



**HAL**  
open science

# Infinite Variation Tempered Stable Ornstein-Uhlenbeck Processes with Discrete Observations

Reiichiro Kawai, Hiroki Masuda

► **To cite this version:**

Reiichiro Kawai, Hiroki Masuda. Infinite Variation Tempered Stable Ornstein-Uhlenbeck Processes with Discrete Observations. *Communications in Statistics - Simulation and Computation*, 2011, 41 (01), pp.125-139. 10.1080/03610918.2011.582561 . hal-00727116

**HAL Id: hal-00727116**

**<https://hal.science/hal-00727116>**

Submitted on 2 Sep 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



**Infinite Variation Tempered Stable Ornstein-Uhlenbeck Processes with Discrete Observations**

Journal:	<i>Communications in Statistics - Simulation and Computation</i>
Manuscript ID:	LSSP-2010-0376.R3
Manuscript Type:	Original Paper
Date Submitted by the Author:	12-Apr-2011
Complete List of Authors:	Kawai, Reiichiro; University of Leicester, Department of Mathematics Masuda, Hiroki; Kyushu University
Keywords:	acceptance-rejection sampling, L\'evy process, Ornstein-Uhlenbeck processes, selfdecomposability, transition law, tempered stable process
Abstract:	We investigate transition law between consecutive observations of Ornstein-Uhlenbeck processes of infinite variation with tempered stable stationary distribution. Thanks to the Markov autoregressive structure, the transition law can be written in the exact sense as a convolution of three random components; a compound Poisson distribution and two independent tempered stable distributions, one with stability index in $(0,1)$ and the other with index in $(1,2)$ .
<p>Note: The following files were submitted by the author for peer review, but cannot be converted to PDF. You must view these files (e.g. movies) online.</p> <p>LSSP-2010-0376.zip</p>	

1  
2  
3  
4  
5  
6  
7  
8  
9  
10  
11  
12  
13  
14  
15  
16  
17  
18  
19  
20  
21  
22  
23  
24  
25  
26  
27  
28  
29  
30  
31  
32  
33  
34  
35  
36  
37  
38  
39  
40  
41  
42  
43  
44  
45  
46  
47  
48  
49  
50  
51  
52  
53  
54  
55  
56  
57  
58  
59  
60

SCHOLARONE™  
Manuscripts

For Peer Review Only

# Infinite Variation Tempered Stable Ornstein-Uhlenbeck Processes with Discrete Observations

REIICHIRO KAWAI\* AND HIROKI MASUDA<sup>†</sup>

## Abstract

We investigate transition law between consecutive observations of Ornstein-Uhlenbeck processes of infinite variation with tempered stable stationary distribution. Thanks to the Markov autoregressive structure, the transition law can be written in the exact sense as a convolution of three random components; a compound Poisson distribution and two independent tempered stable distributions, one with stability index in  $(0, 1)$  and the other with index in  $(1, 2)$ . We discuss simulation techniques for those three random elements. With the exact transition law and proposed simulation techniques, sample paths simulation proves significantly more efficient, relative to the known approximative technique based on infinite shot noise series representation of tempered stable Lévy processes.

*Keywords:* acceptance-rejection sampling, Lévy process, Ornstein-Uhlenbeck processes, selfdecomposability, transition law, tempered stable process.

*2010 Mathematics Subject Classification:* 60J75, 62E15, 65C10, 68U20.

## 1 Introduction

The class of non-Gaussian Ornstein-Uhlenbeck (OU, in short) processes is closely related to the selfdecomposability of the infinitely divisible distribution. Several interesting properties are known, such as the explicit relation between Lévy measures of the stationary distribution and the underlying Lévy process and the representation of entire trajectory based on shot noise series representation of Lévy processes, to mention just a few. (For details, see Section 17 of Sato [15], Masuda [11] and references therein.) Also, due to the growing practical interest, many authors have proposed statistical inference methods for non-Gaussian OU processes. (See, for example, Brockwell et al. [4], Jongbloed et al. [8] and Sun and Zhang [16].)

---

\*Email Address: reiichiro.kawai@gmail.com. Postal Address: Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK.

<sup>†</sup>Email Address: hiroki@math.kyushu-u.ac.jp. Postal Address: Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, Japan.

1  
2 In the class of non-Gaussian OU processes, the class of tempered stable OU (TS-OU, in short) process  
3 of finite variation has been of particular interest from both theoretical and practical points of view. In terms  
4 of mathematical tractability, the transition law between consecutive observations can be written in the exact  
5 sense as a convolution of one compound Poisson and one tempered stable distributions. It is known that  
6 exact, yet simple, simulation methods are available for both random elements. The particular setting of  
7 inverse-Gaussian OU processes was studied in Zhang and Zhang [18], while the general setting in [9]. Also,  
8 it was shown in Zhang and Zhang [19] that the transition law is selfdecomposable when the stability index is  
9 no less than  $1/2$ . In practice, due to its distributional flexibility and the positivity of sample paths, they have  
10 been used in financial economics and mathematical finance (for example, Barndorff-Nielsen and Shephard  
11 [2] and Benth et al. [3]).

12  
13 In this paper, we study the class of TS-OU processes of *infinite variation*, that is, OU processes with  
14 a tempered stable stationary distribution with stability index in  $(1, 2)$ . This can be thought of as a natu-  
15 ral alternative of finite variation TS-OU processes, while the extension is not straightforward. In fact, the  
16 structure of transition law turns out to be significantly different, in the sense that for example, the transition  
17 law is a convolution of two independent tempered stable and one compound Poisson components. Also,  
18 the support of sample paths is necessarily the whole real line, while only the positive half line in the finite  
19 variation setting if no negative jumps exist. We will here only deal with a unilateral setting with no negative  
20 jumps. Nevertheless, the bilateral setting is also within our scope as it can be treated simply by superpo-  
21 sitioning another similar convolution of three independent random components. In addition, the bilateral  
22 framework can produce more distributional flexibility through combinations of positive and negative jump  
23 components in terms of, for example, stability index and even finite and infinite variations. They may widen  
24 the applicability of OU processes in a variety of fields.

25  
26 The rest of this paper is organized as follows. Section 2 summarizes background material on stable and  
27 tempered stable distributions and on OU processes with tempered stable stationary distribution. In Section  
28 3, we derive the transition law in closed form, consisting of three random components; a compound Poisson  
29 distribution and two independent tempered stable distributions, one with stability index  $\alpha \in (1, 2)$ , while the  
30 other with index  $\alpha - 1 \in (0, 1)$ . In Section 4, we discuss simulation methods for the three random elements,  
31 all of which are based on acceptance-rejection sampling techniques. We also provide numerical results to  
32 illustrate the effectiveness of our exact transition law and proposed simulation techniques in sample paths  
33 generation, relative to the existing approximative method with infinite shot noise series representation of  
34 tempered stable Lévy processes. Finally, Section 5 concludes.

## 2 Preliminaries

Let us begin this preliminary section with the notations which will be used throughout the paper. We denote by  $\mathbb{R}$  the one dimensional Euclidean space with the norm  $|\cdot|$  and  $\mathbb{R}_+ := (0, +\infty)$ . Let  $\mathbb{N}$  be the collection of positive integers with  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . We denote by  $\stackrel{\mathcal{L}}{=}$  the identity in law. We denote by  $\Gamma(a, b)$  the gamma distribution with density  $b^a/\Gamma(a)z^{a-1}e^{-bz}$ . We write  $f_L(z)$  for a probability density function of a distribution  $L$ . (For example,  $f_{\Gamma(a,b)}(z) = b^a/\Gamma(a)z^{a-1}e^{-bz}$ .) We fix  $(\Omega, \mathcal{F}, \mathbb{P})$  as our underlying probability space. Finally, note that  $\Gamma(-s) < 0$  for  $s \in (0, 1)$ , while  $\Gamma(-s) > 0$  for  $s \in (1, 2)$ .

### 2.1 Spectrally Positive Stable Processes

Let  $\{L_t^{(s)} : t \geq 0\}$  be a *totally positively skewed* stable (Lévy) process satisfying

$$\begin{aligned} \mathbb{E} \left[ e^{iyL_t^{(s)}} \right] &= \exp \left[ ta\Gamma(-\alpha) \cos \left( \frac{\pi\alpha}{2} \right) |y|^\alpha \left( 1 - i \tan \frac{\pi\alpha}{2} \operatorname{sgn}(y) \right) \right] \\ &= \begin{cases} \exp \left[ t \int_{\mathbb{R}_+} (e^{iyz} - 1) \frac{a}{z^{\alpha+1}} dz \right], & \text{if } \alpha \in (0, 1), \\ \exp \left[ t \int_{\mathbb{R}_+} (e^{iyz} - 1 - iyz) \frac{a}{z^{\alpha+1}} dz \right], & \text{if } \alpha \in (1, 2), \end{cases} \end{aligned} \tag{2.1}$$

with some  $a > 0$ . Throughout this paper, we exclude the case  $\alpha = 1$ . We write  $S(\alpha, a) := \mathcal{L}(L_1^{(s)})$ . The  $C^{+\infty}$ -density of the distribution  $S(\alpha, a)$  is given in the form of convergent power series

$$f_{S(\alpha,b)}(x) = \begin{cases} \frac{(-a\Gamma(-\alpha))^{-1/\alpha}}{\pi} \sum_{k=1}^{+\infty} (-1)^{k-1} \sin(\pi k\alpha) \frac{\Gamma(k\alpha+1)}{k!} \left( \frac{x}{(-a\Gamma(-\alpha))^{1/\alpha}} \right)^{-k\alpha-1}, & \text{if } \alpha \in (0, 1), \\ \frac{(a\Gamma(-\alpha))^{-1/\alpha}}{\pi} \sum_{k=1}^{+\infty} \sin(\pi k \frac{1-\alpha}{\alpha}) \frac{\Gamma(k/\alpha+1)}{k!} \left( -\frac{x}{(a\Gamma(-\alpha))^{1/\alpha}} \right)^{k-1}, & \text{if } \alpha \in (1, 2). \end{cases} \tag{2.2}$$

Note that the above density is defined on  $\mathbb{R}_+$  if  $\alpha \in (0, 1)$ , while on  $\mathbb{R}$  if  $\alpha \in (1, 2)$ . It holds that for each  $t > 0$ ,  $\mathcal{L}(L_t^{(s)}) = S(\alpha, ta)$ , and by the scaling property,  $\mathcal{L}(t^{-1/\alpha}L_t^{(s)}) = S(\alpha, a)$ . Note that the distribution  $S(\alpha, ta)$  has density  $t^{-1/\alpha}f_{S(\alpha,a)}(t^{-1/\alpha}x)$ . The distribution  $S(\alpha, a)$  can be simulated in the exact sense through the well known representation, due to Chambers et al. [5],

$$S(\alpha, a) \stackrel{\mathcal{L}}{=} (-a\Gamma(-\alpha) \cos(\pi\alpha/2))^{1/\alpha} \frac{\sin(\alpha V + \theta)}{(\cos V \cos \theta)^{1/\alpha}} \left( \frac{\cos((1-\alpha)V - \theta)}{E} \right)^{\frac{1-\alpha}{\alpha}}, \tag{2.3}$$

where  $\theta := \arctan(\tan(\pi\alpha/2))$ ,  $V$  is a uniform random variable on  $(-\pi/2, \pi/2)$  and  $E$  is a standard exponential random variable independent of  $V$ . See Zolotarev [20] for more details on the stable distribution.

### 2.2 Spectrally Positive Tempered Stable Processes

Let  $\{L_t^{(ts)} : t \geq 0\}$  be a *centered* and *totally positively skewed* tempered stable (Lévy) process satisfying

$$\mathbb{E} \left[ e^{iyL_t^{(ts)}} \right] = \exp \left[ t \int_{\mathbb{R}_+} (e^{iyz} - 1 - iyz) a \frac{e^{-bz}}{z^{\alpha+1}} dz \right] = \exp \left[ ta\Gamma(-\alpha) \left( (b-iy)^\alpha - b^\alpha + iy\alpha b^{\alpha-1} \right) \right].$$

Note that as indicated by “centered”, the tempered stable process here is centered even in the case  $\alpha \in (0, 1)$ , unlike in the case of the stable process characterized by (2.1). As a matter of course, when  $\alpha \in (0, 1)$ , by adding back the centering term as  $L_t^{(\text{ts})} + t\Gamma(1 - \alpha)ab^{\alpha-1}$ , we can recover the associated tempered stable subordinator. Throughout the paper, we will use the notations

$$TS(\alpha, a, b) := \mathcal{L}(L_1^{(\text{ts})}), \quad (2.4)$$

and

$$TS'(\alpha, a, b) := \mathcal{L}(L_1^{(\text{ts})} + \Gamma(1 - \alpha)ab^{\alpha-1}). \quad (2.5)$$

It is known that

$$\frac{e^{-bz}}{\mathbb{E}[e^{-bL_1^{(\text{ts})}]}] f_{S(\alpha, a)}(z) = e^{-bz - a\Gamma(-\alpha)b^\alpha} f_{S(\alpha, a)}(z) = f_{TS'(\alpha, a, b)}(z). \quad (2.6)$$

The class of tempered stable distributions was first proposed by Tweedie [17]. Several featuring properties of tempered stable distributions and processes were revealed by Rosiński [14], such as a stable-like behavior over short intervals, the absolute continuity with respect to its short-range limiting stable process, an aggregational Gaussianity and an infinite shot noise series representation in closed form

$$\left\{ L_t^{(\text{ts})} : t \in [0, T] \right\} \stackrel{\mathcal{L}}{=} \left\{ \sum_{k=1}^{+\infty} \left[ \left( \frac{\alpha\Gamma_k}{Ta} \right)^{-1/\alpha} \wedge \frac{V_k U_k^{1/\alpha}}{b} \right] \mathbb{1}_{[0, t]}(T_k) - \frac{t}{T} \left( \frac{\alpha k}{Ta} \right)^{-1/\alpha} \mathbb{1}_{(1, 2)}(\alpha) + \frac{t}{T} \left( \frac{Ta}{\alpha} \right)^{1/\alpha} \zeta(1/\alpha) \mathbb{1}_{(1, 2)}(\alpha) - t\Gamma(1 - \alpha)ab^{\alpha-1} : t \in [0, T] \right\}, \quad (2.7)$$

where  $\{\Gamma_k\}_{k \in \mathbb{N}}$  are arrival times of a standard Poisson process,  $\{T_k\}_{k \in \mathbb{N}}$  is a sequence of iid uniform random variables on  $[0, T]$ ,  $\{V_k\}_{k \in \mathbb{N}}$  is a sequence of iid standard exponential random variables and  $\{U_k\}_{k \in \mathbb{N}}$  is a sequence of iid uniform random variables on  $[0, 1]$ . All those random sequences are mutually independent. Note that the kernel of series representation is not unique. In fact, there are different series representations derived in Imai and Kawai [7] through the thinning and rejection methods and yet another representation numerically through the inverse Lévy measure method. (For details about the methods, see Rosiński [13].)

### 2.3 Ornstein-Uhlenbeck Processes with Tempered Stable Stationary Distribution

Consider the stochastic process  $\{X_t : t \geq 0\}$  defined in form of the stochastic differential equation

$$dX_t = \lambda(\mu - X_t)dt + dZ_{\lambda t}, \quad (2.8)$$

where  $\lambda > 0$ ,  $\mu \in \mathbb{R}$  and  $\{Z_t : t \geq 0\}$  is a Lévy process (not necessarily a subordinator), or in canonical form

$$X_t = e^{-\lambda t} X_0 + \mu \left( 1 - e^{-\lambda t} \right) + \int_0^t e^{-\lambda(t-s)} dZ_{\lambda s}. \quad (2.9)$$

Processes of this type are often called non-Gaussian OU processes, or Lévy-driven OU processes. With  $\{Z_t : t \geq 0\}$  being a subordinator, they have been used, for example, to model the squared volatility in a stochastic volatility model of Barndorff-Nielsen and Shephard [2], due to the non-negativity of sample paths.

In this paper, we consider the class of OU processes (2.8) where its invariant law  $\mathcal{L}(\lim_{t \rightarrow +\infty} X_t)$  is  $TS(\alpha, a, b) * \delta_\mu$  with  $\alpha \in (1, 2)$ , where  $*$  denotes the convolution and  $\delta_\mu$  is the degenerate distribution at  $\mu$ . The invariant law is clearly self-decomposable and has Lévy density

$$u(z) = a \frac{e^{-bz}}{z^{\alpha+1}}, \quad z \in \mathbb{R}_+. \quad (2.10)$$

In fact, since the law has finite moments of every order due to the exponential tempering in (2.10), it follows that regardless of the choice of the parameter  $\lambda > 0$ , there exists an ergodic Lévy-driven OU process having  $TS(\alpha, a, b) * \delta_\mu$  as its invariant law. (We refer the reader to Masuda [11] and the references therein for details about Lévy-driven OU processes.) In particular, OU process with inverse Gaussian invariant law ( $\alpha = 1/2$ ) was applied in Benth [3] to stochastic volatility modeling of [2] for volatility and variance swap valuations.

Let  $w(z)$  be the Lévy density of the marginal  $Z_1$  of the background driving Lévy process. Since  $u(z)$  is differentiable, the Lévy densities  $w(z)$  and  $u(z)$  are related by

$$w(z) = -u(z) - z \frac{\partial}{\partial z} u(z) = a\alpha \frac{e^{-bz}}{z^{\alpha+1}} + ab \frac{e^{-bz}}{z^{(\alpha-1)+1}}. \quad (2.11)$$

(See, for example, [2].) Therefore, on the one hand, if  $\alpha \in (0, 1)$ , then the underlying process  $\{Z_t : t \geq 0\}$  is the superposition of a tempered stable process with  $TS'(\alpha, a\alpha, b)$  and a compound Poisson process with Lévy density  $abz^{-\alpha}e^{-bz}$ . Sample paths can be written in the exact sense, using the infinite shot noise series representation (2.7) as

$$\{X_t : t \in [0, T]\} \stackrel{\mathcal{L}}{=} \left\{ e^{-\lambda t} X_0 + \mu \left( 1 - e^{-\lambda t} \right) + \sum_{k=1}^{+\infty} e^{-\lambda(t-T_k)} \left[ \left( \frac{\Gamma_k}{aT} \right)^{-1/\alpha} \wedge \frac{V_k U_k^{1/\alpha}}{b} \right] \mathbb{1}_{[0,t]}(T_k) + \sum_{k=1}^{+\infty} e^{\tilde{\Gamma}_k - \lambda t} G_k \mathbb{1}_{[0,\lambda t]}(\tilde{\Gamma}_k) : t \in [0, T] \right\}, \quad (2.12)$$

where  $\{\tilde{\Gamma}_k\}_{k \in \mathbb{N}}$  are arrival times of a standard Poisson process, independent of  $\{\Gamma_k\}_{k \in \mathbb{N}}$ , with intensity  $\Gamma(1 - \alpha)ab^\alpha (= \int_{\mathbb{R}_+} abz^{-\alpha}e^{-bz}dz)$ , and  $\{G_k\}_{k \in \mathbb{N}}$  is a sequence of iid random variables with gamma distribution  $\Gamma(1 - \alpha, b)$ .

On the other hand, if  $\alpha \in (1, 2)$ , then the equation (2.11) implies that the underlying process  $\{Z_t : t \geq 0\}$  is a superposition of two independent tempered stable processes with  $TS(\alpha, a\alpha, b)$  and  $TS(\alpha - 1, ab, b)$ . Sample paths can also be written in the exact sense with infinite shot noise series representation (2.7). This may however be not very sensible, at least for the following three reasons; (i) there are too many random sequences to be generated, (ii) the series representation for  $TS(\alpha, a\alpha, b)$  contains intricate centering terms as seen in (2.7), and (iii) the issue of truncation error has to be addressed for two infinite shot noise series.



### 3 Transition Law of Tempered Stable Ornstein-Uhlenbeck Processes of Infinite Variation

In this section, we derive the transition law between consecutive observations of discrete time skeleton

$$X_0, X_\Delta, X_{2\Delta}, \dots,$$

of infinite variation TS-OU processes (2.8), with a fixed time stepsize  $\Delta > 0$ . (In principle, the stepsize does not need to be equidistant and can be set different positive values for different steps.) The difference from the finite variation setting [9, 19] lies in the integrability of Lévy density of the transition law around the origin. As a consequence, the Lévy density has to be decomposed twice to extract all infinite activity part, while only once in the finite variation case. For better presentation, we will use the following notations

$$\begin{aligned} w_{1,\Delta}(z) &:= \left(1 - e^{-\alpha\lambda\Delta}\right) \frac{a}{z^{\alpha+1}} e^{-bz}, \\ w_{2,\Delta}(z) &:= e^{-\alpha\lambda\Delta} \frac{a}{z^{\alpha+1}} \left(e^{-bz} - e^{-be^{\lambda\Delta}z}\right), \\ w_{21,\Delta}(z) &:= e^{-\alpha\lambda\Delta} \left(e^{\lambda\Delta} - 1\right) \frac{ab}{z^{(\alpha-1)+1}} e^{-be^{\lambda\Delta}z}, \\ w_{22,\Delta}(z) &:= ae^{-\alpha\lambda\Delta} \frac{e^{-bz} - e^{-be^{\lambda\Delta}z} - b(e^{\lambda\Delta} - 1)ze^{-be^{\lambda\Delta}z}}{z^{\alpha+1}}. \end{aligned}$$

**Theorem 3.1.** Fix  $\Delta > 0$ . For each  $n \in \mathbb{N}_0$ , it holds that given  $X_{n\Delta}$ ,

$$X_{(n+1)\Delta} \stackrel{\mathcal{L}}{=} e^{-\lambda\Delta} X_{n\Delta} + \mu \left(1 - e^{-\lambda\Delta}\right) + Y_{01} + Y_{02} + \left(\sum_{k=1}^{N_{\kappa_\Delta}} \Theta_k - \Delta\gamma\right), \quad (3.1)$$

where

$$\gamma := ab^{\alpha-1} \Gamma(1 - \alpha) \left(e^{-\alpha\lambda\Delta} - e^{-\lambda\Delta} - \left(e^{-\lambda\Delta} - e^{-2\lambda\Delta}\right) (1 - \alpha)\right),$$

and all the random elements are mutually independent and specified as

- $Y_{01} \sim TS(\alpha, a(1 - e^{-\alpha\lambda\Delta}), b)$ ,
- $Y_{02} \sim TS(\alpha - 1, abe^{-\alpha\lambda\Delta}(e^{\lambda\Delta} - 1), be^{\lambda\Delta})$ ,
- $N_{\kappa_\Delta} \sim \text{Pois}(\kappa_\Delta)$  with  $\kappa_\Delta := ab^\alpha \Gamma(-\alpha) (\alpha(1 - e^{-\lambda\Delta}) + e^{-\alpha\lambda\Delta} - 1)$ ,
- $\{\Theta_k\}_{k \in \mathbb{N}}$  is a sequence of iid random variables in  $\mathbb{R}_+$  with common probability density  $v_{1,\Delta}(z) := \kappa_\Delta^{-1} w_{22,\Delta}(z)$ .

Moreover, the transition law is selfdecomposable.

*Proof.* By the homogeneous Markovian autoregressive structure of (2.9), it holds that for each  $n \in \mathbb{N}_0$ , given

$X_{n\Delta}$ ,

$$\begin{aligned} X_{(n+1)\Delta} &= e^{-\lambda\Delta}X_{n\Delta} + \mu \left(1 - e^{-\lambda\Delta}\right) + \int_{n\Delta}^{(n+1)\Delta} e^{-\lambda((n+1)\Delta-s)} dZ_{\lambda,s} \\ &=: e^{-\lambda\Delta}X_{n\Delta} + \mu \left(1 - e^{-\lambda\Delta}\right) + \varepsilon_{\Delta,n+1} \\ &\stackrel{\mathcal{L}}{=} e^{-\lambda\Delta}X_{n\Delta} + \mu \left(1 - e^{-\lambda\Delta}\right) + \int_0^{\lambda\Delta} e^{-\lambda\Delta+s} dZ_s, \end{aligned}$$

where the identity in law holds by the independence and stationarity of increments of the underlying Lévy process  $\{Z_t : t \geq 0\}$ . This implies that  $\{\varepsilon_{\Delta,k}\}_{k \in \mathbb{N}}$  reduces to a sequence of iid random variables with common distribution  $F_\Delta := \mathcal{L}(\int_0^{\lambda\Delta} e^{-\lambda\Delta+s} dZ_s)$ . It thus suffices to investigate the conditional law  $\mathcal{L}(X_\Delta|X_0)$ , that is, only of the first increment. Note that by definition, this law is infinitely divisible.

Using the Lévy-integral transform, we get the characteristic function of the distribution  $F_\Delta$  as

$$\widehat{F}_\Delta(y) = \exp \left[ \int_{\mathbb{R}_+} (e^{iyz} - 1 - iyz) w_\Delta(z) dz \right], \quad (3.2)$$

where

$$w_\Delta(z) := \int_0^{\lambda\Delta} e^s w(e^s z) ds.$$

By further computing  $w_\Delta(z)$ , we get

$$\begin{aligned} w_\Delta(z) &= \frac{a}{z^{\alpha+1}} \int_0^{\lambda\Delta} (\alpha + be^s z) e^{-\alpha s} e^{-be^s z} ds \\ &= -\frac{a}{z^{\alpha+1}} \int_0^{\lambda\Delta} \frac{\partial}{\partial s} \left( e^{-\alpha s} e^{-be^s z} \right) ds \\ &= \frac{a}{z^{\alpha+1}} \left( e^{-bz} - e^{-\alpha\lambda\Delta - be^{\lambda\Delta} z} \right) \\ &= w_{1,\Delta}(z) + w_{2,\Delta}(z). \end{aligned}$$

Note that  $w_{1,\Delta}$  and  $w_{2,\Delta}$  are positive functions on  $\mathbb{R}_+$ . Clearly,  $w_{1,\Delta}$  is the smooth Lévy density of  $TS(\alpha, (1 - e^{-\alpha\lambda\Delta})a, b)$ . Note that  $w_{2,\Delta}(z) \sim abe^{-\alpha\lambda\Delta}(e^{\lambda\Delta} - 1)z^{-\alpha}$  as  $z \downarrow 0$ . This implies that  $w_{2,\Delta}$  is not integrable and thus cannot be a Lévy density of compound Poisson components.

Let us further decompose  $w_{2,\Delta}$  into two parts. We use the identity

$$e^{-x} - e^{-y} = (y-x)e^{-y} + e^{-x}(y-x)^2 H(y-x), \quad y > x > 0. \quad (3.3)$$

where  $H(z) := z^{-2}(1 - e^{-z}(1+z))$  is positive, bounded and strictly decreasing on  $\mathbb{R}_+$  such that  $\lim_{z \downarrow 0} H(z) = 1/2$ . By applying this identity, we get

$$\begin{aligned} w_{2,\Delta}(z) &= e^{-\alpha\lambda\Delta} \left( e^{\lambda\Delta} - 1 \right) \frac{ab}{z^{(\alpha-1)+1}} e^{-be^{\lambda\Delta} z} \\ &\quad + ab^2 \left( e^{\lambda\Delta} - 1 \right)^2 e^{-\alpha\lambda\Delta} z^{(2-\alpha)-1} e^{-bz} H \left( b \left( e^{\lambda\Delta} - 1 \right) z \right) \\ &= w_{21,\Delta}(z) + w_{22,\Delta}(z). \end{aligned}$$

Clearly,  $w_{21,\Delta}$  is the smooth Lévy density of  $TS(\alpha - 1, abe^{-\alpha\lambda\Delta}(e^{\lambda\Delta} - 1), be^{\lambda\Delta})$ .

Next, observe that

$$\int_{\mathbb{R}_+} w_{22,\Delta}(z) dz = ae^{-\alpha\lambda\Delta} \int_{\mathbb{R}_+} z^{-1-\alpha} \left( e^{-bz} - e^{-be^{\lambda\Delta}z} - b(e^{\lambda\Delta} - 1)ze^{-be^{\lambda\Delta}z} \right) dz = \kappa_\Delta.$$

This shows that  $w_{22,\Delta}$  serves as a Lévy density of compound Poisson components. To realize the centering for the compound Poisson distribution due to (3.2), we need to subtract the constant term  $\int_{\mathbb{R}_+} zw_{22,\Delta}(z) dz = \gamma$ , multiplied by the time stepsize  $\Delta$ .

It remains to show the selfdecomposability of the transition law. Define

$$h_\Delta(z) := \frac{1}{z^\alpha} \int_0^{\lambda\Delta} (a + be^s z) e^{-\alpha s - be^s z} ds, \quad z \in \mathbb{R}_+,$$

so that  $w_\Delta(z) = ah_\Delta(z)/z$ . This function is obviously non-negative and is decreasing on  $\mathbb{R}_+$ , due to

$$\begin{aligned} \frac{d}{dz} h_\Delta(z) &= \int_0^{\lambda\Delta} z^{-1-\alpha} e^{-\alpha s - be^s z} \left( -(be^s)^2 z^2 - (2\alpha - 1)be^s z - \alpha^2 \right) ds \\ &\leq -\frac{\alpha^2}{z^{1+\alpha}} \int_0^{\lambda\Delta} e^{-\alpha s - be^s z} ds < 0. \end{aligned} \quad (3.4)$$

Hence, Corollary 15.11 of Sato [15] yields the claim. The proof is complete.  $\square$

**Remark 3.2.** It is worth noting that the selfdecomposability of the transition law holds for any  $\alpha \in (1, 2)$  in the infinite variation setting, while in the finite variation case, it holds only for the stability index of no less than  $1/2$ . This difference occurs due to the term  $2\alpha - 1$  in the integrand of (3.4), that is, the sign of  $2\alpha - 1$  changes at  $\alpha = 1/2$ . (See Zhang and Zhang [19] for the finite variation case.)  $\square$

**Remark 3.3.** It is difficult to provide an efficient simulation method for the distribution induced by the Lévy density  $w_{2,\Delta}$ . Nevertheless, in the finite variation setting, that is, if  $\alpha \in (0, 1)$ , then  $w_{2,\Delta}$  is integrable

$$\int_{\mathbb{R}_+} w_{2,\Delta}(z) dz = -ab^\alpha \Gamma(-\alpha) \left( 1 - e^{-\alpha\lambda\Delta} \right) \in \mathbb{R}_+. \quad (3.5)$$

This shows that  $w_{2,\Delta}$  acts as the smooth Lévy density of a compound Poisson distribution. Moreover, there is no further need to decompose  $w_{2,\Delta}$ , unlike we did in Theorem 3.1, since the corresponding Poisson distribution can be simulated in the exact sense through an acceptance-rejection sampling method, which will be presented in Section 4.1.  $\square$

## 4 Simulation Methods

It follows from Theorem 3.1 that to simulate sample paths of infinite variation TS-OU processes at discrete timings, it suffices to simulate three random elements, that is, the tempered stable random variables  $Y_{01}$  and  $Y_{02}$ , and the compound Poisson random variable  $\sum_{k=1}^{N_{\kappa_\Delta}} \Theta_k$ . In this section, we discuss simulation methods for those random elements. We begin with relatively straightforward cases of  $Y_{02}$  and of the compound Poisson  $\sum_{k=1}^{N_{\kappa_\Delta}} \Theta_k$ .

### 4.1 Simulation of Tempered Stable with Stability Index $\alpha - 1$

First, we present an exact simulation method for  $Y_{02} \sim TS(\alpha - 1, abe^{-\alpha\lambda\Delta}(e^{\lambda\Delta} - 1), be^{\lambda\Delta})$ , which is centered. Hence, we will first instead simulate  $TS'(\alpha - 1, abe^{-\alpha\lambda\Delta}(e^{\lambda\Delta} - 1), be^{\lambda\Delta})$ , that is,

$$Y'_{02} := Y_{02} + \Gamma(2 - \alpha)ab^\alpha \left(1 - e^{-\lambda\Delta}\right),$$

which takes values solely in  $\mathbb{R}_+$  and then subtract the added constant term. An efficient and exact simulation method for the case  $\alpha - 1 = 0.5$ , that is the inverse Gaussian, is well known due to Michael et al. [12]. For the general case of  $\alpha - 1 \in (0, 1)$ , the best route would be acceptance-rejection sampling based on the representation (2.3) of the stable distribution and the likelihood ratio of the two densities; for each  $z \in \mathbb{R}_+$ ,

$$\frac{f_{TS'(\alpha-1, abe^{-\alpha\lambda\Delta}(e^{\lambda\Delta}-1), be^{\lambda\Delta})}(z)}{f_{S(\alpha-1, abe^{-\alpha\lambda\Delta}(e^{\lambda\Delta}-1))}(z)} = e^{-be^{\lambda\Delta}z - \Gamma(1-\alpha)ab^{\alpha+1}(e^{\lambda\Delta}-1)} \leq e^{-\Gamma(1-\alpha)ab^{\alpha+1}(e^{\lambda\Delta}-1)}, \quad (4.1)$$

where the density functions  $f_{S(\alpha,a)}$  and  $f_{TS'(\alpha,a,b)}$  are given respectively by (2.2) and (2.6). The random variable  $Y_{02}$  can then be generated in the exact sense by the following simple acceptance-rejection sampling algorithm.

**Algorithm 1** ( $Y_{02} \sim TS(\alpha - 1, abe^{-\alpha\lambda\Delta}(e^{\lambda\Delta} - 1), be^{\lambda\Delta})$ ):

Step 1. Generate  $U$  as uniform  $(0, 1)$  and  $V$ , independent of  $U$ , as  $S(\alpha - 1, abe^{-\alpha\lambda\Delta}(e^{\lambda\Delta} - 1))$  through the representation (2.3).

Step 2. If  $U \leq e^{-be^{\lambda\Delta}V}$ , let  $Y_{02} \leftarrow V - \Gamma(2 - \alpha)ab^\alpha(1 - e^{-\lambda\Delta})$ . Otherwise, return to Step 1.

The acceptance rate at Step 2 of Algorithm 1 is given by

$$p_1(\Delta) := \mathbb{P}\left(U \leq e^{-be^{\lambda\Delta}V}\right) = e^{\Gamma(1-\alpha)ab^{\alpha+1}(e^{\lambda\Delta}-1)}.$$

Clearly, the algorithm works more efficiently when the acceptance rate  $p_1(\Delta)$  at Step 2 is closer to 1. Indeed, this happens when  $\Delta \downarrow 0$ .

It is, however, more practical to discuss the effectiveness on the *work-normalized* basis. Since the simulation of  $\mathcal{L}(Y_{02})$  is exact through Algorithm 1, all we need to pay attention to is the computing time required to generate iid increments from  $TS(\alpha - 1, abe^{-\alpha\lambda\Delta}(e^{\lambda\Delta} - 1), be^{\lambda\Delta})$  to fill each sample path. Since we are concerned with sample paths over a finite time horizon, by taking a smaller time stepsize  $\Delta$ , the number of increments for each sample path increases in proportion to  $1/\Delta$ . Next, in Algorithm 1, the number of trials until one acceptance has the geometric distribution with success probability  $p_1(\Delta)$ . The average time to get one sample from Algorithm 1 is thus proportional to  $1/p_1(\Delta)$ . Then, we find that as  $\Delta \downarrow 0$ ,

$$\frac{1}{\Delta \cdot p_1(\Delta)} \simeq \frac{1}{\Delta},$$

which implies that the average computing time related to  $Y_{02}$  for each sample path increases in proportion to  $1/\Delta$  as  $\Delta \downarrow 0$ .

For more details about related acceptance-rejection sampling methods, see Baeumer and Meerschaert [1], Devroye [6] and Kawai and Masuda [9].

## 4.2 Simulation of Compound Poisson Component

Next, we consider simulation of the compound Poisson component. Generation of the Poisson random variable  $N_{\kappa_\Delta}$  is straightforward and is thus omitted. We concentrate on generation of the random sequence  $\{\Theta_k\}_{k \in \mathbb{N}}$ . Recall that  $\mathcal{L}(\Theta_1)$  has a probability density function  $v_{1,\Delta}(z)$  and that the function  $H$  in (3.3) is positive, bounded and strictly decreasing on  $\mathbb{R}_+$  with  $\lim_{z \downarrow 0} H(z) = 1/2$ . We can thus show that

$$v_{1,\Delta}(z) \leq \frac{1}{2\kappa_\Delta} ab^\alpha \left(e^{\lambda\Delta} - 1\right)^2 e^{-\alpha\lambda\Delta} \Gamma(2-\alpha) f_{\Gamma(2-\alpha,b)}(z) =: g_\Delta(z), \quad (4.2)$$

where  $f_{\Gamma(2-\alpha,b)}(z) = b^{2-\alpha} \Gamma(2-\alpha)^{-1} z^{(2-\alpha)-1} e^{-bz}$  defined on  $\mathbb{R}_+$ . Then, it holds that

$$\frac{v_{1,\Delta}(z)}{g_\Delta(z)} = \frac{2}{(e^{\lambda\Delta} - 1)^2 b^2} \frac{e^{-bz} - e^{-be^{\lambda\Delta}z} - b(e^{\lambda\Delta} - 1)ze^{-be^{\lambda\Delta}z}}{z^2} =: v_{2,\Delta}(z), \quad z \in \mathbb{R}_+.$$

This suggests the following acceptance-rejection sampling algorithm for generation of the random variable  $\Theta_1$ .

**Algorithm 2** ( $\Theta_1$  with probability density  $v_{1,\Delta}(z)$ )

Step 1: Generate  $U$  as uniform  $(0, 1)$  and  $V$ , independent of  $U$ , as  $\Gamma(2-\alpha, b)$ .

Step 2: If  $U \leq v_{2,\Delta}(V)$ , let  $\Theta_1 \leftarrow V$ . Otherwise, return to Step 1.

The acceptance rate at Step 2 is given by

$$p_2(\Delta) := \mathbb{P}(U \leq v_{2,\Delta}(V)) = \frac{2}{\alpha(\alpha-1)} \frac{\alpha(1 - e^{-\lambda\Delta}) + e^{-\alpha\lambda\Delta} - 1}{e^{-\alpha\lambda\Delta}(e^{\lambda\Delta} - 1)^2}.$$

We can show that the acceptance rate tends to 1 as  $\Delta \downarrow 0$ .

Since the simulation of  $\{\Theta_k\}_{k \in \mathbb{N}}$  is exact by Algorithm 2 as well, all we need to pay attention to is the computing time required to generate iid increments from the distribution  $\mathcal{L}(\sum_{k=1}^{N_{\kappa_\Delta}} \Theta_k)$  to fill each sample path. Again, by taking a smaller time stepsize  $\Delta$ , the number of increments over a finite time horizon increases in proportion to  $1/\Delta$ . Next, the average number of implementation of Algorithm 2 required to generate one sample from the distribution  $\mathcal{L}(\sum_{k=1}^{N_{\kappa_\Delta}} \Theta_k)$  is proportional to the intensity  $\kappa_\Delta$  of the Poisson random variable  $N_{\kappa_\Delta}$ . The average time to get one sample from Algorithm 2 is proportional to  $1/p_2(\Delta)$ . Therefore, we get

$$\frac{\kappa_\Delta}{\Delta \cdot p_2(\Delta)} = \frac{ab^\alpha \Gamma(-\alpha) \alpha(\alpha-1) e^{-\alpha\lambda\Delta} (e^{\lambda\Delta} - 1)^2}{2\Delta} \simeq \Delta,$$

as  $\Delta \downarrow 0$ . This implies that the computing time for simulation of  $\mathcal{L}(\sum_{k=1}^{N_{\kappa_\Delta}} \Theta_k)$  decreases linearly in  $\Delta$ , as  $\Delta \downarrow 0$ .

### 4.3 Simulation of Tempered Stable with Stability Index $\alpha$

We have seen that the two distributions  $\mathcal{L}(Y_{02})$  and  $\mathcal{L}(\sum_{k=1}^{N_{k\Delta}} \Theta_k)$  can be simulated in the exact sense through acceptance-rejection sampling. To the best of our knowledge, the exact simulation of  $Y_{01} \sim TS(\alpha, a(1 - e^{-\alpha\lambda\Delta}), b)$  seems impossible in practice. For example, the series representation (2.7) looks like an exact method, while it is still approximative as soon as a finite truncation is performed. In this paper, amongst several possible approximative methods, we present a method proposed by Baeumer and Meerschaert [1], which seems to be most suitable for our purpose. (See [10] for a comparison among various approximative simulation methods.) Its procedure is outlined as follows.

**Algorithm 3** (Approximation of  $Y_{01} \sim TS(\alpha, a(1 - e^{-\alpha\lambda\Delta}), b)$ ):

Fix  $c > 0$ .

Step 1. Generate  $U$  as uniform  $(0, 1)$  and  $V(\Delta)$ , independent of  $U$ , as  $S(\alpha, a(1 - e^{-\alpha\lambda\Delta}))$ .

Step 2. If  $U \leq e^{-b(V(\Delta)+c)}$ , let  $Y'_{01,c} \leftarrow V(\Delta)$ . Otherwise, return to Step 1.

Step 3. Return  $Y_{01,c} \leftarrow Y'_{01,c} - \Gamma(1 - \alpha)a(1 - e^{-\alpha\lambda\Delta})b^{\alpha-1}$ .

Note that the random variable  $Y'_{01,c}$  in Step 2 approximates  $TS'(\alpha, a(1 - e^{-\alpha\lambda\Delta}), b)$ , while  $Y_{01,c}$  approximates  $TS(\alpha, a(1 - e^{-\alpha\lambda\Delta}), b)$ . In fact, the constant shift  $-\Gamma(1 - \alpha)a(1 - e^{-\alpha\lambda\Delta})b^{\alpha-1}$  in Step 3 accounts for the difference between (2.4) and (2.5).

Let us briefly review basic properties of Algorithm 3 derived in [1]. The acceptance rate at Step 2 of Algorithm 3 is

$$p_3(\Delta, c) := \mathbb{E} \left[ e^{-b(V(\Delta)+c)} \mid V(\Delta) > -c \right] \mathbb{P}(V(\Delta) > -c) + \mathbb{P}(V(\Delta) \leq -c).$$

Moreover, we have

$$\mathbb{P}(Y'_{01,c} \leq z) = \frac{1}{p_3(\Delta, c)} \left( \mathbb{P}(V(\Delta) \leq \min(z, -c)) + \int_{\min(z, -c)}^z e^{-b(y+c)} f_{S(\alpha, a(1 - e^{-\alpha\lambda\Delta}))}(y) dy \right),$$

$$f_{\mathcal{L}(Y'_{01,c})}(z) = \begin{cases} p_3(\Delta, c)^{-1} f_{S(\alpha, a(1 - e^{-\alpha\lambda\Delta}))}(z), & \text{if } z \in (-\infty, -c], \\ p_3(\Delta, c)^{-1} e^{-b(z+c)} f_{S(\alpha, a(1 - e^{-\alpha\lambda\Delta}))}(z), & \text{if } z \in (-c, +\infty). \end{cases}$$

In view of the expression

$$f_{TS'(\alpha, a(1 - e^{-\alpha\lambda\Delta}), b)}(z) = e^{-b(z+a(1 - e^{-\alpha\lambda\Delta})\Gamma(-\alpha)b^{\alpha-1})} f_{S(\alpha, a(1 - e^{-\alpha\lambda\Delta}))}(z), \quad z \in \mathbb{R},$$

the parameter  $c$  in Algorithm 3 acts as a truncation of the entire real line  $\mathbb{R}$  to the domain on which the exponential tempering  $e^{-bz}$  is performed. To sum up, we get

$$f_{\mathcal{L}(Y_{01,c})}(z) = f_{\mathcal{L}(Y'_{01,c})}(z + \Gamma(1 - \alpha)a(1 - e^{-\alpha\lambda\Delta})b^{\alpha-1}), \quad z \in \mathbb{R}.$$

It is also proved in Theorem 8 [1] that the density  $f_{\mathcal{L}(Y_{01,c})}$  converges in  $L^1(\mathbb{R})$  to the true density  $f_{TS(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)}$  as  $c \uparrow +\infty$ , and consequently the Kolmogorov-Smirnov distance  $D_{KS}(\Delta, c)$  between two distributions  $\mathcal{L}(Y_{01,c})$  and  $TS(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)$  converges to zero as well, where

$$\begin{aligned} D_{KS}(\Delta, c) &:= \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^x \left( f_{\mathcal{L}(Y_{01,c})}(z) - f_{TS(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)}(z) \right) dz \right| \\ &= \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^x \left( f_{\mathcal{L}(Y'_{01,c})}(z) - f_{TS'(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)}(z) \right) dz \right| \\ &= \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^x \left( \frac{1 \wedge e^{-b(z+c)}}{p_3(\Delta, c)} - e^{-b(z+a(1-e^{-\alpha\lambda\Delta})\Gamma(-\alpha)b^{\alpha-1})} \right) f_{S(\alpha, a(1-e^{-\alpha\lambda\Delta}))}(z) dz \right|. \end{aligned}$$

Nevertheless, it is not sensible to simply look for a smaller distribution error by taking  $c \uparrow +\infty$ , since then the consequent low acceptance rate makes Algorithm 3 extremely inefficient, due to  $\lim_{c \uparrow +\infty} p_3(\Delta, c) = 0$  for each  $\Delta > 0$ . Meanwhile, it holds that for each  $c > 0$ ,  $\lim_{\Delta \downarrow 0} p_3(\Delta, c) = e^{-bc}$ . Thus, there exists the issue of trade-off between the distribution error and the computing effort in terms of  $\Delta$  and  $c$ . Concerning the computing effort, on the one hand, we wish to find  $(\Delta, c)$  minimizing  $1/(\Delta \cdot p_3(\Delta, c))$ , just as in the previous two subsections. It should be noted that asymptotic behaviors of  $p_3(\Delta, c)$  in  $\Delta$  and  $c$  are difficult to analyze. Next, on the other hand, there exist several appropriate criteria to measure the distribution error. Natural candidates include  $L^1(\mathbb{R})$ ,  $L^2(\mathbb{R})$ -distances between densities  $f_{\mathcal{L}(Y_{01,c})}$  and  $f_{TS(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)}$ , while the Kolmogorov-Smirnov distance  $D_{KS}(\Delta, c)$  above is certainly valid. None of them are tractable in an explicit manner. For illustrative purpose, we present numerical results in Table 1 with parameter set  $(\alpha, a, b, \lambda) = (1.8, 1.0, 1.0, 0.2)$  and  $\Delta = 0.1$ .

$c$	0.0	0.3	0.6	0.8	1.0	1.2	1.4	1.6
$D_{KS}(\Delta, c)$	1.10E-1	6.35E-2	2.14E-2	7.00E-3	1.63E-3	2.69E-4	3.13E-5	2.51E-6
$p_3(\Delta, c)$	0.875	0.753	0.599	0.499	0.411	0.337	0.276	0.226

Table 1: Numerical results of distribution error and acceptance rate for different truncation points  $c$ .

We draw in Figure 1 the resulting density functions  $f_{\mathcal{L}(Y_{01,c})}$  with a few different choices of  $c$ , together with the true tempered stable density function  $f_{TS(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)}$ . For clear comparison, we also provide vertical lines  $x = -c - \Gamma(1-\alpha)a(1-e^{-\alpha\lambda\Delta})b^{\alpha-1}$ . (We need the constant shift here because the truncation  $c$  is performed on the distribution  $TS'(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)$ , rather than on  $TS(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)$ .) Observe that two densities are sufficiently close when  $c = 1.4$ . It is not quite worth setting  $c > 1.4$  in this specific example, as the acceptance rate decreases still steadily. In principle, one would expect a small  $b$  in any application, since otherwise an exponentially tailed distribution would be chosen instead. This fact favors a relatively large acceptance rate  $p_3(\Delta, c)$ .

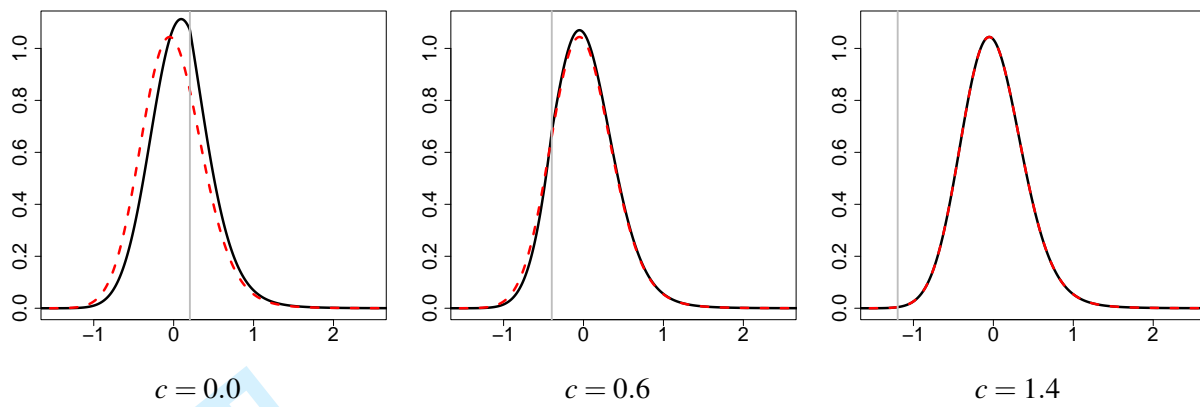


Figure 1: Comparison of two density functions  $f_{\mathcal{L}(Y_{01},c)}$  (solid) and  $f_{TS(\alpha,a(1-e^{-\alpha\lambda\Delta}),b)}$  (dotted) under  $(\alpha, a, b, \lambda) = (1.8, 1.0, 1.0, 0.2)$ . The horizontal line indicates  $x = -c - \Gamma(1 - \alpha)a(1 - e^{-\alpha\lambda\Delta})b^{\alpha-1}$ .

#### 4.4 Sample Paths

We provide in Figure 2 typical sample paths of TS-OU processes of infinite variation with discrete observations, based on the transition law we have obtained in Theorem 3.1 and the simulation methods described in Algorithm 1, 2 and 3. The model parameters are set  $\lambda = 0.2$ ,  $\mu = 0$ ,  $a = b = 1$  and  $\alpha = 1.2, 1.5$  and  $1.8$ . For simplicity, we set the initial state  $X_0 = 0$ , that is the mean of the stationary distribution  $TS(\alpha, a, b) * \delta_\mu$ . Sample paths are generated over the time interval either  $[0, 100]$  or  $[0, 200]$ , where the time stepsize is kept  $\Delta = 0.1$  all over in common. This means that 1000 and 2000 recursive increments are needed, respectively, for the intervals  $[0, 100]$  and  $[0, 200]$ .

In our parameter setting, acceptance rates in Algorithm 1 (to generate one sample of  $Y_{02}$ ) are 0.889, 0.931 and 0.891, respectively, while in Algorithm 2 (to generate one sample of  $\Theta_1$ ), acceptance rates are, respectively, 0.990, 0.993 and 0.997. In Algorithm 3 (to generate one sample of  $Y_{01}$ ), we have chosen the truncation point  $c = 0.3, 0.6$  and  $1.4$ , respectively, for  $\alpha = 1.2, 1.5$  and  $1.8$ , on the basis of the criterion  $D_{KS}(\Delta, c)/(\Delta \cdot p_3(\Delta, c))$ , as discussed in Section 4.3. With the choice of truncation point  $c$ , acceptance rates in Algorithm 3 are 0.831, 0.588 and 0.276. Even in the case  $\alpha = 1.8$  with the lowest acceptance rate 0.276 in Algorithm 3, each sample path can be generated within 0.1 second by Scilab software on a computer with recent regular spec. (The computing time can easily be reduced by employing a low-level language, such as C.)

## 5 Concluding Remarks

We have derived exact transition law between consecutive observations of TS-OU processes of infinite variation as a convolution of three random components; a compound Poisson distribution and two tempered



1 stable distributions, one with stability index in  $(0, 1)$  and the other with index in  $(1, 2)$ . We have adopted  
2 acceptance-rejection sampling techniques to simulate exactly the compound Poisson component and the  
3 tempered stable distribution with index in  $(0, 1)$ . For simulation of the tempered stable distribution with  
4 index in  $(1, 2)$ , we have presented an approximative acceptance-rejection sampling method of [1] with dis-  
5 cussion on the issue of trade-off between distribution error and computing time. Sample paths simulation is  
6 significantly more efficient with our explicit transition law and simulation techniques, relative to the known  
7 approximative method based on infinite shot noise series representation of tempered stable Lévy processes.  
8

9  
10  
11  
12  
13 As mentioned in Section 4.3, we could think of several approximative simulation techniques for the  
14 tempered stable distribution with stability index in  $(1, 2)$ . Those techniques are investigated in [10]. Also,  
15 with the explicit transition density functions of TS-OU processes, it is certainly worthwhile to investigate  
16 related statistical issues, such as the local asymptotic behavior of the likelihood ratio statistics, efficient  
17 parameter estimation, and so on. These topics will be investigated in subsequent papers.  
18  
19  
20  
21  
22

## 23 Acknowledgements

24  
25  
26 The authors would like to appreciate an anonymous referee for valuable suggestions on the computation  
27 part.  
28  
29

## 30 References

- 31  
32  
33 [1] Baeumer, B., Meerschaert, M.M. (2010) Tempered stable Lévy motion and transit super-diffusion, *Journal of Computational*  
34 *and Applied Mathematics*, **233**(10) 2438-2448.  
35  
36 [2] Barndorff-Nielsen, O.E., Shephard, N. (2001) Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in  
37 financial economics (with discussion), *J. R. Statist. Soc. B*, **63**(2) 167-241.  
38  
39 [3] Benth, F.E., Gorth, M., Kufakunesu, R. (2007) Valuing volatility and variance swaps for a non-Gaussian Ornstein-Uhlenbeck  
40 stochastic volatility model, *Applied Mathematical Finance*, **14**(4) 347-363.  
41  
42 [4] Brockwell, P.J., Davis, R.A., Yang, Y. (2007) Estimation for nonnegative Lévy-driven Ornstein-Uhlenbeck processes, *Jour-*  
43 *nal of Applied Probability*, **44**(4) 977-989.  
44  
45 [5] Chambers, J.M., Mallows, C.L., Stuck, B.W. (1976) A method for simulating stable random variables, *Journal of the Amer-*  
46 *ican Statistical Association*, **71**(354) 340-344.  
47  
48 [6] Devroye, L. (2009) Random variate generation for exponential and polynomially tilted stable distributions, *ACM Transac-*  
49 *tions on Modeling and Computer Simulation*, **19**(4) Article No. 18.  
50  
51 [7] Imai, J., Kawai, R., On finite truncation of infinite shot noise series representation of tempered stable laws, preprint.  
52  
53 [8] Jongbloed, G., van der Meulen, F.H., van der Vaart, A.W. (2005) Nonparametric inference for Lévy-driven Ornstein-  
54 Uhlenbeck processes, *Bernoulli*, **11**(5) 759-791.  
55  
56 [9] Kawai, R., Masuda, H., Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes, available at  
57 <http://hdl.handle.net/2324/15635>  
58  
59  
60

- 1  
2 [10] Kawai, R., Masuda, H. (2011) On simulation of tempered stable random variates, *Journal of Computational and Applied*  
3 *Mathematics*, **235**(8) 2873-2887.  
4  
5 [11] Masuda, H. (2004) On multidimensional Ornstein-Uhlenbeck processes driven by a general Lévy process, *Bernoulli*, **10**(1)  
6 1-24.  
7  
8 [12] Michael, J.R., Schucany, W.R., Haas, R.W. (1976) Generating random variates using transformations with multiple roots,  
9 *The American Statistician*, **30**, 88-90.  
10  
11 [13] Rosiński, J. (2001) Series representations of Lévy processes from the perspective of point processes, In: *Lévy Processes -*  
12 *Theory and Applications*, Eds. Barndorff-Nielsen, O.-E., Mikosch, T., Resnick, S.I., Birkhäuser, 401-415.  
13  
14 [14] Rosiński, J. (2007) Tempering stable processes, *Stochastic Processes and their Applications*, **117**(6) 677-707.  
15  
16 [15] Sato, K. (1999) *Lévy processes and infinitely divisible distributions*, Cambridge University Press.  
17  
18 [16] Sun, S., Zhang, X. (2009) Empirical likelihood estimation of discretely sampled processes of OU type, *Science in China*  
19 *Series A: Mathematics*, **52**(5) 908-931.  
20  
21 [17] Tweedie, M.C.K. (1984) An index which distinguishes between some important exponential families, In: *Statistics: Appli-*  
22 *cations and New Directions: Proc. Indian Statistical Institute Golden Jubilee International Conference* (eds. J. Ghosh and J.  
23 Roy) 579-604.  
24  
25 [18] Zhang, S., Zhang, X. (2008) Exact simulation of IG-OU processes, *Methodology and Computing in Applied Probability*,  
26 **10**(3) 337-355.  
27  
28 [19] Zhang, S., Zhang, X. (2009) On the transition law of tempered stable Ornstein-Uhlenbeck processes, *Journal of Applied*  
29 *Probability*, **46**(3) 721-731.  
30  
31 [20] Zolotarev, V.M. (1986) *One-Dimensional Stable Distributions*, American Mathematical Society, Providence, RI.  
32  
33  
34  
35  
36  
37  
38  
39  
40  
41  
42  
43  
44  
45  
46  
47  
48  
49  
50  
51  
52  
53  
54  
55  
56  
57  
58  
59  
60

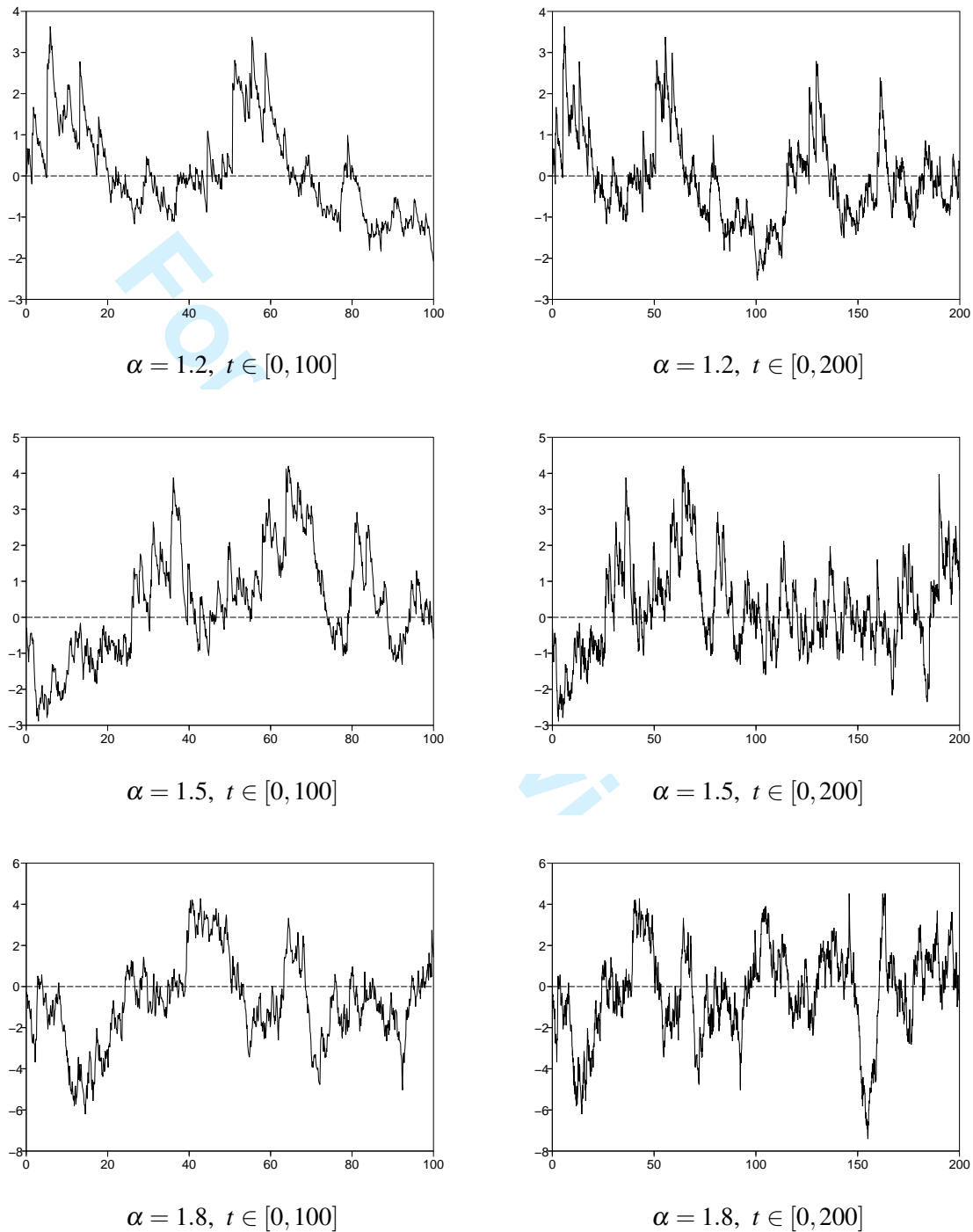


Figure 2: Typical sample paths of tempered stable Ornstein-Uhlenbeck processes through exact simulation algorithm. The horizontal dotted lines indicate  $X_0 = 0 (= \lim_{t \uparrow +\infty} \mathbb{E}[X_t])$ .

1  
2  
3  
4  
5  
6  
7  
8  
9  
10  
11  
12  
13  
14  
15  
16  
17  
18  
19  
20  
21  
22  
23  
24  
25  
26  
27  
28  
29  
30  
31  
32  
33  
34  
35  
36  
37  
38  
39  
40  
41  
42  
43  
44  
45  
46  
47  
48  
49  
50  
51  
52  
53  
54  
55  
56  
57  
58  
59  
60

# Response to Referee Comments on “Infinite Variation Tempered Stable Ornstein-Uhlenbeck Processes with Discrete Observations”

Reiichiro Kawai and Hiroki Masuda

Dear Prof. Balakrishnan and the anonymous reviewer,

Please find enclosed the revised version of the manuscript entitled “Infinite Variation Tempered Stable Ornstein-Uhlenbeck Processes with Discrete Observations” written by R. Kawai and H. Masuda (LSSP-2010-0376.R2) that we submitted to *Communications in Statistics - Simulation and Computation*. In preparing the revision, we have tried to take care of all the points raised in the referee reports. Let us address the issues related to the referee comments.

We admit that our computation is not as robust at a very high precision level as the referee’s. Indeed, by expanding the integration region and by refining the discretization further, we obtained the presented values for  $c = 1.4$  and  $c = 1.6$ . With these values, our discussion on this part breaks down. As the referee pointed out, finding a minimal  $c$  (and numerical results for  $c > 1.4$  as well) are not directly relevant to the main scope of our manuscript. Hence, following the referee’s recommendation, we have decided to delete the part about finding an optimal threshold and have amended the section accordingly. We appreciate the referee for a careful examination and valuable suggestions on this part.

Finally, we denote by  $\mathbb{E}[e^{-b(V(\Delta)+c)}; V(\Delta) > -c]$  the expectation of  $e^{-b(V(\Delta)+c)}$  over the event  $\{V(\Delta) > -c\}$ , rather than a conditional expectation. This notation however might not be completely clear to every reader, since the referee has been confused with

$$\mathbb{E} \left[ e^{-b(V(\Delta)+c)}; V(\Delta) > -c \right] = \mathbb{P}(V(\Delta) > -c) \mathbb{E} \left[ e^{-b(V(\Delta)+c)} \mid V(\Delta) > -c \right].$$

To avoid unnecessary confusion, we have made a modification as above.