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On the adaptive wavelet estimation of a multidimensional
regression function under α -mixing dependence:
Beyond the boundedness assumption on the noise

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Abstract

We investigate the wavelet estimation of a multidimensional regression function f from n observations of a α -mixing process (Y, X) , where $Y = f(X) + \xi$, X represents the design and ξ denotes the noise. In most papers considering this problem either the proposed wavelet estimator is not adaptive (i.e., it depends on the knowledge of the smoothness of f in its construction) or it is supposed that the noise is bounded (excluding the Gaussian case). In this paper we go far beyond this classical framework. Under no boundedness assumption on the noise, we construct two kinds of adaptive term-by-term thresholding wavelet estimators enjoying powerful mean integrated squared error (MISE) properties. More precisely, we prove that they achieve "sharp" rates of convergence under the MISE over a wide class of functions f . "Sharp" in the sense that they coincide with the optimal rate of convergence in the standard *i.i.d.* case up to extra logarithmic terms.

Key words and phrases: Nonparametric regression, α -mixing dependence, Adaptive estimation, Wavelet methods, Rates of convergence.

AMS 2000 Subject Classifications: 62G07, 62G20.

1 Introduction

The nonparametric multidimensional regression model with uniform design can be described as follows. Let $(Y_t, X_t)_{t \in \mathbb{Z}}$ be a strictly stationary random process defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, where

$$Y_t = f(X_t) + \xi_t, \quad (1.1)$$

$f : [0, 1]^d \rightarrow \mathbb{R}$ is the unknown regression function, d is a positive integer, X_1 follows the uniform distribution on $[0, 1]^d$ and $(\xi_t)_{t \in \mathbb{Z}}$ is a strictly stationary centered random process independent of $(X_t)_{t \in \mathbb{Z}}$. Given n observations $(Y_1, X_1), \dots, (Y_n, X_n)$ drawn from $(Y_t, X_t)_{t \in \mathbb{Z}}$, we aim to estimate f globally on $[0, 1]^d$. The performance of an estimator \hat{f} of f can be evaluated by the mean integrated squared error (MISE) defined by

$$\mathbf{R}(\hat{f}, f) = \mathbf{E} \left(\int_{[0, 1]^d} (\hat{f}(x) - f(x))^2 dx \right),$$

where \mathbf{E} denotes the expectation. The smaller $\mathbf{R}(\hat{f}, f)$ is for a large class of f , the better \hat{f} is.

In this study, we focus our attention on the wavelet methods because of their spatial adaptivity, computational efficiency and asymptotic optimality properties under the MISE. For exhaustive discussions of wavelets and their applications in nonparametric statistics, see, e.g., Antoniadis (1997), Vidakovic (1999) and Härdle *et al.* (1998). We consider (1.1) under following general setting:

- (i) $(Y_t, X_t)_{t \in \mathbb{Z}}$ is a dependent process following a α -mixing structure,
- (ii) Y_1 is not necessarily bounded (allowing a Gaussian distribution for ξ_1),

(the precise definitions and more details are given in Section 2). In order to clarify the interest of considering (i) and (ii), let us now present a brief review on the wavelet estimation of f under various dependence structures on the observations. In the common case where $(Y_1, X_1), \dots, (Y_n, X_n)$ are *i.i.d.*, various wavelet methods have been developed. The most famous of them can be found in, e.g., Donoho and Johnstone (1994, 1995), Donoho *et al.* (1995), Delyon and Jusitky (1996), Hall and Turlach (1997), Antoniadis *et al.* (1997, 2001, 2002), Clyde and George (1999), Zhang and Zheng (1999), Cai (1999, 2002), Cai and Brown (1998, 1999), Kerkyacharian and Picard (2004), Chesneau (2007), Pensky and Sapatinas (2007) and Bochkina and Sapatinas (2009). Relaxations of the independence assumption have been considered in several studies. If we focus our attention on the α -mixing dependence, which is reasonably weak and particularly interesting for (1.1) thanks to its applications in dynamic economic systems (see, e.g., Härdle (1990) and White and Domowitz (1984)), recent wavelet methods and results can be found in, e.g., Masry (2000), Patil and

Truong (2001), Doosti *et al.* (2008), Doosti and Niroumand (2009) and Chesneau (2012). However, in most of these studies, either the proposed wavelet estimator is not adaptive (i.e., its construction depends on the knowledge of the smoothness of f) or it is supposed that ξ_1 is bounded. In fact, to the best of our knowledge Chesneau (2012) is the only work which deals with such an adaptive wavelet regression function estimation problem under **(i)** and **(ii)** (with $d = 1$). However, the construction of the proposed wavelet estimator deeply depends on a parameter θ related to the α -mixing dependence. Since θ is unknown a priori, this estimator can be considered as non adaptive.

In this paper we go far beyond this classical framework by providing theoretical contributions to the full adaptive wavelet estimation of f under **(i)** and **(ii)**. We develop two adaptive wavelet estimators \hat{f}_δ and \hat{f}_δ^* both using a term-by-term thresholding rule δ (as the hard thresholding rule or the soft thresholding rule). We evaluate their performances under the MISE over a wide class of functions f : the Besov balls. In a first part, under mild assumptions on (1.1), we show that the rate of convergence achieved by \hat{f}_δ is exactly the one of the standard term-by-term wavelet thresholding estimator for f in the classical *i.i.d.* framework. It corresponds to the optimal one in the minimax sense within a logarithmic term. In a second part, with less assumptions on (1.1) (only moments of order 2 is required on ξ_1), we show that \hat{f}_δ^* achieves the same rate of convergence to \hat{f}_δ up to a logarithmic term. Thus \hat{f}_δ^* is somewhat less efficient than \hat{f}_δ in terms of MISE but can be used under very mild assumptions on (1.1). For proving our main theorems, we establish a general result on the performance of wavelet term-by-term thresholding estimators which may be of independent interest.

The rest of this work is organized as follows. Section 2 clarifies the assumptions on the model and introduces some notations. Section 3 describes the considered wavelet basis on $[0, 1]^d$ and the Besov balls. Section 4 is devoted to our adaptive wavelet estimators and their MISE properties over Besov balls. The technical proofs are postponed in Section 5.

2 Assumptions

We make the following assumptions on the model (1.1).

Assumptions on the noise

Let us recall that $(\xi_t)_{t \in \mathbb{Z}}$ is a strictly stationary random process independent of $(X_t)_{t \in \mathbb{Z}}$ such that $\mathbf{E}(\xi_1) = 0$.

H1. We suppose that there exist three constants $v > 0$, $\sigma > 0$ and $\omega > 0$ such that, for any $t \in \mathbb{R}$,

$$\mathbf{E}(e^{t\xi_1}) \leq \omega e^{-t^2\sigma^2/2}.$$

H2. We suppose that $\mathbf{E}(\xi_1^2) < \infty$.

Obviously **H1** allows a Gaussian distribution on ξ_1 and **H1** implies **H2**.

Remark 2.1 *It follows from H1 that*

- for any $p \geq 1$, we have $\mathbf{E}(|\xi_1|^p) < \infty$,
- for any $\lambda > 0$ we have

$$\mathbf{P}(|\xi_1| \geq \lambda) \leq 2\omega e^{-\lambda^2/(2\sigma^2)}. \quad (2.1)$$

α -mixing assumption

H3. For any $m \in \mathbb{Z}$, we define the m -th strongly mixing coefficient of $(Y_t, X_t)_{t \in \mathbb{Z}}$ by

$$\alpha_m = \sup_{(A,B) \in \mathcal{F}_{-\infty,0}^{(Y,X)} \times \mathcal{F}_{m,\infty}^{(Y,X)}} |\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)|,$$

where $\mathcal{F}_{-\infty,0}^{(Y,X)}$ is the σ -algebra generated by $\dots, (Y_{-1}, X_{-1}), (Y_0, X_0)$ and $\mathcal{F}_{m,\infty}^{(Y,X)}$ is the σ -algebra generated by $(Y_m, X_m), (Y_{m+1}, X_{m+1}), \dots$

We suppose that there exist two constants $\gamma > 0$ and $\beta > 0$ such that, for any integer $m \geq 1$,

$$\alpha_m \leq \gamma e^{-\beta m}.$$

Further details on the α -mixing dependence can be found in, e.g., Bradley (2007), Doukhan (1994) and Carrasco and Chen (2002). Applications on (1.1) under α -mixing dependence can be found in, e.g., White and Domowitz (1984), Härdle (1990) and Lütkepohl (1992).

Boundedness assumptions

H4. We suppose that there exists a constant $K > 0$ such that

$$\sup_{x \in [0,1]^d} |f(x)| \leq K.$$

H5. For any $m \in \{1, \dots, n\}$, let $g_{(X_0, X_m)}$ be the density of (X_0, X_m) . We suppose that there exists a constant $L > 0$ such that

$$\sup_{m \in \{1, \dots, n\}} \sup_{(x, x_*) \in [0,1]^{2d}} g_{(X_0, X_m)}(x, x_*) \leq L. \quad (2.2)$$

These boundedness assumptions are standard for (1.1) under α -mixing dependence. See, e.g., Masry (2000) and Patil and Truong (2001).

3 Preliminaries on wavelets

This section contains some facts about the wavelet tensor-product basis on $[0, 1]^d$ and the considered function space in terms of wavelet coefficients that will be used in the sequel.

3.1 Wavelet tensor-product basis on $[0, 1]^d$

For any $p \geq 1$, set

$$\mathbb{L}_p([0, 1]^d) = \left\{ h : [0, 1]^d \rightarrow \mathbb{R}; \|h\|_p = \left(\int_{[0, 1]^d} |h(x)|^p dx \right)^{1/p} < \infty \right\}.$$

For the purpose of this paper, we use a compactly supported wavelet-tensor product basis on $[0, 1]^d$ based on the Daubechies wavelets. Let N be a positive integer, ϕ be "father" Daubechies-type wavelet and ψ be a "mother" Daubechies-type wavelet of the family $db2N$. In particular, mention that ϕ and ψ have compact supports (see Mallat (2009)).

Then, for any $x = (x_1, \dots, x_d) \in [0, 1]^d$ we construct 2^d functions as follows:

- a scale function

$$\Phi(x) = \prod_{u=1}^d \phi(x_u)$$

- $2^d - 1$ wavelet functions

$$\Psi_u(x) = \begin{cases} \psi(x_u) \prod_{\substack{v=1 \\ v \neq u}}^d \phi(x_v) & \text{when } u \in \{1, \dots, d\}, \\ \prod_{v \in A_u} \psi(x_v) \prod_{v \notin A_u} \phi(x_v) & \text{when } u \in \{d+1, \dots, 2^d - 1\}, \end{cases}$$

where $(A_u)_{u \in \{d+1, \dots, 2^d - 1\}}$ forms the set of all non void subsets of $\{1, \dots, d\}$ of cardinality greater or equal to 2.

We set

$$D_j = \{0, \dots, 2^j - 1\}^d,$$

for any $j \geq 0$ and $k = (k_1, \dots, k_d) \in D_j$,

$$\Phi_{j,k}(x) = 2^{jd/2} \Phi(2^j x_1 - k_1, \dots, 2^j x_d - k_d)$$

and, for any $u \in \{1, \dots, 2^d - 1\}$,

$$\Psi_{j,k,u}(x) = 2^{jd/2} \Psi_u(2^j x_1 - k_1, \dots, 2^j x_d - k_d).$$

Then there exists an integer τ such that the collection

$$\mathcal{B} = \{\Phi_{\tau,k}, k \in D_\tau; (\Psi_{j,k,u})_{u \in \{1, \dots, 2^d-1\}}, j \in \mathbb{N} - \{0, \dots, \tau-1\}, k \in D_j\}$$

(with appropriated treatments at the boundaries) forms an orthonormal basis of $\mathbb{L}_2([0, 1]^d)$. Let j_* be an integer such that $j_* \geq \tau$. A function $h \in \mathbb{L}_2([0, 1]^d)$ can be expanded into a wavelet series as

$$h(x) = \sum_{k \in D_{j_*}} c_{j_*,k} \Phi_{j_*,k}(x) + \sum_{u=1}^{2^d-1} \sum_{j=j_*}^{\infty} \sum_{k \in D_j} d_{j,k,u} \Psi_{j,k,u}(x),$$

where

$$c_{j,k} = \int_{[0,1]^d} h(x) \Phi_{j,k}(x) dx, \quad d_{j,k,u} = \int_{[0,1]^d} h(x) \Psi_{j,k,u}(x) dx. \quad (3.1)$$

For further results and details about this wavelet basis on $[0, 1]^d$, we refer the reader to Meyer (1992), Cohen *et al.* (1993) and Mallat (2009).

3.2 Besov balls

Let $M > 0$, $s \in (0, N)$, $p \geq 1$ and $r \geq 1$. A function $h \in \mathbb{L}_2([0, 1]^d)$ belongs to the Besov balls $B_{p,r}^s(M)$ if and only if there exists a constant $M^* > 0$ such that the associated wavelet coefficients (3.1) satisfy

$$\left(\sum_{k \in D_\tau} |c_{\tau,k}|^p \right)^{1/p} + \left(\sum_{j=\tau}^{\infty} \left(2^{j(s+d(1/2-1/p))} \left(\sum_{u=1}^{2^d-1} \sum_{k \in D_j} |d_{j,k,u}|^p \right)^{1/p} \right)^r \right)^{1/r} \leq M^*$$

and with the usual modifications for $p = \infty$ or $q = \infty$.

For a particular choice of parameters s , p and r , these sets contain Sobolev and Hölder balls as well as function classes of significant spatial inhomogeneity (such as the Bump Algebra and Bounded Variations balls). Details about Besov balls can be found in, e.g., Devore and Popov (1988), Meyer (1992) and Härdle *et al.* (1998).

4 Wavelet estimators and results

4.1 Introduction

We consider the model (1.1) and we adopt the notations introduced in Sections 2 and 3. We expand the unknown regression function f as

$$f(x) = \sum_{k \in D_{j_0}} c_{j_0,k} \Phi_{j_0,k}(x) + \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{\infty} \sum_{k \in D_j} d_{j,k,u} \Psi_{j,k,u}(x), \quad (4.1)$$

where $j_0 \geq \tau$, $c_{j,k} = \int_{[0,1]^d} f(x)\Phi_{j,k}(x)dx$ and $d_{j,k,u} = \int_{[0,1]^d} f(x)\Psi_{j,k,u}(x)dx$. In the next section, we will construct two different wavelet estimators for f according to the two following groups of assumptions:

Group 1: H1, H3, H4 and H5,

Group 2: H2, H3 and H4.

4.2 Wavelet estimator I and result

Suppose that **H1, H3, H4** and **H5** hold. We define the hard thresholding estimator \hat{f}_δ by

$$\hat{f}_\delta(x) = \sum_{k \in D_{j_0}} \hat{c}_{j_0,k} \Phi_{j_0,k}(x) + \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_1} \sum_{k \in D_j} \delta(\hat{d}_{j,k,u}, \kappa \lambda_n) \Psi_{j,k,u}(x), \quad (4.2)$$

where $j_0 \geq \tau$ is an integer satisfying

$$\frac{1}{2}(\ln n)^2 < 2^{j_0 d} \leq (\ln n)^2,$$

$\hat{c}_{j,k}$ and $\hat{d}_{j,k,u}$ are the empirical wavelet coefficients estimators of $c_{j,k}$ and $d_{j,k,u}$, i.e.,

$$\hat{c}_{j,k} = \frac{1}{n} \sum_{i=1}^n Y_i \Phi_{j,k}(X_i), \quad \hat{d}_{j,k,u} = \frac{1}{n} \sum_{i=1}^n Y_i \Psi_{j,k,u}(X_i), \quad (4.3)$$

$\delta : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ is a term-by-term thresholding rule satisfying: there exists a constant $C > 0$ such that, for any $(u, v, \lambda) \in \mathbb{R}^2 \times (0, \infty)$,

$$|\delta(v, \lambda) - u| \leq C(\min(|u|, \lambda) + |v - u| \mathbf{1}_{\{|v-u| > \lambda/2\}}) \quad (4.4)$$

κ is a large enough constant,

$$\lambda_n = \sqrt{\frac{\ln n}{n}} \quad (4.5)$$

and j_1 is an integer satisfying

$$\frac{1}{2} \frac{n}{(\ln n)^4} < 2^{j_1 d} \leq \frac{n}{(\ln n)^4}.$$

Remark 4.1 *The estimators $\hat{c}_{j,k}$ and $\hat{d}_{j,k,u}$ (4.3) are unbiased. Indeed the independence of X_1 and ξ_1 , and $\mathbf{E}(\xi_1) = 0$ imply that*

$$\mathbf{E}(\hat{c}_{j,k}) = \mathbf{E}(Y_1 \Phi_{j,k}(X_1)) = \mathbf{E}(f(X_1) \Phi_{j,k}(X_1)) = \int_{[0,1]^d} f(x) \Phi_{j,k}(x) dx = c_{j,k}.$$

Similarly we prove that $\mathbf{E}(\hat{d}_{j,k,u}) = d_{j,k,u}$.

Remark 4.2 Among the thresholding rules δ satisfying (4.4), there are

- the hard thresholding rule defined by $\delta(v, \lambda) = v\mathbf{1}_{\{|v| \geq \lambda\}}$, where $\mathbf{1}$ denotes the indicator function,
- the soft thresholding rule defined by $\delta(v, \lambda) = \text{sign}(v) \max(|v| - \lambda, 0)$, where sign denotes the sign function.

The technical details can be found in (Delyon and Jusitky, 1996, Lemma 1).

The idea behind the term-by-term thresholding rule δ in \hat{f}_δ is to only estimate the “large” wavelet coefficients of f (and to remove the others). The reason is that wavelet coefficients having small absolute value are considered to encode mostly noise whereas the important information of f is encoded by the coefficients having large absolute value. This term-by-term selection gives to \hat{f}_δ an extraordinary local adaptability in handling discontinuities. For further details on such estimators in various statistical framework, we refer the reader to, e.g., Donoho and Johnstone (1994, 1995), Donoho *et al.* (1995), Delyon and Jusitky (1996), Antoniadis *et al.* (1997) and Härdle *et al.* (1998).

The considered threshold λ_n (4.5) corresponds to the universal one determined in the standard Gaussian *i.i.d.* case (see Donoho and Johnstone (1994, 1995)).

It is important to underline that \hat{f}_δ is adaptive; its construction does not depend on the smoothness of f .

Theorem 4.1 below explores the performance of \hat{f}_δ under the MISE over Besov balls.

Theorem 4.1 *Let us consider the model (1.1) under **H1**, **H3**, **H4** and **H5**. Let \hat{f}_δ be (4.2). Suppose that $f \in B_{p,r}^s(M)$ with $r \geq 1$, $\{p \geq 2 \text{ and } s \in (0, N)\}$ or $\{p \in [1, 2) \text{ and } s \in (d/p, N)\}$. Then there exists a constant $C > 0$ such that*

$$\mathbf{R}(\hat{f}_\delta, f) \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+d)}.$$

The proof of Theorem 4.1 is based on a general result on the performance of the wavelet term-by-term thresholding estimators (see Theorem 5.1 below) and some statistical properties on (4.3) (see Proposition 5.1 below).

The rate of convergence $(\ln n/n)^{2s/(2s+d)}$ is the near optimal one (in the minimax sense) for the standard Gaussian *i.i.d.* case (see, e.g., Härdle *et al.* (1998) and Tsybakov (2004)). “Near” is due to the extra logarithmic term $(\ln n)^{2s/(2s+d)}$. Also, following the terminology of Härdle *et al.* (1998), note that this rate of convergence is attained over both the homogeneous zone of the Besov balls (corresponding to $p \geq 2$) and the inhomogeneous zone (corresponding to $p \in [1, 2)$). This shows that the performance of \hat{f}_δ is unaffected by the presence of discontinuities in f .

In view of Theorem 4.1, it is natural to address the following question: is it possible to construct an adaptive wavelet estimator reaching the two following objectives:

- relax some assumptions on the model,
- attain a suitable rate of convergence, i.e., as close as possible to the optimal one $n^{-2s/(2s+d)}$.

An answer is provided in the next section.

4.3 Wavelet estimator II and result

Suppose that **H2**, **H3** and **H4** hold (only moments of order 2 are required on ξ_1 and we have no a priori assumption on (2.2)). We define the hard thresholding estimator \hat{f}_δ^* by

$$\hat{f}_\delta^*(x) = \sum_{k \in D_{j_0}} \hat{c}_{j_0,k}^* \Phi_{j_0,k}(x) + \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_1} \sum_{k \in D_j} \delta(\hat{d}_{j,k,u}^*, \kappa \lambda_n) \Psi_{j,k,u}(x), \quad (4.6)$$

where $j_0 = \tau$, $\hat{c}_{j,k}^*$ and $\hat{d}_{j,k,u}^*$ are the wavelet coefficients estimators of $c_{j,k}$ and $d_{j,k,u}$ defined by

$$\hat{c}_{j,k}^* = \frac{1}{n} \sum_{i=1}^n A_{i,j,k}, \quad \hat{d}_{j,k,u}^* = \frac{1}{n} \sum_{i=1}^n B_{i,j,k,u}, \quad (4.7)$$

where

$$A_{i,j,k} = Y_i \Phi_{j,k}(X_i) \mathbf{1}_{\{|Y_i \Phi_{j,k}(X_i)| \leq \frac{\sqrt{n}}{\ln n}\}}, \quad B_{i,j,k,u} = Y_i \Psi_{j,k,u}(X_i) \mathbf{1}_{\{|Y_i \Psi_{j,k,u}(X_i)| \leq \frac{\sqrt{n}}{\ln n}\}},$$

κ is a large enough constant,

$$\lambda_n = \frac{\ln n}{\sqrt{n}},$$

$\delta : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ is a term-by-term thresholding rule satisfying (4.4) and j_1 is an integer such that

$$\frac{1}{2} \frac{n}{(\ln n)^2} < 2^{j_1 d} \leq \frac{n}{(\ln n)^2}.$$

The role of the thresholding selection in (4.7) is to remove the large $|Y_i|$. This allows us to replace **H1** by the less restrictive assumption **H2**. Such an observations thresholding technique has already used in various contexts of wavelet regression function estimation in Delyon and Jusitky (1996), Chesneau and Fadili (2012), Chesneau (2012) and Chesneau and Shirazi (2013).

Again, let us mention that \hat{f}_δ^* is adaptive.

Theorem 4.2 below investigates the performance of \hat{f}_δ^* under the MISE over Besov balls.

Theorem 4.2 *Let us consider the regression model (1.1) under **H2**, **H3** and **H4**. Let \hat{f}_δ^* be (4.6). Suppose that $f \in B_{p,r}^s(M)$ with $r \geq 1$, $\{p \geq 2 \text{ and } s \in (0, N)\}$ or $\{p \in [1, 2) \text{ and } s \in (d/p, N)\}$. Then there exists a constant $C > 0$ such that*

$$\mathbf{R}(\hat{f}_\delta^*, f) \leq C \left(\frac{(\ln n)^2}{n} \right)^{2s/(2s+d)}.$$

The proof of Theorem 4.2 is based on a general result on the performance of the wavelet term-by-term thresholding estimators (see Theorem 5.1 below) and some statistical properties on (4.7) (see Proposition 5.2 below).

Theorem 4.1 improves (Chesneau, 2012, Theorem 1) in terms of rates of convergence and provides an extension to the multidimensional setting.

Remark 4.3 *Obviously the assumptions **H1** and **H2** includes the bounded noise. If ξ is bounded, the only interest of Theorem 4.2, and a fortiori \hat{f}_δ^* , is to relax **H5**.*

Remark 4.4 *Our work can be extended to random design X_1 such that $X_1(\Omega) = [a, b]^d$ with $a < b$, $(a, b) \in \mathbb{R}^2$, having a known density g bounded from below. In this case, it suffices to replace Y_i by $Y_i/g(X_i)$ in the definitions of \hat{f}_δ and \hat{f}_δ^* (more precisely, in (4.3) and (4.7)) to be able to prove Theorems 4.1 and 4.2. When g is unknown, a possible approach following the idea of Patil and Truong (2001) is to consider \hat{f}_δ (or \hat{f}_δ^*) to estimate fg , then estimate the unknown density g by a term-by-term wavelet thresholding estimator \hat{g} (as the one in Härdle et al. (1998)) and finally construct $\hat{f}^\dagger = \hat{f}_\delta/\hat{g}$. However, we do not claim it to be near optimal in the minimax sense.*

Conclusion and discussion

This paper provides some theoretical contributions to the adaptive wavelet estimation of a multidimensional regression function from the α -mixing sequence $(Y_t, X_t)_{t \in \mathbb{Z}}$ defined by (1.1). Two different wavelet term-by-term thresholding estimators \hat{f}_δ and \hat{f}_δ^* are constructed. Under mild assumptions on (1.1), we determine their rates of convergence under the MISE over Besov balls $B_{p,r}^s(M)$. To be more specific, we prove that

result	assumptions	estimator	rate of convergence
Theorem 4.1	H1, H3, H4, H5	\hat{f}_δ (4.2)	$(\ln n/n)^{2s/(2s+d)}$
Theorem 4.2	H2, H3, H4	\hat{f}_δ^* (4.6)	$((\ln n)^2/n)^{2s/(2s+d)}$

Since $n^{-2s/(2s+d)}$ is the optimal rate of convergence (in the minimax sense) in the standard *i.i.d.* framework, these results show the good performances of \hat{f}_δ and \hat{f}_δ^* . Let us now discuss several aspects of our study.

- Some useful assumptions in Theorem 4.1 are relaxed in Theorem 4.2 and the rate of convergence attained by \hat{f}_δ^* is close to the one of \hat{f}_δ (up to the logarithmic term $(\ln n)^{2s/(2s+d)}$).
- Stricto sensu \hat{f}_δ is more efficient to \hat{f}_δ^* . Moreover the construction of \hat{f}_δ^* is more complicated to the one of \hat{f}_δ due to the presence of the thresholding in (4.7). This could be an obstacle from a practical point of view.

Possible perspectives of this work are to

- determine the optimal lower bound for (1.1) under the α -mixing dependence,
- consider a random design X_1 with unknown density or/and unbounded,
- relax the exponential decay assumption of α_m in **H3**,
- improve the rates of convergence by perhaps using a group thresholding rule (see, e.g., Cai (1999, 2002)).

All these aspects needs further investigations that we leave for a future work.

5 Proofs

In the following, the quantity C denotes a generic constant that does not depend on j , k and n . Its value may change from one term to another.

5.1 A general result

Theorem 5.1 below is derived from (Kerkycharian and Picard, 2000, Theorem 3.1) and (Delyon and Jusitky, 1996, Theorem 1). The main contributions of this result is to provide more flexibility on the choices of j_0 and j_1 (which will be crucial in our dependent framework) and to clarified the minimal assumptions on the wavelet coefficients estimators to ensure a “suitable” rate of convergence for the corresponding wavelet term-by-term thresholding estimator (see **(a)** and **(b)** in Theorem 5.1). This result may be of independent interest.

Theorem 5.1 *Let us consider a (general) nonparametric model where an unknown function $f \in \mathbb{L}_2([0, 1]^d)$ needs to be estimated from n observations of a random process defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Using its wavelet series expansion (4.1), we define the hard thresholding estimator \hat{f}_δ^\diamond by*

$$\hat{f}_\delta^\diamond(x) = \sum_{k \in D_{j_0}} \hat{c}_{j_0, k}^\diamond \Phi_{j_0, k}(x) + \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_1} \sum_{k \in D_j} \delta(\hat{d}_{j, k, u}^\diamond, \kappa \lambda_n) \Psi_{j, k, u}(x),$$

where $j_0 \geq \tau$ is an integer such that

$$\frac{1}{2}2^{\tau d}(\ln n)^\nu < 2^{j_0 d} \leq 2^{\tau d}(\ln n)^\nu,$$

with $\nu \geq 0$, $\hat{c}_{j_0, k}^\diamond$ and $\hat{d}_{j, k, u}^\diamond$ are wavelet coefficients estimators of $c_{j_0, k}$ and $d_{j, k, u}$ respectively, κ is a large enough constant, λ_n is a threshold, $\delta : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ is a term-by-term thresholding satisfying (4.4) and j_1 is an integer such that

$$\frac{1}{2} \frac{1}{\lambda_n^2 (\ln n)^\varrho} \leq 2^{j_1 d} \leq \frac{1}{\lambda_n^2 (\ln n)^\varrho},$$

with $\varrho \geq 0$.

It is understood that

$$\lim_{n \rightarrow \infty} (\ln n)^{\max(\nu, \varrho)} \lambda_n^{2(1-\nu)} = 0$$

for any $\nu \in (0, 1)$.

We suppose that ν , $\hat{c}_{j, k}^\diamond$, $\hat{d}_{j, k, u}^\diamond$, κ , λ_n and ϱ satisfy the following inequalities:

(a) there exists a constant $C > 0$ such that, for any $k \in D_j$,

$$\mathbf{E}((\hat{c}_{j_0, k}^\diamond - c_{j_0, k})^2) \leq C \lambda_n^2.$$

(b) there exists a constant $C > 0$ such that, for any $j \in \{j_0, \dots, j_1\}$, $k \in D_j$ and $u \in \{1, \dots, 2^d - 1\}$,

$$\mathbf{P} \left(|\hat{d}_{j, k, u}^\diamond - d_{j, k, u}| \geq \frac{\kappa}{2} \lambda_n \right) \leq C \frac{\lambda_n^8}{\varpi_n},$$

where ϖ_n is such that

$$\mathbf{E}((\hat{d}_{j, k, u}^\diamond - d_{j, k, u})^4) \leq \varpi_n.$$

Furthermore we suppose that $f \in B_{p, r}^s(M)$ with $r \geq 1$, $\{p \geq 2$ and $s \in (0, N)\}$ or $\{p \in [1, 2)$ and $s \in (d/p, N)\}$. Then there exists a constant $C > 0$ such that

$$\mathbf{R}(\hat{f}_\delta^\diamond, f) \leq C (\lambda_n^2)^{2s/(2s+d)}.$$

Proof of Theorem 5.1. The orthonormality of the considered wavelet basis yields

$$\mathbf{R}(\hat{f}_\delta^\diamond, f) = R_1 + R_2 + R_3, \tag{5.1}$$

where

$$R_1 = \sum_{k \in D_{j_0}} \mathbf{E}((\hat{c}_{j_0, k}^\diamond - c_{j_0, k})^2), \quad R_2 = \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_1} \sum_{k \in D_j} \mathbf{E}((\delta(\hat{d}_{j, k, u}^\diamond, \kappa \lambda_n) - d_{j, k, u})^2)$$

and

$$R_3 = \sum_{u=1}^{2^d-1} \sum_{j=j_1+1}^{\infty} \sum_{k \in D_j} d_{j,k,u}^2.$$

Bound for R_1 : By **(a)** we have

$$R_1 \leq C 2^{j_0 d} \lambda_n^2 \leq C (\ln n)^\nu \lambda_n^2 \leq C (\lambda_n^2)^{2s/(2s+d)}. \quad (5.2)$$

Bound for R_2 : The feature of the term-by-term thresholding δ (i.e., (4.4)) yields

$$R_2 \leq C(R_{2,1} + R_{2,2}), \quad (5.3)$$

where

$$R_{2,1} = \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_1} \sum_{k \in D_j} (\min(|d_{j,k,u}|, \kappa \lambda_n))^2$$

and

$$R_{2,2} = \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_1} \sum_{k \in D_j} \mathbf{E} \left(|\hat{d}_{j,k,u}^\diamond - d_{j,k,u}|^2 \mathbf{1}_{\{|\hat{d}_{j,k,u}^\diamond - d_{j,k,u}| \geq \kappa \lambda_n / 2\}} \right).$$

Bound for $R_{2,1}$: Let j_2 an integer satisfying

$$\frac{1}{2} \left(\frac{1}{\lambda_n^2} \right)^{1/(2s+d)} < 2^{j_2} \leq \left(\frac{1}{\lambda_n^2} \right)^{1/(2s+d)}.$$

First of all, let us consider the case $p \geq 2$. Since $f \in B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$, we have

$$\begin{aligned} R_{2,1} &= \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_2} \sum_{k \in D_j} (\min(|d_{j,k,u}|, \kappa \lambda_n))^2 + \sum_{u=1}^{2^d-1} \sum_{j=j_2+1}^{j_1} \sum_{k \in D_j} (\min(|d_{j,k,u}|, \kappa \lambda_n))^2 \\ &\leq \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_2} \sum_{k \in D_j} \kappa^2 \lambda_n^2 + \sum_{u=1}^{2^d-1} \sum_{j=j_2+1}^{j_1} \sum_{k \in D_j} d_{j,k,u}^2 \\ &\leq C \left(\lambda_n^2 \sum_{j=\tau}^{j_2} 2^{j d} + \sum_{j=j_2+1}^{\infty} 2^{-2j s} \right) \leq C (\lambda_n^2 2^{j_2 d} + 2^{-2j_2 s}) \leq C (\lambda_n^2)^{2s/(2s+d)}. \end{aligned}$$

Let us now explore the case $p \in [1, 2)$. The facts that $f \in B_{p,r}^s(M)$ with $s > d/p$ and $(2s + d)(2 - p)/2 + (s + d(1/2 - 1/p))p = 2s$ lead to

$$\begin{aligned}
R_{2,1} &= \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_2} \sum_{k \in D_j} (\min(|d_{j,k,u}|, \kappa \lambda_n))^2 + \sum_{u=1}^{2^d-1} \sum_{j=j_2+1}^{j_1} \sum_{k \in D_j} (\min(|d_{j,k,u}|, \kappa \lambda_n))^{2-p+p} \\
&\leq \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_2} \sum_{k \in D_j} \kappa^2 \lambda_n^2 + \sum_{u=1}^{2^d-1} \sum_{j=j_2+1}^{j_1} \sum_{k \in D_j} |d_{j,k,u}|^p (\kappa \lambda_n)^{2-p} \\
&\leq C \left(\lambda_n^2 \sum_{j=\tau}^{j_2} 2^{jd} + (\lambda_n^2)^{(2-p)/2} \sum_{j=j_2+1}^{\infty} 2^{-j(s+d(1/2-1/p))p} \right) \\
&\leq C \left(\lambda_n^2 2^{j_2 d} + (\lambda_n^2)^{(2-p)/2} 2^{-j_2(s+d(1/2-1/p))p} \right) \leq C (\lambda_n^2)^{2s/(2s+d)}.
\end{aligned}$$

Therefore, for any $r \geq 1$, $\{p \geq 2 \text{ and } s \in (0, N)\}$ or $\{p \in [1, 2) \text{ and } s \in (d/p, N)\}$, we have

$$R_{2,1} \leq C (\lambda_n^2)^{2s/(2s+d)}. \quad (5.4)$$

Bound for $R_{2,2}$: It follows from the Cauchy-Schwarz inequality and **(b)** that

$$\begin{aligned}
R_{2,2} &\leq C \sum_{u=1}^{2^d-1} \sum_{j=j_0}^{j_1} \sum_{k \in D_j} \sqrt{\mathbf{E} \left((\hat{d}_{j,k,u}^\circ - d_{j,k,u})^4 \right) \mathbf{P} \left(|\hat{d}_{j,k,u}^\circ - d_{j,k,u}| > \kappa \lambda_n / 2 \right)} \\
&\leq C \lambda_n^4 \sum_{j=\tau}^{j_1} 2^{jd} \leq C \lambda_n^4 2^{j_1 d} \leq C \lambda_n^4 \frac{1}{\lambda_n^2 (\ln n)^\varrho} \leq C \lambda_n^2 \leq C (\lambda_n^2)^{2s/(2s+d)}. \quad (5.5)
\end{aligned}$$

Putting (5.3), (5.4) and (5.5) together, for any $r \geq 1$, $\{p \geq 2 \text{ and } s \in (0, N)\}$ or $\{p \in [1, 2) \text{ and } s \in (d/p, N)\}$, we obtain

$$R_2 \leq C (\lambda_n^2)^{2s/(2s+d)}. \quad (5.6)$$

Bound for R_3 : In the case $p \geq 2$, we have $f \in B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$ which implies that

$$R_3 \leq C \sum_{j=j_1+1}^{\infty} 2^{-2js} \leq C 2^{-2j_1 s} \leq C (\lambda_n^2 (\ln n)^\varrho)^{2s/d} \leq C (\lambda_n^2)^{2s/(2s+d)}.$$

On the other hand, when $p \in [1, 2)$, we have $f \in B_{p,r}^s(M) \subseteq B_{2,\infty}^{s+d(1/2-1/p)}(M)$. Observing that $s > d/p$ leads to $(s + d(1/2 - 1/p))/d > s/(2s + d)$, we have

$$\begin{aligned}
R_3 &\leq C \sum_{j=j_1+1}^{\infty} 2^{-2j(s+d(1/2-1/p))} \leq C 2^{-2j_1(s+d(1/2-1/p))} \\
&\leq C (\lambda_n^2 (\ln n)^\varrho)^{2(s+d(1/2-1/p))/d} \leq C (\lambda_n^2)^{2s/(2s+d)}.
\end{aligned}$$

Hence, for $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > d/p\}$, we have

$$R_3 \leq C (\lambda_n^2)^{2s/(2s+d)}. \quad (5.7)$$

Combining (5.1), (5.2), (5.6) and (5.7), we arrive at, for $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > d/p\}$,

$$\mathbf{R}(f_\delta^\diamond, f) \leq C (\lambda_n^2)^{2s/(2s+d)}.$$

The proof of Theorem 5.1 is completed. □

5.2 Proof of Theorem 4.1

The proof of Theorem 4.1 is a consequence of Theorem 5.1 above and Proposition 5.1 below. To be more specific, Proposition 5.1 shows that (a) and (b) of Theorem 5.1 are satisfied under the following configuration:

- $\nu = 2$,
- $\hat{c}_{j_0, k}^\diamond = \hat{c}_{j_0, k}$ and $\hat{d}_{j, k, u}^\diamond = \hat{d}_{j, k, u}$ from (4.3),
- $\lambda_n = \sqrt{\ln n/n}$,
- κ is a large enough constant,
- $\varrho = 3$.

Proposition 5.1 *Suppose that **H1**, **H3**, **H4** and **H5** hold. Let $\hat{c}_{j, k}$ and $\hat{d}_{j, k, u}$ be defined by (4.3), and*

$$\lambda_n = \sqrt{\frac{\ln n}{n}}.$$

Then

- (i) *there exists a constant $C > 0$ such that, for any j satisfying $(\ln n)^2 \leq 2^{jd} \leq n$ and $k \in D_j$,*

$$\mathbf{E}((\hat{c}_{j, k} - c_{j, k})^2) \leq C \frac{1}{n} \quad (\leq C \lambda_n^2).$$

- (ii) *there exists a constant $C > 0$ such that, for any j satisfying $2^{jd} \leq n$, $k \in D_j$ and $u \in \{1, \dots, 2^d - 1\}$,*

$$\mathbf{E}((\hat{d}_{j, k, u} - d_{j, k, u})^4) \leq C n \quad (= \varpi_n).$$

(iii) for $\kappa > 0$ large enough, there exists a constant $C > 0$ such that, for any j satisfying $(\ln n)^2 \leq 2^{jd} \leq n/(\ln n)^4$, $k \in D_j$ and $u \in \{1, \dots, 2^d - 1\}$,

$$\mathbf{P} \left(|\hat{d}_{j,k,u} - d_{j,k,u}| \geq \frac{\kappa}{2} \lambda_n \right) \leq C \frac{1}{n^5} \quad (\leq C \lambda_n^8 / \varpi_n).$$

Proof of Proposition 5.1. The technical ingredients in our proof are suitable covariance decompositions, a covariance inequality for α -mixing processes (see Lemma 5.3 in Appendix) and a Bernstein-type exponential inequality for α -mixing processes (see Lemma 5.4 in Appendix).

(i) Since $\mathbf{E}(Y_1 \Phi_{j,k}(X_1)) = c_{j,k}$, we have

$$\hat{c}_{j,k} - c_{j,k} = \frac{1}{n} \sum_{i=1}^n U_{i,j,k},$$

where

$$U_{i,j,k} = Y_i \Phi_{j,k,u}(X_i) - \mathbf{E}(Y_1 \Phi_{j,k}(X_1)).$$

Considering the event $\mathcal{A}_i = \{|Y_i| \geq \kappa_* \sqrt{\ln n}\}$, where κ_* denotes a constant which will be chosen later, we can split $U_{i,j,k}$ as follows:

$$U_{i,j,k} = V_{i,j,k} + W_{i,j,k},$$

where

$$V_{i,j,k} = Y_i \Phi_{j,k}(X_i) \mathbf{1}_{\mathcal{A}_i} - \mathbf{E}(Y_1 \Phi_{j,k}(X_1) \mathbf{1}_{\mathcal{A}_i})$$

and

$$W_{i,j,k} = Y_i \Phi_{j,k}(X_i) \mathbf{1}_{\mathcal{A}_i^c} - \mathbf{E}(Y_1 \Phi_{j,k}(X_1) \mathbf{1}_{\mathcal{A}_i^c}).$$

The inequality $(x + y)^2 \leq 2(x^2 + y^2)$, $(x, y) \in \mathbb{R}^2$, yields

$$\mathbf{E}((\hat{c}_{j,k} - c_{j,k})^2) = \frac{1}{n^2} \mathbf{E} \left(\left(\sum_{i=1}^n V_{i,j,k} + \sum_{i=1}^n W_{i,j,k} \right)^2 \right) \leq \frac{2}{n^2} (S + T), \quad (5.8)$$

where

$$S = \mathbf{V} \left(\sum_{i=1}^n Y_i \Phi_{j,k}(X_i) \mathbf{1}_{\mathcal{A}_i} \right), \quad T = \mathbf{V} \left(\sum_{i=1}^n Y_i \Phi_{j,k}(X_i) \mathbf{1}_{\mathcal{A}_i^c} \right).$$

Bound for S: Let us now introduce a result which will be useful in the rest of study.

Lemma 5.1 *Let $p \geq 1$. Consider (1.1). Suppose that $\mathbf{E}(|\xi_1|^p) < \infty$ and **H4** holds. Then*

- there exists a constant $C > 0$ such that, for any $j \geq \tau$ and $k \in D_j$,

$$\mathbf{E}(|Y_1 \Phi_{j,k}(X_1)|^p) \leq C 2^{jd(p/2-1)}.$$

- there exists a constant $C > 0$ such that, for any $j \geq \tau$, $k \in D_j$ and $u \in \{1, \dots, 2^d - 1\}$,

$$\mathbf{E}(|Y_1 \Psi_{j,k,u}(X_1)|^p) \leq C 2^{jd(p/2-1)}.$$

Using the inequality $(\sum_{i=1}^m a_i)^2 \leq m \sum_{i=1}^m a_i^2$, $a = (a_1, \dots, a_m) \in \mathbb{R}^m$, Lemma 5.1 with $p = 4$ (thanks to **H1** implying $\mathbf{E}(|\xi_1|^p) < \infty$ for $p \geq 1$) and $2^{jd} \leq n$, we arrive at

$$\begin{aligned} S &\leq \mathbf{E} \left(\left(\sum_{i=1}^n Y_i \Phi_{j,k}(X_i) \mathbf{1}_{\mathcal{A}_i} \right)^2 \right) \leq n^2 \mathbf{E} \left((Y_1 \Phi_{j,k}(X_1))^2 \mathbf{1}_{\mathcal{A}_1} \right) \\ &\leq n^2 \sqrt{\mathbf{E} \left((Y_1 \Phi_{j,k}(X_1))^4 \right) \mathbf{P}(\mathcal{A}_1)} \leq C n^2 2^{jd/2} \sqrt{\mathbf{P}(\mathcal{A}_1)} \\ &\leq C n^{5/2} \sqrt{\mathbf{P}(\mathcal{A}_1)}. \end{aligned}$$

Now, using **H4**, **H1** (implying (2.1)) and taking κ_* large enough, we obtain

$$\begin{aligned} \mathbf{P}(\mathcal{A}_1) &\leq \mathbf{P}(|\xi_1| \geq \kappa_* \sqrt{\ln n} - K) \leq \mathbf{P} \left(|\xi_1| \geq \frac{\kappa_*}{2} \sqrt{\ln n} \right) \\ &\leq 2\omega e^{-\kappa_*^2 \ln n / (8\sigma^2)} = 2\omega n^{-\kappa_*^2 / (8\sigma^2)} \leq C \frac{1}{n^3}. \end{aligned}$$

Hence

$$S \leq C n^{5/2} \frac{1}{n^{3/2}} = Cn. \quad (5.9)$$

Bound for T: Observe that

$$T \leq T_1 + T_2, \quad (5.10)$$

where

$$T_1 = n \mathbf{V} (Y_1 \Phi_{j,k}(X_1) \mathbf{1}_{\mathcal{A}_1^c})$$

and

$$T_2 = \left| \sum_{v=2}^n \sum_{\ell=1}^{v-1} \mathbf{C}_{ov} (Y_v \Phi_{j,k}(X_v) \mathbf{1}_{\mathcal{A}_v^c}, Y_\ell \Phi_{j,k}(X_\ell) \mathbf{1}_{\mathcal{A}_\ell^c}) \right|.$$

Bound for T₁: Lemma 5.1 with $p = 2$ yields

$$T_1 \leq n \mathbf{E} \left((Y_1 \Phi_{j,k}(X_1))^2 \mathbf{1}_{\mathcal{A}_1^c} \right) \leq n \mathbf{E} \left((Y_1 \Phi_{j,k}(X_1))^2 \right) \leq Cn. \quad (5.11)$$

Bound for T_2 : It follows from the stationarity of $(Y_t, X_t)_{t \in \mathbb{Z}}$ and $2^{jd} \leq n$ that

$$\begin{aligned} T_2 &= \left| \sum_{m=1}^n (n-m) \mathbf{C}_{ov} (Y_0 \Phi_{j,k}(X_0) \mathbf{1}_{\mathcal{A}_0^c}, Y_m \Phi_{j,k}(X_m) \mathbf{1}_{\mathcal{A}_m^c}) \right| \\ &\leq n \sum_{m=1}^n |\mathbf{C}_{ov} (Y_0 \Phi_{j,k}(X_0) \mathbf{1}_{\mathcal{A}_0^c}, Y_m \Phi_{j,k}(X_m) \mathbf{1}_{\mathcal{A}_m^c})| = n(T_{2,1} + T_{2,2}), \end{aligned} \quad (5.12)$$

where

$$\begin{aligned} T_{2,1} &= \sum_{m=1}^{\lfloor \ln n / \beta \rfloor - 1} |\mathbf{C}_{ov} (Y_0 \Phi_{j,k}(X_0) \mathbf{1}_{\mathcal{A}_0^c}, Y_m \Phi_{j,k}(X_m) \mathbf{1}_{\mathcal{A}_m^c})|, \\ T_{2,2} &= \sum_{m=\lfloor \ln n / \beta \rfloor}^n |\mathbf{C}_{ov} (Y_0 \Phi_{j,k}(X_0) \mathbf{1}_{\mathcal{A}_0^c}, Y_m \Phi_{j,k}(X_m) \mathbf{1}_{\mathcal{A}_m^c})| \end{aligned}$$

and $\lfloor \ln n / \beta \rfloor$ is the integer part of $\ln n / \beta$ (where β is the one in **H3**).

Bound for $T_{2,1}$: First of all, for any $m \in \{1, \dots, n\}$, let $h_{(Y_0, X_0, Y_m, X_m)}$ be the density of (Y_0, X_0, Y_m, X_m) and $h_{(Y_0, X_0)}$ the density of (Y_0, X_0) . We set

$$\begin{aligned} \theta_m(y, x, y_*, x_*) &= h_{(Y_0, X_0, Y_m, X_m)}(y, x, y_*, x_*) - h_{(Y_0, X_0)}(y, x) h_{(Y_0, X_0)}(y_*, x_*), \\ &\quad (y, x, y_*, x_*) \in \mathbb{R} \times [0, 1]^d \times \mathbb{R} \times [0, 1]^d. \end{aligned} \quad (5.13)$$

For any $(x, x_*) \in [0, 1]^{2d}$, since the density of X_0 is 1 over $[0, 1]^d$ and using **H5**, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\theta_m(y, x, y_*, x_*)| dy dy_* \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{(Y_0, X_0, Y_m, X_m)}(y, x, y_*, x_*) dy dy_* + \left(\int_{-\infty}^{\infty} h_{(Y_0, X_0)}(y, x) dy \right)^2 \\ &= g_{(X_0, X_m)}(x, x_*) + 1 \leq L + 1. \end{aligned} \quad (5.14)$$

By a standard covariance equality, the definition of (5.13), (5.14) and Lemma 5.1 with

$p = 1$, we obtain

$$\begin{aligned}
& \left| \mathbf{C}_{ov} \left(Y_0 \Phi_{j,k}(X_0) \mathbf{1}_{\mathcal{A}_0^c}, Y_m \Phi_{j,k}(X_m) \mathbf{1}_{\mathcal{A}_m^c} \right) \right| \\
&= \left| \int_{-\kappa_* \sqrt{\ln n}}^{\kappa_* \sqrt{\ln n}} \int_{[0,1]^d} \int_{-\kappa_* \sqrt{\ln n}}^{\kappa_* \sqrt{\ln n}} \int_{[0,1]^d} \theta_m(y, x, y_*, x_*) (y \Phi_{j,k}(x) y_* \Phi_{j,k}(x_*)) dy dx dy_* dx_* \right| \\
&\leq \int_{[0,1]^d} \int_{[0,1]^d} \left(\int_{-\kappa_* \sqrt{\ln n}}^{\kappa_* \sqrt{\ln n}} \int_{-\kappa_* \sqrt{\ln n}}^{\kappa_* \sqrt{\ln n}} |y| |y_*| |\theta_m(y, x, y_*, x_*)| dy dy_* \right) |\Phi_{j,k}(x)| |\Phi_{j,k}(x_*)| dx dx_* \\
&\leq \kappa_*^2 \ln n \int_{[0,1]^d} \int_{[0,1]^d} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\theta_m(y, x, y_*, x_*)| dy dy_* \right) |\Phi_{j,k}(x)| |\Phi_{j,k}(x_*)| dx dx_* \\
&\leq C \ln n \left(\int_{[0,1]^d} |\Phi_{j,k}(x)| dx \right)^2 \leq C \ln n 2^{-jd}.
\end{aligned}$$

Therefore, since $2^{jd} \geq (\ln n)^2$,

$$T_{2,1} \leq C (\ln n)^2 2^{-jd} \leq C. \quad (5.15)$$

Bound for $T_{2,2}$: By the Davydov inequality (see Lemma 5.3 in Appendix with $p = q = 4$), Lemma 5.1 with $p = 4$, $2^{jd} \leq n$ and **H3**, we have

$$\begin{aligned}
\left| \mathbf{C}_{ov} \left(Y_0 \Phi_{j,k}(X_0) \mathbf{1}_{\mathcal{A}_0^c}, Y_m \Phi_{j,k}(X_m) \mathbf{1}_{\mathcal{A}_m^c} \right) \right| &\leq C \sqrt{\alpha_m} \sqrt{\mathbf{E} \left((Y_0 \Phi_{j,k}(X_0))^4 \mathbf{1}_{\mathcal{A}_0^c} \right)} \\
&\leq C \sqrt{\alpha_m} \sqrt{\mathbf{E} \left((Y_0 \Phi_{j,k}(X_0))^4 \right)} \\
&\leq C \sqrt{\alpha_m} 2^{jd/2} \leq C e^{-\beta m/2} \sqrt{n}.
\end{aligned}$$

The previous inequality implies that

$$T_{2,2} \leq C \sqrt{n} \sum_{m=\lceil \ln n / \beta \rceil}^n e^{-\beta m/2} \leq C \sqrt{n} e^{-\ln n/2} \leq C. \quad (5.16)$$

Combining (5.12), (5.15) and (5.16), we arrive at

$$T_2 \leq n(T_{2,1} + T_{2,2}) \leq Cn. \quad (5.17)$$

Putting (5.10), (5.11) and (5.17) together, we have

$$T \leq T_1 + T_2 \leq Cn. \quad (5.18)$$

Finally, (5.8), (5.9) and (5.18) lead to

$$\mathbf{E}((\hat{c}_{j,k} - c_{j,k})^2) \leq \frac{2}{n^2} (S + T) \leq C \frac{1}{n^2} n \leq C \frac{1}{n}.$$

This ends the proof of **(i)**.

(ii) Using $\mathbf{E}(Y_1 \Psi_{j,k,u}(X_1)) = d_{j,k,u}$ the inequality $(\sum_{i=1}^m a_i)^4 \leq m^3 \sum_{i=1}^m a_i^4$, $a = (a_1, \dots, a_m) \in \mathbb{R}^m$, the Hölder inequality, Lemma 5.1 with $p = 4$ and $2^{jd} \leq n$, we obtain

$$\begin{aligned} \mathbf{E}((\hat{d}_{j,k,u} - d_{j,k,u})^4) &= \frac{1}{n^4} \mathbf{E} \left(\left(\sum_{i=1}^n (Y_i \Psi_{j,k,u}(X_i)) - \mathbf{E}(Y_1 \Psi_{j,k,u}(X_1)) \right)^4 \right) \\ &\leq C \frac{1}{n^4} n^4 \mathbf{E}((Y_1 \Psi_{j,k,u}(X_1))^4) \leq C 2^{jd} \leq Cn. \end{aligned}$$

The proof of (ii) is completed.

Remark 5.1 *This bound can be improved using more sophisticated moment inequalities for α -mixing processes (as (Yang, 2007, Theorem 2.2)). However, the obtained bound in (ii) is sufficient in the rest of our study.*

(iii) Since $\mathbf{E}(Y_1 \Psi_{j,k,u}(X_1)) = d_{j,k,u}$, we have

$$\hat{d}_{j,k,u} - d_{j,k,u} = \frac{1}{n} \sum_{i=1}^n P_{i,j,k,u},$$

where

$$P_{i,j,k,u} = Y_i \Psi_{j,k,u}(X_i) - \mathbf{E}(Y_1 \Psi_{j,k,u}(X_1)).$$

Considering again the event $\mathcal{A}_i = \{|Y_i| \geq \kappa_* \sqrt{\ln n}\}$, where κ_* denotes a constant which will be chosen later, we can split $P_{i,j,k,u}$ as follows:

$$P_{i,j,k,u} = Q_{i,j,k,u} + R_{i,j,k,u},$$

where

$$Q_{i,j,k,u} = Y_i \Psi_{j,k,u}(X_i) \mathbf{1}_{\mathcal{A}_i} - \mathbf{E}(Y_1 \Psi_{j,k,u}(X_1) \mathbf{1}_{\mathcal{A}_i})$$

and

$$R_{i,j,k,u} = Y_i \Psi_{j,k,u}(X_i) \mathbf{1}_{\mathcal{A}_i^c} - \mathbf{E}(Y_1 \Psi_{j,k,u}(X_1) \mathbf{1}_{\mathcal{A}_i^c}).$$

Therefore

$$\mathbf{P} \left(|\hat{d}_{j,k,u} - d_{j,k,u}| \geq \frac{\kappa}{2} \lambda_n \right) \leq I_1 + I_2, \quad (5.19)$$

where

$$I_1 = \mathbf{P} \left(\frac{1}{n} \left| \sum_{i=1}^n Q_{i,j,k,u} \right| \geq \frac{\kappa}{4} \lambda_n \right), \quad I_2 = \mathbf{P} \left(\frac{1}{n} \left| \sum_{i=1}^n R_{i,j,k,u} \right| \geq \frac{\kappa}{4} \lambda_n \right).$$

Bound for I_1 : The Markov inequality, the Cauchy-Schwarz inequality and Lemma 5.1 with $p = 2$ yield

$$\begin{aligned} I_1 &\leq \frac{4}{\kappa n \lambda_n} \mathbf{E} \left(\left| \sum_{i=1}^n Q_{i,j,k,u} \right| \right) \leq C\sqrt{n} \mathbf{E}(|Q_{1,j,k,u}|) \leq C\sqrt{n} \mathbf{E}(|Y_1 \Psi_{j,k,u}(X_1)| \mathbf{1}_{\mathcal{A}_1}) \\ &\leq C\sqrt{n} \sqrt{\mathbf{E} \left((Y_1 \Psi_{j,k,u}(X_1))^2 \right) \mathbf{P}(\mathcal{A}_1)} \leq C\sqrt{n} \sqrt{\mathbf{P}(\mathcal{A}_1)}. \end{aligned}$$

Now, using **H4**, **H1** (implying (2.1)) and taking κ_* large enough, we obtain

$$\begin{aligned} \mathbf{P}(\mathcal{A}_1) &\leq \mathbf{P}(|\xi_1| \geq \kappa_* \sqrt{\ln n} - K) \leq \mathbf{P} \left(|\xi_1| \geq \frac{\kappa_*}{2} \sqrt{\ln n} \right) \\ &\leq 2\omega e^{-\kappa_*^2 \ln n / (8\sigma^2)} = 2\omega n^{-\kappa_*^2 / (8\sigma^2)} \leq C \frac{1}{n^{11}}. \end{aligned}$$

Hence

$$I_1 \leq C\sqrt{n} \frac{1}{n^{11/2}} \leq C \frac{1}{n^5}. \quad (5.20)$$

Bound for I_2 : We will bound I_2 via the Bernstein inequality for α -mixing process described in Lemma 5.4 (see Appendix).

We have $\mathbf{E}(R_{1,j,k,u}) = 0$ and, since $|Y_1| \mathbf{1}_{\mathcal{A}_1^c} \leq \kappa_* \sqrt{\ln n}$ and $|\Psi_{j,k,u}(x)| \leq C2^{jd/2} \leq C\sqrt{n}/(\ln n)^2$,

$$|R_{i,j,k,u}| \leq C\sqrt{\ln n} \sup_{x \in [0,1]^d} |\Psi_{j,k,u}(x)| \leq C\sqrt{\ln n} \frac{\sqrt{n}}{(\ln n)^2} = C\sqrt{\frac{n}{(\ln n)^3}}.$$

Using arguments similar to the proofs of the bounds for T_1 and $T_{2,1}$ in **(i)**, for any $l \leq C \ln n$, since $2^{jd} \geq (\ln n)^2$, we have

$$\mathbf{V} \left(\sum_{i=1}^l R_{i,j,k,u} \right) = \mathbf{V} \left(\sum_{i=1}^l Y_i \Psi_{j,k,u}(X_i) \mathbf{1}_{\mathcal{A}_1^c} \right) \leq C(l + l^2 \ln n 2^{-jd}) \leq Cl.$$

Hence

$$D_m = \max_{l \in \{1, \dots, 2m\}} \mathbf{V} \left(\sum_{i=1}^l R_{i,j,k,u} \right) \leq Cm.$$

Lemma 5.4 applied with the α -mixing random variables $R_{1,j,k,u}, \dots, R_{n,j,k,u}$, $\lambda = \kappa \lambda_n / 4$, $\lambda_n = \sqrt{\ln n / n}$, $m = \lceil u \ln n \rceil$ with $u > 0$ chosen later, $M = C\sqrt{n}/(\ln n)^3$ and

H3 gives

$$\begin{aligned}
I_2 &\leq C \left(\exp \left(-C \frac{\kappa^2 \lambda_n^2 n}{D_m/m + \kappa \lambda_n m M} \right) + \frac{M}{\lambda_n} n e^{-\beta m} \right) \\
&\leq C \left(\exp \left(-C \frac{\kappa^2 \ln n}{1 + \kappa \sqrt{\ln n/n} \ln n \sqrt{n/(\ln n)^3}} \right) + \sqrt{\frac{n/(\ln n)^3}{\ln n/n}} n e^{-\beta u \ln n} \right) \\
&\leq C \left(n^{-C\kappa^2/(1+\kappa u)} + n^{2-\beta u} \right).
\end{aligned}$$

Therefore, taking $u = \sqrt{\kappa}$ (for instance) and κ large enough, we have

$$I_2 \leq C \frac{1}{n^5}. \quad (5.21)$$

It follows from (5.19), (5.20) and (5.21) that

$$\mathbf{P} \left(|\hat{d}_{j,k,u} - d_{j,k,u}| \geq \frac{\kappa}{2} \lambda_n \right) \leq I_1 + I_2 \leq C \frac{1}{n^5}.$$

This completes the proof of **(iii)**.

This ends the proof of Proposition 5.1. □

5.3 Proof of Theorem 4.2

The proof of Theorem 4.2 is a consequence of Theorem 5.1 above and Proposition 5.2 below. To be more specific, Proposition 5.2 shows that **(a)** and **(b)** of Theorem 5.1 can be applied under the following configuration:

- $\nu = 0$,
- $\hat{c}_{j_0,k}^\diamond = \hat{c}_{j_0,k}^*$ and $\hat{d}_{j,k,u}^\diamond = \hat{d}_{j,k,u}^*$ from (4.7),
- $\lambda_n = \ln n / \sqrt{n}$,
- κ is a large enough constant,
- $\varrho = 0$.

Proposition 5.2 *Suppose that **H2**, **H3** and **H4** hold. Let $\hat{c}_{j,k}^*$ and $\hat{d}_{j,k,u}^*$ be defined by (4.7), and*

$$\lambda_n = \frac{\ln n}{\sqrt{n}}.$$

Then

(i) there exists a constant $C > 0$ such that, for any j satisfying $2^{jd} \leq n$ and $k \in D_j$,

$$\mathbf{E}((\hat{c}_{j,k}^* - c_{j,k})^2) \leq C \frac{(\ln n)^2}{n} \quad (\leq C\lambda_n^2).$$

(ii) there exists a constant $C > 0$ such that, for any j such that $2^{jd} \leq n$, $k \in D_j$ and $u \in \{1, \dots, 2^d - 1\}$,

$$\mathbf{E}((\hat{d}_{j,k,u}^* - d_{j,k,u})^4) \leq C \quad (= \varpi_n).$$

(iii) for $\kappa > 0$ large enough, there exists a constant $C > 0$ such that, for any j satisfying $2^{jd} \leq n/(\ln n)^4$, $k \in D_j$ and $u \in \{1, \dots, 2^d - 1\}$,

$$\mathbf{P}\left(|\hat{d}_{j,k,u}^* - d_{j,k,u}| \geq \frac{\kappa}{2}\lambda_n\right) \leq C \frac{1}{n^4} \quad (\leq C\lambda_n^8/\varpi_n).$$

Proof of Proposition 5.2. Again the technical tools in our proof are suitable covariance decompositions, a covariance inequality for α -mixing processes (see Lemma 5.3 in Appendix) and a Bernstein-type exponential inequality for α -mixing processes (see Lemma 5.4 in Appendix).

The following result will be useful in the sequel.

Lemma 5.2 Let $\hat{c}_{j,k}^*$ and $\hat{d}_{j,k,u}^*$ be defined by (4.7). Suppose that **H2** and **H4** hold. Then

- there exists a constant $C > 0$ such that, for any $j \geq \tau$ and $k \in D_j$,

$$|\hat{c}_{j,k}^* - c_{j,k}| \leq \frac{1}{n} \left| \sum_{i=1}^n (A_{i,j,k} - \mathbf{E}(A_{1,j,k})) \right| + C \frac{\ln n}{\sqrt{n}},$$

- there exists a constant $C > 0$ such that, for any $j \geq \tau$, $k \in D_j$ and $u \in \{1, \dots, 2^d - 1\}$,

$$|\hat{d}_{j,k,u}^* - d_{j,k,u}| \leq \frac{1}{n} \left| \sum_{i=1}^n (B_{i,j,k,u} - \mathbf{E}(B_{1,j,k,u})) \right| + C \frac{\ln n}{\sqrt{n}}.$$

(i) Lemma 5.2 and the inequality $(x^2 + y^2) \leq 2(x^2 + y^2)$, $(x, y) \in \mathbb{R}^2$, yield

$$\begin{aligned} \mathbf{E}((\hat{c}_{j,k}^* - c_{j,k})^2) &\leq C \left(\frac{1}{n^2} \mathbf{V} \left(\sum_{i=1}^n A_{i,j,k} \right) + \frac{(\ln n)^2}{n} \right) \\ &\leq C \frac{1}{n^2} (S + T + n(\ln n)^2), \end{aligned} \quad (5.22)$$

where

$$S = n\mathbf{V}(A_{1,j,k}), \quad T = \left| \sum_{v=2}^n \sum_{\ell=1}^{v-1} \mathbf{C}_{ov}(A_{v,j,k}, A_{\ell,j,k}) \right|.$$

Bound for S: It follows from Lemma 5.1 with $p = 2$ that

$$S \leq n\mathbf{E}((A_{1,j,k})^2) \leq n\mathbf{E}\left((Y_1\Phi_{j,k}(X_1))^2\right) \leq Cn. \quad (5.23)$$

Bound for T: It follows from the stationarity of $(Y_t, X_t)_{t \in \mathbb{Z}}$ that

$$\begin{aligned} T &= \left| \sum_{m=1}^n (n-m) \mathbf{C}_{ov}(A_{0,j,k}, A_{m,j,k}) \right| \leq n \sum_{m=1}^n |\mathbf{C}_{ov}(A_{0,j,k}, A_{m,j,k})| \\ &= n(T_1 + T_2), \end{aligned} \quad (5.24)$$

where

$$T_1 = \sum_{m=1}^{\lfloor \ln n / \beta \rfloor - 1} |\mathbf{C}_{ov}(A_{0,j,k}, A_{m,j,k})|, \quad T_2 = \sum_{m=\lfloor \ln n / \beta \rfloor}^n |\mathbf{C}_{ov}(A_{0,j,k}, A_{m,j,k})|$$

and $\lfloor \ln n / \beta \rfloor$ is the integer part of $\ln n / \beta$ (where β is the one in **H3**).

Bound for T_1 : The covariance inequality: $\mathbf{C}_{ov}(U, V) \leq \mathbf{E}(U^2)$, where U and V are identically distributed random variables admitting moments of order 2, and Lemma 5.1 with $p = 2$ lead to

$$T_1 \leq C \sum_{m=1}^{\lfloor \ln n / \beta \rfloor - 1} \mathbf{E}((A_{0,j,k})^2) \leq C \ln n \mathbf{E}((Y_0\Phi_{j,k}(X_0))^2) \leq C \ln n. \quad (5.25)$$

Bound for T_2 : By the Davydov inequality (see Lemma 5.3 in Appendix with $p = q = 4$), the Hölder inequality, $(A_{0,j,k})^4 \leq n(Y_0\Phi_{j,k}(X_0))^2$, Lemma 5.1 with $p = 2$ and **H3**, we have

$$\begin{aligned} |\mathbf{C}_{ov}(A_{0,j,k}, A_{m,j,k})| &\leq C\sqrt{\alpha_m} \sqrt{\mathbf{E}((A_{0,j,k})^4)} \\ &\leq C\sqrt{\alpha_m} \sqrt{n} \sqrt{\mathbf{E}((Y_0\Phi_{j,k}(X_0))^2)} \leq Ce^{-\beta m/2} \sqrt{n}. \end{aligned}$$

Owing to the previous inequality, we arrive at

$$T_2 \leq C\sqrt{n} \sum_{m=\lfloor \ln n / \beta \rfloor}^n e^{-\beta m/2} \leq C\sqrt{n} e^{-\ln n/2} = C. \quad (5.26)$$

Combining (5.24), (5.25) and (5.26), we obtain

$$T \leq n(T_1 + T_2) \leq Cn \ln n. \quad (5.27)$$

Finally, putting (5.22), (5.23) and (5.27) together, we have

$$\mathbf{E}((\hat{c}_{j,k}^* - c_{j,k})^2) \leq C \frac{1}{n^2} (S + T + n(\ln n)^2) \leq C \frac{(\ln n)^2}{n}.$$

This ends the proof of **(i)**.

(ii) Using $|d_{j,k,u}| \leq C$ (since $f \in \mathbb{L}_2([0, 1]^d)$) and $|B_{i,j,k,u}| \leq \sqrt{n}/\ln n$, we have

$$|\hat{d}_{j,k,u}^* - d_{j,k,u}| \leq |\hat{d}_{j,k,u}^*| + |d_{j,k,u}| \leq \frac{1}{n} \sum_{i=1}^n |B_{i,j,k,u}| + C \leq \frac{\sqrt{n}}{\ln n} + C \leq C \frac{\sqrt{n}}{\ln n}.$$

Moreover, proceeding as in the proof of **(i)** but with $\Psi_{j,k,u}$ instead of $\Phi_{j,k}$, we obtain

$$\mathbf{E}((\hat{d}_{j,k,u}^* - d_{j,k,u})^2) \leq C \frac{(\ln n)^2}{n}.$$

Therefore

$$\mathbf{E}((\hat{d}_{j,k,u}^* - d_{j,k,u})^4) \leq C \frac{n}{(\ln n)^2} \mathbf{E}((\hat{d}_{j,k,u}^* - d_{j,k,u})^2) \leq C \frac{n}{(\ln n)^2} \frac{(\ln n)^2}{n} \leq C.$$

This finishes the proof of **(ii)**.

(iii) For any $j \geq \tau$, $k \in D_j$ and $u \in \{1, \dots, 2^d - 1\}$, set

$$W_{i,j,k,u} = B_{i,j,k,u} - \mathbf{E}(B_{1,j,k,u}).$$

Lemma 5.2 and $\lambda_n = \ln n/\sqrt{n}$ imply that, for κ large enough,

$$\begin{aligned} \mathbf{P} \left(|\hat{d}_{j,k,u}^* - d_{j,k,u}| \geq \frac{\kappa}{2} \lambda_n \right) &\leq \mathbf{P} \left(\frac{1}{n} \left| \sum_{i=1}^n W_{i,j,k,u} \right| \geq \frac{\kappa}{2} \lambda_n - C \frac{\ln n}{\sqrt{n}} \right) \\ &\leq \mathbf{P} \left(\frac{1}{n} \left| \sum_{i=1}^n W_{i,j,k,u} \right| \geq \frac{\kappa}{4} \lambda_n \right). \end{aligned} \quad (5.28)$$

We will bound this probability term via the Bernstein inequality for α -mixing process (see Lemma 5.4 in Appendix)

We have $\mathbf{E}(W_{1,j,k,u}) = 0$ and, since $|B_{i,j,k,u}| \leq \sqrt{n}/\ln n$,

$$|W_{i,j,k,u}| \leq 2 \frac{\sqrt{n}}{\ln n}.$$

Similar arguments to the proofs of the bounds of S and T in **(i)** with $1 \leq l \leq C \ln n$ lead to

$$\mathbf{V} \left(\sum_{i=1}^l W_{i,j,k,u} \right) = \mathbf{V} \left(\sum_{i=1}^l B_{i,j,k,u} \right) \leq C(l + l^2) \leq l^2.$$

Hence

$$D_m = \max_{l \in \{1, \dots, 2m\}} \mathbf{V} \left(\sum_{i=1}^l W_{i,j,k,u} \right) \leq Cm^2.$$

Lemma 5.4 applied with the α -mixing random variables $W_{1,j,k,u}, \dots, W_{n,j,k,u}$, $\lambda = \kappa \lambda_n / 4$, $\lambda_n = \ln n / \sqrt{n}$, $m = \lceil u \ln n \rceil$ with $u > 0$ chosen later, $M = C \sqrt{n} / \ln n$ and **H3** gives

$$\begin{aligned} & \mathbf{P} \left(\frac{1}{n} \left| \sum_{i=1}^n W_{i,j,k,u} \right| \geq \frac{\kappa}{4} \lambda_n \right) \\ & \leq C \left(\exp \left(-C \frac{\kappa^2 \lambda_n^2 n}{D_m/m + \kappa \lambda_n m M} \right) + \frac{M}{\lambda_n} n e^{-\beta m} \right) \\ & \leq C \left(\exp \left(-C \frac{\kappa^2 (\ln n)^2}{u \ln n + \kappa (\ln n / \sqrt{n}) u \ln n (\sqrt{n} / \ln n)} \right) + \frac{\sqrt{n} / \ln n}{\ln n / \sqrt{n}} n e^{-\beta u \ln n} \right) \\ & \leq C \left(\exp \left(-C \frac{\kappa^2 (\ln n)^2}{u \ln n (1 + \kappa)} \right) + n^2 e^{-\beta u \ln n} \right) \\ & \leq C \left(n^{-C \kappa^2 / u (1 + \kappa)} + n^{2 - \beta u} \right). \end{aligned}$$

Therefore, taking $u = \sqrt{\kappa}$ (for instance) and κ large enough, we have

$$\mathbf{P} \left(\frac{1}{n} \left| \sum_{i=1}^n W_{i,j,k,u} \right| \geq \frac{\kappa}{4} \lambda_n \right) \leq C \frac{1}{n^4}. \quad (5.29)$$

It follows from (5.28) and (5.29) that

$$\mathbf{P} \left(|\hat{d}_{j,k,u} - d_{j,k,u}| \geq \frac{\kappa}{2} \lambda_n \right) \leq C \frac{1}{n^4}.$$

This completes the proof of **(iii)**.

This ends the proof of Proposition 5.2. □

5.4 Proof of the auxiliary results

Proof of Lemma 5.1. Owing to $\mathbf{E}(|\xi_1|^p) < \infty$, **H4**, the inequality $|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p)$, $(x, y) \in \mathbb{R}^2$, $p \geq 1$, the independence between X_1 and ξ_1 and the change of variables $y = 2^j x - k$, we obtain

$$\begin{aligned} \mathbf{E}(|Y_1 \Phi_{j,k}(X_1)|^p) &\leq C \mathbf{E}((K^p + |\xi_1|^p) |\Phi_{j,k}(X_1)|^p) \\ &= C(K^p + \mathbf{E}(|\xi_1|^p)) \mathbf{E}(|\Phi_{j,k}(X_1)|^p) \leq C \int_{[0,1]^d} |\Phi_{j,k}(x)|^p dx \\ &= C 2^{jd(p-1)} \left(\int_{[0,1]^d} |\phi(2^j x - k)|^p dx \right)^d \leq C 2^{jd(p/2-1)}. \end{aligned}$$

The proof of the other point is similar; it is enough to replace $\Phi_{j,k}$ by $\Psi_{j,k,u}$. This ends the proof of Lemma 5.1. □

Proof of Lemma 5.2. Since $\mathbf{E}(Y_1 \Phi_{j,k}(X_1)) = c_{j,k}$, we have

$$c_{j,k} = \mathbf{E}(A_{1,j,k}) + \mathbf{E} \left(Y_1 \Phi_{j,k}(X_1) \mathbf{1}_{\{|Y_1 \Phi_{j,k}(X_1)| > \frac{\sqrt{n}}{\ln n}\}} \right).$$

Therefore

$$|c_{j,k}^* - c_{j,k}| \leq \frac{1}{n} \left| \sum_{i=1}^n (A_{i,j,k} - \mathbf{E}(A_{1,j,k})) \right| + \mathbf{E} \left(|Y_1 \Phi_{j,k}(X_1)| \mathbf{1}_{\{|Y_1 \Phi_{j,k}(X_1)| > \frac{\sqrt{n}}{\ln n}\}} \right).$$

Let us now bound the last term. The Markov inequality and Lemma 5.1 with $p = 2$ yield

$$\mathbf{E} \left(|Y_1 \Phi_{j,k}(X_1)| \mathbf{1}_{\{|Y_1 \Phi_{j,k}(X_1)| > \frac{\sqrt{n}}{\ln n}\}} \right) \leq \frac{\ln n}{\sqrt{n}} \mathbf{E}((Y_1 \Phi_{j,k}(X_1))^2) \leq C \frac{\ln n}{\sqrt{n}},$$

that ends the proof of the first point. The proof of the second point is identical; it is enough to replace $\Phi_{j,k}$ by $\Psi_{j,k,u}$. Lemma 5.2 is proved. □

Appendix

In this section we give some preliminary lemmas which have been used in the proofs of our main results.

Lemma 5.3 (Davydov (1970)) Let $(A_t)_{t \in \mathbb{Z}}$ be a stationary α -mixing process with mixing coefficient α_m , $m \geq 0$, and h and k be two measurable functions. Let $p > 0$ and $q > 0$ satisfying $1/p + 1/q < 1$, such that $\mathbf{E}(|h(A_0)|^p)$ and $\mathbf{E}(|k(A_0)|^q)$ exist. Then there exists a constant $C > 0$ such that

$$|\mathbf{C}_{ov}(h(A_0), k(A_m))| \leq C \alpha_m^{1-1/p-1/q} (\mathbf{E}(|h(A_0)|^p))^{1/p} (\mathbf{E}(|k(A_0)|^q))^{1/q}.$$

Lemma 5.4 (Liebscher (2001)) Let $(A_t)_{t \in \mathbb{Z}}$ be a stationary process with the m -th strongly mixing coefficient α_m , $m \geq 0$, n be a positive integer, $h : \mathbb{R} \rightarrow \mathbb{C}$ be a measurable function and, for any $t \in \mathbb{Z}$, $U_t = h(A_t)$. We assume that $\mathbf{E}(U_1) = 0$ and there exists a constant $M > 0$ satisfying $|U_1| \leq M$. Then, for any $m \in \{1, \dots, [n/2]\}$ and $\lambda > 0$, we have

$$\mathbf{P} \left(\frac{1}{n} \left| \sum_{i=1}^n U_i \right| \geq \lambda \right) \leq 4 \exp \left(- \frac{\lambda^2 n}{16(D_m/m + \lambda M m/3)} \right) + 32 \frac{M}{\lambda} n \alpha_m,$$

where

$$D_m = \max_{l \in \{1, \dots, 2m\}} \mathbf{V} \left(\sum_{i=1}^l U_i \right).$$

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