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► **To cite this version:**

Jyothiniranjana Pillay, Camille Wacal. On Curvature and the Navier Stoke Equations. (A Letter). 2012. hal-00727151v7

**HAL Id: hal-00727151**

**<https://hal.science/hal-00727151v7>**

Preprint submitted on 21 May 2013

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# On Curvature and the Navier Stoke Equations. (A Letter)

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## Abstract.

We introduce herein ideas surrounding curvature and its relation to the Navier Stoke equations. We will show the link that exists between the mechanism involved in the formulation of a solution to such PDE's and conditions necessary thereof, in the production of a blow up solution.

## Introduction.

The major concern surrounding the Navier Stoke Equations (NS), which we remind the reader follows as:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + f \quad (1)$$

where  $(f_o, \rho_o, \mathbf{u}_o)$  represent arbitrary initial forces, pressure and the initial velocity vector field respectively, Is the possibility of there being blow up solutions associated with  $(\rho, \mathbf{u})$  the functions obtained upon the solution of the NS equations for the pressure and vlocity evolutions respectively.

A blow up solution occurs when the solution to a system of partial differential equations produces functions that are asymptotic over some region of its domain. As these equations are of significance in fluid mechanics, smoothness requires that such blow up solutions are impossible[1]. Understanding what might cause such solutions is one part of understanding what it is that might be required of the initial conditions, in order to produce such solutions. Knowing this one can check whether such initial conditions are even possible.

There are many works that are related to understanding generalized solutions to such PDE's[2][3]. Attempts at understanding 3D-case of (NS) include Solutions of the Navier-Stokes and Euler equations with initial conditions (Cauchy problem) for 2D and 3D cases which were obtained in the converging series form by analytical iterative method using Fourier and Laplace transforms in a paper [4].

Our aim is to formulate the problem in a sense of what might be required of the mechanical building blocks involved in the production of a blow up solution. More clearly, we will lay foundations for the investigation into the

nature of such solutions via its mechanical association with curvature.

### Outline

Many entangled factors come into play when considering the possibility of a blow-up solution. The nature of  $f_0$  (arbitrary body forces), the initial velocity vector field and also the nature and setup of the PDE's itself.

To investigate this, assume a blow up solution that we associate with either of  $(\rho, \mathbf{u})$ , the functions obtained upon the solution of the NS equations for the pressure and velocity evolutions respectively. For this to be the case, the following criterion is necessary.

Given a finite continuous region  $R := \{\mathbf{r}_i \in \mathbb{R}^n\}$  over which either  $(\rho, \mathbf{u})$  blows up, it is required that:

$$\frac{\mathbf{u}(\mathbf{r}_1) - \mathbf{u}(\mathbf{r}_2)}{\mathbf{r}_1 - \mathbf{r}_2} > \epsilon(\mathbf{r}) \quad (2)$$

$\forall \mathbf{r}_i, \mathbf{r}_j \in \Phi^{-1} | \Phi = \mathbf{u} \cap P$  where  $P$  is a vertical plane through  $\mathbf{r}_k | \lim_{r \rightarrow r_k} \Phi(\mathbf{r}) \rightarrow \infty$ .

Here, we associate  $\epsilon(\mathbf{r})$  with some minimal gradient  $\kappa$  required of either  $(\rho, \mathbf{u})$  over  $R$ .

The nature of such pressure and vector fields will naturally depend on its initial form and its incremental change in the initial fields expressed in terms of a  $t$ -variable. More specifically, the differential above pertains to  $t$ , and as such the requirements will be of the form:  $(\mathbf{u}_x, \mathbf{u}_t)$  where it is the  $t$ -component that the incremental change-requirement is relayed into.

### A Logical Discourse

The behavior of such PDE's can be described as follows:

Any magnitude  $\mathcal{M}$  at  $t_0$  adjoined to a differential  $\frac{\partial \mathbf{u}}{\partial t}$ ; in this case:

$$\mathcal{M} = (\mathbf{u} \cdot \nabla) \mathbf{u} - \left(-\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + f\right) \quad (3)$$

is expressed as a functional of the variable with respect to which the functional is being differentiated, in this instance  $t$ . The nature of the functional  $F(t)$  (i.e. the solution at increment  $(t + \delta, t)$ ) depends solely on the magnitude of the change  $\sum_{\forall i} \partial^i \mathcal{M}$ . This magnitude we know is entangled with  $F(i)$  of neighboring points  $x, x + \delta x$ . More specifically, the expression involving  $\mathbf{u}_\rho, t$  that describes the variation over  $x, x + \delta x$  is incorporated into the magnitudes over  $x + \delta x, x + \delta x + \delta x$  as well. It is this entanglement that makes ascertaining the necessary curvatures required of  $(\mathbf{u}^\rho, \mathbf{f})$  in order for

the existence of a blow up solution difficult.

The mechanism associated with any differential is simple, it describes a difference as measured of functionals over neighboring points in some  $R \in \mathbb{R}^n$  via the use of a functional of the form  $F(t)$ . For instance, if the magnitudes  $\sum_{\forall i} \partial^i \mathcal{M}$  were expressible via  $t$  as  $t^4$  then the functional  $F(t)$  would roughly be expressed as:  $ct^5$ . What we are interested in is a blow up initiator point, specifically one that results in a change that is expressed as some asymptotic functional  $\frac{1}{t-c}$ .

In any one instance should the incremental change be expressible as some asymptotic functional, then let us consider the ramifications under the conditions as outlined for the NS above. Should such a change result (with all previous changes expressible as non asymptotic changes), then as explained above, this change (expressed in terms of the variable  $t$ ) would incorporate itself into the PDE for use in the evaluation in its next evaluation. Consider then the many ways in which this can happen if all the terms that such an increment  $k$  adjoins to is non asymptotic, for instance polynomial like (by this we mean that the values associated with such polynomial like terms when differentiated w.r.t  $t$  over any interval yields non asymptotic results.). We can expect  $\mathcal{M}$  to involve terms of the form :  $D(p_1k + p_2k) + p_3k + k$ , for some differential  $D$ , involving its previous non-asymptotic changes (perhaps polynomial like)  $p_i$ . As differentials express change , using simple algebraic techniques, one can deduce that for expressions such as the one associated with  $\mathcal{M}$  above, the change will in he next increment be asymptotic, and for the same reasons, this will be true of all further increments; unless new parameters (asymptotic in nature) associated with  $(f_o, \rho_0, \mathbf{u}_0)$  occur.

To reiterate, any non-reciprocal (for instance) polynomial like  $F(t)$  adjoined with  $\mathbf{u}^o$  will again result (logically speaking) in another polynomial like expression expressing the difference over neighboring points. This can be seen more clearly if one notices that there are only so many ways in which one can adjoin a polynomial like  $F(t)$  to  $\mathbf{u}^o$ , and once accomplished, a difference, which is essentially what a differential is, is incapable of reducing polynomial expressions to reciprocal ones. Thus it is easy to see that blow up solutions can only occur in such constructions if  $\sum_{\forall i} \partial^i \mathcal{M}$  at some point is specifically of the general magnitude of the form  $S := \{\frac{1}{t}\} | t \in R \in \mathbb{R}^n$ . knowing the magnitudes of  $\mathcal{M}(x) - \mathcal{M}(x + \delta x)$  i.e. specifically of  $\mathbf{u}^o, \mathbf{f}$ , one can ascertain the general nature of the functional  $F(t)$ . Logically, this will tell us a lot about whether a blow up ripple boundary of points having differences in the proportion of  $\frac{i}{k-c}$  exists, and if so we need only notice that in such

setups, since differences of the form :  $u_x P(t) + .. - u_x \frac{1}{k-c}$  can never result in polynomial forms which indicates that at such ripples blow up solutions are possible.

”At present, all known methods for obtaining global smooth solutions to a (deterministic) nonlinear PDE Cauchy problem require either:

- 1) Exact and explicit solutions (or at least an exact, explicit transformation to a significantly simpler PDE or ODE);
- 2) Perturbative hypotheses (e.g. small data, data close to a special solution, or more generally a hypothesis which involves an  $\epsilon$  somewhere); or
- 3) One or more globally controlled quantities (such as the total energy) which are both coercive and either critical or subcritical.” [5]

### Argument in a nutshell.

A mechanistic approach involves analysis of the nature and actual intrinsic method by which a functional solution in  $(t)$  of a PDE is formed. As we saw from our previous discourse, over every interval  $(t, t + \epsilon)$ , a functional  $F(t)$  is adjoined to  $P_{DE}(\bar{X}, t_o)|\bar{X}$  forming a new functional which expresses the change  $\Delta t Q$ , for some changing quantity  $Q$ . We need a clear understanding how exactly this happens will be the task that follows. Our intensions are precisely to show that a solution formed at each interval<sup>1</sup> (under certain conditions) leads again to a solution, which when convoluted into the constituents of  $\mathcal{M}$  will again lead to a non-asymptotic solution over the next interval, having properties similar to those of its preceeding solution.

Given an arbitraty initial velocity field  $\vec{\mathbf{u}}_0 = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ .

We have

$$\frac{\partial \mathbf{u}}{\partial t} = \mathcal{M} \tag{4}$$

The variable  $t$ , is used in conjunction with constants  $k_i$  and  $\vec{u}_0$  to produce a function, that when differentiated produces  $\mathcal{M}$  at  $t$ , or over the interval  $(t, t + \delta)$ .

This is done by arithmetically convoluting  $(t, k_i, \vec{u}_0)$ , written :  $(\vec{u}_0) \otimes t \otimes k_i$ . An example of this is :  $x(t) + t\mathbf{i} + y(t)/t\mathbf{j} + z(t)2t\mathbf{k}$ .

Given that at  $t = 0$  the field  $\vec{u}$  is uniform and not asymptotic over any region, (and this is the crux of the matter) we have then that at the boundry  $(t = 0)$ ,  $\frac{\partial \mathbf{u}}{\partial t}$ , yields a non-asymptotic gradient, then the convolution of the variable  $t$  into  $(\vec{u}_0) \otimes t \otimes k_i$ , need not be in such a manner that produces a

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<sup>1</sup>infinitesimally small

field which when differentiated, yields an asymptotic difference.

It is almost certain then, that terms such as  $\vec{u}\nabla\vec{u}$  forming the constituents of  $\mathcal{M}$  will not be asymptotic, thus  $\mathcal{M}$ , evaluated over the next immediate interval  $(t + \delta, t + 2\delta)$  used for the evaluation of  $\frac{\partial\mathbf{u}}{\partial t}$  over this interval which inturn is used to find the convolution :  $(\vec{u}_0) \otimes t \otimes k_i$ , will again produce a non-asymptotic  $\frac{\partial\mathbf{u}}{\partial t}, (\vec{u}_0) \otimes t \otimes k_i$ , respectively over  $(t + \delta, t + 2\delta)$ . Since  $t$  here is arbitrary, this argument is everywhere applicable.

The nature of the possible convolutions  $(\vec{u}_0) \otimes t \otimes k_i$  lie within a field, and it is the study of this field, that will be the focus of a followup article.

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## Acknowledgements

We would like to thank the University of Pretoria Dept Physics and ActuarialAcuity for their support and Prof. Adattu (PSBA) for his support over the years.