\varepsilon}(u) = \begin{cases} \int_{\Omega_{\varepsilon}} f(\nabla u) \, dx + \varepsilon^2 \int_B g(\widehat{\nabla}\tau_{\varepsilon}u, \frac{1}{\varepsilon} \frac{\partial \tau_{\varepsilon}u}{\partial x_N}) \, dx - \langle \mathcal{S}_{\varepsilon}, \tau_{\varepsilon}u \rangle \text{ if } u \in W^{1,2}_{\Gamma_0}(\Omega) \\ +\infty \text{ otherwise.} \end{cases}$$

3 The variational asymptotic model

Let $H: V(B) \longrightarrow \mathbb{R}$ be the functional defined by

$$H(\theta) := \int_{B} g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_{N}}) dx - \langle \mathcal{S}, \theta \rangle$$

= $H_{in}(\theta) - \langle \mathcal{S}, \theta \rangle.$ (3.1)

We refer the functional H_{in} as the *internal part* of H. We claim that, when $L^2(\Omega)$ is equipped with its strong topology, the functional F_{ε} Γ -converges to the functional $F_0: L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$F_{0}(u) = \begin{cases} \int_{\Omega} f(\nabla u) \, dx + \inf_{\theta \in X(u)} H(\theta) \text{ if } u \in W^{1,2}_{\Gamma_{0}}(\Omega \setminus \Sigma), \\ +\infty \text{ otherwise,} \end{cases}$$

where $X(u):=\big\{\theta\in V(B): \theta(.,\pm\frac{1}{2})=u^{\pm}\big\}.$

Before addressing the variational convergence process, we begin by establishing some compactness properties for sequences with bounded energy. Let us introduce the ε -translate operator T_{ε} from $W^{1,2}(\Omega)$ into $W^{1,2}(\Omega \setminus \Sigma)$. For any function $w \in W^{1,2}(\Omega)$, \tilde{w} stands for its extension by reflexion on $\Sigma \times (-2r, -r) \cup$ (r, 2r) and we define the ε -translate $T_{\varepsilon}w$ of w by

$$T_{\varepsilon}w(\hat{x}, x_N) = \begin{cases} \widetilde{w}(\hat{x}, x_N + \frac{\varepsilon}{2}) & \text{if } x \in \Omega^+;\\ \widetilde{w}(\hat{x}, x_N - \frac{\varepsilon}{2}) & \text{if } x \in \Omega^-. \end{cases}$$

Lemma 3.1 (compactness). Let $(u_{\varepsilon})_{\varepsilon>0}$ be a sequence in $L^2(\Omega)$ such that $\sup_{\varepsilon>0} F_{\varepsilon}(u_{\varepsilon}) < +\infty$. Then (i)

$$\int_{B_{\varepsilon}} |u_{\varepsilon}|^2 \, dx \le C\varepsilon \Big(\int_{\Omega_{\varepsilon}} |\nabla u_{\varepsilon}|^2 \, dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial u_{\varepsilon}}{\partial x_N} \right|^2 \, dx \Big); \tag{3.2}$$

(ii)

$$\sup_{\varepsilon > 0} \left(\int_{\Omega_{\varepsilon}} \left| \nabla u_{\varepsilon} \right|^2 dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial u_{\varepsilon}}{\partial x_N} \right|^2 dx \right) < +\infty;$$
(3.3)

- (iii) there exist $u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$ and a subsequence of $(u_{\varepsilon})_{\varepsilon>0}$ such that $u_{\varepsilon} \to u$ in $L^2(\Omega)$ and $u_{\varepsilon} \to u$ in $W^{1,2}_{\Gamma_0}(\Omega_{\eta})$ for all $\eta > 0$;
- (iv) there exist $\theta \in V(B)$ and a subsequence such that $\tau_{\varepsilon} u_{\varepsilon} \rightharpoonup \theta$ in V(B), i.e.

$$\frac{\tau_{\varepsilon} u_{\varepsilon} \rightharpoonup \theta \ in \ L^2(B)}{\frac{\partial \tau_{\varepsilon} u_{\varepsilon}}{\partial x_N}} \rightharpoonup \frac{\partial \theta}{\partial x_N} \ in \ L^2(B);$$

moreover, $\varepsilon \widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon} \rightharpoonup 0$ in $L^2(B, \mathbb{R}^{N-1});$

$$(v) \ \theta(.,\pm\frac{1}{2}) = u^{\pm}$$

Proof. Proof of (i). Without loss of generality, we may assume that the N - 1-dimensional Hausdorff measure of the intersection of Γ_0 with $[x_N > 0]$ is positive so that (i) is a mere consequence of the following Poincaré-like inequality:

$$\exists C > 0, \ \int_{B_{\varepsilon}} |\varphi|^2 \ dx \le C\varepsilon \Big(\int_{\Omega_{\varepsilon}^+} |\nabla \varphi|^2 \ dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial \varphi}{\partial x_N} \right|^2 \ dx \Big) \ \forall \varphi \in W^{1,2}_{\Gamma_0}(\Omega).$$
(3.4)

Indeed, because

$$\varphi\left(\hat{x}, x_N\right) = T_{\varepsilon}\varphi(\hat{x}, 0) + \int_{\frac{\varepsilon}{2}}^{x_N} \frac{\partial}{\partial x_N} \varphi\left(\hat{x}, t\right) dt \qquad \forall x \in B_{\varepsilon},$$

for all smooth function $\varphi \in W^{1,2}_{\Gamma_0}(\Omega)$, we get

$$|\varphi(\hat{x}, x_N)|^2 \le 2\Big(|T_{\varepsilon}\varphi(\hat{x}, 0)|^2 + \varepsilon \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left|\frac{\partial}{\partial x_N}\varphi(\hat{x}, t)\right|^2 dt\Big).$$

Hence, integrating on B_{ε} and using trace inequality and Poincaré inequality in Ω^+ give the desired inequality (3.4) for smooth φ , thus for all φ in $W^{1,2}_{\Gamma_0}(\Omega)$ by a density argument.

Proof of (ii). From the coercivity conditions satisfied by f and g, estimate (3.2), and the strong convergence of S_{ε} in V'(B), one has

$$\begin{aligned} \alpha \Big(\int_{\Omega_{\varepsilon}} |\nabla u_{\varepsilon}|^{2} dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial u_{\varepsilon}}{\partial x_{N}} \right|^{2} dx \Big) &\leq C + |\langle \mathcal{S}^{\varepsilon}, u_{\varepsilon} \rangle_{\varepsilon}| \\ &= C + |\langle \mathcal{S}_{\varepsilon}, \tau_{\varepsilon} u_{\varepsilon} \rangle| \\ &\leq C + ||\mathcal{S}_{\varepsilon}||_{V'(B)} ||\tau_{\varepsilon} u_{\varepsilon}||_{V(B)} \\ &= C + ||\mathcal{S}_{\varepsilon}||_{V'(B)} \left(\frac{1}{\varepsilon} \int_{B_{\varepsilon}} |u_{\varepsilon}|^{2} dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial u_{\varepsilon}}{\partial x_{N}} \right|^{2} dx \right)^{\frac{1}{2}} \\ &\leq C + C \Big(\int_{\Omega_{\varepsilon}} |\nabla u_{\varepsilon}|^{2} dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial u_{\varepsilon}}{\partial x_{N}} \right|^{2} dx \Big)^{1/2}. \end{aligned}$$

Then, setting $X_{\varepsilon} := \left(\int_{\Omega_{\varepsilon}} |\nabla u_{\varepsilon}|^2 dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial u_{\varepsilon}}{\partial x_N} \right|^2 dx \right)^{1/2}$, (3.3) follows from the estimate $\alpha X_{\varepsilon}^2 \leq C + C X_{\varepsilon}$.

Proof of (iii).

Step 1. We claim that there exist $z \in W^{1,2}(\Omega \setminus \Sigma)$ and a subsequence of $(u_{\varepsilon})_{\varepsilon>0}$ such that $T_{\varepsilon}u_{\varepsilon} \rightharpoonup z$ in $W^{1,2}(\Omega \setminus \Sigma)$ and strongly in $L^2(\Omega \setminus \Sigma)$. Clearly,

$$T_{\varepsilon}u_{\varepsilon} \in W^{1,2}(\Omega \setminus \Sigma) \text{ and } \frac{\partial}{\partial x_i}T_{\varepsilon}u_{\varepsilon} = T_{\varepsilon}\frac{\partial}{\partial x_i}u_{\varepsilon} \text{ for all } \varepsilon > 0.$$
 (3.5)

Combining the Poincaré inequality, (3.3) and (3.5), we deduce

$$\sup_{\varepsilon>0} \|T_{\varepsilon}u_{\varepsilon}\|^{2}_{W^{1,2}(\Omega\setminus\Sigma)} \leq C \sup_{\varepsilon>0} \left(\int_{\Omega_{\varepsilon}} |\nabla u_{\varepsilon}(x)|^{2} dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial u_{\varepsilon}}{\partial x_{N}}(x) \right|^{2} dx \right) < +\infty.$$

Therefore, $(T_{\varepsilon}u_{\varepsilon})_{\varepsilon>0}$ is bounded in $W^{1,2}(\Omega\setminus\Sigma)$ and the claim follows immediately.

Step 2. We establish that there exists u in $L^2(\Omega)$ such that we can extract from the previous subsequence $(u_{\varepsilon})_{\varepsilon>0}$ a subsequence strongly converging to u in $L^2(\Omega)$. We can write

$$\int_{\Omega_{\varepsilon}} |u_{\varepsilon}(x)|^2 dx = \int_{\Omega^+ \cup \Omega^-} |T_{\varepsilon}u_{\varepsilon}(x)|^2 dx - \int_{\Sigma \times \left(\left(r - \frac{\varepsilon}{2}, r\right) \cup \left(-r, -r + \frac{\varepsilon}{2}\right)\right)} |T_{\varepsilon}u_{\varepsilon}(x)|^2 dx,$$

so that

$$\|u_{\varepsilon}\|_{L^{2}(\Omega)}^{2} = \int_{\Omega^{+}\cup\Omega^{-}} |T_{\varepsilon}u_{\varepsilon}(x)|^{2} dx + \int_{B_{\varepsilon}} |u_{\varepsilon}(x)|^{2} dx - \int_{\Sigma\times\left(\left(r-\frac{\varepsilon}{2},r\right)\cup\left(-r,-r+\frac{\varepsilon}{2}\right)\right)} |T_{\varepsilon}u_{\varepsilon}(x)|^{2} dx.$$
(3.6)

From step 1 and (3.2), we deduce that $\sup_{\varepsilon>0} ||u_{\varepsilon}||_{L^{2}(\Omega)} < +\infty$. Thus there exist $u \in L^{2}(\Omega)$ and a not relabelled subsequence such that $u_{\varepsilon} \rightharpoonup u$ in $L^{2}(\Omega)$. Let us prove that u = z. Since $u_{\varepsilon} \rightharpoonup u$ in $L^{2}(\Omega)$ and $T_{\varepsilon}u_{\varepsilon} \rightharpoonup z$ in $W^{1,2}(\Omega \setminus \Sigma)$, we have for any $\varphi \in C_{c}^{\infty}(\Omega)$,

$$\begin{split} \int_{\Omega} u(x)\varphi(x)dx &= \lim_{\varepsilon \to 0} \int_{\Omega} u_{\varepsilon}(x)\varphi(\widehat{x}, x_N - \frac{\varepsilon}{2})dx \\ &= \lim_{\varepsilon \to 0} \int_{\Omega} T_{\varepsilon} u_{\varepsilon}(x)\varphi(x)dx \\ &= \int_{\Omega} z(x)\varphi(x)dx. \end{split}$$

Thus u = z almost everywhere in Ω and we deduce that $u \in W^{1,2}(\Omega \setminus \Sigma)$. Moreover, from (3.2) we have that $\int_{B_{\varepsilon}} |u_{\varepsilon}(x)|^2 dx \to 0$ as $\varepsilon \to 0$. On the other hand, since $T_{\varepsilon}u_{\varepsilon} \to z$ in $L^2(\Omega)$, we infer $\int_{\Omega^+\cup\Omega^-} |T_{\varepsilon}u_{\varepsilon}(x)|^2 dx \to \int_{\Omega} |z(x)|^2 dx$ and $\int_{\Sigma\times\left(\left(r-\frac{\varepsilon}{2},r\right)\cup\left(-r,-r+\frac{\varepsilon}{2}\right)\right)} |T_{\varepsilon}u_{\varepsilon}(x)|^2 dx \to 0$. Then we deduce that $||u_{\varepsilon}||_{L^2(\Omega)} \to ||z||_{L^2(\Omega)} = ||u||_{L^2(\Omega)}$ and thus that $(u_{\varepsilon})_{\varepsilon>0}$ strongly converges to u in $L^2(\Omega)$.

Step 3. It remains to establish that for any $\eta > 0$, there exists a subsequence of $(u_{\varepsilon})_{\varepsilon>0}$ such that $u_{\varepsilon \mid \Omega_{\eta}} \rightharpoonup u_{\mid \Omega_{\eta}}$ in $W_{\Gamma_{0}}^{1,2}(\Omega_{\eta})$. It will immediately result that $u \in W_{\Gamma_{0}}^{1,2}(\Omega \setminus \Sigma)$. Let $\eta > 0$. Clearly, there exists $0 < \varepsilon_{1} < \eta$ such that $\Omega_{\eta} \subseteq \Omega_{\varepsilon}$ for all $\varepsilon \leq \varepsilon_{1}$. By the Poincaré inequality we have

$$\sup_{\varepsilon>0} \|u_{\varepsilon}\|_{W^{1,2}(\Omega_{\eta})}^{2} \leq C \sup_{\varepsilon>0} \left(\int_{\Omega_{\varepsilon}} |\nabla u_{\varepsilon}(x)|^{2} dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial u_{\varepsilon}}{\partial x_{N}}(x) \right|^{2} dx \right) < +\infty.$$

Thus, $(u_{\varepsilon})_{\varepsilon>0}$ is bounded in $W^{1,2}_{\Gamma_0}(\Omega_{\eta})$, and there exist $w \in W^{1,2}_{\Gamma_0}(\Omega_{\eta})$ and a not relabelled subsequence of $(u_{\varepsilon})_{\varepsilon>0}$ satisfying $u_{\varepsilon} \to w$ in $L^2(\Omega_{\eta})$ and $u_{\varepsilon} \rightharpoonup w$ in $W^{1,2}_{\Gamma_0}(\Omega_{\eta})$. It is easily seen that in fact $w = u_{\lfloor \Omega_{\eta}}$.

Proof of (iv). The weak convergence of $\tau_{\varepsilon} u_{\varepsilon}$ to some θ in V(B) follows from (3.2) and (3.3). Indeed

$$\sup_{\varepsilon>0} \|\tau_{\varepsilon} u_{\varepsilon}\|_{V(B)} = \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int_{B_{\varepsilon}} |u_{\varepsilon}|^2 dx + \varepsilon \int_{B_{\varepsilon}} \left|\frac{\partial u_{\varepsilon}}{\partial x_N}\right|^2 dx\right)^{\frac{1}{2}} \\ \leq C \sup_{\varepsilon>0} X_{\varepsilon} < +\infty.$$

Now we deduce that $\widehat{\nabla}\tau_{\varepsilon}u_{\varepsilon} \rightarrow \widehat{\nabla}\theta$ in the distributional sense so that $\varepsilon\widehat{\nabla}\tau_{\varepsilon}u_{\varepsilon} \rightarrow 0$ in the distributional sense. On the other hand, from the coercivity of g, $\varepsilon\widehat{\nabla}\tau_{\varepsilon}u_{\varepsilon}$ weakly converges to some $L^{2}(B, \mathbb{R}^{N-1})$ function. Hence, $\varepsilon\widehat{\nabla}\tau_{\varepsilon}u_{\varepsilon} \rightarrow 0$ in $L^{2}(B, \mathbb{R}^{N-1})$.

Proof of (v). Note that $\theta(., \pm \frac{1}{2})$ is well defined. Indeed, one has

$$V(B) \subset W^{1,2}\left((-\frac{1}{2},\frac{1}{2}), L^2(\Sigma)\right) \subset \mathcal{C}\left([-\frac{1}{2},\frac{1}{2}], L^2(\Sigma)\right).$$

Clearly, $\tau_{\varepsilon} u_{\varepsilon}(\hat{x}, \pm \frac{1}{2}) = (T_{\varepsilon} u_{\varepsilon})^{\pm}(\hat{x})$ (in the sense of traces on Σ of $W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ -functions) so that $\tau_{\varepsilon} u_{\varepsilon}(\hat{x}, \pm \frac{1}{2}) \to u^{\pm}$ in $L^2(\Sigma)$. On the other hand, since

$$\tau_{\varepsilon} u_{\varepsilon}(\hat{x}, x_N) = \tau_{\varepsilon} u_{\varepsilon}(\hat{x}, \pm \frac{1}{2}) + \int_{\pm \frac{1}{2}}^{x_N} \frac{\partial \tau_{\varepsilon} u_{\varepsilon}}{\partial x_N}(\hat{x}, s) \ ds$$

for a.e. x in B, we infer that for all $\varphi \in \mathcal{C}_c(\Sigma)$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \tau_{\varepsilon} u_{\varepsilon}(\hat{x}, x_N) \varphi(\hat{x}) \, dx = \int_{\Sigma} (T_{\varepsilon} u_{\varepsilon})^{\pm} (\hat{x}) \varphi(\hat{x}) d\hat{x} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \int_{\pm\frac{1}{2}}^{x_N} \frac{\partial \tau_{\varepsilon} u_{\varepsilon}}{\partial x_N} (\hat{x}, s) \varphi(\hat{x}) ds dx. \quad (3.7)$$

By passing to the limit in (3.7), we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \theta(\hat{x}, x_N) \varphi(\hat{x}) \, dx = \int_{\Sigma} u^{\pm}(\hat{x}) \varphi(\hat{x}) d\hat{x} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \int_{\pm\frac{1}{2}}^{x_N} \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \varphi(\hat{x}) ds dx$$

from which we deduce

$$\int_{\Sigma} u^{\pm}(\hat{x})\varphi(\hat{x})d\hat{x} = \int_{\Sigma} \theta(\hat{x}, \pm \frac{1}{2})\varphi(\hat{x})d\hat{x}.$$

Thus $\theta(., \pm \frac{1}{2}) = u^{\pm}$ almost everywhere in Σ .

Lemma 3.2. For every $u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, $\inf_{\theta \in X(u)} H(\theta) > -\infty$ and there exists $\theta(u) \in X(u)$ such that $\inf_{\theta \in X(u)} H(\theta) = H(\theta(u))$.

Proof. The proof follows from standard arguments used in the direct method of the Calculus of Variation. \Box

As a consequence of Lemma 3.2, in its domain $W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, the functional F_0 may be written

$$F_0(u) = \int_{\Omega} f(\nabla u) \, dx + H(\theta(u))$$

Theorem 3.3 is the main result of this section.

Theorem 3.3. The sequence $(F_{\varepsilon})_{\varepsilon>0}$ Γ -converges to the functional F_0 when $L^2(\Omega)$ is equipped with its strong topology.

The proof results from the following two propositions.

Proposition 3.4. For every $u \in L^2(\Omega)$ and every $(u_{\varepsilon})_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ one has

$$F_0(u) \leq \liminf_{\varepsilon \to 0} F_{\varepsilon}(u_{\varepsilon}).$$

Proposition 3.5. For every $u \in L^2(\Omega)$ there exists $(v_{\varepsilon})_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ satisfying $F_0(u) \ge \limsup_{\varepsilon \to 0} F_{\varepsilon}(v_{\varepsilon}).$

Proof of Proposition 3.4. We may assume $\liminf_{\varepsilon \to 0} F_{\varepsilon}(u_{\varepsilon}) < +\infty$. From Lemma 3.1 $u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$ and there exists $\theta \in X(u)$ such that $\tau_{\varepsilon}u_{\varepsilon} \rightharpoonup \theta$ in V(B). Since $\mathcal{S}_{\varepsilon} \to \mathcal{S}$ in V'(B), one has

$$\lim_{\varepsilon \to 0} \langle \mathcal{S}_{\varepsilon}, \tau_{\varepsilon} u_{\varepsilon} \rangle = \langle \mathcal{S}, \theta \rangle.$$
(3.8)

On the other hand, since from Lemma 3.1, $u_{\varepsilon} \rightharpoonup u$ in $W_{\Gamma_0}^{1,2}(\Omega_{\eta})$ for all $\eta > 0$, one has

$$\liminf_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} f(\nabla u_{\varepsilon}) \, dx \ge \int_{\Omega} f(\nabla u) \, dx.$$
(3.9)

Finally from (iv) of Lemma 3.1 and a standard lower semicontinuity argument

$$\lim_{\varepsilon \to 0} \inf \varepsilon^{2} \int_{B} g(\widehat{\nabla}\tau_{\varepsilon}u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon}u_{\varepsilon})}{\partial x_{N}}) dx$$

$$\geq \lim_{\varepsilon \to 0} \inf \left(\varepsilon^{2} \int_{B} g(\widehat{\nabla}\tau_{\varepsilon}u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon}u_{\varepsilon})}{\partial x_{N}}) dx - \int_{B} g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_{\varepsilon}u_{\varepsilon}, \frac{\partial(\tau_{\varepsilon}u_{\varepsilon})}{\partial x_{N}}) dx \right)$$

$$+ \lim_{\varepsilon \to 0} \inf \int_{B} g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_{\varepsilon}u_{\varepsilon}, \frac{\partial(\tau_{\varepsilon}u_{\varepsilon})}{\partial x_{N}}) dx$$

$$\geq \lim_{\varepsilon \to 0} \left(\varepsilon^{2} \int_{B} g(\widehat{\nabla}\tau_{\varepsilon}u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon}u_{\varepsilon})}{\partial x_{N}}) dx - \int_{B} g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_{\varepsilon}u_{\varepsilon}, \frac{\partial(\tau_{\varepsilon}u_{\varepsilon})}{\partial x_{N}}) dx \right)$$

$$+ \int_{B} g^{\infty,2}(\widehat{0}, \frac{\partial\theta}{\partial x_{N}}) dx$$
(3.10)

provided that we establish

$$\lim_{\varepsilon \to 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx - \int_B g^{\infty, 2}(\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx \right) = 0. \tag{3.11}$$

Since $g^{\infty,2}$ is positively homogeneous of degree 2, and from (2.1), we have

$$\begin{split} & \int_{B} \left| \varepsilon^{2} g(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_{N}}) - g^{\infty, 2}(\varepsilon \widehat{\nabla} \tau_{\varepsilon}^{-1} u_{\varepsilon}, \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_{N}}) \right| dx \\ &= \varepsilon^{2} \int_{B} \left| g(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_{N}}) - g^{\infty, 2}(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_{N}}) \right| dx \\ &\leq C \varepsilon^{2} \int_{B} \left[1 + \left| \widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon} \right|^{2-\delta} + \left| \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_{N}} \right|^{2-\delta} \right] dx. \end{split}$$

Thus, by using Hölder's inequality (take $p = \frac{2}{2-\delta}, q = \frac{2}{\delta}$) we deduce

$$\int_{B} \left| \varepsilon^{2} g(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial \tau_{\varepsilon} u_{\varepsilon}}{\partial x_{N}}) - g^{\infty, 2} (\varepsilon \widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{\partial (\tau_{\varepsilon} u_{\varepsilon})}{\partial x_{N}}) \right| dx \leq C \varepsilon^{\delta}$$

which proves (3.11). The conclusion of Proposition 3.4 follows by collecting (3.8), (3.9) and (3.10). \Box

Proof of Proposition 3.5. Let $u \in L^2(\Omega)$. We have to construct a sequence $(v_{\varepsilon})_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ such that $\limsup_{\varepsilon \to 0} F_{\varepsilon}(v_{\varepsilon}) \leq F_0(u)$. If $F_0(u) = +\infty$, then $u \in L^2(\Omega) \setminus W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$, and clearly, for any sequence $(v_{\varepsilon})_{\varepsilon>0}$ converging to u, $\limsup_{\varepsilon \to 0} F_{\varepsilon}(v_{\varepsilon}) \leq F_0(u)$ is true. Now, for the harder part, we assume $F_0(u) < +\infty$. Then $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ and

$$F_0(u) = \int_{\Omega} f(\nabla u(x)) dx + \inf_{\theta \in X(u)} H(\theta).$$

To complete the proof, from $\overline{\theta} := \theta(u)$, i.e. $H(\overline{\theta}) = \inf_{\theta \in X(u)} H(\theta)$, we construct a sequence $(v_{\varepsilon})_{\varepsilon > 0}$ strongly converging to u in $L^2(\Omega)$ and satisfying

$$F_0(u) \ge \limsup_{\varepsilon \to 0} F_{\varepsilon}(v_{\varepsilon}).$$

The proof is divided into four steps:

Step 1. Let us extend u and $\overline{\theta}$ by 0 into $(\mathbb{R}^{N-1} \setminus \Sigma) \times (-r, r)$ and write these extensions \widetilde{u} and $\tilde{\overline{\theta}}$. For a sequence δ of positive numbers intended to go to 0, consider a standard sequence of molifier $(\rho_{\delta})_{\delta}$ and set

$$u_{\delta} := \rho_{\delta} * \widetilde{u} \text{ defined by } \rho_{\delta} * \widetilde{u}(\widehat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_{\delta}(\widehat{x} - \widehat{y})\widetilde{u}(\widehat{y}, x_N)d\widehat{y} \text{ for all } (\widehat{x}, x_N) \in \Omega;$$

$$\theta_{\delta} := \rho_{\delta} * \widetilde{\overline{\theta}} \text{ defined by } \rho_{\delta} * \widetilde{\overline{\theta}}(\widehat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_{\delta}(\widehat{x} - \widehat{y})\widetilde{\overline{\theta}}(\widehat{y}, x_N)d\widehat{y} \text{ for all } (\widehat{x}, x_N) \in \Omega.$$

Clearly,

$$\theta_{\delta}(\widehat{x}, \pm \frac{1}{2}) = u_{\delta}(\widehat{x}, 0) \text{ for all } \widehat{x} \in \Sigma,$$

$$u_{\delta} \in W^{1,2}(\Omega \setminus \Sigma), \theta_{\delta} \in W^{1,2}(B),$$

$$u_{\delta} \to u \text{ in } W^{1,2}(\Omega \setminus \Sigma), \ \theta_{\delta} \to \overline{\theta} \text{ in } V(B).$$
(3.12)

Next, for each $\delta > 0$, we define the sequence $(v_{\delta,\varepsilon})_{\varepsilon>0}$ as follows:

$$v_{\delta,\varepsilon}(\widehat{x}, x_N) = \begin{cases} u_{\delta}(\widehat{x}, x_N \pm \frac{\varepsilon}{2}) & \text{on} \quad \Omega_{\varepsilon}^{\mp} \\ \\ \theta_{\delta}(\widehat{x}, \frac{x_N}{\varepsilon}) & \text{on} \quad B_{\varepsilon}. \end{cases}$$
(3.13)

Obviously $v_{\delta,\varepsilon}(\hat{x}, x_N)$ belongs to $W^{1,2}(\Omega)$ and strongly converges to u_{δ} in $L^2(\Omega)$.

Step 2. We we claim that

$$\lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} f(\nabla v_{\delta,\varepsilon})(x) dx = \int_{\Omega} f(\nabla u_{\delta})(x) dx$$
(3.14)

$$\lim_{\varepsilon \to 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon v_{\delta,\varepsilon}, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon v_{\delta,\varepsilon}}{\partial x_N})(x) dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon v_{\delta,\varepsilon} \rangle \right) = H(\theta_\delta).$$
(3.15)

Proof of (3.14): one has

$$\begin{split} \lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} f(\nabla v_{\delta,\varepsilon})(x) dx &= \lim_{\varepsilon \to 0} \left(\int_{\Omega_{\varepsilon}^{+}} f(\nabla u_{\delta})(\widehat{x}, x_{N} - \frac{\varepsilon}{2}) dx + \int_{\Omega_{\varepsilon}^{-}} f(\nabla u_{\delta}(\widehat{x}, x_{N} + \frac{\varepsilon}{2})) dx \right) \\ &= \int_{\Omega^{+}} f(\nabla u_{\delta})(x) dx + \int_{\Omega^{-}} f(\nabla u_{\delta})(x) dx \\ &= \int_{\Omega} f(\nabla u_{\delta})(x) dx. \end{split}$$

Proof of (3.15): Since $g^{\infty,2}$ is positively homogeneous of degree 2 and S_{ε} strongly converges to S in V'(B), one has

$$\lim_{\varepsilon \to 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla} \theta_{\delta}, \frac{1}{\varepsilon} \frac{\partial \theta_{\delta}}{\partial x_N})(x) dx - \langle \mathcal{S}_{\varepsilon}, \theta_{\delta} \rangle \right) = \int_B g^{\infty, 2}(\widehat{0}, \frac{\partial \theta_{\delta}}{\partial x_N}) dx - \langle \mathcal{S}, \theta_{\delta} \rangle = H(\theta_{\delta}).$$

Step 3. We establish that $\lim_{\delta \to 0} \int_{\Omega} f(\nabla u_{\delta}) dx + H(\theta_{\delta}) = F_0(u)$. Since

$$\int_{\Omega} f(\nabla u_{\delta}) dx + H(\theta_{\delta}) = \int_{\Omega} f(\nabla u_{\delta}) dx + \int_{B} g^{\infty,2}(\widehat{0}, \frac{\partial \theta_{\delta}}{\partial x_{N}}) dx - \langle \mathcal{S}, \theta_{\delta} \rangle$$

the result is a straightforward consequence of (3.12).

Step 4. By using a standard diagonalization argument, from step 2 and step 3, there exists a mapping $\varepsilon \mapsto \delta(\varepsilon)$ such that $v_{\delta(\varepsilon)} \longrightarrow u$ in $L^2(\Omega)$ and

$$\lim_{\varepsilon \to 0} \left(\int_{\Omega} f(\nabla v_{\delta(\varepsilon)})(x) dx + \varepsilon^2 \int_{B} g(\widehat{\nabla} \tau_{\varepsilon} v_{\delta(\varepsilon)}, \frac{1}{\varepsilon} \frac{\partial \tau_{\varepsilon} v_{\delta(\varepsilon)}}{\partial x_N})(x) dx - \left\langle \mathcal{S}_{\varepsilon}, \tau_{\varepsilon} v_{\delta(\varepsilon)} \right\rangle \right) = F_0(u).$$

The sequence $(v_{\varepsilon})_{\varepsilon>0}$ where $v_{\varepsilon} := v_{\delta(\varepsilon)}$ fulfils all the conditions except the boundary condition on Γ_0 . From assumption dist $(\overline{\Gamma}_0, \overline{\partial B_{\varepsilon} \cap \partial \Omega}) > 0$, and by using a standard slicing method due to De Giorgi in a neighborhood of Γ_0 (see [4]), one can modify v_{ε} in Ω_{ε} into a function \tilde{v}_{ε} equal to v_{ε} in B_{ε} , satisfying the boundary condition on Γ_0 , and $\limsup_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} f(\nabla v_{\varepsilon}) dx = \limsup_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} f(\nabla \tilde{v}_{\varepsilon}) dx$. Still denoting by v_{ε} this new function, we have $\lim_{\varepsilon \to 0} F_{\varepsilon}(v_{\varepsilon}) = F_0(u)$ and the proof is complete.

Remark 3.6. In order to give an interpretation of the limit energy functional, it is worthwhile to write

$$\inf_{\theta \in X(u)} H(\theta) = \inf_{\theta \in V_0(B)} \left\{ \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}(x) + [u](\hat{x})) \, dx - \langle \mathcal{S}, \theta \rangle \right\} - \langle \mathcal{S}, \tilde{u} \rangle \tag{3.16}$$

where $[u] = u^+ - u^-$, $V_0(B) = \{\theta \in V(B) : \theta = 0 \text{ on } \Sigma \times \{\pm \frac{1}{2}\}\}$ and $\tilde{u}(x) = x_N[u](\hat{x}) + \frac{u^+(\hat{x}) + u^-(\hat{x})}{2}$. Therefore when the limit source \mathcal{S} vanishes on V(B), by using Jensen's inequality, $\inf_{\theta \in X(u)} H(\theta)$ reduces to

$$\bar{H}(u) = \int_{\Sigma} g^{\infty,2} \left(\hat{0}, [u](\hat{x}) \right) \, d\hat{x}$$

which is nothing but the surface energy of the model obtained in [9]. When the limit source is not trivial, by using the Euler equation associated with (3.16), it is easily seen that \bar{H} is a surface energy on Σ of the form

$$\bar{H}(u) = \int_{\Sigma} \bar{h}(\hat{x}, [u](\hat{x}), \frac{u^+ + u^-}{2}(\hat{x})) \ d\hat{x}.$$

In this case we note that the energy density depends explicitly of the mean of the traces and that the surface energy \bar{H} mixes the internal energy and the work of the loading.

4 Examples of measure sources S_{ε} concentrated in B_{ε}

The general form of elements of V'(B) is given for every θ in V(B) by $\langle \mathcal{S}, \theta \rangle = \int_B s_0 \theta \, dx + \int_B s_1 \frac{\partial \theta}{\partial x_N} \, dx$ where $(s_0, s_1) \in L^2(B) \times L^2(B)$. The limit sources \mathcal{S} considered in this section are generated by measures $\mathcal{S}^{\varepsilon}$ in $\mathbb{M}(B_{\varepsilon})$ whose slicing structure $\mathcal{H}^{N-1} \lfloor \Sigma \otimes \mathcal{S}^{\varepsilon}_{\hat{x}}$ is such that their slicing components $\mathcal{S}^{\varepsilon}_{\hat{x}}$ do not present a diffuse singular part in their Lebesgue-Nikodym decomposition in $\mathbb{M}(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, i.e., are of the general form

$$S_{\hat{x}}^{\varepsilon} = \frac{1}{\varepsilon} a_{\varepsilon}(\hat{x}, \varepsilon t) \ dt + \sum_{n = -\infty}^{+\infty} b_{\varepsilon,n}(\hat{x}) \delta_{\varepsilon t_n(\hat{x})}$$

where

$$\left\{ \begin{array}{l} a_{\varepsilon} \in L^{2}(B), \ b_{\varepsilon,n} \in L^{2}(\Sigma), \\ t_{n}: \Sigma \longrightarrow (-\frac{1}{2}, \frac{1}{2}) \ \text{is a Borel measurable map.} \end{array} \right.$$

Roughly, such sources S^{ε} are sums of a function in $L^2(B)$ and a countable sum of surface sources, each of them being concentrated in the N-1-dimensional surface included in B_{ε} whose graph is εt_n . We make the following additional assumptions:

- (H1) there exists $a \in L^2(B)$ such that $a_{\varepsilon} \to a$ in $L^2(B)$;
- (H2) there exists $b_n \in L^2(\Sigma)$ such that $b_{\varepsilon,n} \to b_n$ in $L^2(\Sigma)$ when $\varepsilon \to 0$;
- (H3) there exists $c_n \in \mathbb{R}^+$ such that $\|b_{\varepsilon,n}\|_{L^2(\Sigma)} \leq c_n$ and $\sum_{n=-\infty}^{+\infty} c_n < +\infty$;

It is easy to check that the measure $S_{\varepsilon} = \tau_{\varepsilon} S^{\varepsilon}$ of $\mathbb{M}(B)$ is given by: $S_{\varepsilon} = \mathcal{H}^{N-1} \lfloor \Sigma \otimes (S_{\varepsilon})_{\hat{x}}$ where

$$(\mathcal{S}_{\varepsilon})_{\hat{x}} = a_{\varepsilon}(\hat{x}, t) \ dt + \sum_{n = -\infty}^{+\infty} b_{\varepsilon, n}(\hat{x}) \delta_{t_n(\hat{x})}.$$

Proposition 4.1. The measure S_{ε} strongly converges in V'(B) to the measure S defined for every $\theta \in V(B)$ by

$$\langle \mathcal{S}, \theta \rangle = \int_{B} a(x)\theta(x) \, dx + \sum_{n=-\infty}^{+\infty} \int_{\Sigma} b_n(\hat{x})\theta(\hat{x}, t_n(\hat{x})) \, d\hat{x}$$

Therefore, the functional F_{ε} Γ -converges to the functional $F_0: L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$F_{0}(u) = \begin{cases} \int_{\Omega} f(\nabla u) \ dx + \inf_{\theta \in X(u)} \left\{ \int_{B} g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_{N}}) \ dx - \int_{B} a\theta \ dx - \sum_{n=-\infty}^{+\infty} \int_{\Sigma} b_{n}(\hat{x})\theta(\hat{x}, t_{n}(\hat{x})) \ d\hat{x} \right\} \\ +\infty \ otherwise. \end{cases}$$

Proof. The second assertion is a straightforward consequence of Theorem 3.3 provided that we establish the strong convergence of S_{ε} to S in V'(B). For every $\theta \in V(B)$ we have

$$\langle \mathcal{S}_{\varepsilon} - \mathcal{S}, \theta \rangle = \int_{B} (a_{\varepsilon} - a)\theta \, dx + \int_{\Sigma} \sum_{n = -\infty}^{+\infty} (b_{\varepsilon,n} - b_n)\theta(\hat{x}, t_n(\hat{x})) \, d\hat{x},$$

thus

$$\left|\left\langle \mathcal{S}_{\varepsilon} - \mathcal{S}, \theta \right\rangle\right| \le \left\|\theta\right\|_{L^{2}(B)} \left\|a_{\varepsilon} - a\right\|_{L^{2}(B)} + \sum_{n = -\infty}^{+\infty} \left[\left\|b_{\varepsilon, n} - b_{n}\right\|_{L^{2}(\Sigma)} \left(\int_{\Sigma} \left|\theta(\hat{x}, t_{n}(\hat{x})\right|^{2} d\hat{x}\right)^{\frac{1}{2}}\right].$$
(4.1)

But it is easy to establish that there exists a non negative constant C such that

$$\left(\int_{\Sigma} |\theta(\hat{x}, t_n(\hat{x}))|^2 d\hat{x}\right)^{\frac{1}{2}} \le C \, \|\theta\|_{V(B)}$$

so that (4.1) yields

$$\|\mathcal{S}_{\varepsilon} - \mathcal{S}\|_{V'(B)} \le \|a_{\varepsilon} - a\|_{L^{2}(B)} + C \sum_{n = -\infty}^{+\infty} \|b_{\varepsilon,n} - b_{n}\|_{L^{2}(\Sigma)}.$$

The conclusion follows from assumptions (H1), (H2) and (H3).

5 The gradient concentration phenomenon

We first recall the notion of gradient Young-concentration measure introduced in [3]. Let us denote the unit sphere $\{-1,1\}$ of \mathbb{R} by \mathbb{S}^0 , and consider $\Sigma' \subset \subset \Sigma$, $B'_{\varepsilon} := \Sigma' \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$.

Definition 5.1. A pair $(v, \mu_{\Sigma'}) \in L^2(\Omega) \times \mathbb{M}^+(\overline{\Omega} \times \mathbb{S}^0)$ is a gradient Young-concentration measure (localized on Σ') iff there exists a sequence $(v_{\varepsilon})_{\varepsilon>0}$ in $W^{1,2}_{\Gamma_0}(\Omega)$ satisfying

$$\begin{cases} \sup_{\varepsilon>0} \int_{\Omega\setminus B_{\varepsilon}} |\nabla v_{\varepsilon}|^2 \, dx < +\infty, \\ v_{\varepsilon} \to v \ in \ L^2(\Omega), \\ \mu_{\varepsilon} := \delta_{\frac{\partial v_{\varepsilon}}{\partial x_N} / \left| \frac{\partial v_{\varepsilon}}{\partial x_N} \right| \ (x)} \otimes \varepsilon \mathbb{1}_{B'_{\varepsilon}} |\frac{\partial v_{\varepsilon}}{\partial x_N}|^2 \, dx \stackrel{*}{\rightharpoonup} \mu_{\Sigma'}. \end{cases}$$

We say that the sequence $(v_{\varepsilon})_{\varepsilon>0}$ generates the gradient Young-concentration measure $(v, \mu_{\Sigma'})$. We denote the set of gradient Young-concentration measures localized on Σ' by $\mathcal{YC}(\Sigma')$.

Recall that the weak convergence $\stackrel{*}{\rightharpoonup}$ above is defined by

$$\int_{\bar{\Omega}} \int_{\mathbb{S}^0} \theta(x) \varphi(\zeta) d\mu_{\varepsilon} = \int_{B'_{\varepsilon}} \varepsilon \theta(x) \tilde{\varphi}(\frac{\partial v_{\varepsilon}}{\partial x_N}) \ dx \to \int_{\bar{\Omega}} \int_{\mathbb{S}^0} \theta(x) \varphi(\zeta) \ d\mu_{\Sigma'}$$

for all $\theta \in \mathcal{C}(\bar{\Omega})$ and all $\varphi \in \mathcal{C}(\mathbb{S}^0)$, where the 2-homogeneous extension $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of $\varphi \in \mathcal{C}(\mathbb{S}^0)$ is defined for all $\zeta \in \mathbb{R}^m$ by

$$\tilde{\varphi}(\zeta) = \begin{cases} |\zeta|^2 \varphi(\frac{\zeta}{|\zeta|}), & \text{if } \zeta \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In [3], Theorem 3.1, the gradient Young-concentration measures was characterized as follows.

Theorem 5.2 (Characterization). A pair $(v, \mu_{\Sigma'} = \mu_x \otimes \pi)$ belongs to $\mathcal{YC}(\Sigma')$ if and only if $v \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, π is concentrated on $\bar{\Sigma}'$ and, for every $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$,

$$\frac{d\pi}{d\mathcal{H}^{N-1}\lfloor\Sigma'}(x)\int_{\mathbb{S}^0}\varphi(\zeta)\ d\mu_x \ge \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1} \ a. \ e. \ x \in \Sigma'$$

$$\int_{\mathbb{S}^0}\varphi(\zeta)\ d\mu_x \ge 0 \quad \text{for } \pi_s \ a. \ e. \ x \in \bar{\Sigma}'$$
(5.1)

where $\pi = \frac{d\pi}{d\mathcal{H}^{N-1}\lfloor \Sigma'} \mathcal{H}^{N-1} \lfloor \Sigma' + \pi_s$ is the Radon-Nikodym decomposition of π with respect to the measure $\mathcal{H}^{N-1} \lfloor \Sigma'$.

Remark 5.3. Although from (3.3), $\delta_{\frac{\partial v_{\varepsilon}}{\partial x_N}} |_{\frac{\partial v_{\varepsilon}}{\partial x_N}} \otimes \varepsilon \mathbb{1}_{B_{\varepsilon}} |_{\frac{\partial v_{\varepsilon}}{\partial x_N}} |^2 dx$ possesses weak cluster points in the sense of the weak convergence $\stackrel{*}{\longrightarrow}$ made precise above, for technical reason (proof of the sufficient conditions in Proposition 3.5 in [3]), it was not possible to state such a characterization for these cluster points because of possible concentration effects on the boundary of Σ . This is the reason why we deal with gradient Young-concentration measures localized on $\Sigma' \subset \subset \Sigma$.

Taking into account that the 2-homogeneous extension $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of $\varphi \in \mathcal{C}(\mathbb{S}^0)$ satisfying $\varphi^{**} > -\infty$ is of the form

$$\varphi(\zeta) = \begin{cases} c\zeta^2 \text{ if } \zeta \ge 0\\ d\zeta^2 \text{ if } \zeta \le 0, \end{cases}$$

with $(c, d) \in \mathbb{R}^+ \times \mathbb{R}^+$, the above characterization theorem can be reduced to the following (cf Corollary 3.6 in [3])

Corollary 5.4. A measure $(v, \mu_{\Sigma'} = (a(x)\delta_1 + b(x)\delta_{-1}) \otimes \pi)$ belongs to $\mathcal{YC}(\Sigma')$ if and only if $v \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, π is concentrated on $\overline{\Sigma}'$ and

$$\frac{d\pi}{d\mathcal{H}^{N-1}\lfloor\Sigma'}(x)\big(a(x)c+b(x)d\big) \ge \varphi([v](x)) \text{ for } \mathcal{H}^{N-1}\lfloor\Sigma' \text{ a.e. } x \text{ and for all } (c,d) \in \mathbb{R}^+ \times \mathbb{R}^+$$

where $\varphi(\zeta) = \begin{cases} c\zeta^2 & \text{if } \zeta \ge 0 \\ d\zeta^2 & \text{if } \zeta \le 0 \end{cases}$

As stated in [3] Remark 2.5, every sequence $(u_{\varepsilon})_{\varepsilon>0}$ satisfying (3.3) generates a gradient Youngconcentration measure. Therefore every sequence $(\bar{u}_{\varepsilon})_{\varepsilon>0}$, $\bar{u}_{\varepsilon} \in \operatorname{argmin} F_{\varepsilon}$, generates a measure $\bar{\mu}_{\Sigma'} \in \mathcal{YC}(\Sigma')$. Let \bar{u} be a strong limit of $(\bar{u}_{\varepsilon})_{\varepsilon>0}$ in $L^2(\Omega)$, then, under the condition $g^{\infty,2}(\hat{\xi},\xi_3) \geq g^{\infty,2}(\hat{0},\xi_3)$, the next theorem states that the internal term $H_{in}(\bar{\theta})$ (cf (3.1)) where $\bar{\theta}$ is the solution of $\inf_{\theta \in X(\bar{u})} H(\theta)$, possesses an integral representation with respect to the Young-concentration measure $\bar{\mu}_{\Sigma'}$. In some sense we localize H_{in} on $S \times \{\pm 1\}$. Moreover, by using Theorem 5.2 we will deduce some bounds on $\bar{\mu}_{\Sigma'}$.

Theorem 5.5. Let \bar{u}_{ε} be a minimizer of min $\{F_{\varepsilon}(v) : v \in L^2(\Omega)\}$ and, for every $\Sigma' \subset \subset \Sigma$, $(\bar{u}, \bar{\mu}_{\Sigma'})$ be a gradient Young-concentration measure localized on Σ' generated by the sequence $(\bar{u}_{\varepsilon})_{\varepsilon>0}$. Then the two following assertions hold:

- i) $\bar{u}_{\varepsilon} \to \bar{u} \text{ in } L^2(\Omega), \ F_{\varepsilon}(\bar{u}_{\varepsilon}) \to F_0(\bar{u}) = \min\left\{F_0(u) : u \in L^2(\Omega)\right\};$
- ii) Let \mathcal{F} be a countable familly of $\Sigma' \subset \subset \Sigma$, then there exists $\bar{\mu} \in \mathbb{M}(\bar{\Omega} \times \mathbb{S}^0)$, $\bar{\mu} = \bar{\mu}_{\hat{x}} \otimes \bar{\pi}$ with $\bar{\pi}$ concentrated on $\bar{\Sigma}$ such that for all $\Sigma' \in \mathcal{F}$, $\bar{\mu} \lfloor \bar{\Sigma}' \times \mathbb{S}^0 = \bar{\mu}_{\Sigma'}$. Assume furthermore that $g^{\infty,2}$ satisfies the condition

$$\forall \xi \in \mathbb{R}^3, \ g^{\infty,2}(\hat{\xi},\xi_3) \ge g^{\infty,2}(\hat{0},\xi_3).$$
(5.2)

Then, every weak cluster point $\bar{\theta}$ of the sequence $(\tau_{\varepsilon}\bar{u}_{\varepsilon})_{\varepsilon>0}$ in V(B) satisfies $H(\bar{\theta}) = \inf_{\theta \in X(\bar{u})} H(\theta)$ and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\hat{0}, \frac{\partial\bar{\theta}}{\partial x_N})(\hat{x}, s) \, ds = \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \, \text{for a.e. } \hat{x} \, \text{in } \Sigma';$$

$$H_{in}(\bar{\theta}) = \int_{\Sigma} \left[\frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right] \, d\hat{x}.$$
(5.3)

Proof. According to the variational nature of the Γ -convergence, for a subsequence one has

$$\begin{aligned}
\bar{u}_{\varepsilon} &\to \bar{u} \text{ in } L^{2}(\Omega) \\
\lim_{\varepsilon \to 0} F_{\varepsilon}(\bar{u}_{\varepsilon}) &= F_{0}(\bar{u}) = \min\left\{F_{0}(v) : v \in L^{2}(\Omega)\right\} \\
&= \int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\theta \in X(\bar{u})} H(\theta).
\end{aligned}$$
(5.4)

Fix $\Sigma' \subset \subset \Sigma$. From (3.3), for the subsequence (possibly dependent on Σ') associated with the gradient Young-concentration measure $(\bar{u}, \bar{\mu}_{\Sigma'})$, there exist a subsequence and a measure $\bar{\mu} = \bar{\mu}_{\hat{x}} \otimes \bar{\pi}$ in $\mathbb{M}(\bar{\Omega} \times \mathbb{S}^0)$ with $\bar{\pi}$ concentrated in $\bar{\Sigma}$, such that

$$\delta_{\frac{\partial \bar{u}_{\varepsilon}}{\partial x_N}/\left|\frac{\partial \bar{u}_{\varepsilon}}{\partial x_N}\right|}(x)} \otimes \varepsilon \mathbb{1}_{B_{\varepsilon}} |\frac{\partial \bar{u}_{\varepsilon}}{\partial x_N}|^2 \ dx \rightharpoonup \bar{\mu}$$

Thus, from (3.11) and (5.2) we infer

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_B g\left(\widehat{\nabla}\tau_\varepsilon \bar{u}_\varepsilon, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}\right) \, dx = \lim_{\varepsilon \to 0} \int_B g^{\infty,2} \left(\varepsilon \widehat{\nabla}\tau_\varepsilon \bar{u}_\varepsilon, \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}\right) \, dx$$
$$\geq \lim_{\varepsilon \to 0} \int_{B_\varepsilon} g^{\infty,2} \left(\hat{0}, \frac{\partial \bar{u}_\varepsilon}{\partial x_N}\right) \, dx$$
$$= \int_{\widehat{\Sigma}} \left(\int_{\mathbb{S}^0} g^{\infty,2} (\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}}\right) \, d\bar{\pi}. \tag{5.5}$$

Let $\bar{\theta}$ be the weak limit of $\tau_{\varepsilon} \bar{u}_{\varepsilon}$ in V(B) for the considered subsequence. Then, from (5.5) and since

$$\liminf_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} f(\nabla \bar{u}_{\varepsilon}) \ dx \ge \int_{\Omega} f(\nabla \bar{u}) \ dx \text{ and } \lim_{\varepsilon \to 0} \langle \mathcal{S}_{\varepsilon}, \tau_{\varepsilon} \bar{u}_{\varepsilon} \rangle = \langle \mathcal{S}, \bar{\theta} \rangle,$$

we infer

$$\lim_{\varepsilon \to 0} F_{\varepsilon}(\bar{u}_{\varepsilon}) \ge \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\bar{\Sigma}} \left(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi_3) \, d\bar{\mu}_{\hat{x}} \right) \, d\bar{\pi} - \langle \mathcal{S},\bar{\theta} \rangle.$$
(5.6)

Collecting (5.4) and (5.6) we obtain

$$\int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\theta \in X(\bar{u})} H(\theta) \ge \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\bar{\Sigma}} \left(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi_3) \, d\bar{\mu}_{\hat{x}} \right) \, d\bar{\pi} - \langle \mathcal{S},\bar{\theta} \rangle,$$

in particular

$$\int_{\Omega} f(\nabla \bar{u}) \, dx + H(\bar{\theta}) \ge \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\bar{\Sigma}} \left(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi_3) \, d\bar{\mu}_{\hat{x}} \right) \, d\bar{\pi} - \langle \mathcal{S},\bar{\theta} \rangle,$$

 ${\rm thus}$

$$\int_{B} g^{\infty,2} \left(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_{N}} \right) dx \geq \int_{\bar{\Sigma}} \left(\int_{\mathbb{S}^{0}} g^{\infty,2} (\hat{0}, \xi_{3}) d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} \\
\geq \int_{\bar{\Sigma}} \frac{d\bar{\pi}}{d\hat{x}} (\hat{x}) \left(\int_{\mathbb{S}^{0}} g^{\infty,2} (\hat{0}, \xi_{3}) d\bar{\mu}_{\hat{x}} \right) d\hat{x}.$$
(5.7)

On the other hand, by a standard lower semicontinuity argument, for every $\varphi \in \mathcal{C}_c(\Sigma), \varphi \geq 0$,

$$\liminf_{\varepsilon \to 0} \int_{B} \varphi(\hat{x}) g^{\infty,2} \left(\hat{0}, \frac{\partial \tau_{\varepsilon} \bar{u}_{\varepsilon}}{\partial x_{N}} \right) \, dx = \int_{\Sigma} \varphi(\hat{x}) \left(\int_{\mathbb{S}^{0}} g^{\infty,2}(\hat{0}, \xi_{3}) \, d\bar{\mu}_{\hat{x}} \right) \, d\bar{\pi}$$
$$\geq \int_{B} \varphi(\hat{x}) g^{\infty,2} \left(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_{N}} \right) \, dx$$

so that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2} (\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N}) (\hat{x}, s) \, ds \le \frac{d\bar{\pi}}{d\hat{x}} (\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2} (\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \quad \text{for a.e. } \hat{x} \in \Sigma.$$

$$(5.8)$$

Combining (5.7) and (5.8) we deduce

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\hat{0},\frac{\partial\bar{\theta}}{\partial x_N})(\hat{x},s) \ ds = \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi_3) \ d\bar{\mu}_{\hat{x}} \quad \text{for a.e. } \hat{x} \in \Sigma.$$

Clearly, $\bar{\mu}\lfloor\bar{\Sigma}'\times\mathbb{S}^0=\bar{\mu}_{\Sigma'}$. Now, by using a standard Cantor's diagonal process, the same equality holds for all Σ' of the countable family \mathcal{F} . It remains to show that $H(\bar{\theta}) = \inf_{\theta \in X(\bar{u})} H(\theta)$. It's enough to notice that

$$\begin{split} \lim_{\varepsilon \to 0} F_{\varepsilon}(\bar{u}_{\varepsilon}) &= \int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\theta \in X(\bar{u})} H(\theta) \\ &\geq \liminf_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} f(\nabla \bar{u}_{\varepsilon}) \, dx + \liminf_{\varepsilon \to 0} \left(\varepsilon^{2} \int_{B} g(\widehat{\nabla} \tau_{\varepsilon} \bar{u}_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial \tau_{\varepsilon} \bar{u}_{\varepsilon}}{\partial x_{N}} \right) \, dx - \langle \mathcal{S}_{\varepsilon}, \tau_{\varepsilon} \bar{u}_{\varepsilon} \rangle \Big) \\ &\geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{B} g^{\infty, 2} (\hat{0}, \frac{\partial \bar{\theta}}{\partial x_{N}}) \, dx - \langle \mathcal{S}, \bar{\theta} \rangle \\ &= \int_{\Omega} f(\nabla \bar{u}) \, dx + H(\bar{\theta}) \end{split}$$

which completes the proof.

We define the following two constants associated with the function g:

$$c(g) := \min\Big(\frac{g^{\infty,2}(\widehat{0},-1)}{g^{\infty,2}(\widehat{0},1)}, \frac{g^{\infty,2}(\widehat{0},1)}{g^{\infty,2}(\widehat{0},-1)}\Big), \ C(g) = \frac{1}{c(g)} = \max\Big(\frac{g^{\infty,2}(\widehat{0},-1)}{g^{\infty,2}(\widehat{0},1)}, \frac{g^{\infty,2}(\widehat{0},1)}{g^{\infty,2}(\widehat{0},-1)}\Big).$$

Recall that

$$g^{\infty,2}(\widehat{0},\xi) = \begin{cases} g^{\infty,2}(\widehat{0},-1) \, |\xi|^2 & \text{if} \quad \xi \le 0\\ g^{\infty,2}(\widehat{0},1) \, |\xi|^2 & \text{if} \quad \xi > 0 \end{cases}$$

Moreover, from the assumption on the function g, clearly,

$$g^{\infty,2}(\widehat{0},1) > 0$$
 and $g^{\infty,2}(\widehat{0},-1) > 0$.

We make precise the probability measure $\bar{\mu}_{\hat{x}}$ localized on $\Sigma' \subset \subset \Sigma$ as follows:

$$\bar{\mu}_{\hat{x}} := p(\hat{x})\delta_1 + q(\hat{x})\delta_{-1}$$
 with $p(\hat{x}) + q(\hat{x}) = 1$ a.e. $\hat{x} \in \Sigma'$.

Corollary 5.6. Under the assumptions of Theorem 5.5, the three following estimates hold:

(i) for a.e. \hat{x} in Σ'

$$c(g)\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds \le \frac{d\overline{\pi}}{dx_N}(\hat{x}) \le C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds, \tag{5.9}$$

$$and \ \frac{d\bar{\pi}}{dx_N}(\hat{x}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 \ ds \quad when \ g^{\infty, 2}(\hat{0}, -1) = g^{\infty, 2}(\hat{0}, 1);$$

$$(ii) \ \frac{c(g) \ |[\bar{u}](\hat{x})|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 \ ds} \le p(\hat{x}) \le 1 \ for \ a.e. \ \hat{x} \ such \ that \ [\bar{u}](\hat{x}) > 0;$$

$$(iii) \quad \frac{c(g) |[\bar{u}](\hat{x})|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial \bar{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds} \le q(\hat{x}) \le 1 \text{ for a.e. } \hat{x} \text{ such that } [\bar{u}](\hat{x}) < 0.$$

Proof. Since $\bar{\mu}_{\hat{x}} = p(\hat{x})\delta_1 + q(\hat{x})\delta_{-1}$, we have $\int_{\mathbb{S}^0} g^{\infty,2}(\xi)d\bar{\mu}_{\hat{x}} = p(\hat{x})g^{\infty,2}(\hat{0},1) + q(\hat{x})g^{\infty,2}(\hat{0},-1)$ with $p(\hat{x}) + q(\hat{x}) = 1$ a.e. \hat{x} in Σ' so that from (5.3), one has

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0}, \frac{\partial \overline{\theta}}{\partial x_N}(\widehat{x}, s)) ds = \left(\int_{\mathbb{S}^0} g^{\infty,2}(\xi) d\overline{\mu}_{\widehat{x}} \right) \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})$$
$$= \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \left\{ p(\widehat{x})g^{\infty,2}(\widehat{0}, 1) + q(\widehat{x})g^{\infty,2}(\widehat{0}, -1) \right\} \text{ a.e. } \widehat{x} \in \Sigma'.$$
(5.10)

We are going to establish

$$c(g)\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \le \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \le C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds$$

From (5.10) we deduce that

$$\begin{split} \min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_{N}}(\widehat{x},s)\right|^{2}ds &\leq \int_{-\frac{1}{2}}^{\frac{1}{2}}g^{\infty,2}(\widehat{0},\frac{\partial\overline{\theta}}{\partial x_{N}}(\widehat{x},s))ds \\ &= \left(\int_{\mathbb{S}^{0}}g^{\infty,2}(\xi)d\overline{\mu}_{x}\right)\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ &= \left\{p(\widehat{x})g^{\infty,2}(\widehat{0},1)+q(\widehat{x})g^{\infty,2}(\widehat{0},-1)\right\}\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ &\leq \max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \quad (5.11) \end{split}$$

and

$$\min\left\{g^{\infty,2}(\widehat{0},-1), \ g^{\infty,2}(\widehat{0},1)\right\} \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) = \min\left\{g^{\infty,2}(\widehat{0},-1), \ g^{\infty,2}(\widehat{0},1)\right\} \{p(\widehat{x})+q(\widehat{x})\} \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ \leq \left\{p(\widehat{x})g^{\infty,2}(\widehat{0},1)+q(\widehat{x})g^{\infty,2}(\widehat{0},-1)\right\} \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ = \left(\int_{\mathbb{S}^{0}} g^{\infty,2}(\xi)d\overline{\mu}_{x}\right) \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ = \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0},\frac{\partial\overline{\theta}}{\partial x_{N}}(\widehat{x},s))ds \\ \leq \max\left\{g^{\infty,2}(\widehat{0},-1), \ g^{\infty,2}(\widehat{0},1)\right\} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_{N}}(\widehat{x},s)\right|^{2} d\xi(5.12)$$

Then, from (5.11) and (5.12) we have

$$c(g)\int_{\frac{-1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds = \frac{\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}{\max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \le \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x},s)$$

and

$$\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \le \frac{\max\left\{g^{\infty,2}(\widehat{0},-1), \ g^{\infty,2}(\widehat{0},1)\right\}}{\min\left\{g^{\infty,2}(\widehat{0},-1), \ g^{\infty,2}(\widehat{0},1)\right\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds = C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds$$

from which we deduce,

$$c(g)\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \leq \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \leq C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds.$$

Let us prove (ii) and (iii). According to Theorem 5.2, for every $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$,

$$\frac{d\bar{\pi}}{d\mathcal{H}_{\lfloor\Sigma'}^{N-1}}(x)\int_{\mathbb{S}^0}\varphi(\zeta)\ d\bar{\mu}_x \ge \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1}\text{a.e. } x \in \Sigma',$$
(5.13)

where $\bar{\pi} = \frac{d\bar{\pi}}{d\mathcal{H}_{\lfloor\Sigma'}^{N-1}}\mathcal{H}_{\lfloor\Sigma'}^{N-1} + \bar{\pi}_s$ is the Radon-Nikodym decomposition of $\bar{\pi}$ with respect to the measure $\mathcal{H}_{\lfloor\Sigma'}^{N-1}$. We assume that $[\bar{u}](\hat{x}) > 0$ and show that

$$\begin{split} & \frac{c(g) \ \left| \left[\bar{u} \right] (\hat{x}) \right|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N} (\hat{x}, s) \right|^2 \ ds} \leq p(\hat{x}) \leq 1. \\ & \varphi(\xi) = \begin{cases} & \varphi(1) \left| \xi \right|^2 & \text{if} \quad \xi \geq 0 \\ & 0 & \text{if} \quad \xi < 0 \end{cases} . \end{split}$$

(5.14)

Let

Clearly,
$$\varphi^{**}([\bar{u}](\hat{x})) = \varphi([\bar{u}](\hat{x})) = \varphi(1) ||u|(\hat{x})|^2$$
. From the inequality (5.13), it follows that

$$\begin{split} \varphi(1) \ \left| \left[\bar{u} \right](\hat{x}) \right|^2 &= \varphi^{**}(\left[\bar{u} \right](\hat{x})) \\ &\leq \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \left(\int_{\mathbb{S}^0} \varphi(\xi) d\overline{\mu}_x \right) \\ &\leq C(g) \Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds \Big) \Big(\int_{\mathbb{S}^0} \varphi(\xi) d\overline{\mu}_x \Big) \\ &= C(g) p(\widehat{x}) \varphi(1) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds. \end{split}$$

Then, we obtain

$$\frac{\left|\left[\bar{u}\right](\hat{x})\right|^2}{C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial \overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds} = \frac{c(g) \left|\left[\bar{u}\right](\hat{x})\right|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial \overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds} \le p(\hat{x}) \le 1$$

The proof of (iii) is similar.

6 References

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