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On the iterated Lévy transforms of Brownian motion

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Abstract

The Lévy transform of a Brownian motion B is given by $T(B)_t = \int_0^t \mathrm{sgn}(B_s)dB_s$. Csáki and Vincze have defined in 1961 a discrete analogous $\mathcal{T}$ of T which applies to simple random walks and is measure preserving. Although their result is quite important and optimal, its relationship with the Lévy transform has not been explored enough until now. In this paper, we prove that $\mathcal{T}$ is exact : $\bigcap_{k\geq 1} \sigma(\mathcal{T}^k(S))$ is trivial for each simple random walk S and give a precise description of the lost information at each step k. We then consider a suitable simple random walk S^n embedded in B depending on n and prove that for each k, $\mathcal{T}^k(S^n)$ suitably normalized and time scaled converges to $T^k(B)$ in probability in $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ as $n \to \infty$. We also derive some interesting consequences of this result. Finally, we give a sufficient condition under which the Lévy transform is ergodic.

1 Introduction and main results.

Let B be a Brownian motion, then $T(B)_t = \int_0^t \mathrm{sgn}(B_s)dB_s$ is a Brownian motion too. Iterating T yields a family of Brownian motions $(B^n)_n$ given by

$$B^0 = B, \quad B^{n+1} = T(B^n).$$

We call B^n the n-iterated Lévy transformation of B. Let $\mathbb{W} = \mathbb{C}(\mathbb{R}_+, \mathbb{R})$ be the Wiener space equipped with the Borel σ-field $\mathcal{F}$ associated to the distance of uniform convergence on compact sets

$$d_U(w, w') = \sum_{n\geq 1} 2^{-n}\big(\sup_{0\leq t\leq n} |w(t) - w'(t)| \wedge 1\big).$$

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Let μ be the Wiener measure and $B_t(\omega) = \omega(t)$ be the canonical Brownian motion defined on $\mathbb{W}$. The Lévy transformation on $\mathbb{W}$

$$\begin{aligned} T_{\mathbb{W}}: \ \mathbb{W} &\longrightarrow \mathbb{W} \\ \omega &\longmapsto \hat{\omega}, \end{aligned}$$

where

$$\hat{\omega}(t) = \int_0^t \operatorname{sgn}(B_s) dB_s(\omega) = |B_t(\omega)| - L_t(B)(\omega) = |\omega(t)| - \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|\omega(u)| \le \epsilon\}} du.$$

leaves invariant μ. Whether $T_{\mathbb{W}}$ is ergodic, i.e.

$$\forall E \in \mathcal{F}, \ T_{\mathbb{W}}^{-1}(E) = E \Rightarrow \mu(E) \in \{0, 1\},$$

is a very known open question raised by Marc Yor since the late 70's (see [9] page 257). The ergodicity of $T_{\mathbb{W}}$ is an immediate consequence of : $\bigcap_{n \ge 0} T_{\mathbb{W}}^{-n}(\mathcal{F})$ is μ-trivial (see [7] Corollary 4). Dubins and Smorodinsky [2] gave a proof of ergodicity in the modified discrete case. For a simple random walk (SRW) S, their definition of the Lévy transform of S is the SRW obtained by skipping plat paths from

$$n \longmapsto |S_n| - L_n$$

where L is a discrete analogous of local time. Another discrete Lévy transformation was also given in Fujita [4]. However, what was probably missing in [2] and [4] is the relationship with all the iterated Lévy transforms and the question of ergodicity in the spirit of the results below.

Before stating our results, let us introduce some notations. We equip $\mathbb{E} = \mathbb{W}^{\mathbb{N}}$ with the product metric defined for each $x = (x_k)_{k \ge 0}, y = (y_k)_{k \ge 0}$ by

$$d(x, y) = \sum_{k \ge 0} 2^{-k} (d_U(x_k, y_k) \wedge 1).$$

Thus $(\mathbb{E}, d)$ is a separable complete metric space.

For each SRW S and $h \ge 0$, we denote by $\mathcal{T}^h(S)$ the h-iterated Csáki-Vincze transformation (to be defined in Section 2.1) of S with the convention $\mathcal{T}^0(S) = S$.

Let B be a Brownian motion defined on $(\Omega, \mathcal{A}, \mathbb{P})$. For each $n \ge 1$, define $T_0^n = 0$ and for all $k \ge 0$,

$$T_{k+1}^n = \inf\left\{ t \ge T_k^n : |B_t - B_{T_k^n}| = \frac{1}{\sqrt{n}} \right\}.$$

Then $S_k^n = \sqrt{n} B_{T_k^n}, \ k \ge 0$, is a SRW and we have the following

Theorem 1. *(i) For each SRW S (and in particular for S^n for each $n \geq 1$) and $h \geq 0$, $\mathcal{T}^h(S)$ is independent of $(S_j, j \leq h)$ and a fortiori*

$$\bigcap_{h\geq 0} \sigma(\mathcal{T}^h(S))$$

is trivial.

(ii) For each $n \geq 1$, $h \geq 0$ and $t \geq 0$, define

$$S^{n,h}(t) = \frac{1}{\sqrt{n}}\mathcal{T}^h(S^n)_{\lfloor nt \rfloor} + \frac{(nt - \lfloor nt \rfloor)}{\sqrt{n}}\big(\mathcal{T}^h(S^n)_{\lfloor nt \rfloor + 1} - \mathcal{T}^h(S^n)_{\lfloor nt \rfloor}\big).$$

Then

$$(S^{n,0}, S^{n,1}, S^{n,2}, \cdots)$$

converges to

$$(B^0, B^1, B^2, \cdots)$$

in probability in $\mathbb{E}$ as $n \to \infty$.

A sufficient condition for the ergodicity of the Lévy transformation is the convergence in law of (B^n, B) (or (B^{α_n}, B) where $(\alpha_n)_n$ is a nonegative sequence with values in $\mathbb{N}$ and such that $\alpha_n \longrightarrow +\infty$ as $n \to \infty$) to a 2-dimensional Brownian motion. This fact has been proved recently by Prokaj [10] (Proposition 16). We prove that this property is satisfied in discrete time.

Corollary 1. *Fix $p \geq 2$ and let $\alpha^i = (\alpha^i_n)_{n\geq 1}$, $i \in [1, p]$ be p nonegative sequences such that*

$$\alpha^1_n \longrightarrow +\infty, \quad \alpha^i_n - \alpha^{i-1}_n \longrightarrow +\infty \text{ as } n \to \infty \text{ for all } i \in [2, p].$$

Let $X_0, X_1, \cdots, X_p$ be $p + 1$ independent Brownian motions.

(i) Let S be a SRW and for $n \geq 1, h \geq 0, t \in \mathbb{R}_+$, define

$$S^h_n(t) = \frac{1}{\sqrt{n}}\mathcal{T}^h(S)_{\lfloor nt \rfloor} + \frac{(nt - \lfloor nt \rfloor)}{\sqrt{n}}\big(\mathcal{T}^h(S)_{\lfloor nt \rfloor + 1} - \mathcal{T}^h(S)_{\lfloor nt \rfloor}\big). \tag{1}$$

Then

$$\Big(S^0_n, S^{\lfloor n\alpha^1_n \rfloor}_n, \cdots, S^{\lfloor n\alpha^p_n \rfloor}_n\Big) \xrightarrow[n \to +\infty]{law} \big(X_0, X_1, \cdots, X_p\big) \text{ in } \mathbb{W}^{p+1}.$$

(ii) With the same notations of Theorem 1, we have

$$\left(B, S^{n,\lfloor n\alpha_n^1\rfloor}, \cdots, S^{n,\lfloor n\alpha_n^p\rfloor}\right) \xrightarrow[n\to+\infty]{law} \left(X_0, X_1, \cdots, X_p\right) \text{ in } \mathbb{W}^{p+1}.$$

We were not able to deduce the ergodicity of the Lévy transformation. The following proposition gives a sufficient condition to prove this result.

Proposition 1. *Suppose there exists a nonegative sequence $(\alpha_n)_n$ with values in $\mathbb{N}$ tending to $+\infty$ and a subsequence $(a_n)_n$ of $n \mapsto n$ such that for all $t \geq 0$,*

$$\lim_{n\to\infty}\left(S^{a_n,a_n\alpha_n}(t) - B_t^{a_n\alpha_n}\right) = 0 \text{ in probability,} \tag{2}$$

then as $n \to \infty$, $(B, B^{a_n\alpha_n})$ converges in law to (X, Y) in $\mathbb{W}^2$ where X, Y are two independent Brownian motions and a fortiori the Lévy transformation is ergodic.

From Theorem 1, we can deduce that for each subsequence $(a_n)_n$ there exists a nonegative bounded sequence $(\alpha_n)_n$ such that (2) holds. We just write the result for $a_n = n$ as follows.

Corollary 2. *With the same notations of Theorem 1, there exists a family $(\alpha^i)_{i\in\mathbb{N}}$ of nondecreasing sequences $\alpha^i = (\alpha_n^i)_{n\in\mathbb{N}}$ with values in $\mathbb{N}$ such that*

$$\alpha_n^0 \longrightarrow +\infty, \quad \alpha_n^i - \alpha_n^{i-1} \longrightarrow +\infty \text{ as } n \to \infty \text{ for all } i \geq 1$$

and moreover

$$\lim_{n\to\infty}\left(S^{n,\alpha_n^i} - B^{\alpha_n^i}\right) = 0 \text{ in probability in } \mathbb{W}$$

for all $i \in \mathbb{N}$.

The sequences $(\alpha_i)_{i\in\mathbb{N}}$ will be constructed such that $0 \leq \alpha_n^0 \leq n$ and $0 \leq \alpha_n^i - \alpha_n^{i-1} \leq n$ for all i and n and it is not easy to deduce from Corollary 2 that (2) is satisfied (for $a_n = n$).
In the next section we prove the previous results. We first review the Csáki-Vincze transformation, establish part (i) of Theorem 1 and show that $(S, \mathcal{T}(S), \cdots, \mathcal{T}^h(S), \cdots)$ "converges" in law to $(B, T(B), \cdots, T^h(B), \cdots)$. To prove part (ii) of Theorem 1, we use the simple idea : if Z_n converges in law to a constant c, then the convergence holds also in probability. The other proofs are based on the crucial property of the transformation $\mathcal{T}$: $\mathcal{T}^h(S)$ is independent of $\sigma(S_j, j \leq h)$ for each h.

2 Proofs.

2.1 The Csáki-Vincze transformation and convergence in law.

For the sequel, we recommand the lecture of the pages 109 and 110 in [8] (Theorem 2 below). Some consequences (see Proposition 2 below) have been drawn in [5] (Sections 2.1 and 2.2) to give a discrete approach to some stochastic flows associated to Tanaka's SDE. We also notice that our stating of this result is slightly different from [8]. We leave to the reader to make the obvious analogy.

Theorem 2. *([8] page 109) Let $S = (S_n)_{n\geq 0}$ be a SRW defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and $X_i = S_i - S_{i-1}, i \geq 1$. Define $\tau_0 = 0$ and for $l \geq 0$,*

$$\tau_{l+1} = \min\left\{i > \tau_l : S_{i-1}S_{i+1} < 0\right\}.$$

Set

$$\overline{X}_j = \sum_{l\geq 0} (-1)^l X_1 X_{j+1} 1_{\{\tau_l+1 \leq j \leq \tau_{l+1}\}}.$$

Then $\overline{S}_0 = 0,\ \overline{S}_n = \overline{X}_1 + \cdots + \overline{X}_n, n \geq 1$ is a SRW. Moreover if $Y_n := \overline{S}_n - \min_{k\leq n} \overline{S}_k$, then for all $n \in \mathbb{N}$,

$$|Y_n - |S_n|| \leq 2. \tag{3}$$

We call $\overline{S} = \mathcal{T}(S)$, the Csáki-Vincze transformation of S (see the figures 1 and 2 below).

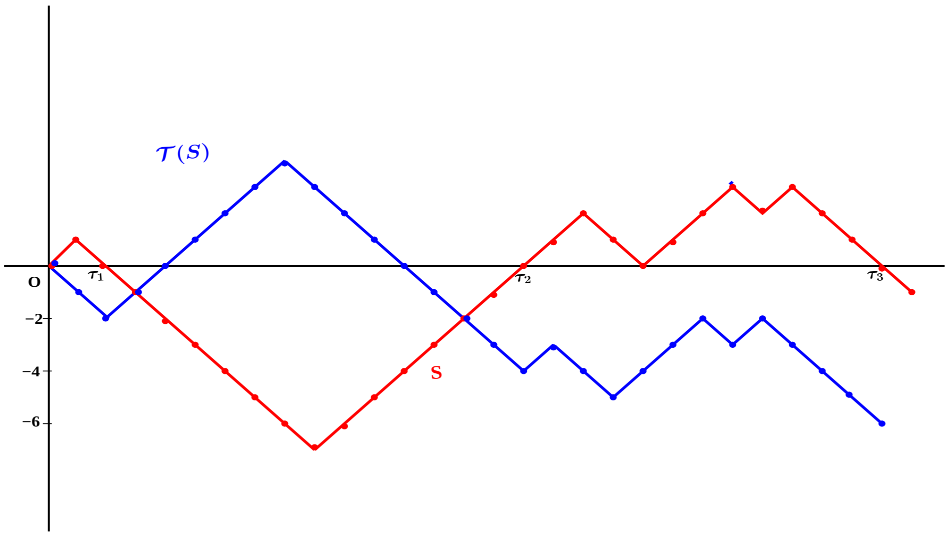

Figure 1: S and $\mathcal{T}(S)$.

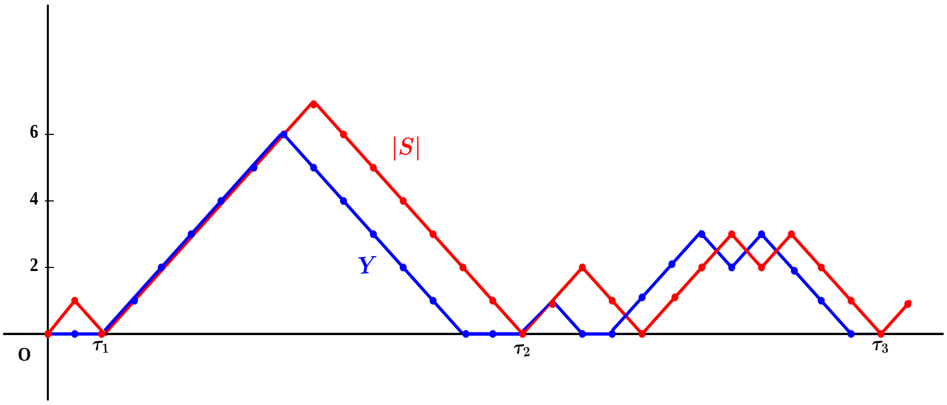

Figure 2: $|S|$ and Y.

Note that $(-1)^l X_1$ is simply equal to $\mathrm{sgn}(S)_{|[\tau_l+1,\tau_{l+1}]}(:= X_{\tau_l+1})$ which can easily be checked by induction on l. Thus for all $j \in [\tau_l + 1, \tau_{l+1}]$,

$$\overline{X}_j = \mathrm{sgn}(S)_{|[\tau_l+1,\tau_{l+1}]}(S_{j+1} - S_j)$$

and one can expect that $(S, \mathcal{T}(S)$ will "converge" to (B, B^1) in a suitable sense. The following proposition has been established in [5]. We give its proof for completeness.

Proposition 2. *With the same notations of Theorem 2, we have*

(i) For all $n \geq 0$, $\sigma(\mathcal{T}(S)_j, j \leq n) \vee \sigma(S_1) = \sigma(S_j, j \leq n+1)$.

(ii) S_1 is independent of $\sigma(\mathcal{T}(S))$.

Proof. (i) To prove the inclusion $\subset$, we only need to check that $\{\tau_l + 1 \leq j \leq \tau_{l+1}\} \in \sigma(S_h, h \leq n+1)$ for a fixed $j \leq n$. This is clear since $\{\tau_l = m\} \in \sigma(S_h, h \leq m+1)$ for all $l, m \in \mathbb{N}$.
Now, for all $1 \leq j \leq n$, we have $X_{j+1} = \sum_{l \geq 0} (-1)^l X_1 \overline{X}_j 1_{\{\tau_l+1 \leq j \leq \tau_{l+1}\}}$. As a consequence of (iii) and (iv) [8] (page 110), for all $l \geq 0$,

$$\tau_l = \min\{n \geq 0, \mathcal{T}(S)_n = -2l\}.$$

Thus τ_l is a stopping time with respect to the natural filtration of $\mathcal{T}(S)$ and as a result $\{\tau_l + 1 \leq j \leq \tau_{l+1}\} \in \sigma(\mathcal{T}(S)_h, h \leq j-1)$ which proves the inclusion $\supset$.
(ii) We may write for all $l \geq 1$,

$$\tau_l = \min\{i > \tau_{l-1} : X_1 S_{i-1} X_1 S_{i+1} < 0\}.$$

This shows that $\mathcal{T}(S)$ is $\sigma(X_1X_{j+1}, j \geq 0)$-measurable and (ii) is proved.

$\square$

Note that

$$\mathcal{T}(S) = \mathcal{T}(-S), \quad \sigma(\mathcal{T}^{h+1}(S)) \subset \sigma(\mathcal{T}^h(S)),$$

which is the analogous of

$$T(B) = T(-B), \quad \sigma(T^{h+1}(B)) \subset \sigma(T^h(B)).$$

The previous proposition yields the following

Corollary 3. *For all $n \geq 0$,*

(i) $\sigma(S) = \sigma(\mathcal{T}^n(S)) \vee \sigma(S_k, k \leq n)$.

(ii) $\sigma(\mathcal{T}^n(S))$ *and* $\sigma(S_k, k \leq n)$ *are independent.*

(iii) The σ*-field*

$$\mathcal{G}^\infty = \bigcap_{n \geq 0} \sigma(\mathcal{T}^n(S))$$

is $\mathbb{P}$*-trivial.*

Proof. Set $X_i = S_i - S_{i-1}, i \geq 1$.
(i) We apply successively Proposition 2 (i) so that for all $n \geq 1$,

$$\begin{aligned}
\sigma(S) &= \sigma(\mathcal{T}(S)) \vee \sigma(S_1) \\
&= \sigma(\mathcal{T}^2(S)) \vee \sigma(\mathcal{T}(S)_1) \vee \sigma(S_1) \\
&= \cdots \\
&= \sigma(\mathcal{T}^n(S)) \vee \sigma(\mathcal{T}^{n-1}(S)_1) \vee \cdots \vee \sigma(\mathcal{T}(S)_1) \vee \sigma(S_1).
\end{aligned}$$

To deduce (i), it suffices to prove that

$$\sigma(S_k, k \leq n) = \sigma(\mathcal{T}^{n-1}(S)_1) \vee \cdots \vee \sigma(\mathcal{T}(S)_1) \vee \sigma(S_1). \tag{4}$$

Again Proposition 2 (i), yields

$$\begin{aligned}
\sigma(S_k, k \leq n) &= \sigma(\mathcal{T}(S)_j, j \leq n-1) \vee \sigma(S_1) \\
&= \sigma(\mathcal{T}^2(S)_j, j \leq n-2) \vee \sigma(\mathcal{T}(S)_1) \vee \sigma(S_1) \\
&= \cdots \\
&= \sigma(\mathcal{T}^{n-1}(S)_1) \vee \sigma(\mathcal{T}^{n-2}(S)_1) \cdots \vee \sigma(\mathcal{T}(S)_1) \vee \sigma(S_1)
\end{aligned}$$

which proves (4) and allows to deduce (i).

(ii) will be proved by induction on n. For $n = 0$, this is clear. Suppose the result holds for n, then $S_1, \mathcal{T}^1(S)_1, \cdots, \mathcal{T}^{n-1}(S)_1, \mathcal{T}^n(S)$ are independent (recall (4)). Let prove that $S_1, \mathcal{T}^1(S)_1, \cdots, \mathcal{T}^n(S)_1, \mathcal{T}^{n+1}(S)$ are independent which will imply (ii) by (4). Note that $\mathcal{T}^n(S)_1$ and $\mathcal{T}^{n+1}(S)$ are $\sigma(\mathcal{T}^n(S))$-measurable. By the induction hypothesis, this show that $(S_1, \mathcal{T}^1(S)_1, \cdots, \mathcal{T}^{n-1}(S)_1)$ and $(\mathcal{T}^n(S)_1, \mathcal{T}^{n+1}(S))$ are independent. But $\mathcal{T}^n(S)_1$ and $\mathcal{T}^{n+1}(S)$ are also independent by Proposition 2 (ii). Hence (ii) holds for $n + 1$ and thus for all n.

(iii) Let $A \in \mathcal{G}^\infty$ and fix $n \geq 1$. Then $A \in \sigma(\mathcal{T}^n(S))$ and we deduce from (ii) that A is independent of $\sigma(S_k, k \leq n)$. Since this holds for all n, A is independent of $\sigma(S)$. As $\mathcal{G}^\infty \subset \sigma(S)$, A is therefore independent of itself. □

Let S be a SRW defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and recall the definition of $S_n^h(t)$ from (1). On $\mathbb{E}$, define

$$Z^n(t_0, t_1, \cdots, t_h, \cdots) = \Big(S_n^0(t_0), S_n^1(t_1), \cdots, S_n^h(t_h), \cdots\Big)$$

and let $\mathbb{P}_n$ be the law of Z^n.

Lemma 1. *The family $\{\mathbb{P}_n, n \geq 1\}$ is tight on $\mathbb{E}$.*

Proof. By Donsker theorem for each h, S_n^h converges in law to standard Brownian motion as $n \to \infty$. Thus the law of each coordinate of Z^n is tight on $\mathbb{W}$ which is sufficient to get the result (see [3] page 107). □

The limit process. Fix a sequence $(m_n, n \in \mathbb{N})$ such that $Z^{m_n} \xrightarrow[n \to +\infty]{\text{law}} Z$ in $\mathbb{E}$ where

$$Z = \Big(B^{(0)}, B^{(1)}, \cdots, B^{(h)}, \cdots\Big)$$

is the limit process. Note that $B^{(0)}$ is a Brownian motion. From (3), we have $\forall n \geq 1, t \geq 0$

$$\left| |S_n^0(t)| - (S_n^1(t) - \min_{0 \leq u \leq t} S_n^1(u)) \right| \leq \frac{2012}{\sqrt{n}}.$$

Letting $n \to \infty$, we get

$$|B_t^{(0)}| = B_t^{(1)} - \min_{0 \leq u \leq t} B_u^{(1)}.$$

Tanaka's formula for local time gives

$$|B_t^{(0)}| = \int_0^t sgn(B_u^{(0)}) dB_u^{(0)} + L_t(B^{(0)}) = B_t^{(1)} - \min_{0 \leq u \leq t} B_u^{(1)},$$

where $L_t(B^{(0)})$ is the local time at 0 of $B^{(0)}$ and so

$$B_t^{(1)} = \int_0^t \mathrm{sgn}(B_s^{(0)}) dB_s^{(0)}.$$

The same reasoning shows that for all $h \geq 1$,

$$B_t^{(h+1)} = \int_0^t \mathrm{sgn}(B_s^{(h)}) dB_s^{(h)}.$$

Thus the law of Z is independent of the sequence $(m_n, n \in \mathbb{N})$ and therefore

$$\left(S_n^0, S_n^1, \cdots, S_n^h, \cdots\right) \xrightarrow[n \to +\infty]{\text{law}} \left(B, B^1, \cdots, B^h, \cdots\right) \text{ in } \mathbb{E} \tag{5}$$

where B is a Brownian motion.

2.2 Convergence in probability.

Let B a Brownian motion and recall the notations in Theorem 1. For each $n \geq 1$, define

$$U^n = \left(B, S^{n,0}, B^1, S^{n,1}, \cdots, B^h, S^{n,h}, \cdots\right).$$

and let $\mathbb{Q}_n$ be the law of U^n. Since $\mathcal{T}^h(S^n)$ is a simple random walk for each (h, n), a similar argument as in the proof of Lemma 1 shows that $\{\mathbb{Q}_n, n \geq 1\}$ is tight on $\mathbb{E}$. Fix a sequence $(m_n, n \in \mathbb{N})$ such that $U^{m_n} \xrightarrow[n \to +\infty]{\text{law}} U$ in $\mathbb{E}$. Using (5), we see that there exist two Brownian motions X and Y such that

$$U = \left(X, Y, X^1, Y^1, \cdots, X^h, Y^h, \cdots\right).$$

It is easy to check that if $\varphi : \mathbb{W} \longrightarrow \mathbb{R}$ is bounded uniformly continuous, then $\Psi(f, g) = \varphi(f - g)$ defined for all $(f, g) \in \mathbb{W}^2$ is bounded uniformly continuous which comes from

$$d_U(f - f', g - g') = d_U(f - g, f' - g') \text{ for all } f, f', g, g' \in \mathbb{W}.$$

Thus if (F_n, G_n) converges in law to (F, G) in $\mathbb{W}^2$, then $F_n - G_n$ converges in law to $F - G$ in $\mathbb{W}$. Applying this, we see that $B - S^{n,0}$ converges in law to $X - Y$. On the other hand, $B - S^{n,0}$ converges to 0 (in $\mathbb{W}$) in probability (see [6] page 39). Consequently $X = Y$ and

$$U^n \xrightarrow[n \to +\infty]{\text{law}} \left(B, B, B^1, B^1, \cdots, B^h, B^h, \cdots\right) \text{ in } \mathbb{E}.$$

In particular for each h, $S^{n,h} - B^h$ converges in law to 0 as $n \to \infty$, that is $S^{n,h}$ converges to B^h in probability as $n \to \infty$. Now the following equivalences are classical

(i) $\lim_{n\to\infty} U^n = (B, B^1, \cdots)$ in probability in $\mathbb{E}$.

(ii) $\lim_{n\to\infty} E[d(U^n, U) \wedge 1] = 0$.

(iii) For each h, $\lim_{n\to\infty} E[d_U(S^{n,h}, B^h) \wedge 1] = 0$.

(iv) For each h, $\lim_{n\to\infty} S^{n,h} = B^h$ in probability in $\mathbb{W}$.

Since we have proved (iv), Theorem 1 holds.

2.3 Proof of Corollary 1.

(i) Let S be a SRW and $X_0, X_1, \cdots, X_p$ be $p+1$ independent Brownian motions (not necessarily defined on the same probability space as S). Fix

$$0 \le t_1^0 \le \cdots \le t_{i_0}^0, \quad 0 \le t_1^1 \le \cdots \le t_{i_1}^1, \cdots, 0 \le t_1^p \le \cdots \le t_{i_p}^p.$$

By Corollary 3 (ii), for n large enough (such that $\lfloor nt_{i_0}^0 \rfloor + 1 \le \lfloor n\alpha_n^1 \rfloor$), $\big(S_n^0(t_1^0), \cdots, S_n^0(t_{i_0}^0)\big)$ which is $\sigma(S_j, j \le \lfloor nt_{i_0}^0 \rfloor + 1)$-measurable, is independent of $\mathcal{T}^{\lfloor n\alpha_n^1 \rfloor}(S)$. Thus $(S_n^0(t_1^0), \cdots, S_n^0(t_{i_0}^0))$ is independent of $(S_n^{\lfloor n\alpha_n^1 \rfloor}, \cdots, S_n^{\lfloor n\alpha_n^p \rfloor})$ and similarly $\mathcal{T}^{\lfloor n\alpha_n^2 \rfloor}(S)$ is independent of $\sigma(\mathcal{T}^{\lfloor n\alpha_n^1 \rfloor}(S)_j, j \le \lfloor n\alpha_n^2 \rfloor - \lfloor n\alpha_n^1 \rfloor)$. Again, for n large (such that $\lfloor nt_{i_1}^1 \rfloor + 1 \le \lfloor n\alpha_n^2 \rfloor - \lfloor n\alpha_n^1 \rfloor$), $\big(S_n^{\lfloor n\alpha_n^1 \rfloor}(t_1^1), \cdots, S_n^{\lfloor n\alpha_n^1 \rfloor}(t_{i_1}^1)\big)$ is $\sigma(\mathcal{T}^{\lfloor n\alpha_n^1 \rfloor}(S)_j, j \le \lfloor n\alpha_n^2 \rfloor - \lfloor n\alpha_n^1 \rfloor)$-measurable and therefore is independent of $\big(S_n^{\lfloor n\alpha_n^2 \rfloor}, \cdots, S_n^{\lfloor n\alpha_n^p \rfloor}\big)$. By induction on p, for n large enough,

$$\big(S_n^0(t_1^0), \cdots, S_n^0(t_{i_0}^0)\big), \quad \big(S_n^{\lfloor n\alpha_n^1 \rfloor}(t_1^1), \cdots, S_n^{\lfloor n\alpha_n^1 \rfloor}(t_{i_1}^1)\big), \cdots, \big(S_n^{\lfloor n\alpha_n^p \rfloor}(t_1^p), \cdots, S_n^{\lfloor n\alpha_n^p \rfloor}(t_{i_p}^p)\big)$$

are independent and this yields the convergence in law of

$$\Big(S_n^0(t_1^0), \cdots, S_n^0(t_{i_0}^0), S_n^{\lfloor n\alpha_n^1 \rfloor}(t_1^1), \cdots, S_n^{\lfloor n\alpha_n^1 \rfloor}(t_{i_1}^1), \cdots, S_n^{\lfloor n\alpha_n^p \rfloor}(t_1^p), \cdots, S_n^{\lfloor n\alpha_n^p \rfloor}(t_{i_p}^p)\Big)$$

to

$$\Big(X_0(t_1^0), \cdots, X_0(t_{i_0}^0), X_1(t_1^1), \cdots, X_1(t_{i_1}^1), \cdots, X_p(t_1^p), \cdots, X_p(t_{i_p}^p)\Big).$$

Thus the convergence of the finite dimensional marginals holds and the proof is completed.
(ii) is easy to check using (i) and the convergence in probability of $S^{n,0}$ to B.

2.4 Proof of Proposition 1.

To simplify notations, we will take $a_n = n$. Again, we only need to check the convergence of the finite dimensional marginals. So, let $0 \leq t_1 \leq \cdots \leq t_p, 0 \leq u_1 \leq \cdots \leq u_q$ and $f : \mathbb{R}^{p+q} \longrightarrow \mathbb{R}$ be a bounded uniformly continuous function. We can suppose that $\lfloor nt_p \rfloor + 1 \leq n\alpha_n$ for all n. We have

$$E\big[f\big(B_{t_1}, \cdots, B_{t_p}, B^{n\alpha_n}_{u_1}, \cdots, B^{n\alpha_n}_{u_q}\big)\big] = \varepsilon_n + E[f(S^{n,0}(t_1), \cdots, S^{n,0}(t_p), B^{n\alpha_n}_{u_1}, \cdots, B^{n\alpha_n}_{u_q})] \tag{6}$$

with $\lim_{n\to\infty} \varepsilon_n = 0$ using the uniform continuity of f and the convergence in probability of $\big(S^{n,0}(t_i)\big)_{1\leq i\leq p}$ to $\big(B_{t_i}\big)_{1\leq i\leq p}$. The right hand side of (6) may be written as

$$\varepsilon_n + \eta_n + E\big[f\big(S^{n,0}(t_1), \cdots, S^{n,0}(t_p), S^{n,n\alpha_n}(u_1), \cdots, S^{n,n\alpha_n}(u_q)\big)\big]$$

with $\lim_{n\to\infty} \eta_n = 0$ using (2) and the uniform continuity of f. Recall that $(S^{n,0}(t_1), \cdots, S^{n,0}(t_p))$ is measurable with respect to $\sigma(S^n_j, j \leq \lfloor nt_p \rfloor + 1) \subset \sigma(S^n_j, j \leq n\alpha_n)$. Since $S^{n,n\alpha_n}$ is independent of $\sigma(S^n_j, j \leq n\alpha_n)$ and

$$\Big(S^{n,0}(t_1), \cdots, S^{n,0}(t_p), S^{n,n\alpha_n}(u_1), \cdots, S^{n,n\alpha_n}(u_q)\Big) \xrightarrow[n\to+\infty]{\text{law}} \big(X_{t_1}, \cdots, X_{t_p}, Y_{u_1}, \cdots, Y_{u_q}\big),$$

the proof is now completed.

2.5 Proof of Corollary 2.

We need the following lemma which may be found in [1] page 32 in more generality:

Lemma 2. *If $(u_{k,n})_{k,n\in\mathbb{N}}$ is a nonnegative and bounded doubly indexed sequence such that for all $k, \lim_{n\to\infty} u_{k,n} = 0$, then there exists a nondecreasing sequence $(k_n)_n$ such that $\lim_{n\to\infty} k_n = +\infty$ and $\lim_{n\to\infty} u_{k_n,n} = 0$.*

Proof. By induction on p, we construct an increasing sequence $(n_p)_{p\in\mathbb{N}}$ such that $u_{p,n} < 2^{-p}$ for all $n \geq n_p$. Now define

$$k_n = \begin{cases} n & \text{if } 0 \leq n \leq n_0 \\ p & \text{if } n_p \leq n < n_{p+1} \text{ for some } p \in \mathbb{N}. \end{cases}$$

Clearly $n \longmapsto k_n$ is nondecreasing and $\lim_{n\to\infty} k_n = +\infty$. Moreover for all p and $n \geq n_p$, we have $u_{k_n,n} < 2^{-p}$. Thus for all p, $0 \leq \limsup_{n\to\infty} u_{k_n,n} \leq 2^{-p}$ and since p is arbitrary, the lemma is proved. □

Lemma 2 applied to

$$u_{k,n} = E[d_U(S^{n,k}, B^k) \wedge 1],$$

guarantees the existence of a nondecreasing sequence $(\alpha_n^0)_n$ with values in $\mathbb{N}$ such that $\lim_{n\to\infty} \alpha_n^0 = +\infty$ and

$$\lim_{n\to\infty} \left(S^{n,\alpha_n^0} - B^{\alpha_n^0}\right) = 0 \text{ in probability in } \mathbb{W}. \tag{7}$$

Now set

$$V^n = \left(B^{\alpha_n^0}, S^{n,\alpha_n^0}, B^{\alpha_n^0+1}, S^{n,\alpha_n^0+1}, \cdots, B^{\alpha_n^0+h}, S^{n,\alpha_n^0+h}, \cdots\right).$$

Using the same idea as in Section 2.2 and the relation (7), we prove that for all $j \in \mathbb{N}$,

$$\lim_{n\to\infty} \left(S^{n,\alpha_n^0+j} - B^{\alpha_n^0+j}\right) = 0 \text{ in probability in } \mathbb{W}. \tag{8}$$

This is also equivalent to : for all $j \in \mathbb{N}$,

$$\lim_{n\to\infty} u_{j,n}^0 = 0 \quad \text{where} \quad u_{j,n}^0 = E\big[d_U(S^{n,\alpha_n^0+j}, B^{\alpha_n^0+j}) \wedge 1\big].$$

By Lemma 2 again, there exists a nondecreasing sequence $(\beta_n^0)_n$ with values in $\mathbb{N}$ such that $\lim_{n\to\infty} \beta_n^0 = +\infty$ and

$$\lim_{n\to\infty} \left(S^{n,\alpha_n^0+\beta_n^0} - B^{\alpha_n^0+\beta_n^0}\right) = 0 \text{ in probability in } \mathbb{W}. \tag{9}$$

Define $\alpha_n^1 = \alpha_n^0 + \beta_n^0$. Now using (9) and the same preceding idea, we construct α^2 and all the $(\alpha^i)_i$ by the same way.

Remark 1. *Note that in Lemma 2, we have $k_n \leq n$ for all n and it is not always possible to construct k_n such that $\lim_{n\to\infty} \frac{k_n}{n} = +\infty$ (take $u_{k,n} = \frac{k}{n}$). This result cannot therefore be applied to Proposition 1.*

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